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μ-CONSTANT MONODROMY GROUPS AND MARKED SINGULARITIES

by Claus HERTLING (*)

ABSTRACT. — μ-constant families of holomorphic function germs with isolated singularities are considered from a global perspective. First, a monodromy group from all families which contain a fixed singularity is studied. It consists of automorphisms of the Milnor lattice which respect not only the intersection form, but also the Seifert form and the monodromy. We conjecture that it contains all such automorphisms, modulo $\pm$ id. Second, marked singularities are defined and global moduli spaces for right equivalence classes of them are established. The conjecture on the group would imply that these moduli spaces are connected. The relation with Torelli type problems is discussed and a new global Torelli type conjecture for marked singularities is formulated. All conjectures are proved for the simple and 22 of the 28 exceptional singularities.

RÉSUMÉ. — Nous considérons d'un point de vue global les familles μ-constantes de germes de fonctions holomorphes à singularités isolées. Tout d'abord, nous étudions un groupe de monodromie des familles contenant une singularité fixée. Ce groupe est constitué d'automorphismes du réseau de Milnor qui respectent non seulement la forme d'intersection, mais aussi la forme de Seifert et la monodromie. Nous conjecturons qu'il contient tous les automorphismes de ce type, modulo $\pm$ id. Ensuite, nous définissons les singularités marquées et construisons leurs espaces de modules globaux pour leurs classes d'équivalence à droite. La conjecture sur le groupe impliquerait que ces espaces de modules sont connexes. Nous discutons de la relation avec les problèmes de type Torelli et nous formulons une nouvelle conjecture de type Torelli global pour les singularités marquées. Toutes ces conjectures sont montrées pour les singularités simples et pour 22 des 28 singularités exceptionnelles.

1. Introduction

This paper studies local objects from a global perspective. The local objects are holomorphic function germs $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an

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isolated singularity at 0 (short: singularity). Two types of global objects for them are considered. The first are new monodromy groups, the *μ-constant monodromy groups*. The second are moduli spaces for *marked singularities*. They are related. And both are important for the study of period maps to spaces of Brieskorn lattices, that is, regular singular TERP-structures or non-commutative Hodge structures.

The Milnor lattice of a singularity f is $Ml(f) := H_n(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ (reduced homology if $n = 0$), here μ is the Milnor number, $\tau > 0$, and $f^{-1}(\tau)$ is a regular fiber in a suitable representative of the function germ f. It comes equipped with two pairings, the intersection form I and the Seifert form L, and with the monodromy $M_h \in \mathrm{Aut}(Ml(f), L, I)$. We put them together in one tuple $ML(f) := (Ml(f), L, M_h, I)$. In fact, L determines M_h and I, so

$$G_{\mathbb{Z}}(f) := \mathrm{Aut}(ML(f)) = \mathrm{Aut}(Ml(f), L).$$

We consider two kinds of μ-constant families, either C^∞-families F of singularities over a base space X which is a C^∞-manifold, or holomorphic families F where the base space X is a reduced complex space. In either case the Milnor lattices $Ml(F_t)$, $t \in X$, of the members of the family F glue to a local system of $\mathbb{Z}$-lattices of rank μ with Seifert form, monodromy automorphism and intersection form. After fixing one point $t_0 \in X$, the monodromy group $G(F, X, t_0)$ of such a family is the image of the natural homomorphism $\pi_1(X, t_0) \to G_{\mathbb{Z}}(F_{t_0})$ (Definition 3.1 (a)).

For a singularity f, the *μ-constant monodromy group* $G^{\mathrm{smar}}(f)$ is the subgroup of $G_{\mathbb{Z}}(f)$ generated by all monodromy groups of all μ-constant families which contain f (Definition 3.1 (b)). But using k-jets and the finite determinacy of singularities, it is not hard to construct one global holomorphic μ-constant family whose monodromy group is $G^{\mathrm{smar}}(f)$ (Lemma 3.5 (c)).

This global μ-constant family was the starting point in [11, Theorem 13.15] for the construction of a global moduli space $M_\mu(f_0)$ for right equivalence classes of singularities in the μ-homotopy class of a fixed singularity f_0. Here we will adapt this construction and establish a moduli space $M_\mu^{\mathrm{mar}}(f_0)$ [respectively $M_\mu^{\mathrm{smar}}(f_0)$] of [strongly] marked singularities (Theorem 4.3).

Fix one singularity f_0. A *[strongly] marked singularity* is a pair $(f, \pm\rho)$ [respectively (f, ρ)] where f is a singularity in the μ-homotopy class of f_0 and $\rho : ML(f) \to ML(f_0)$ is an isomorphism. Here $\pm\rho$ means the set $\{\rho, -\rho\}$, so neither ρ nor $-\rho$ is distinguished. Two [strongly] marked singularities $(f_1, \pm\rho_1)$ and $(f_2, \pm\rho_2)$ [(f_1, ρ_1) and (f_2, ρ_2)] are *right equivalent*

if a coordinate change $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ exists with $f_1 = f_2 \circ \varphi$ and $\rho_1 = \pm\rho_2 \circ \varphi_{\hom}$ [respectively $\rho_1 = \rho_2 \circ \varphi_{\hom}$], here $\varphi_{\hom} : ML(f_1) \to ML(f_2)$ is the induced isomorphism.

A surprising fact is that the strongly marked singularities (f, ρ) and $(f, -\rho)$ are right equivalent if and only if mult $f = 2$ (Theorem 3.3 (e) and (g)). This leads to potential problems for the space M_μ^{smar}: If there would exist a μ-homotopy class which contains singularities with multiplicity 2 and singularities with multiplicity $\geqslant 3$ (which I don't believe), then its moduli space M_μ^{smar} of strongly marked singularities would not be Hausdorff (Theorem 4.3 (e)). The moduli space M_μ^{mar} is not sensitive to this, it exists always as an analytic geometric quotient.

This is one reason why we consider not only strongly marked singularities, but also marked singularities. The other is that the period map $M_\mu^{\mathrm{smar}}(f_0) \to D_{BL}(f_0)$ to a classifying space for Brieskorn lattices factors through $M_\mu^{\mathrm{mar}}(f_0)$.

Locally M_μ^{mar} and M_μ^{smar} (if it is Hausdorff) are isomorphic to the μ-constant stratum of a singularity (Theorem 4.3 (b)). The group $G_{\mathbb{Z}}(f_0)$ acts on M_μ^{mar} and M_μ^{smar} by $\psi : [(f, \pm\rho)] \mapsto [(f, \pm\psi \circ \rho)]$ [respectively $\psi : [(f, \rho)] \mapsto [(f, \psi \circ \rho)]$]. The action is properly discontinuous (on M_μ^{smar} if it is Hausdorff), the quotient is M_μ (Theorem 4.3 (d)).

The μ-constant monodromy group $G^{\mathrm{smar}}(f_0)$ turns out to be the subgroup of $G_{\mathbb{Z}}(f_0)$ which acts on that topological component $(M_\mu^{\mathrm{smar}})^0$ which contains $[(f_0, \mathrm{id})]$ (if M_μ^{smar} is Hausdorff), and likewise, the group

$$G^{\mathrm{mar}}(f_0) := \{\pm\psi \,|\, \psi \in G^{\mathrm{smar}}(f_0)\}$$

is the subgroup of $G_{\mathbb{Z}}(f_0)$ which acts on the component $(M_\mu^{\mathrm{mar}})^0$ which contains $[(f_0, \pm\,\mathrm{id})]$ (Theorem 4.4 (a) and (b)). Especially, there is a 1–1 correspondence

$$\begin{array}{rcl} G_{\mathbb{Z}}(f_0)/G^{\mathrm{mar}}(f_0) & \to & \{\text{topological components of } M_\mu^{\mathrm{mar}}\} \\ \psi \cdot G^{\mathrm{mar}}(f_0) & \mapsto & \psi((M_\mu^{\mathrm{mar}})^0) \end{array}$$

[and similarly for $G^{\mathrm{smar}}(f_0)$ and M_μ^{smar} if it is Hausdorff].

Conjecture. — *(3.2)*

(a) $G^{\mathrm{mar}}(f_0) = G_{\mathbb{Z}}(f_0)$, *equivalent:* M_μ^{mar} *is connected.*

(b) *If all singularities in the* μ*-homotopy class of* f_0 *have multiplicity* $\geqslant 3$ *then* $G^{\mathrm{mar}}(f_0) = G^{\mathrm{smar}}(f_0) \times \{\pm\,\mathrm{id}\}$, *equivalent:* $-\,\mathrm{id} \notin G^{\mathrm{smar}}(f_0)$.

Part (a) is a fragile conjecture. If it is true, it points at hidden properties which distinguish the lattice $Ml(f_0)$ from other monodromy invariant lattices in $Ml(f_0) \otimes_{\mathbb{Z}} \mathbb{Q}$. For example, it implies that any basis which has the

same Coxeter-Dynkin diagram as a distinguished basis is also distinguished (Remark 3.4).

Part (b) leads to the question how in cases where it is true, the index 2 subgroup $G^{\mathrm{smar}}(f_0) \subset G^{\mathrm{mar}}(f_0)$ can be described a priori. Both conjectures are proved in Section 8 for the simple and 22 of the 28 families of exceptional singularities. In another paper they will be proved for the remaining 6 families of exceptional singularities, for the simple-elliptic and the hyperbolic singularities.

In [10] a classifying space $D_{BL}(f_0)$ for (candidates of) Brieskorn lattices was constructed. It is a complex manifold, and $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on it. Now one obtains a holomorphic period map

$$BL : M_{\mu}^{\mathrm{mar}}(f_0) \to D_{BL}(f_0),$$

which is $G_{\mathbb{Z}}(f_0)$-equivariant. An infinitesimal Torelly type result is that it is an immersion ([11, Theorem 12.8]). The following is a global Torelli type conjecture for marked singularities.

Conjecture. — *(5.3)* $BL : M_{\mu}^{\mathrm{mar}}(f_0) \to D_{BL}(f_0)$ *is injective.*

It is equivalent to two Torelli type conjectures which I had proposed earlier. One is that the period map after taking the quotient by $G_{\mathbb{Z}}(f_0)$

$$LBL : M_{\mu}^{\mathrm{mar}}(f_0)/G_{\mathbb{Z}}(f_0) = M_{\mu}(f_0) \to D_{BL}(f_0)/G_{\mathbb{Z}}(f_0)$$

is injective. It says that the right equivalence class of a singularity is determined by its Brieskorn lattice (up to isomorphism). I worked on it in [7]–[11]. The other is that for any $[(f, \pm\rho)] \in M_{\mu}^{\mathrm{mar}}(f_0)$

$$\mathrm{Stab}_{G_{\mathbb{Z}}(f_0)}([(f, \pm\rho)]) = \mathrm{Stab}_{G_{\mathbb{Z}}(f_0)}(BL([(f, \pm\rho)])),$$

this is [11, Conjecture 13.12]. Obvious is only $\subset$ and that both groups are finite, because $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on $M_{\mu}^{\mathrm{mar}}(f_0)$ and $D_{BL}(f_0)$.

Nevertheless, the isotropy group $\mathrm{Stab}_{G_{\mathbb{Z}}(f_0)}([(f, \pm\rho)])$ and also the subgroup $\mathrm{Stab}_{G_{\mathbb{Z}}(f_0)}([(f, \rho)])$ are much better understood than the monodromy groups $G^{\mathrm{mar}}(f_0)$ and $G^{\mathrm{smar}}(f_0)$. The isotropy groups had been studied from the point of view of symmetries of singularities in [11, 13.1 and 13.2]. Section 6 reviews the results.

The isotropy group $\mathrm{Stab}_{G_{\mathbb{Z}}(f_0)}([(f, \rho)])$ can also be seen as a μ-constant monodromy group, but for μ-constant families where all members are right equivalent to f (Theorem 4.4 (d)).

This paper deals almost exclusively with μ-constant families of singularities. Semiuniversal unfoldings are only used in the discussion of symmetries

of singularities and in the construction of M_μ^{mar}. But later I hope to extend M_μ^{mar} to a manifold of dimension μ which is locally a semiuniversal unfolding and which allows to consider distinguished bases and Stokes data of deformations which are not μ-constant from a global perspective.

Section 2 reviews the topology of singularities. Section 3 defines and studies the μ-constant monodromy groups. Section 4 establishes the moduli spaces for [strongly] marked singularities, though the main proof is given in Section 7. Section 5 discusses the period maps BL and LBL. Section 6 reviews the symmetries of singularities. Section 8 proves all conjectures for the simple and 22 of the 28 exceptional singularities.

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2. Review on the topology of isolated hypersurface singularities

First, we recall some classical facts and fix some notations. An *isolated hypersurface singularity* (short: *singularity*) is a holomorphic function germ $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at 0. Its *Milnor number*

$$\mu := \dim \mathcal{O}_{\mathbb{C}^{n+1},0} / \Big(\frac{\partial f}{\partial x_i} \Big)$$

is finite. A *Milnor fibration* for f is constructed as follows [19]. Choose $\varepsilon > 0$ such that f is defined on the ball $B_\varepsilon^{2n+2} := \{x \in \mathbb{C}^{n+1} \,|\, |x| < \varepsilon\}$ and $f^{-1}(0)$ is transversal to $\partial B_{\widetilde{\varepsilon}}^{2n+2}$ for all $\widetilde{\varepsilon} \leqslant \varepsilon$. Choose $\delta > 0$ such that $f^{-1}(\tau)$ is transversal to $\partial B_\varepsilon^{2n+2}$ for all $\tau \in T_\delta := \{\tau \in \mathbb{C} \,|\, |\tau| < \delta\}$. Define $T'_\delta := T_\delta - \{0\}$, $Y(\varepsilon, \delta) := B_\varepsilon^{2n+2} \cap f^{-1}(T_\delta)$ and $Y'(\varepsilon, \delta) := Y(\varepsilon, \delta) - f^{-1}(0)$. Then $f : Y'(\varepsilon, \delta) \to T'_\delta$ is a locally trivial C^∞-fibration, the *Milnor fibration*, and each fiber has the homotopy type of a bouquet of μ n-spheres [19].

Therefore the (reduced for $n = 0$) middle homology groups are $H_n^{(\mathrm{red})}(f^{-1}(\tau), \mathbb{Z}) \cong \mathbb{Z}^\mu$ for $\tau \in T'_\delta$. Each comes equipped with an intersection form I, which is a datum of one fiber, a monodromy M_h and a Seifert form L, which come from the Milnor fibration, see [3, I.2.3] for their definitions (for the Seifert form, there are several conventions in the literature, we follow [3]). M_h is a quasiunipotent automorphism, I and L are bilinear forms with values in $\mathbb{Z}$, I is $(-1)^n$-symmetric and L is unimodular. L determines M_h and I because of the formulas [3, I.2.3]

$$L(M_h a, b) = (-1)^{n+1} L(b, a), \quad I(a, b) = -L(a, b) + (-1)^{n+1} L(b, a).$$

If $f : Y'(\widetilde{\varepsilon}, \widetilde{\delta}) \to T'_{\widetilde{\delta}}$ is a Milnor fibration with $\widetilde{\varepsilon} < \varepsilon$ and $\widetilde{\delta} < \delta$ then the inclusion

$$Y'(\widetilde{\varepsilon}, \widetilde{\delta}) \cap f^{-1}(\partial T'_{\widetilde{\delta}}) \hookrightarrow Y'(\varepsilon, \delta) \cap f^{-1}(\partial T'_{\widetilde{\delta}})$$

is a fiber homotopy equivalence between the restrictions to $\partial T'_{\widetilde{\delta}}$ of the new and the old Milnor fibration ([19] or [26, Lemma 2.2]). Therefore the Milnor lattices $H_n(f^{-1}(\tau), \mathbb{Z})$ for all Milnor fibrations and all $\tau \in \mathbb{R}_{>0} \cap T'_\delta$ are canonically isomorphic, and the isomorphisms respect M_h, I and L. These lattices are identified and called $Ml(f)$, the tuple $(Ml(f), L, M_h, I)$ is called $ML(f)$ (for MiLnor and Lattice and L = Seifert form). Remark that $\mathrm{Aut}(ML(f)) = \mathrm{Aut}(Ml(f), L)$ because L determines M_h and I. This group is called $G_{\mathbb{Z}}(f)$.

The function germ $f(x_0, ..., x_n) + x_{n+1}^2 \in \mathcal{O}_{\mathbb{C}^{n+2},0}$ is called *stabilization* or *suspension* of f. There is a canonical isomorphism $Ml(f) \otimes Ml(x_{n+1}^2) \to Ml(f + x_{n+1}^2)$ [3, I.2.7]. As there are only two isomorphisms $Ml(x_{n+1}^2) \to \mathbb{Z}$, and they differ by a sign, there are two equally canonical isomorphisms $Ml(f) \to Ml(f + x_{n+1}^2)$, and they differ just by a sign. Therefore automorphisms and bilinear forms on $Ml(f)$ can be identified with automorphisms and bilinear forms on $Ml(f + x_{n+1}^2)$. In this sense

$$L(f + x_{n+1}^2) = (-1)^n \cdot L(f) \qquad \text{and} \qquad M_h(f + x_{n+1}^2) = -M_h(f)$$

[3, I.2.7], and $G_{\mathbb{Z}}(f + x_{n+1}^2) = G_{\mathbb{Z}}(f)$.

The group of biholomorphic map germs $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ is called $\mathcal{R}$, its elements are called *coordinate changes*. Two singularities f and $g \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ are *right equivalent*, if $f = g \circ \varphi$ for some $\varphi \in \mathcal{R}$, notation: $f \sim_{\mathcal{R}} g$. In that case φ induces an isomorphism

$$\varphi_{\mathrm{hom}} : ML(f) \to ML(g).$$

The *multiplicity* of f is $\mathrm{mult}\, f := \max(k \mid f \in \mathbf{m}^k)$, here $\mathbf{m} \subset \mathcal{O}_{\mathbb{C}^{n+1},0}$ is the maximal ideal. The *splitting lemma* says for isolated hypersurface singularities $f, f_1, f_2 \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ [2]

$$\begin{aligned} \mathrm{mult}\, f = 2 \quad &\Longleftrightarrow \quad f \sim_{\mathcal{R}} g(x_0, ..., x_{n-1}) + x_n^2 \quad \text{for some} \quad g \in \mathbf{m}^2_{\mathbb{C}^n,0} \\ f_1 \sim_{\mathcal{R}} f_2 \quad &\Longleftrightarrow \quad f_1 + x_{n+1}^2 \sim_{\mathcal{R}} f_2 + x_{n+1}^2 \end{aligned}$$

(in the first equivalence $\Leftarrow$ is trivial, in the second $\Rightarrow$).

The next definition and the theorem after it are preparations for Section 3.

Definition 2.1.

(a) *A C^∞ μ-constant family consists of a number $\mu \in \mathbb{Z}_{\geqslant 1}$, a connected C^∞-manifold X, possibly with boundary (e.g., $X = [0, 1]$), an open*

neighborhood $Y \subset \mathbb{C}^{n+1} \times X$ of $\{0\} \times X$ and a C^∞-function $F : Y \to \mathbb{C}$, such that $F_t := F_{|Y \cap \mathbb{C}^{n+1} \times \{t\}}$ for any $t \in X$ is holomorphic and has an isolated singularity with Milnor number μ at 0.

(b) *A* holomorphic μ-constant family *consists of a number $\mu \in \mathbb{Z}_{\geqslant 1}$, a connected reduced complex space X, an open neighborhood $Y \subset \mathbb{C}^{n+1} \times X$ of $\{0\} \times X$ and a holomorphic function $F : Y \to \mathbb{C}$, such that $F_t := F_{|Y \cap \mathbb{C}^{n+1} \times \{t\}}$ for any $t \in X$ has an isolated singularity with Milnor number μ at 0.*

(c) *The* μ-homotopy class *of f consists of all singularities g such that a C^∞ μ-constant family exists which contains f and g.*

THEOREM 2.2. — *In both cases ((a) and (b) in Definition 2.1) the Milnor lattices $Ml(F_t)$ and the tuples $ML(F_t)$ for $t \in X$ are locally canonically isomorphic. They glue to a local system $Ml(F)$ of free $\mathbb{Z}$-modules of rank μ on X with a flat unimodular pairing L, a flat automorphism M_h and a flat intersection form I. The tuple $(Ml(F), L, M_h, I)$ is called $ML(F)$.*

Proof. — (a) For any $t \in X$ one can choose $\varepsilon(t)$ and $\delta(t)$ such that $F_t : Y'(\varepsilon(t), \delta(t)) \times \{t\} \to T'_{\delta(t)}$ is a Milnor fibration. But it may happen that $\epsilon(t)$ and $\delta(t)$ cannot be chosen as continuous functions (a *vanishing fold* might exist). Luckily [26, Lemma 2.2] says that for F_t with t close to t_0 and $\varepsilon(t) \leqslant \varepsilon(t_0)$, $\delta(t) \leqslant \delta(t_0)$, the inclusion

$$Y'(\varepsilon(t), \delta(t)) \times \{t\} \cap F_t^{-1}(\partial T'_{\delta(t)}) \hookrightarrow B^{2n+2}_{\epsilon(t_0)} \times \{t\} \cap F_t^{-1}(\partial T'_{\delta(t)})$$

is a fiber homotopy equivalence over $\partial T'_{\delta(t)}$. And the second fibration is obviously diffeomorphic to the restriction of the Milnor fibration of F_{t_0} to $\partial T'_{\delta(t)}$. This proves (a).

(b) This follows from (a). □

3. μ-constant monodromy groups

Definition 3.1 presents the first main subject of this paper, the μ-constant monodromy groups and some subgroups.

DEFINITION 3.1. — *Let $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ have an isolated singularity at 0.*

(a) *For any C^∞ or holomorphic μ-constant family (X, Y, F) (Definition 2.1) with $F_{t_0} = f$ for some $t_0 \in X$, the local system $Ml(F)$ over X yields a homomorphism $\pi_1(X, t_0) \to G_{\mathbb{Z}}(f)$. The image is the* μ-constant monodromy group *$G(F, t_0) \subset G_{\mathbb{Z}}(f)$ of this μ-constant family.*

If $X = S^1$ we call the image of the standard generator of $\pi_1(S^1, t_0)$ the monodromy *of the μ-constant family.*

(b) *We define four subgroups of $G_{\mathbb{Z}}(f)$. The first two are called* μ-constant monodromy groups *of f.*

$$\begin{aligned}
G^{\mathrm{smar}}(f) &:= \{\text{the subgroup generated by all } G(F, t_0) \text{ as in (a)}\},\\
G^{\mathrm{mar}}(f) &:= \{\pm\psi \,|\, \psi \in G^{\mathrm{smar}}(f)\},\\
G^{\mathrm{smar}}_{\mathcal{R}}(f) &:= \{\text{the subgroup generated by all } G(F, t_0) \text{ as in (a)}\\
&\qquad \text{where } F_t \sim_{\mathcal{R}} f \text{ for all } t \in X\},\\
G^{\mathrm{mar}}_{\mathcal{R}}(f) &:= \{\pm\psi \,|\, \psi \in G^{\mathrm{smar}}_{\mathcal{R}}(f)\}.
\end{aligned}$$

In Lemma 3.5 (c) and in Theorem 3.3 (e) other more compact descriptions of $G^{\mathrm{smar}}(f)$ and $G^{\mathrm{smar}}_{\mathcal{R}}(f)$ will be given. The indices "smar" and "mar" stand for *strongly marked* and *marked*. They are motivated by Theorem 4.4 (a) and (b). Theorem 4.4 will put $G^{\mathrm{smar}}(f)$ and $G^{\mathrm{mar}}(f)$ into action. Obviously

$$\begin{array}{ccccc}
G^{\mathrm{smar}}_{\mathcal{R}}(f) & \subset & G^{\mathrm{mar}}_{\mathcal{R}}(f) & & \\
\cap & & \cap & & \\
G^{\mathrm{smar}}(f) & \subset & G^{\mathrm{mar}}(f) & \subset & G_{\mathbb{Z}}(f).
\end{array}$$

The two groups $G^{\mathrm{smar}}_{\mathcal{R}}(f)$ and $G^{\mathrm{mar}}_{\mathcal{R}}(f)$ are finite (Theorem 6.1 (f)). They depend on the right equivalence class of f. They were studied already in [11, ch. 13]. We cite some results about them in Theorem 3.3 and discuss them in Section 6. Conjecture 5.1 would give complete control on them through the Brieskorn lattice.

The two groups $G^{\mathrm{smar}}(f)$ and $G^{\mathrm{mar}}(f)$ depend up to conjugacy only on the μ-homotopy class of f. They are hard to calculate. I propose the following two conjectures.

CONJECTURE 3.2. — *Let $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ have an isolated singularity at 0.*

(a)
$$G^{\mathrm{mar}}(f) = G_{\mathbb{Z}}(f).$$

(b) *If all singularities in the μ-homotopy class of f have multiplicity $\geqslant 3$ then $-\mathrm{id} \notin G^{\mathrm{smar}}(f)$, equivalent: then $G^{\mathrm{mar}}(f) = G^{\mathrm{smar}}(f) \times \{\pm \mathrm{id}\}$.*

At first sight, Conjecture 3.2 (a) might look safe as all monodromy groups of all μ-constant families together should give a large subgroup of $G_{\mathbb{Z}}(f)$. At second sight, it turns out to be a fragile conjecture. Often there are other $\mathbb{Z}$-lattices $V_{\mathbb{Z}}$ of maximal rank in $Ml(f) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathrm{Aut}(V_{\mathbb{Z}}, L) \supsetneq G_{\mathbb{Z}}(f)$. Conjecture 3.2 (a) is related to hidden properties

which distinguish $(Ml(f), L)$ from other $\mathbb{Z}$-lattices in $Ml(f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Section 8 will give examples. See also Remark 3.4.

Conjecture 3.2 (b) is even more mysterious. If both conjectures are true then $G_{\mathbb{Z}}(f) = G^{\mathrm{smar}}(f) \times \{\pm \mathrm{id}\}$ for f as in (b). Is there an a priori way to distinguish such a subgroup of index 2 in $G_{\mathbb{Z}}(f)$?

Theorem 3.3 collects some evidence for the conjectures and some results about the four groups. The singularities with modality $\leqslant 2$ are given in [2].

THEOREM 3.3. — *Let $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ have an isolated singularity at 0.*

(a) *The conjectures 3.2 (a) and (b) are true for all singularities with modality $\leqslant 1$, that means, simple (ADE), simple-elliptic (=parabolic, $\widetilde{E}_6 = P_8$, $\widetilde{E}_7 = X_9$, $\widetilde{E}_8 = J_{10}$), hyperbolic (T_{pqr}) and exceptional unimodal. They are also true for the 14 families of exceptional bimodal singularities (for the other bimodal singularities I did not yet make enough calculations), and for the Brieskorn-Pham singularities $\sum_{i=0}^n x_i^{a_i}$ with pairwise coprime exponents.*

(b) *If some singularity in the μ-homotopy class of f has multiplicity 2 then $-\mathrm{id} \in G^{\mathrm{smar}}(f)$, equivalent: then $G^{\mathrm{smar}}(f) = G^{\mathrm{mar}}(f)$.*

(c) *$M_h \in G^{\mathrm{smar}}(f)$. If f is quasihomogeneous then $M_h \in G^{\mathrm{smar}}_{\mathcal{R}}(f)$.*

(d) *If $\mathrm{mult}\, f \geqslant 3$ then $M_h^k \neq -\mathrm{id}$ for any $k \in \mathbb{Z}$.*

(e)
$$G^{\mathrm{smar}}_{\mathcal{R}}(f) = \{\varphi_{\mathrm{hom}} \in G_{\mathbb{Z}}(f) \,|\, \varphi \in \mathcal{R} \text{ with } f = f \circ \varphi\}.$$

(f) *$G^{\mathrm{smar}}_{\mathcal{R}}(f)$ and $G^{\mathrm{mar}}_{\mathcal{R}}(f)$ are finite.*

(g)
$$-\mathrm{id} \notin G^{\mathrm{smar}}_{\mathcal{R}}(f) \iff \mathrm{mult}\, f \geqslant 3.$$
Equivalent: $G^{\mathrm{mar}}_{\mathcal{R}}(f) = G^{\mathrm{smar}}_{\mathcal{R}}(f)$ if $\mathrm{mult}\, f = 2$, and $G^{\mathrm{mar}}_{\mathcal{R}}(f) = G^{\mathrm{smar}}_{\mathcal{R}}(f) \times \{\pm \mathrm{id}\}$ if $\mathrm{mult}\, f \geqslant 3$.

(h) *$G^{\mathrm{mar}}_{\mathcal{R}}(f) = G^{\mathrm{mar}}_{\mathcal{R}}(f + x_{n+1}^2)$.*

(i) *If all singularities in the μ-homotopy class of $f + x_{n+1}^2$ have multiplicity 2 then $G^{\mathrm{mar}}(f) = G^{\mathrm{mar}}(f + x_{n+1}^2)$.*

Proof. — (a) See Theorem 8.3, Theorem 8.4 and Remark 8.5 (i) for the simple, 22 of the 28 exceptional and the Brieskorn-Pham singularities. The remaining 6 families of unimodal and bimodal exceptional singularities, the simple-elliptic and the hyperbolic singularities will be treated in another paper.

(b) Let $g(x_0, ..., x_{n-1}) + x_n^2$ be in the μ-homotopy class of f. Then
$$g + t \cdot x_n^2, \quad t \in S^1 =: X,$$

is a C^∞ μ-constant family. With the canonical isomorphism $Ml(g+x_n^2) \cong Ml(g) \otimes Ml(x_n^2)$ one sees that its monodromy is $\mathrm{id} \otimes (-\mathrm{id}) = -\mathrm{id} \in G_{\mathbb{Z}}(g+x_n^2)$. Any path from f to $g+x_n^2$ in a C^∞ μ-constant family induces an isomorphism $B : ML(f) \to ML(g+x_n^2)$ with $G^{\mathrm{smar}}(f) = B^{-1} \circ G^{\mathrm{smar}}(g+x_n^2) \circ B$.

(c) The monodromy of the C^∞ μ-constant family

$$F_t := t^{-1} \cdot f, \quad t \in S^1,$$

is M_h, because $F_t^{-1}(\tau) = f^{-1}(t \cdot \tau)$ for $\tau > 0$. If f is quasihomogeneous then $F_t \sim_{\mathcal{R}} f$.

(d) Théorème 1 in [1] (and already a letter from Deligne to A'Campo, see [1]) shows

$$\operatorname{trace} M_h^k = (-1)^{n+1} \qquad \text{if } 0 < k < \operatorname{mult} f.$$

Let Φ_m for $m \in \mathbb{Z}_{\geqslant 1}$ be the cyclotomic polynomial of primitive unit roots of order m. It is well known that

$$\sum_{\Phi_m(\lambda)=0} \lambda = \begin{cases} 0 & \text{if } p^2 | m \text{ for some prime number } p, \\ (-1)^s & \text{if } m = p_1 \cdot \ldots \cdot p_s \text{ with different} \\ & \text{prime numbers } p_1, \ldots, p_s. \end{cases}$$

Now suppose $\operatorname{mult} f \geqslant 3$. Then $\mu \geqslant 2$ and $\operatorname{trace} M_h = \operatorname{trace} M_h^2 = (-1)^{n+1}$. Further suppose $M_h^k = -\mathrm{id}$ for some $k \in \mathbb{Z}$. Then there exist $a \in \mathbb{Z}_{\geqslant 1}$ and odd numbers $b_1, \ldots, b_r \in \mathbb{Z}_{\geqslant 1}$ with

$$\{\operatorname{ord} \lambda \,|\, \lambda \text{ eigenvalue of } M_h\} = \{2^a \cdot b_1, \ldots, 2^a \cdot b_r\}.$$

Now $\operatorname{trace} M_h \neq 0$ implies $a = 1$. Then $\{\operatorname{ord} \lambda \,|\, \lambda \text{ eigenvalue of } M_h^2\} = \{b_1, \ldots, b_r\}$ and $\operatorname{trace} M_h^2 = -\operatorname{trace} M_h$, a contradiction.

(e) See Theorem 4.4 (d).

(f)+(g)+(h) See Theorem 6.1 (i)

(i) See Theorem 4.4 (e). □

Remarks 3.4. — There is a set $\mathcal{B}^* \subset Ml(f)^\mu$ of *distinguished bases*, see [3] or [5] for the definition.

Claim: The elements of $G^{\mathrm{mar}}(f)$ respect this set, so

$$G^{\mathrm{mar}}(f) \subset \operatorname{Aut}(Ml(f), L, \mathcal{B}^*) \subset G_{\mathbb{Z}}(f).$$

If one knows how distinguished bases arise, it is not hard to see this claim. I will discuss it in another paper. Here I just want to point to an implication of Conjecture 3.2 (a): It would imply equalities. Equivalent to the second equality $\operatorname{Aut}(Ml(f), L, \mathcal{B}^*) = G_{\mathbb{Z}}(f)$ is that any basis of $Ml(f)$ which has the same Coxeter-Dynkin diagram as some distinguished basis is also distinguished. This is true for the singularities in Theorem 3.3 (a). For the

simple and the simple-elliptic singularities there are older proofs. It seems to be hard to establish it in any case.

Now we will describe a holomorphic μ-constant family which in a certain sense induces any μ-constant family of singularities in a fixed μ-homotopy class and whose monodromy group is the group G^{smar}. This is based on the theory of Tougeron and Mather of jets and finite determinacy of singularities [17] (see also [4]).

Write $\mathcal{O} = \mathcal{O}_{\mathbb{C}^{n+1},0}$ and $\mathbf{m} = \mathbf{m}_{\mathbb{C}^{n+1},0}$. The k-jet of a function germ $f \in \mathcal{O}$ is the class $j_k f \in \mathcal{O}/\mathbf{m}^{k+1}$, the k-jet of a coordinate change $\varphi = (\varphi_0, ..., \varphi_n) \in \mathcal{R}$ is $j_k\varphi = (j_k\varphi_0, ..., j_k\varphi_n)$. The action of $\mathcal{R}$ on $\mathbf{m}^2$ pushes down to an action of the algebraic group $j_k\mathcal{R}$ on $\mathbf{m}^2/\mathbf{m}^{k+1}$ as a smooth affine algebraic variety.

By a result of Tougeron and Mather [17, Theorem (3.5)] a singularity $f \in \mathbf{m}^2$ with Milnor number μ is μ+1-determined, that means, any function germ $g \in \mathbf{m}^2$ with $j_{\mu+1}g = j_{\mu+1}f$ is right equivalent to f.

Fix μ and $k \geqslant \mu + 1$. For any singularity $g \in \mathbf{m}^2$ with $\mu(g) \leqslant k - 1$ the codimension of the orbit $j_k\mathcal{R} \cdot j_k g$ in $\mathbf{m}^2/\mathbf{m}^{k+1}$ is $\mu(g) - 1$. The union of all orbits with codimension $\geqslant \mu - 1$ is an algebraic subvariety of $\mathbf{m}^2/\mathbf{m}^{k+1}$. The set $\{j_k g \,|\, g \in \mathbf{m}^2, \mu(g) = \mu\}$ is Zariski open in it and thus a quasiaffine variety.

For a fixed singularity $f \in \mathbf{m}^2$ with $\mu(f) = \mu$ denote by $C(k, f)$ the topological component of it which contains $j_k f$. It is also a quasiaffine variety. For any $t \in C(k, f)$ denote by F_t the unique polynomial of degree $\leqslant k$ with $j_k F_t = t$. These polynomials glue to a regular function $F : \mathbb{C}^{n+1} \times C(k, f) \to \mathbb{C}$.

Lemma 3.5.

(a) *$(C(k, f), \mathbb{C}^{n+1} \times C(k, f), F)$ is a holomorphic μ-constant family.*

(b) *For any C^∞ μ-constant family (X, Y, E) in the μ-homotopy class of f the local system $ML(E)$ over X is obtained from $ML(F)$ by pull back via the jet map $X \to C(k, f),\ t \mapsto j_k E_t$.*

(c) *Denote by $B : ML(f) \to ML(j_k f)$ the isomorphism induced from the C^∞ μ-constant family $f + t \cdot (j_k f - f)$, $t \in [0, 1]$. Then*

$$G^{\mathrm{smar}}(f) = B^{-1} \circ G(F, j_k f) \circ B,$$
$$G^{\mathrm{smar}}_{\mathcal{R}}(f) = B^{-1} \circ G(F_{|j_k\mathcal{R} \cdot j_k f}, j_k f) \circ B.$$

Proof. — (a) $\mu(F_t) = \mu$ due to the finite determinacy.

(b) Again due to the finite determinacy, the family

$$\widetilde{E}_{(s,t)} := E_s + t \cdot (j_k E_s - E_s), \quad (s, t) \in X \times [0, 1],$$

is a C^∞ μ-constant family. Its restriction to $X \times \{1\}$ is induced by F via the natural map $X \times \{1\} \to C(k, f)$.

(c) This follows from (b) and an analogous statement for μ-constant families (X, Y, G) with $G_s \sim_{\mathcal{R}} f$ for any $s \in X$ and the restriction of the family F to $j_k\mathcal{R} \cdot j_k f$. □

Remarks 3.6. — (i) In Definition 3.1 (b) it is sufficient to consider C^∞ μ-constant families over S^1. Replacing C^∞ by continuous and piecewise C^∞ would not give more, because one can smoothen a curve at a point where it is only piecewise C^∞ by a reparametrization using a (monotonous) C^∞-function $[0,1] \to [0,1]$ with $0 \mapsto 0$, $\frac{1}{2} \mapsto \frac{1}{2}$ and $1 \mapsto 1$ which is constant near $\frac{1}{2}$.

(ii) Also a μ-constant family $G : Y \to \mathbb{C}$ over S^1 where $Y \to S^1$ is a priori only locally isomorphic to ((a neighborhood of $\{0\} \times S^1$ in $\mathbb{C}^{n+1} \times S^1$) $\to S^1$) would not give more. Then $Y \to S^1$ is globally isomorphic to ((a neighborhood of $\{0\} \times S^1$ in $\mathbb{C}^{n+1} \times S^1$) $\to S^1$), as a C^∞-family of neighborhoods of 0 in $\mathbb{C}^{n+1}$.

4. Moduli space of marked singularities

Now we come to the second main subject of this paper, (strongly) marked singularities and moduli spaces for them.

DEFINITION 4.1. — *Let $f_0 \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ be a function germ with an isolated singularity at 0 with Milnor number μ (short: a* singularity*). Recall $ML(f) = (Ml(f), L, M_h, I)$ from Section 2.*

(a) *A* strongly marked singularity *is a tuple (f, ρ) where $f \in \mathbf{m}^2$ is a singularity in the μ-homotopy class of f_0 and $\rho : ML(f) \to ML(f_0)$ is an isomorphism.*

(b) *A* marked singularity *is a tuple $(f, \pm\rho)$ with f and ρ as in (a) (writing $\pm\rho$ we mean the set $\{\rho, -\rho\}$, neither ρ nor $-\rho$ is preferred, so $(f, \pm\rho) = (f, \pm(-\rho))$).*

(c) *Two strongly marked singularities (f_1, ρ_1) and (f_2, ρ_2) are* right equivalent *(notation: $(f_1, \rho_1) \sim_{\mathcal{R}} (f_2, \rho_2)$) if a coordinate change $\varphi \in \mathcal{R}$ exists with*

$$f_1 = f_2 \circ \varphi \quad \text{and} \quad \rho_1 = \rho_2 \circ \varphi_{\text{hom}}.$$

(d) *Two marked singularities $(f_1, \pm\rho_1)$ and $(f_2, \pm\rho_2)$ are* right equivalent *(notation: $(f_1, \pm\rho_1) \sim_{\mathcal{R}} (f_2, \pm\rho_2)$) if a coordinate change*

$\varphi \in \mathcal{R}$ *exists with*

$$f_1 = f_2 \circ \varphi \quad \text{and} \quad (\rho_1 = \rho_2 \circ \varphi_{\hom} \quad \text{or} \quad \rho_1 = -\rho_2 \circ \varphi_{\hom}).$$

Remarks 4.2. — (a) The notions *strongly marked* and *marked* are closely related, but the first looks more natural than the second. We use also the second notion, for two reasons:

(i) (f, ρ) and $(f, -\rho)$ have the same value under the period map to D_{BL} considered in Section 5.

(ii) $(f, \rho) \sim_{\mathcal{R}} (f, -\rho)$ if $\operatorname{mult} f = 2$ and $(f, \rho) \not\sim_{\mathcal{R}} (f, -\rho)$ if $\operatorname{mult} f \geqslant 3$, by Theorem 3.3 (e) and (g). This implies that the moduli space for strongly marked singularities in Theorem 4.3 is not Hausdorff if a μ-homotopy class contains singularities with multiplicity $\geqslant 3$ and singularities with multiplicity 2. The moduli space for marked singularities is not affected by this.

(b) Because of (ii), we will sometimes make one of the following two assumptions.

(4.1) Assumption (4.1) : Any singularity in the μ-homotopy class of f_0 has multiplicity $\geqslant 3$.

(4.2) Assumption (4.2) : Any singularity in the μ-homotopy class of f_0 has multiplicity 2.

For $n \neq 2$ the topological type of a singularity is constant within a μ-homotopy class [26, Theorem (2.1)]. Then one of the two assumptions would follow from Zariski's multiplicity conjecture. But Zariski's multiplicity conjecture is proved essentially only for curve singularities and quasihomogeneous singularities. For curve singularities and quasihomogeneous singularities (4.1) or (4.2) holds.

Theorem 4.3 and Theorem 4.4 (and Theorem 3.3) are the main results of the paper. Theorem 4.3 is related to [11, Theorem 13.15]. It will be proved in Section 7.

THEOREM 4.3. — *Let $f_0 \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ be a singularity with Milnor number μ and $j_k f_0 = f_0$ for some $k \geqslant \mu + 1$. Fix this k. Define the sets*

$$\begin{aligned} M_\mu^{\text{smar}}(f_0) &:= \{\text{strongly marked } (f, \rho) \mid \\ &\qquad f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathcal{R}}, \\ M_\mu^{\text{mar}}(f_0) &:= \{\text{marked } (f, \pm\rho) \mid \\ &\qquad f \text{ in the } \mu\text{-homotopy class of } f_0\}/\sim_{\mathcal{R}}. \end{aligned}$$

(a) *Recall the set* $C(k,f_0) \subset \mathbf{m}^2/\mathbf{m}^{k+1}$ *discussed before Lemma 3.5. The sets*

$$\begin{aligned} C^{\mathrm{smar}}(k,f_0) := \{&(f,\rho) \,|\, f \in C(k,f_0),\\ &\rho : ML(f) \to ML(f_0) \text{ an isomorphism}\},\\ C^{\mathrm{mar}}(k,f_0) := \{&(f,\pm\rho) \,|\, f \in C(k,f_0),\\ &\rho : ML(f) \to ML(f_0) \text{ an isomorphism}\} \end{aligned}$$

are reduced complex spaces and locally isomorphic to $C(k,f_0)$. *As sets* $M_\mu^{\mathrm{smar}} = C^{\mathrm{smar}}(k,f_0)/j_k\mathcal{R}$, $M_\mu^{\mathrm{mar}} = C^{\mathrm{mar}}(k,f_0)/j_k\mathcal{R}$, *here* $j_k\mathcal{R}$ *acts by* $j_k\varphi : (f,\rho) \mapsto (f\circ j_k\varphi^{-1}, \rho\circ\varphi_{\mathrm{hom}}^{-1})$ *on* $C^{\mathrm{smar}}(k,f_0)$ *and similarly on* $C^{\mathrm{mar}}(k,f_0)$.

(b) $C^{\mathrm{mar}}(k,f_0)/j_k\mathcal{R}$ *is an analytic geometric quotient. The induced reduced complex structure on* M_μ^{mar} *is independent of* k. *The germ* $(M_\mu^{\mathrm{mar}}, [(f,\pm\rho)])$ *is isomorphic to the* μ*-constant stratum in a semi-universal unfolding of* f *(see Section 7 for the* μ*-constant stratum). The canonical complex structure on* μ*-constant strata from* [11, Theorem 12.4] *induces a canonical complex structure on* M_μ^{mar}. *If not said otherwise,* M_μ^{mar} *will be considered with the canonical complex structure.*

(c) *For any* $\psi \in G_{\mathbb{Z}}(f_0) =: G_{\mathbb{Z}}$, *the map*

$$\psi_{\mathrm{mar}} : M_\mu^{\mathrm{mar}} \to M_\mu^{\mathrm{mar}}, \quad [(f,\pm\rho)] \to [(f,\pm\psi\circ\rho)]$$

is an automorphism of M_μ^{mar}. *The action*

$$G_{\mathbb{Z}} \times M_\mu^{\mathrm{mar}}, \quad (\psi, [(f,\pm\rho)] \mapsto \psi_{\mathrm{mar}}([(f,\pm\rho)])$$

is a group action from the left.

(d) *The action of* $G_{\mathbb{Z}}$ *on* M_μ^{mar} *is properly discontinuous. The quotient* $M_\mu^{\mathrm{mar}}/G_{\mathbb{Z}}$ *is the moduli space* M_μ *from* [11, Theorem 13.5] *for right equivalence classes in the* μ*-homotopy class of* f_0, *with its canonical complex structure. Especially* $[(f_1,\pm\rho_1)]$ *and* $[(f_2,\pm\rho_2)]$ *are in one* $G_{\mathbb{Z}}$*-orbit if and only if* f_1 *and* f_2 *are right equivalent.*

(e) *If Assumption* (4.1) *or* (4.2) *holds then (b), (c) and (d) are also true for* $C^{\mathrm{smar}}(k,f_0)$, M_μ^{smar} *and* ψ_{smar} *with* $\psi_{\mathrm{smar}}([(f,\rho)]) := [(f,\psi\circ\rho)]$. *If neither* (4.1) *nor* (4.2) *holds then the quotient topology on* $C^{\mathrm{smar}}(k,f_0)/j_k\mathcal{R}$ *is not Hausdorff.*

THEOREM 4.4. — *Consider the same data as in Theorem 4.3.*

(a) *Let* $(M_\mu^{\mathrm{mar}})^0$ *be the topological component of* M_μ^{mar} *(with its reduced complex structure) which contains* $[(f_0,\pm\mathrm{id})]$. *Then*

$$G^{\mathrm{mar}}(f_0) = \{\psi \in G_{\mathbb{Z}} \,|\, \psi \text{ maps } (M_\mu^{\mathrm{mar}})^0 \text{ to itself}\},$$

and the map

$$G_{\mathbb{Z}}/G^{\mathrm{mar}}(f_0) \to \{\text{topological components of } M_\mu^{\mathrm{mar}}\}$$
$$\psi \cdot G^{\mathrm{mar}}(f_0) \mapsto (\text{the component } \psi_{\mathrm{mar}}((M_\mu^{\mathrm{mar}})^0)$$

is a bijection.

(b) *If Assumption* (4.1) *or* (4.2) *holds then (a) is also true for* M_μ^{smar} *and* $G^{\mathrm{smar}}(f_0)$.

(c) *For any* $[(f, \pm\rho)] \in M_\mu^{\mathrm{mar}}$

$$\mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \pm\rho)]) = \rho \circ G_{\mathcal{R}}^{\mathrm{mar}}(f) \circ \rho^{-1}$$

(this does not use Theorem 4.3 (b)-(d)).

(d) *For any* $[(f, \rho)] \in M_\mu^{\mathrm{smar}}$

$$\begin{aligned}\mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \rho)]) &= \rho \circ G_{\mathcal{R}}^{\mathrm{smar}}(f) \circ \rho^{-1} \\ &= \rho \circ \{\varphi_{\mathrm{hom}} \,|\, \varphi \in \mathcal{R} \text{ with } f = f \circ \varphi\} \circ \rho^{-1}\end{aligned}$$

(this does not require Assumption (4.1) *or* (4.2)*, and it does not use Theorem 4.3 (e)).*

(e) $-\mathrm{id} \in G_{\mathbb{Z}}$ *acts trivially on* $M_\mu^{\mathrm{mar}}(f_0)$. *Suppose that Assumption* (4.2) *holds and that* $f_0 = g_0(x_0, ..., x_{n-1}) + x_n^2$. *Then* $-\mathrm{id}$ *acts trivially on* $M_\mu^{\mathrm{smar}}(f_0)$ *and*

$$\begin{array}{ccccc} M_\mu^{\mathrm{smar}}(f_0) & = & M_\mu^{\mathrm{mar}}(f_0) & = & M_\mu^{\mathrm{mar}}(g_0), \\ G^{\mathrm{mar}}(f_0) & = & G^{\mathrm{smar}}(f_0) & = & G^{\mathrm{mar}}(g_0). \end{array}$$

Suppose additionally that Assumption (4.1) *holds for* g_0 *(instead of* f_0 *in* (4.1)*). Then* $\{\pm\mathrm{id}\}$ *acts freely on* $M_\mu^{\mathrm{smar}}(g_0)$*, and the quotient map*

$$M_\mu^{\mathrm{smar}}(g_0) \overset{/\{\pm\mathrm{id}\}}{\longrightarrow} M_\mu^{\mathrm{mar}}(g_0), \quad [(f, \rho)] \mapsto [(f, \pm\rho)]$$

is a double covering.

Proof. — (a) $C(k, f_0)$ is the base space for the holomorphic μ-constant family $(C(k, f_0), \mathbb{C}^{n+1} \times C(k, f_0), F)$ considered in Lemma 3.5 (a). By Lemma 3.5 (c) and $f_0 = j_k f_0$

$$G^{\mathrm{mar}}(f_0) = \{\pm\psi \,|\, \psi \in G(F, f_0)\}.$$

The space $C^{\mathrm{mar}}(k, f_0)$ contains the set $\{f_0\} \times G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$. The component $(C^{\mathrm{mar}}(k, f_0))^0$ of $C^{\mathrm{mar}}(k, f_0)$ which contains $(f_0, \pm\mathrm{id})$ intersects $\{f_0\} \times G_{\mathbb{Z}}/\{\pm\mathrm{id}\}$ precisely in the set $\{f_0\} \times G^{\mathrm{mar}}/\{\pm\mathrm{id}\}$.

The group $G_{\mathbb{Z}}$ acts on $C^{\mathrm{mar}}(k, f_0)$ by $\psi : (f, \pm\rho) \mapsto (f, \pm\psi \circ \rho)$ for $\psi \in G_{\mathbb{Z}}$. The subgroup which maps the component $(C^{\mathrm{mar}}(k, f_0))^0$ to itself is $G^{\mathrm{mar}}(f_0)$. The quotient map $C^{\mathrm{mar}}(k, f_0) \to M_\mu^{\mathrm{mar}}$ is $G_{\mathbb{Z}}$-equivariant.

As $j_k\mathcal{R}$ is connected, the component $(M_\mu^{\mathrm{mar}}(f_0))^0$ is the quotient of the component $(C^{\mathrm{mar}}(k,f_0))^0$. Therefore $G^{\mathrm{mar}}(f_0)$ is the subgroup which maps $(M_\mu^{\mathrm{mar}}(f_0))^0$ to itself. The bijective correspondence is clear.

(b) Similar to (a).

(c) Similar to (a). Instead of $C^{\mathrm{mar}}(k,f_0)$ one considers the (smooth) analytic subvariety

$$\{(g,\pm\sigma)\,|\,g\in C(k,f_0), g\sim_{\mathcal{R}} f, \sigma: ML(g)\to ML(f_0) \text{ an isomorphism}\}.$$

The action of $j_k\mathcal{R}$ on $C^{\mathrm{mar}}(k,f_0)$ restricts to a transitive action on each component of this subvariety. The components are not permuted as $j_k\mathcal{R}$ is connected. The components are mapped to different points in M_μ^{mar}.

The component which contains $(f,\pm\rho)$ intersects $\{f\}\times(G_{\mathbb{Z}}/\{\pm\mathrm{id}\})\circ\rho$ precisely in $\{f\}\times\rho\circ(G_{\mathcal{R}}^{\mathrm{mar}}(f)/\{\pm\mathrm{id}\})$, because of Lemma 3.5 (c). Therefore the subgroup which maps this component to itself is $\rho\circ G_{\mathcal{R}}^{\mathrm{mar}}(f)\circ\rho^{-1}$. As the quotient map is $G_{\mathbb{Z}}$-equivariant, this subgroup coincides with $\mathrm{Stab}_{G_{\mathbb{Z}}}([(f,\pm\rho)])$.

(d) The first equality follows as in (c). The equality of the first and the third term follows immediately from the definition of right equivalence classes of strongly marked singularities.

(e) By Assumption (4.2) any $[(f,\rho)]\in M_\mu^{\mathrm{smar}}(f_0)$ satisfies $\mathrm{mult}\,f=2$, so $(f,\rho)\sim_{\mathcal{R}}(f,-\rho)$ by part (d) and Theorem 3.3 (g). Therefore $-\mathrm{id}$ acts trivially on $M_\mu^{\mathrm{smar}}(f_0)$, and $M_\mu^{\mathrm{smar}}(f_0)=M_\mu^{\mathrm{mar}}(f_0)$. Theorem 3.3 (b) says $G^{\mathrm{smar}}(f_0)=G^{\mathrm{mar}}(f_0)$. With the two isomorphisms

$$Ml(g)\to Ml(g)\otimes Ml(x_n^2)=Ml(g+x_n^2)$$

which just differ by a sign, the map

$$M_\mu^{\mathrm{mar}}(g_0)\to M_\mu^{\mathrm{mar}}(f_0),\quad [(g,\pm\sigma)]\mapsto[(g+x_n^2,\pm\sigma\otimes\mathrm{id})]$$

is well defined. It is surjective because of the splitting lemma and Assumption (4.2) for f_0. It is bijective because of part (c) and Theorem 3.3 (h). Part (a) and $M_\mu^{\mathrm{mar}}(f_0)=M_\mu^{\mathrm{mar}}(g_0)$ show $G^{\mathrm{mar}}(f_0)=G^{\mathrm{mar}}(g_0)$.

By Assumption (4.1) for g_0, $\mathrm{mult}\,g\geqslant 3$ for any $(g,\sigma)\in M_\mu^{\mathrm{smar}}(g_0)$, so $(g,\sigma)\not\sim_{\mathcal{R}}(g,-\sigma)$ by part (d) and Theorem 3.3 (g). Therefore $\pm\mathrm{id}$ acts freely on $M_\mu^{\mathrm{smar}}(g_0)$ and the map $M_\mu^{\mathrm{smar}}(g_0)\to M_\mu^{\mathrm{mar}}(g_0)$ is a double covering. □

Remarks 4.5. — (i) Theorem 4.4 (a) shows that Conjecture 3.2 (a) is equivalent to the connectedness of $M_\mu^{\mathrm{mar}}(f_0)$. Theorem 4.4 (b) shows that Conjecture 3.2 (b) is equivalent to $[(f,\rho)]$ and $[(f,-\rho)]$ being in different

components of $M_\mu^{\rm smar}(f_0)$ if Assumption (4.1) holds. Together the conjectures say that under Assumption (4.1) $M_\mu^{\rm smar}(f_0)$ has two components, each isomorphic to $M_\mu^{\rm mar}(f_0)$, and that they are permuted by $-\,{\rm id}$.

(ii) Theorem (4.4) (a) and (b) use that $M_\mu^{\rm mar}(f_0)$ respectively $M_\mu^{\rm smar}(f_0)$ is an analytic quotient. But Theorem 4.4 (c) and (d) use only the (trivial) equality $C^{\rm mar}(k,f_0)/j_k\mathcal{R} = M_\mu^{\rm mar}$ as sets (respectively for smar). In part (e) only the equality $G^{\rm mar}(f_0) = G^{\rm mar}(g_0)$ uses that $M_\mu^{\rm mar}(f_0)$ and $M_\mu^{\rm mar}(g_0)$ are analytic geometric quotients. The other statements can be understood and proved without this.

5. Period maps and Torelli type problems

In [7] I had defined an analytic invariant $LBL(f)$ of the right equivalence class of a singularity f and had formulated the Torelli type conjecture that $LBL(f)$ determines f up to right equivalence. I worked on it in [7]–[11]. It is reformulated in Conjecture 5.4. Using $M_\mu^{\rm mar}$, now a stronger conjecture for marked singularities can be proposed, Conjecture 5.3.

First, the invariant $LBL(f)$ will be described, but with the minimum of details necessary to appreciate it. More detailed accounts can be found in [7]–[11, ch. 10]. It builds on the Brieskorn lattice and the Gauss-Manin connection, which had been studied in many ways, *e.g.,* [16][27][3, III][23][21] [22][15].

Fix a singularity $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ and a Milnor fibration $f : Y'(\varepsilon,\delta) \to T'_\delta$ for it as in Section 2. The cohomology bundle

$$H^n_{\mathbb{C}} := \bigcup_{\tau\in T'_\delta} H^n(f^{-1}(\tau),\mathbb{C}) \supset H^n_{\mathbb{Z}} := \bigcup_{\tau\in T'_\delta} H^n(f^{-1}(\tau),\mathbb{Z})$$

is a flat vector bundle of rank μ and contains a flat $\mathbb{Z}$-lattice bundle. Denote by $H^\infty \supset H^\infty_{\mathbb{Z}}$ the spaces of global flat multivalued sections in $H^n_{\mathbb{C}}$ respectively in $H^n_{\mathbb{Z}}$.

A holomorphic section can be written as a linear combination of a basis of $H^\infty_{\mathbb{C}}$ (or $H^\infty_{\mathbb{Z}}$) with multivalued holomorphic coefficients. A section in a punctured neighborhood of 0 is said to have moderate growth respectively vanishing growth if these coefficients have moderate growth respectively vanishing growth. The spaces of germs at 0 of such sections are denoted $V^{>-\infty}$ and $V^{>0}$. Write also $V^{>\alpha} := \tau^\alpha V^{>0}$ for $\alpha \in \mathbb{Z}$. Then $V^{>-\infty}$ is a $\mathbb{C}\{\tau\}[\tau^{-1}]$-vector space of dimension μ and $V^{>\alpha}$ is a free $\mathbb{C}\{\tau\}$-module of rank μ in it with $V^{>\alpha} \otimes_{\mathbb{C}\{\tau\}} \mathbb{C}\{\tau\}[\tau^{-1}] = V^{>-\infty}$.

The Brieskorn lattice $H_0''(f)$ is a free $\mathbb{C}\{\tau\}$-module of rank μ whose (germs of) sections come from differential forms as follows: For $\omega \in \Omega^{n+1}_{Y(\varepsilon,\delta)}$ the section $s[\omega]$ on T'_δ with value

$$s[\omega](\tau) := \left[\frac{\omega}{\mathrm{d}f}_{|f^{-1}(\tau)}\right] \in H^n(f^{-1}(\tau),\mathbb{C})$$

is holomorphic, its germ $s[\omega]_0$ at 0 turns out to be in $V^{>-1}$ [16]. $H_0''(f)$ is generated by such germs. Therefore $V^{>-1} \supset H_0''(f)$. Also $H_0''(f) \supset V^{>n-1}$ holds.

The Brieskorn lattice is a rich invariant. It induces a (sum of two) polarized mixed Hodge structure(s) on $H^\infty \supset H^\infty_{\mathbb{Z}}$ (the mixed Hodge structure: [27][23][21], its polarization: [10], but see [11, Remark 10.25] for a sign mistake in [10]).

Any fiber $H^n_{\mathbb{Z},\tau}$ for $\tau > 0$ is canonically isomorphic to the dual of the Milnor lattice $Ml(f)$. Therefore the Milnor lattice $Ml(f)$ and its monodromy M_h determine uniquely $H^n_{\mathbb{Z}}$, $H^\infty_{\mathbb{Z}}$, $V^{>-\infty}$, $V^{>\alpha}$. Any automorphism of $(Ml(f), M_h)$ induces automorphisms of $H^n_{\mathbb{Z}}$, $H^\infty_{\mathbb{Z}}$, $V^{>-\infty}$, $V^{>\alpha}$, which will be denoted by the same letter as the original automorphism (here $\psi \in \mathrm{Aut}(Ml(f), M_h)$ induces $\psi(\gamma) := \gamma \circ \psi^{-1}$ for $\gamma \in H^n_{\mathbb{Z},\tau}$). Like $Ml(f)$, the germ at 0 of the bundle $H^n_{\mathbb{Z}}$ and the spaces $H^\infty_{\mathbb{Z}}$, $V^{>-\infty}$, $V^{>\alpha}$, $H_0''(f)$ are independent of the choice of the Milnor fibration. The tuple

$$(ML(f),\ \text{the germ at 0 of } H^n_{\mathbb{Z}}, V^{>-\infty}, H_0''(f))$$

will be abbreviated $(ML(f), V^{>-\infty}, H_0''(f))$. It is a datum of the germ $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ (and not of some special representative). The invariant $LBL(f)$ is the isomorphism class of $(ML(f), V^{>-\infty}, H_0''(f))$. It is a datum of the right equivalence class of f.

The group

$$\mathrm{Stab}_{G_{\mathbb{Z}}(f)}(H_0''(f)) := \mathrm{Aut}(ML(f), V^{>-\infty}, H_0''(f)) \subset G_{\mathbb{Z}}(f)$$

is finite because of the polarized mixed Hodge structure on $H^\infty \supset H^\infty_{\mathbb{Z}}$ [10]. If $\varphi \in \mathcal{R}$ with $f = f \circ \varphi$ then $\varphi_{\mathrm{hom}} \in \mathrm{Stab}_{G_{\mathbb{Z}}(f)}(H_0''(f))$, because $H_0''(f)$ is defined by the geometry and is independent of the choice of coordinates. Therefore (and because of Theorem 3.3 (e))

$$G^{\mathrm{mar}}_{\mathcal{R}}(f) \subset \mathrm{Stab}_{G_{\mathbb{Z}}(f)}(H_0''(f)),$$

and $G^{\mathrm{mar}}_{\mathcal{R}}(f)$ is also finite. Conjecture 5.1 for $G^{\mathrm{mar}}_{\mathcal{R}}(f)$ is similar to Conjecture 3.2 (a) for $G^{\mathrm{mar}}(f)$.

Conjecture 5.1. — [11, Conjecture 13.12]

$$G^{\mathrm{mar}}_{\mathcal{R}}(f) = \mathrm{Stab}_{G_{\mathbb{Z}}(f)}(H_0''(f)).$$

See Theorem 5.6 for some cases in which it holds.

In a μ-constant family (X, Y, F) as in Definition 2.1, locally the Milnor lattices with Seifert forms $ML(F_t)$ are canonically isomorphic (Theorem 2.2). Therefore also the germs of bundles $H^n_{\mathbb{Z}}(F_t)$ and the spaces $H^\infty_{\mathbb{Z}}(F_t), V^{>-\infty}(F_t), V^{>\alpha}(F_t)$ are canonically isomorphic. But the Brieskorn lattices vary holomorphically.

Now fix one singularity $f_0 \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$. In [10] a classifying space $D_{BL}(f_0)$ for $\mathbb{C}\{\tau\}$-lattices in $V^{>-\infty}(f_0)$ which have many properties of Brieskorn lattices was constructed, a *classifying space for Brieskorn lattices*. It is a complex manifold.

Let $(f, \pm\rho)$ be a marked singularity, so f is in the μ-homotopy class of f_0 and $\rho : ML(f) \to ML(f_0)$ is an isomorphism. Then ρ induces an isomorphism $\rho : V^{>-\infty}(f) \to V^{>-\infty}(f_0)$, and $\rho(H''_0(f))$ is a point in $D_{BL}(f_0)$. One obtains a period map

$$BL : M^{\mathrm{mar}}_\mu(f_0) \to D_{BL}(f_0), \quad [(f, \pm\rho)] \mapsto \rho(H''_0(f)).$$

Locally, $M^{\mathrm{mar}}_\mu(f_0)$ is isomorphic to a μ-constant stratum (Theorem 4.3 (b)). Locally, this period map had been studied in [21][22][7]–[11], and it is holomorphic, so BL is holomorphic.

THEOREM 5.2. — [11, Theorem 12.8] *BL is an immersion, here the reduced complex structure on $M^{\mathrm{mar}}_\mu(f_0)$ is considered.*

This improves a slightly weaker result (finite-to-one) in [22]. It is an infinitesimal Torelli type result. I have some evidence that BL is also an immersion with the canonical complex structure on $M^{\mathrm{mar}}_\mu(f_0)$. But if true, this will be subject of another paper.

The following is a global Torelli type conjecture for marked singularities.

CONJECTURE 5.3. — *(New) BL is injective, here the canonical complex structure on $M^{\mathrm{mar}}_\mu(f_0)$ is considered.*

The group $G_{\mathbb{Z}}(f_0)$ acts on $M^{\mathrm{mar}}_\mu(f_0)$ and on $D_{BL}(f_0)$ properly discontinuously, and by its definition the period map BL is $G_{\mathbb{Z}}(f_0)$-equivariant. The quotient $M^{\mathrm{mar}}_\mu(f_0)/G_{\mathbb{Z}}(f_0)$ is the moduli space $M_\mu(f_0)$ of right equivalence classes of singularities in the μ-homotopy class of f_0 (Theorem 4.3 (d)), the quotient $D_{BL}(f_0)/G_{\mathbb{Z}}(f_0)$ is a moduli space for the invariants $LBL(f)$. One obtains a period map

$$LBL : M_\mu(f_0) \to D_{BL}(f_0)/G_{\mathbb{Z}}(f_0), \quad [f] \mapsto LBL(f).$$

The global Torelli type conjecture for right equivalence classes of singularities from [7] says that LBL is injective where the reduced complex structure on $M_\mu(f_0)$ is considered. It can be strengthened as follows.

Conjecture 5.4. — [11, Conjecture 12.7] *LBL is injective, here the canonical complex structure on $M_\mu(f_0)$ is considered.*

Lemma 5.5. — *Conjecture 5.3 $\iff$ Conjecture 5.1 (for all singularities f in the μ-homotopy class of f_0) and Conjecture 5.4.*

Proof. — Conjecture 5.4 says that $\rho(H_0''(f)) \neq \widetilde{\rho}(H_0''(\widetilde{f}))$ for all possible markings $\pm\rho$ and $\pm\widetilde{\rho}$ of f and $\widetilde{f}$ if and only if $f \not\sim_{\mathcal{R}} \widetilde{f}$. In the case $f = \widetilde{f}$, $\rho(H_0''(f)) = \widetilde{\rho}(H_0''(f))$ is equivalent to $\widetilde{\rho}^{-1} \circ \rho \in \mathrm{Stab}_{G_{\mathbb{Z}}(f)}(H_0''(f))$. By Conjecture 5.1 this is equivalent to $(f, \pm\rho) \sim_{\mathcal{R}} (f, \pm\widetilde{\rho})$.

This shows the equivalence in the case of the reduced complex structures. For the canonical complex structures, one observes that the one on $M_\mu(f_0)$ is induced by the one on $M_\mu^{\mathrm{mar}}(f_0)$ by the quotient map $M_\mu^{\mathrm{mar}}(f_0) \to M_\mu(f_0) = M_\mu^{\mathrm{mar}}(f_0)/G_{\mathbb{Z}}(f_0)$ (Theorem 4.3 (d)). □

Theorem 5.6.

(a) *Conjecture 5.4 is true for all singularities with modality $\leqslant 2$ possibly with the exception of the subseries $Z_{1,14k}$, $S_{1,10k}$, $S^\sharp_{1,10k}$ ($k \geqslant 1$)* [7][8]*. It is true for the μ-homotopy classes of the Brieskorn-Pham singularities $\sum_{i=0}^n x_i^{a_i}$ with pairwise coprime exponents and for the μ-homotopy class of the singularity $x_0^3 + x_1^3 + x_2^3 + x_3^3$* [9].

(b) [12] $\mathrm{Stab}_{G_{\mathbb{Z}}}(H_0''(f)) = \{\pm\mathrm{id}\}$ *for generic semiquasihomogeneous singularities with $\frac{n+1}{2} - \sum_{i=0}^n w_i \geqslant 4$.*

(c) [12] *The period map LBL is generically injective for the semiquasihomogeneous singularities with $\frac{n+1}{2} - \sum_{i=0}^n w_i \geqslant 4$.*

(d) *Conjecture 5.3 (and by Lemma 5.5 also the conjectures 5.1 and 5.4, but the last one is known since* [7]*) is true for all singularities listed in Theorem 3.3 (a).*

The new part of this theorem is part (d). For the simple, 22 of the 28 exceptional and the Brieskorn-Pham singularities it will be proved in Section 8. The remaining 6 exceptional, the simple-elliptic and the hyperbolic singularities will be treated in another paper. In all these cases one can build on the study of the period map $LBL : M_\mu \to D_{BL}/G_{\mathbb{Z}}$ in [7]. The crucial new point is to determine M_μ^{mar}. And central for this is to prove Conjecture 3.2 (a). Then M_μ^{mar} has only one component by Theorem 4.4 (a).

One may expect that $\mathrm{Stab}_{G_{\mathbb{Z}}}(H_0''(f)) = \{\pm\mathrm{id}\}$ for generic singularities in one μ-homotopy class, if that class is not too small. Part (b) shows this for semiquasihomogeneous singularities with $\frac{n+1}{2} - \sum_{i=0}^n w_i \geqslant 4$. But special members may have a large finite isotropy group of their Brieskorn lattice.

6. Symmetries of singularities

Here we will review some results on symmetries of singularities from [11, 13.1 and 13.2] and simplify the proofs. The results will imply Theorem 3.3 (f)+(g)+(h). They will also be used in the proof of Theorem 4.3 in Section 7. They build on work of Slodowy [24] and Wall [28][29].

An *unfolding* of a singularity $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$(a germ with an isolated singularity at 0) is a holomorphic function germ $F : (\mathbb{C}^{n+1} \times M, 0) \to (\mathbb{C}^{n+1}, 0)$ with $F_{|(\mathbb{C}^{n+1}\times\{0\},0)} = f$ and $(M, 0)$ a germ of a manifold. It is *versal* if it induces any unfolding $G : (\mathbb{C}^{n+1} \times N, 0) \to (\mathbb{C}, 0)$, that means, a morphism $\varphi : (N, 0) \to (M, 0)$ and a morphism $\Phi : (\mathbb{C}^{n+1} \times N, 0) \to (\mathbb{C}^{n+1} \times M, 0)$ exist such that

$$\mathrm{pr}_M \circ \Phi = \varphi \circ \mathrm{pr}_N \quad \text{and} \quad G = F \circ \Phi \quad \text{and} \quad \Phi_{|(\mathbb{C}^{n+1}\times\{0\},0)} = \mathrm{id}\,.$$

It is *semiuniversal* if it is versal und if $\dim(M, 0)$ is minimal.

Semiuniversal unfoldings exist (see *e.g.*, [17][2]). Denote by $J_F := (\partial F/\partial x_i) \subset \mathcal{O}_{\mathbb{C}^{n+1}\times M,0}$ the *Jacobi ideal* of F, and by J_f the one of f. An unfolding is semiuniversal if and only if the map

$$\mathbf{a}_C : \mathcal{T}_{M,0} \to \mathcal{O}_{\mathbb{C}^{n+1}\times M,0}/J_F, \quad \frac{\partial}{\partial t_i} \mapsto \left[\frac{\partial F}{\partial t_i}\right]$$

is an isomorphism. Equivalent is that the map

$$\mathbf{a}_0 : T_0M \to \mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$$

is an isomorphism. Then $\mathbf{a}_C$ induces a multiplication $\circ$ on $\mathcal{T}_{M,0}$, a unit vector field $e := \mathbf{a}_C^{-1}(1)$ and an Euler field $E := \mathbf{a}_C^{-1}([F])$. Then $((M, 0), \circ, e, E)$ is the germ of an *F-manifold with Euler field*, see [13][11] for its definition.

One can choose good representatives F and M. Then for all $t \in M$ the sum of the Jacobi algebras of the critical points of F_t is isomorphic via $\mathbf{a}_C$ to T_tM as an algebra. For generic $t \in M$ F_t has only A_1-singularities, so the multiplication on TM is generically semisimple. Such an F-manifold is called *massive*. The group $\mathrm{Aut}_M := \mathrm{Aut}((M, 0), \circ, e, E)$ of automorphisms of a germ of a massive F-manifold with Euler field is finite [11, Theorem 4.14].

Denote by

$$\mathcal{R}^f := \{\varphi \in \mathcal{R} \,|\, f = f \circ \varphi\}$$

the group of *symmetries* of f.

Consider a semiuniversal unfolding F of f with base space $(M, 0)$ and a symmetry φ of f. Then $F \circ \varphi^{-1}$ is a semiuniversal unfolding of f with the

same base space $(M,0)$. It is induced by F via a pair (Φ,φ_M) of isomorphisms with

$$F\circ\varphi^{-1}=F\circ\Phi,\quad \Phi_{|(\mathbb{C}^{n+1}\times\{0\},0)}=\mathrm{id},\quad \varphi_M\in\mathrm{Aut}_M\,.$$

Then $\widetilde{\Phi}:=\Phi\circ\varphi$ satisfies

$$\mathrm{pr}_M\circ\widetilde{\Phi}=\varphi_M\circ\mathrm{pr}_M,\quad F=F\circ\widetilde{\Phi},\quad \widetilde{\Phi}_{|(\mathbb{C}^{n+1}\times\{0\},0)}=\varphi. \tag{6.1}$$

Here Φ is not at all unique, but φ_M is unique because Aut_M is finite and $\mathbf{a}_0\circ(\mathrm{d}\varphi_M)_{|0}\circ\mathbf{a}_0^{-1}$ is the automorphism of $\mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$ which is induced by φ. One obtains a group homomorphism

$$()_M:\mathcal{R}^f\to\mathrm{Aut}_M,\quad \varphi\mapsto\varphi_M.$$

The group $\mathcal{R}^f$ is possibly ∞-dimensional, but the group $j_k\mathcal{R}^f$ of k-jets in $\mathcal{R}^f$ is an algebraic group for any k. Let

$$R_f:=j_1\mathcal{R}^f/(j_1\mathcal{R}^f)^0$$

be the finite group of components of $j_1\mathcal{R}^f$. It is easy to see that $R_f=j_k\mathcal{R}^f/(j_k\mathcal{R}^f)^0$ for any $k\geqslant 1$ [11, Lemma 13.8]. The following theorem is contained in [11, Theorem 13.9], except for part (d). Some parts of the proof below are simpler than in [11].

THEOREM 6.1. — *Fix a singularity $f\in\mathbf{m}^2_{\mathbb{C}^{n+1},0}$ and a semiuniversal unfolding F with base space $(M,0)$.*

(a) *The homomorphism $()_M:\mathcal{R}^f\to\mathrm{Aut}_M$ factors through R_f to a homomorphism $()_M:R_f\to\mathrm{Aut}_M$.*

(b) *If $\mathrm{mult}\,f\geqslant 3$ then $()_M:R_f\to\mathrm{Aut}_M$ is an isomorphism. And then $j_1\mathcal{R}^f=R_f$.*

(c) *If $\mathrm{mult}\,f=2$ then $()_M:R_f\to\mathrm{Aut}_M$ is surjective with kernel of order 2. If $f=g(x_0,...,x_{n-1})+x_n^2$ then the kernel is generated by the class of the symmetry $(x\mapsto(x_0,...,x_{n-1},-x_n))$.*

(d) *If $\mathrm{mult}\,f=2$ denote by $\varphi^{(1)}$ and $\varphi^{(2)}$ the (linear) actions of $\varphi\in\mathcal{R}^f$ on $\frac{\mathbf{m}}{\mathbf{m}^2+J_f}$ and on $\frac{\mathbf{m}^2+J_f}{\mathbf{m}^2}$. Then $\det\varphi^{(2)}\in\{\pm1\}$. The homomorphism*

$$\mathcal{R}^f\to\mathrm{Aut}\left(\frac{\mathbf{m}}{\mathbf{m}^2+J_f}\right)\times\{\pm1\},\quad \varphi\mapsto(\varphi^{(1)},\det\varphi^{(2)})$$

factors through R_f to an injective homomorphism

$$R_f\to\mathrm{Aut}\left(\frac{\mathbf{m}}{\mathbf{m}^2+J_f}\right)\times\{\pm1\}.$$

(e) *If $f=g(x_0,..,x_m)+x_{m+1}^2+...+x_n^2$ and $\mathrm{mult}\,g\geqslant 3$ then $R_f=R_g\times(\ker()_M)\cong R_g\times S_2$. If $\mathrm{mult}\,g=2$ then $R_f=R_g$.*

(f) *The homomorphism* $()_{\rm hom} : \mathcal{R}^f \to G_{\mathbb{Z}}(f)$ *factors through* R_f *to a homomorphism* $()_{\rm hom} : R_f \to G_{\mathbb{Z}}(f)$. *The image is*

$$(R_f)_{\rm hom} = G^{\rm smar}_{\mathcal{R}}(f) \subset {\rm Stab}_{G_{\mathbb{Z}}}(H''_0(f)) \subset G_{\mathbb{Z}}(f).$$

(g) *The homomorphism* $()_{\rm hom} : R_f \to G^{\rm smar}_{\mathcal{R}}(f)$ *is an isomorphism.*

(h) *The homomorphism* $()_M \circ ()^{-1}_{\rm hom} : G^{\rm smar}_{\mathcal{R}}(f) \to {\rm Aut}_M$ *is an isomorphism if* $\mu f \geqslant 3$. *It is 2–1 with kernel* $\{\pm {\rm id}\}$ *if* $\mu f = 2$. *In any case it extends to a 2–1 morphism* $()_{{\rm hom}\to M} : G^{\rm mar}_{\mathcal{R}}(f) \to {\rm Aut}_M$ *with kernel* $\{\pm {\rm id}\}$.

(i) *Theorem 3.3 (f)+(g)+(h) is true.*

Proof. — (a) The action of $\varphi \in \mathcal{R}^f$ on $\mathcal{O}_{\mathbb{C}^{n+1},0}/J_f$ depends only on a sufficiently high k-jet of φ, and it depends continuously on it. Because of $\mathcal{O}/J_f \cong T_0M$ and because ${\rm Aut}_M$ is finite, k-jets of symmetries in one component of $j_k\mathcal{R}^f$ induce the same element of ${\rm Aut}_M$.

(b) Surjectivity: The F-manifold $(M, \circ, e, E)$ determines a Lagrange variety in T^*M, and this determines up to isomorphism a semiuniversal unfolding, see [11, Theorem 5.6] for details and [2, 19.3] for the relation between Lagrange maps and unfoldings. Therefore any $\varphi_M \in {\rm Aut}_M$ lifts to an automorphism $(\widetilde{\Phi}, \varphi_M)$ of the unfolding F, with $\widetilde{\Phi}$ as in (6.1).

Injectivity: $\mu f \geqslant 3$ implies $J_f \subset \mathbf{m}^2$ and surjectivity of the map $\mathbf{m}/J_f \to \mathbf{m}/\mathbf{m}^2$. The action of $j_1\mathcal{R}^f$ on $\mathbf{m}/\mathbf{m}^2$ is faithful. Therefore $j_1\mathcal{R}^f = R_f$ and $()_M : R_f \to {\rm Aut}_M$ is injective.

(c)+(d)+(e) Surjectivity in (c): as in (b).

It is sufficient to consider the case $f = g(x_0, ..., x_m) + x^2_{m+1} + ... + x^2_n$ with $\mu g \geqslant 3$, $m < n$. Then

$$\frac{\mathbf{m}}{\mathbf{m}^2 + J_f} = \frac{\mathbf{m}}{\mathbf{m}^2 + (x_{m+1}, ..., x_n)} \quad \text{and} \quad \frac{\mathbf{m}^2 + J_f}{\mathbf{m}^2} = \frac{\mathbf{m}^2 + (x_{m+1}, ..., x_n)}{\mathbf{m}^2}.$$

The kernel of the natural homomorphism

$$j_1\mathcal{R}^f \to {\rm Aut}\left(\frac{\mathbf{m}}{\mathbf{m}^2 + J_f}\right) \times {\rm Aut}\left(\frac{\mathbf{m}^2 + J_f}{\mathbf{m}^2}\right)$$

is unipotent, so connected. The image is $R_g \times O(n-m)$. The second factor is due to $j_2 f = x^2_{m+1} + ... + x^2_n$. For the first factor observe the following: $(M, 0)$ is also the base space of a semiuniversal unfolding of g, $()_M : R_g \to {\rm Aut}_M$ is an isomorphism, $()_M : R_f \to {\rm Aut}_M$ is surjective, so R_g and R_f induce the same automorphisms of T_0M. Also

$$T_0M \cong \frac{\mathcal{O}_{\mathbb{C}^{m+1},0}}{J_g} \supset \frac{\mathbf{m}_{\mathbb{C}^{m+1},0}}{J_g} \twoheadrightarrow \frac{\mathbf{m}_{\mathbb{C}^{m+1},0}}{\mathbf{m}^2_{\mathbb{C}^{m+1},0}} \cong \frac{\mathbf{m}}{(\mathbf{m}^2 + J_f)},$$

and $j_1\mathcal{R}^g = R_g$ acts faithfully on $\mathbf{m}_{\mathbb{C}^{m+1},0}/\mathbf{m}^2_{\mathbb{C}^{m+1},0}$ Therefore the image above is $R_g \times O(n-m)$.

As the kernel of $j_1\mathcal{R}^f \to R_g \times O(n-m)$ is connected

$$\begin{aligned} R_f &= j_1\mathcal{R}^f/(j_1\mathcal{R}^f)^0 = R_g \times O(n-m)/(R_g \times O(n-m))^0 \\ &= R_g \times \{\mathrm{id}, (\text{the class of } (x \mapsto (x_0, ..., x_{n-1}, -x_n) \text{ in } R_f)\}. \end{aligned}$$

The rest is clear now, too.

(f) $G_{\mathbb{Z}}(f)$ is a discrete group. $(R_f)_{\mathrm{hom}} = G^{\mathrm{smar}}(f)$ is Theorem 3.3 (e).

(g) It rests to show that $()_{\mathrm{hom}} : R_f \to G_{\mathbb{Z}}(f)$ is injective. Suppose $\varphi_{\mathrm{hom}} = \mathrm{id}$ for some $\varphi \in \mathcal{R}^f$. Then φ_{hom} acts trivially on $H''_0(f)$. The space

$$\Omega_f := \Omega^{n+1}_{\mathbb{C}^{n+1},0}/\mathrm{d}f \wedge \Omega^n_{\mathbb{C}^{n+1},0}$$

is a quotient of $H''_0(f)$ (due to Brieskorn, see *e.g.*, [16][3][11]). It is a free $\mathcal{O}/J_f$-module of rank 1, generated by the class $[\omega_0]$ of the volume form $\omega_0 := \mathrm{d}x_0 \wedge ... \wedge \mathrm{d}x_n$. The action of φ on Ω_f is trivial, because the action on $H''_0(f)$ is trivial. Therefore $[\varphi^*\omega_0] = [\omega_0]$ and the action of φ on $\mathcal{O}/J_f$ is trivial. Therefore $\det\left(\frac{\partial \varphi_j}{\partial x_i}\right)(0) = 1$ and $\varphi_M = \mathrm{id}$. Because of (b) and (c) the class of φ in R_f is id.

(h) For $f = g(x_0, ..., x_{n-1}) + x_n^2$ the proof of Theorem 3.3 (b) shows

$$(x \mapsto (x_0, ..., x_{n-1}, -x_n))_{\mathrm{hom}} = -\mathrm{id}\,.$$

Thus if $\mathrm{mult}\, f = 2$ then $-\mathrm{id} \in G^{\mathrm{smar}}_{\mathcal{R}}(f)$ and $(\{\pm \mathrm{id}\})^{-1}_{\mathrm{hom}} = \ker()_M$.

If $\mathrm{mult}\, f \geqslant 3$ then $()_M \circ ()^{-1}_{\mathrm{hom}} : G^{\mathrm{smar}}_{\mathcal{R}}(f) \to \mathrm{Aut}_M$ is an isomorphism, and the extension to $G^{\mathrm{smar}}_{\mathcal{R}}(f + x^2_{n+1})$ is 2-1 with kernel $\{\pm \mathrm{id}\}$, thus $-\mathrm{id} \notin G^{\mathrm{smar}}_{\mathcal{R}}(f)$.

(i) Compare (e)–(h). □

In the case of a quasihomogeneous singularity the group R_f has a canonical lift to $\mathcal{R}^f$. It will be useful for the calculation of R_f.

THEOREM 6.2. — [11, Theorem 13.11] *Let $f \in \mathbb{C}[x_0, ..., x_n]$ be a quasihomogeneous polynomial with an isolated singularity at 0 and weights $w_0, ..., w_n \in \mathbb{Q} \cap (0, \frac{1}{2}]$ and weighted degree 1. Suppose that $w_0 \leqslant ... \leqslant w_{n-1} < \frac{1}{2}$ (then $f \in \mathbf{m}^3$ if and only if $w_n < \frac{1}{2}$). Let G_w be the algebraic group of quasihomogeneous coordinate changes, that means, those which respect $\mathbb{C}[x_0, ..., x_n]$ and the grading by the weights $w_0, ..., w_n$ on it. Then*

$$R_f \cong \mathrm{Stab}_{G_w}(f).$$

7. Proof of Theorem 4.3

The proof of Theorem 4.3 will be similar to the proof of Theorem 13.15 in [11]. Like that proof it will use results from [11, 13.3], they are reformulated in Theorem 7.2. But it will use also joint consequences of these results and Theorem 6.1, they are formulated in Corollary 7.3. The proof of Theorem 4.3 comes after it. The results in [11, 13.3] concern the μ-constant stratum.

For a moment, fix a singularity $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ and choose a good representative $F : U \times M \to \mathbb{C}$ with $U \subset \mathbb{C}^{n+1}$ of a semiuniversal unfolding, with base space M. The μ-constant stratum $S_\mu \subset M$ is

$$\begin{aligned} S_\mu &= \{t \in M \,|\, \mathrm{Crit}(F_t) = \{x\} \text{ and } F_t(x) = 0\} \\ &= \{t \in M \,|\, 0 \text{ is the only critical value of } F_t\} \\ &= \{t \in M \,|\, E\circ \text{ is nilpotent on } T_tM\}. \end{aligned}$$

The second equality is due to Gabrielov [6], Lazzeri and Lê. The third equality follows from the definition of multiplication and Euler field on M (see Section 6): The eigenvalues of $E\circ$ on T_tM are the critical values of F_t.

The germ $(S_\mu, 0) \subset (M, 0)$ is a datum of the germ $(M, 0)$ of an F-manifold with Euler field, any automorphism $\psi \in \mathrm{Aut}_M := \mathrm{Aut}((M, 0), \circ, e, E)$ restricts to an automorphism of $(S_\mu, 0)$.

The critical points x of F_t with $t \in S_\mu$ might a priori not be equal to 0. But by a result of Teissier [25, 6.14] there exists a holomorphic section $\sigma : M \to U \times M$ with $\mathrm{Crit}(F_t) = \{\sigma(t)\}$ for $t \in S_\mu$. Because $F(x+\sigma(t), t)$ is also a semiuniversal unfolding of f, we can assume from now on that for $t \in S_\mu$ $\mathrm{Crit}(F_t) = \{0\}$, Then the restriction of F to S_μ is a holomorphic μ-constant family in the sense of Definition 2.1. By Theorem 2.2 it comes equipped with a flat bundle $ML(F_{|S_\mu})$ of Milnor lattices with Seifert forms L.

Any $\varphi \in \mathrm{Aut}_M$ lifts to an automorphism of the (germ of the) unfolding F (see the proof of Theorem 6.1 (b) and [11, Theorem 5.6]). Therefore one may expect that $(S_\mu, 0)/\sim_{\mathcal{R}} = (S_\mu, 0)/\mathrm{Aut}_M$. This is true and part of much stronger results in [11, 13.3]. They are cited in Theorem 7.2. The existence of an unfolding with the properties in Definition 7.1 is part of them.

DEFINITION 7.1. — *Fix a singularity $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$. A representative $F : U \times M \to \mathbb{C}$ of a semiuniversal unfolding, with $U \subset \mathbb{C}^{n+1}$ a neighborhood of 0 and M the base space, is called a* very good representative *if the following holds.*

(i) *Any $\varphi \in \mathrm{Aut}_M$ extends (from the germ) to an automorphism of the F-manifold M.*
(ii) *Any isomorphism $\psi : (M,t) \to (M,\widetilde{t})$ of germs of F-manifolds with Euler fields and with $t, \widetilde{t} \in S_\mu$ is the restriction of an element of Aut_M.*
(iii) *Any $\varphi \in \mathrm{Aut}_M$ lifts to an automorphism (Φ, φ) of the unfolding F, that means, $\Phi : Y_1 \to Y_2$ is an isomorphism of suitable open subsets Y_1 and Y_2 of $U \times M$ which contain all critical points of all F_t, with $\mathrm{pr}_M \circ \Phi = \varphi \circ \mathrm{pr}_M$ and $F = F \circ \Phi$.*
(iv) *S_μ is contractible. Therefore $ML(F_t)$ for $t \in S_\mu$ can and will be identified with $ML(f)$.*

Theorem 7.2 collects the main results of [11, 13.3]. We will not review the proofs here. They use the construction of Frobenius manifolds on the base spaces of semiuniversal unfoldings and an interplay of this with the polarized mixed Hodge structures on the spaces H^∞ from Section 5.

THEOREM 7.2. — *Fix a singularity $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$.*

(a) [11, Theorem 13.18] *A very good representative of a semiuniversal unfolding exists. For such a representative $S_\mu / \sim_{\mathcal{R}} = S_\mu / \mathrm{Aut}_M$.*
(b) [11, Theorem 13.17] *If $\widetilde{f} \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ is a singularity in the μ-homotopy class of f, but not right equivalent to f, then very good representatives F and $\widetilde{F}$ of semiuniversal unfoldings of f and $\widetilde{f}$ exist with μ-constant strata $S_\mu \subset M$ and $\widetilde{S}_\mu \subset \widetilde{M}$ such that $F_t \not\sim_{\mathcal{R}} \widetilde{F}_{\widetilde{t}}$ for any $t \in S_\mu$ and any $\widetilde{t} \in \widetilde{S}_\mu$.*
(c) [11, Theorem 13.15] *Recall the space $C(k,f) \subset \mathbf{m}^2/\mathbf{m}^{k+1}$ from Section 3. For $k \geqslant \mu + 1$, the space $C(k,f)/j_k\mathcal{R}$ is an analytic geometric quotient. It is a moduli space for the right equivalence classes in the μ-homotopy class of f. Locally at $[f]$ it is isomorphic to S_μ / Aut_M where S_μ is the μ-constant stratum of a very good representative of a semiuniversal unfolding of f. A priori it carries the induced reduced complex structure. But it comes also equipped with a canonical complex structure induced by that on μ-constant strata in* [11, Theorem 12.4].

COROLLARY 7.3. — *Fix two singularities f_0 and $f \in \mathbf{m}^2_{\mathbb{C}^{n+1},0}$ in the same μ-homotopy class. Fix a very good representative $F : U \times M \to \mathbb{C}$ of a semiuniversal unfolding of f. Fix two isomorphisms ρ and $\widetilde{\rho} : ML(f) \to ML(f_0)$. Suppose that $(F_t, \pm\rho) \sim_{\mathcal{R}} (F_{\widetilde{t}}, \pm\widetilde{\rho})$ for some $t, \widetilde{t} \in S_\mu$ [respectively that $(F_t, \rho) \sim_{\mathcal{R}} (F_{\widetilde{t}}, \widetilde{\rho})$ and that Assumption* (4.1) *or* (4.2) *holds].*

Then $\widetilde{\rho}^{-1}\circ\rho \in G^{\mathrm{mar}}_{\mathcal{R}}(f)$ *[respectively* $\widetilde{\rho}^{-1}\circ\rho \in G^{\mathrm{smar}}_{\mathcal{R}}(f)$*], and the image* $(\widetilde{\rho}^{-1}\circ\rho)_{\mathrm{hom}\to M}$ *in* Aut_M *(defined in Theorem 6.1 (h)) satisfies: For any* $s,\widetilde{s}\in S_\mu$

$$\widetilde{s}=(\widetilde{\rho}^{-1}\circ\rho)_{\mathrm{hom}\to M}(s) \iff (F_s,\pm\rho)\sim_{\mathcal{R}}(F_{\widetilde{s}},\pm\widetilde{\rho}) \quad [\text{respectively } (F_s,\rho)\sim_{\mathcal{R}}(F_{\widetilde{s}},\widetilde{\rho})].$$

Proof. — A coordinate change $\varphi\in\mathcal{R}$ with $F_t=F_{\widetilde{t}}\circ\varphi$ and $\rho=\pm\widetilde{\rho}\circ\varphi_{\mathrm{hom}}$ [respectively $\rho=\widetilde{\rho}\circ\varphi_{\mathrm{hom}}$] exists. Exactly as in the discussion of the homomorphism $\mathcal{R}^f\to\mathrm{Aut}_M$ before Theorem 6.1, it induces an isomorphism $\varphi_M:(M,t)\to(M,\widetilde{t})$ of germs of F-manifolds with Euler fields. Because F is a very good representative of a semiuniversal unfolding, φ_M is in Aut_M, and φ_M lifts to an automorphism (Φ,φ_M) of the unfolding. Denote $\Phi_s:=\Phi_{|(\mathbb{C}^{n+1}\times\{s\},0)}$ for $s\in S_\mu$. Then $f=f\circ\Phi_0$, $F_t=F_{\widetilde{t}}\circ\Phi_t$, and $\varphi_M=(\Phi_0)_M$. Thus

$$F_t=F_t\circ\varphi^{-1}\circ\Phi_t,\quad (\varphi^{-1}\circ\Phi_t)_M=\mathrm{id},$$

and

$$(\varphi^{-1}\circ\Phi_t)_{\mathrm{hom}}=\pm\,\mathrm{id}$$

by Theorem 6.1 (h) [$(\varphi^{-1}\circ\Phi_t)_{\mathrm{hom}}=\mathrm{id}$ in the case of Assumption (4.1)]. Therefore

$$f=f\circ\Phi_0,\quad \varphi_M=(\Phi_0)_M,\quad \pm\widetilde{\rho}^{-1}\circ\rho=\varphi_{\mathrm{hom}}=\pm(\Phi_0)_{\mathrm{hom}}\in G^{\mathrm{mar}}_{\mathcal{R}}(f)$$

[$(\widetilde{\rho}^{-1}\circ\rho\in G^{\mathrm{smar}}_{\mathcal{R}}(f)$ in the case of Assumption (4.1)]. In the case of Assumption (4.2) $-\mathrm{id}\in G^{\mathrm{smar}}_{\mathcal{R}}(f)$, and also $\widetilde{\rho}^{-1}\circ\rho\in G^{\mathrm{smar}}_{\mathcal{R}}(f)$.

In any case $(\widetilde{\rho}^{-1}\circ\rho)_{\mathrm{hom}\to M}=\varphi_M$.

Going again through the proof, now with s and $\widetilde{s}$ instead of t and $\widetilde{t}$ one obtains $\Leftarrow$. The implication $\Rightarrow$ follows from $F_s=F_{\varphi_M(s)}\circ\Phi_s$ and $(\Phi_s)_{\mathrm{hom}}=\pm\widetilde{\rho}^{-1}\circ\rho$ [respectively $(\Phi_s)_{\mathrm{hom}}=\widetilde{\rho}^{-1}\circ\rho$ in the case of Assumption (4.1), and with $(F_s,\rho)\sim_{\mathcal{R}}(F_s,-\rho)$ in the case of Assumption (4.2)]. □

Proof of Theorem 4.3. — (a) This is clear.

(b) We will use a result of Holmann [14, Satz 17] which shows that the quotient is an analytic geometric quotient if two criteria are satisfied. The first is that the quotient topology is Hausdorff. The second is the existence of holomorphic functions in a neighborhood of a point $(f,\pm\rho)$ in $C^{\mathrm{mar}}(k,f_0)$ which are constant on $j_k\mathcal{R}$ orbits and which separate points in different orbits.

Fix $(f,\pm\rho)\in C^{\mathrm{mar}}(k,f_0)$. A *transversal disk* for f is an embedding $j:\check{M}\to\mathbf{m}^2/\mathbf{m}^{k+1}$ of an open neighborhood of 0 in $\mathbb{C}^{\mu-1}$ into $\mathbf{m}^2/\mathbf{m}^{k+1}$ such that $j(0)=f$ and $j(\check{M})$ intersects $j_k\mathcal{R}\cdot f$ transversally in f.

By a construction of Gabrielov [6] and the result of Teissier [25, 6.14] cited above, the germ $(j(\check{M}) \cap C(k, f_0), f)$ is isomorphic to the μ-constant stratum of f with reduced complex structure in a semiuniversal unfolding, by an isomorphism which maps singularities in $j(\check{M}) \cap C(k, f_0)$ to parameters of right equivalent singularities in the μ-constant stratum (see [11, Proof of Theorem 13.15] for the details).

Now Theorem 7.2 (b) and Corollary 7.3 show that the quotient topology on $C^{\mathrm{mar}}(k, f_0)/j_k\mathcal{R}$ is Hausdorff. This gives the first criterion of Holmann.

Let us choose a small submanifold $R \subset j_k\mathcal{R}$ which contains $\mathrm{id} \in j_k\mathcal{R}$ and which is transversal at id to the stabilizer in $j_k\mathcal{R}$ of f. Then

$$R \times (j(\check{M}) \cap C(k, f_0)) \xrightarrow{\cong} (\text{a certain neighborhood of } f \text{ in } C(k, f_0)) =: S(f).$$

The marking $\pm\rho$ in $(f, \pm\rho)$ induces a marking $(g, \pm\rho)$ for all $g \in S(f)$. $S(f) \times \{\pm\rho\}$ is a neighborhood of $(f, \pm\rho)$ in $C^{\mathrm{mar}}(k, f_0)$.

Because of Corollary 7.3 the $j_k\mathcal{R}$-orbit of $(g, \pm\rho)$ intersects this neighborhood only in $R \cdot g \times \{\pm\rho\}$. The holomorphic functions on $j(\check{M}) \cap C(k, f_0)$ lift to this neighborhood and satisfy the second criterion of Holmann.

Therefore $M_\mu^{\mathrm{mar}} = C^{\mathrm{mar}}(k, f_0)/j_k\mathcal{R}$ is an analytic geometric quotient, and locally it is isomorphic to the μ-constant stratum of a singularity with the reduced complex structure.

The canonical complex structures from [11, Theorem 12.4] on all the μ-constant strata glue together. This follows from their construction: By construction, if $(S_\mu, 0)$ is a germ of a μ-constant stratum and S_μ is a sufficiently small representative then its canonical complex structure from $(S_\mu, 0)$ restricts for any $t \in S_\mu$ to the canonical complex structure on (S_μ, t). Therefore the canonical complex structures glue to a canonical complex structure on M_μ^{mar}.

(c) The map ψ_{mar} is a bijection, and locally it maps one copy of a μ-constant stratum of f to another copy, so it is an isomorphism. The rest is clear.

(d) By definition of ψ_{mar}, $[(f_1, \pm\rho_1)]$ and $[(f_2, \pm\rho_2)] \in M_\mu^{\mathrm{mar}}$ are in one $G_{\mathbb{Z}}$-orbit if and only if $f_1 \sim_{\mathcal{R}} f_2$. Therefore $M_\mu^{\mathrm{mar}}/G_{\mathbb{Z}} = M_\mu$ as a set.

For some $[(f, \pm\rho)] \in M_\mu^{\mathrm{mar}}$ choose a very good representative F of a semiuniversal unfolding with base space M and μ-constant stratum $S_\mu \subset M$ such that a neighborhood of $[(f, \pm\rho)]$ in M_μ^{mar} is isomorphic to S_μ.

Suppose that $\psi_{\mathrm{mar}}([(F_t, \pm\rho)]) = [(F_{\widetilde{t}}, \pm\rho)]$ for some $t, \widetilde{t} \in S_\mu$. Corollary 7.3 shows that $\rho^{-1} \circ \psi \circ \rho \in G_{\mathcal{R}}^{\mathrm{mar}}(f)$, by Theorem 4.4 (c) then $\psi \in \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \pm\rho)])$. Therefore the action of $G_{\mathbb{Z}}$ on M_μ^{mar} is properly discontinuous.

Locally at $[(f,\pm\rho)]$ the quotient is isomorphic to $S_\mu/G^{\mathrm{mar}}_{\mathcal{R}}(f) = S_\mu/\mathrm{Aut}_M$, and this is a neighborhood of $[f]$ in M_μ. Therefore $M^{\mathrm{mar}}_\mu/G_{\mathbb{Z}} = M_\mu$.

(e) If Assumption (4.1) or (4.2) holds, the proofs of (b)–(d) can be repeated for strongly marked singularities.

Suppose that neither (4.1) nor (4.2) holds. Then an $(f,\rho) \in C^{\mathrm{smar}}(k,f_0)$ exists such that $\mathrm{mult}\, f \geqslant 3$, but $\mathrm{mult}\, g = 2$ for arbitrarily close (g,ρ). Then $(g,\rho) \sim_{\mathcal{R}} (g,-\rho)$, but $(f,\rho) \not\sim_{\mathcal{R}} (f,-\rho)$. The quotient topology of $C^{\mathrm{smar}}(k,f_0)/j_k\mathcal{R}$ does not separate the orbits of (f,ρ) and $(f,-\rho)$. So it is not Hausdorff. This finishes the proof of Theorem 4.3. □

8. Examples: Simple and exceptional singularities

Here we will prove Conjecture 3.2 and Conjecture 5.3 for the simple singularities and 22 of the 28 families of exceptional singularities. Conjecture 5.3 will use calculations in [7] of period maps to D_{BL} (for the exceptional singularities) and an analysis of Conjecture 3.2 for the simple singularities and the quasihomogeneous exceptional singularities.

For the remaining 6 families of exceptional singularities, for the simple-elliptic singularities and for the hyperbolic singularities the conjectures are also true. They will be treated in another paper.

We denote $e(a) := e^{2i\pi a} \in \mathbb{C}$ for $a \in \mathbb{C}$.

LEMMA 8.1. — [9, Lemma 6.5] *Let p be a prime number, $k,m \in \mathbb{Z}_{\geqslant 1}$, $c(x) \in \mathbb{Z}[x]$ such that $c(e(\frac{1}{p^k m})) = 1$ and $|c(e(\frac{1}{m}))| = 1$.*

(a) *If $p \geqslant 3$ then $c(e(\frac{1}{m})) = 1$.*
(b) *If $p = 2$ then $c(e(\frac{1}{m})) = \pm 1$.*
(c) *If $p = 2$ and $c(e(\frac{1}{p^l m})) = 1$ for some $l \in \mathbb{Z}_{\geqslant 1} - \{k\}$ then $c(e(\frac{1}{m})) = 1$.*

Lemma 8.2 can be seen as a generalisation of the number theoretic fact: *For any unit root λ*

$$\{g(\lambda) \in \mathbb{Z}[\lambda] \mid |g(\lambda)| = 1\} = \{\pm\lambda^k \mid k \in \mathbb{Z}\}.$$

The proof of Lemma 8.2 uses this fact and Lemma 8.1.

LEMMA 8.2. — *Let H be a free $\mathbb{Z}$-module of finite rank μ, and $H_{\mathbb{C}} := H \otimes_{\mathbb{Z}} \mathbb{C}$. Let $M_h : H \to H$ be an automorphism of finite order, called monodromy, with three properties:*

(i) *Each eigenvalue has multiplicity* 1.
Denote $H_\lambda := \ker(M_h - \lambda \cdot \mathrm{id} : H_{\mathbb{C}} \to H_{\mathbb{C}})$.

(ii) *Denote* $\mathrm{Ord} := \{\mathrm{ord}\,\lambda \,|\, \lambda \text{ eigenvalue of } M_h\} \subset \mathbb{Z}_{\geqslant_1}$. *There exist four sequences* $(m_i)_{i=1,...,|\,\mathrm{Ord}\,|}$, $(j(i))_{i=2,...,|\,\mathrm{Ord}\,|}$, $(p_i)_{i=2,...,|\,\mathrm{Ord}\,|}$, $(k_i)_{i=2,...,|\,\mathrm{Ord}\,|}$ *of numbers in* $\mathbb{Z}_{\geqslant 1}$ *and two numbers* $i_1, i_2 \in \mathbb{Z}_{\geqslant 1}$ *with* $i_1 \leqslant i_2 \leqslant |\,\mathrm{Ord}\,|$ *and with the properties:*
$\mathrm{Ord} = \{m_1, ..., m_{|\,\mathrm{Ord}\,|}\}$,
p_i *is a prime number,* $p_i = 2$ *for* $i_1 + 1 \leqslant i \leqslant i_2$, $p_i \geqslant 3$ *else,*
$j(i) = i - 1$ *for* $i_1 + 1 \leqslant i \leqslant i_2$, $j(i) < i$ *else,*

$$m_i = m_{j(i)}/p_i^{k_i}.$$

(iii) *A cyclic generator* $a_1 \in H$ *exists, that means,*

$$H = \bigoplus_{i=0}^{\mu-1} \mathbb{Z} \cdot M_h^i(a_1).$$

Finally, let I *be an* M_h*-invariant nondegenerate bilinear form (not necessarily* (± 1)*-symmetric) on* $\bigoplus_{\lambda \neq \pm 1} H_\lambda$ *with values in* $\mathbb{C}$. *Then*

$$\mathrm{Aut}(H, M_h, I) = \{\pm M_h^k \,|\, k \in \mathbb{Z}\}.$$

Proof. — For any $A \in \mathrm{Aut}(H, M_h)$ the polynomial $c(x) = \sum_{i=0}^{\mu-1} c_i x^i \in \mathbb{Z}[x]$ with $A(a_1) = c(M_h)(a_1)$ is well-defined because of (iii). Then, also because of (iii), $A = c(M_h)$. The eigenvalue of A on H_λ is $c(\lambda)$. It maps H_1 and H_{-1} to themselves, and $H_{\pm 1} \cap H$ are rank 1 sublattices, so $|c(\pm 1)| = 1$.

Now suppose $A \in \mathrm{Aut}(H, M_h, I)$. As $I : H_\lambda \times H_{\overline{\lambda}} \to \mathbb{C}$ for $\lambda \neq \pm 1$ is nondegenerate, $|c(\lambda)| = 1$ for such λ, hence for all eigenvalues λ.

By the number theoretic fact cited above, there exist $k \in \mathbb{Z}$ and $\varepsilon_1 \in \{\pm 1\}$ such that

$$\varepsilon_1 \cdot e\Big(\frac{1}{m_1}\Big)^k \cdot c\Big(e\Big(\frac{1}{m_1}\Big)\Big) = 1.$$

Define $c^{(2)}(x) := \varepsilon_1 \cdot x^k \cdot c(x)$, so $c^{(2)}(e(\frac{1}{m_1})) = 1$. One finds inductively $c^{(2)}(e(\frac{1}{m_i})) = 1$ for $i = 2, ..., i_1$ by applying lemma 8.1 (a) at each step. Now distinguish two cases.

Case 1, $i_1 = i_2$, so all $p_i \geqslant 3$: Define $c^{(3)}(x) := c^{(2)}(x)$.

Case 2, $i_1 < i_2$: Lemma 8.1 (b) shows

$$c^{(2)}\Big(e\Big(\frac{1}{m_{i_1+1}}\Big)\Big) = \varepsilon_2 \in \{\pm 1\}.$$

Define

$$c^{(3)}(x) := \begin{cases} c^{(2)}(x) & \text{if } \varepsilon_2 = 1 \\ (-x^{m_1/2}) \cdot c^{(2)}(x) & \text{if } \varepsilon_2 = -1. \end{cases}$$

Then

$$c^{(3)}\Big(e\Big(\frac{1}{m_i}\Big)\Big) = 1 \qquad \text{for } 1 \leqslant i \leqslant i_1 + 1.$$

With Lemma 8.1 (c) one finds inductively

$$c^{(3)}\Big(e\Big(\frac{1}{m_i}\Big)\Big) = 1 \qquad \text{for } i = i_1+2, ..., i_2.$$

Now in both cases one finds inductively $c^{(3)}(e(\frac{1}{m_i})) = 1$ for $i = i_2 + 1, ..., |\operatorname{Ord}|$, with Lemma 8.1 (a). Therefore $c^{(3)}(M_h) = \mathrm{id}$ and $A \in \{\pm M_h^k \mid k \in \mathbb{Z}\}$. □

THEOREM 8.3.

(a) *The quasihomogeneous singularities with modality* $\leqslant 2$ *and with one-dimensional eigenspaces (of the monodromy) are the singularities* A_μ, D_{2k+1}, E_μ *and 22 of the 28 quasihomogeneous exceptional unimodal and bimodal singularities, the exceptions are* Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16}.

(b) *For all of them*

$$G_{\mathbb{Z}} := \mathrm{Aut}(\text{Milnor lattice, Seifert form}) = \{\pm M_h^k \mid k \in \mathbb{Z}\}.$$

This is independent of the number of variables. The orders of the groups can be read off from the characteristic polynomials (table in the proof). For the simple singularities they are

A_1	A_μ $(\mu \geqslant 2)$	D_μ $(\mu = 2k+1 \geqslant 5)$	E_6	E_7	E_8
2	$2(\mu+1)$	$4(\mu-1)$	24	18	30

(c) *For all of them,* $M_\mu^{\mathrm{mar}} \cong \mathbb{C}^{\mathrm{mod}(f)}$, *here* $\mathrm{mod}(f) \in \{0,1,2\}$ *is the modality of* f, *and the period map* $M_\mu^{\mathrm{mar}} \to D_{BL}$ *is an isomorphism. Therefore for all of them Conjecture 3.2 (a) and Conjecture 5.3 are true (and thus also Conjecture 5.1 and Conjecture 5.4, though Conjecture 5.4 was shown already in* [7]*). Also Conjecture 3.2 (b) is true for all of them.*

Proof. — (a) The following table lists the characteristic polynomials of all quasihomogeneous surface singularities with modality $\leqslant 2$. It can be extracted from the tables of spectral numbers in [3, 13.3.4] or from [7]. Inspection of the tables gives (a). Φ_m for $m \in \mathbb{Z}_{\geqslant 1}$ denotes the cyclotomic polynomial of primitive unit roots of order m.

A_μ	$\frac{t^{\mu+1}-1}{t-1}$	$E_{3,0}$	$\Phi_{18}^2\Phi_6\Phi_2^2$
D_μ	$(t^{\mu-1}+1)\Phi_2$	$Z_{1,0}$	$\Phi_{14}^2\Phi_2^3$
E_6	$\Phi_{12}\Phi_3$	$Q_{2,0}$	$\Phi_{12}^2\Phi_4^2\Phi_3$
E_7	$\Phi_{18}\Phi_2$	$W_{1,0}$	$\Phi_{12}^2\Phi_6\Phi_4\Phi_3\Phi_2$
E_8	Φ_{30}	$S_{1,0}$	$\Phi_{10}^2\Phi_5\Phi_2^2$
$\widetilde{E}_6$	$\Phi_3^3\Phi_1^2$	$U_{1,0}$	$\Phi_9^2\Phi_3$
$\widetilde{E}_7$	$\Phi_4^2\Phi_2^3\Phi_1^2$		
$\widetilde{E}_8$	$\Phi_6\Phi_3^2\Phi_2^2\Phi_1^2$		

E_{12}	Φ_{42}	E_{18}	$\Phi_{30}\Phi_{15}\Phi_3$
E_{13}	$\Phi_{30}\Phi_{10}\Phi_2$	E_{19}	$\Phi_{42}\Phi_{14}\Phi_2$
E_{14}	$\Phi_{24}\Phi_{12}\Phi_3$	E_{20}	Φ_{66}
Z_{11}	$\Phi_{30}\Phi_6\Phi_2$	Z_{17}	$\Phi_{24}\Phi_{12}\Phi_6\Phi_3\Phi_2$
Z_{12}	$\Phi_{22}\Phi_2^2$	Z_{18}	$\Phi_{34}\Phi_2^2$
Z_{13}	$\Phi_{18}\Phi_9\Phi_2$	Z_{19}	$\Phi_{54}\Phi_2$
Q_{10}	$\Phi_{24}\Phi_3$	Q_{16}	$\Phi_{21}\Phi_3^2$
Q_{11}	$\Phi_{18}\Phi_6\Phi_3\Phi_2$	Q_{17}	$\Phi_{30}\Phi_{10}\Phi_6\Phi_3\Phi_2$
Q_{12}	$\Phi_{15}\Phi_3^2$	Q_{18}	$\Phi_{48}\Phi_3$
W_{12}	$\Phi_{20}\Phi_5$	W_{17}	$\Phi_{20}\Phi_{10}\Phi_5\Phi_2$
W_{13}	$\Phi_{16}\Phi_8\Phi_2$	W_{18}	$\Phi_{28}\Phi_7$
S_{11}	$\Phi_{16}\Phi_4\Phi_2$	S_{16}	Φ_{17}
S_{12}	Φ_{13}	S_{17}	$\Phi_{24}\Phi_8\Phi_6\Phi_3\Phi_2$
U_{12}	$\Phi_{12}\Phi_6\Phi_4^2\Phi_2^2$	U_{16}	$\Phi_{15}\Phi_5^2$

(b) In Section 2 $G_{\mathbb{Z}}(f) = G_{\mathbb{Z}}(f + x_{n+1}^2)$ was shown. Therefore it is sufficient to show (b) for the surface singularities in (a). Lemma 8.2 with $H = Ml(f)$, $I =$ *intersection form* or *Seifert form*, $M_h =$ *monodromy* shall be applied to the surface singularities in (a). Condition (i) is clear. Condition (ii) can be checked by inspection of the table of characteristic polynomials above (only for D_{2k+1}, Q_{11} and Q_{17} one has to choose $i_1 > 1$).

Condition (iii) is a special case of the following conjecture of Orlik [20]:

For a quasihomogeneous singularity consider the unique decomposition of its characteristic polynomial p_{ch} into a product $p_{ch} = p_1 \cdot ... \cdot p_l$ of unitary polynomials with $p_l | p_{l-1} | ... | p_1$, $p_l \neq 1$. Then $Ml(f)$ is a direct sum of cyclic modules,

$$Ml(f) = \bigoplus_{i=1}^{l} \left(\bigoplus_{j=1}^{\deg p_j} \mathbb{Z} \cdot M_h^{j-1}(a_j) \right)$$

for suitable $a_1, ..., a_l \in Ml(f)$ such that the monodromy on the j-th block has characteristic polynomial p_j.

Of course, if the conjecture holds for a singularity $f(x_0, ..., x_n)$ then it holds also for any suspension $f(x_0, ..., x_n) + x_{n+1}^2$. Michel and Weber can prove the conjecture for $n = 1$ [18]. In [7, 3.1] it is proved (using Coxeter-Dynkin diagrams) for those quasihomogeneous surface singularities with modality $\leqslant 2$ which are not suspensions of curve singularities. So, for the singularities in (a) the conjecture is true with $l = 1$. There condition (iii) holds. Lemma 8.2 applies and gives the statement.

(c) By Theorem 3.3 (c) (and Theorem 4.4 (c)), for any quasihomogeneous singularity

$$\{\pm M_h^k \,|\, k \in \mathbb{Z}\} \subset \mathrm{Stab}_{G_\mathbb{Z}(f)}([(f, \pm\mathrm{id})]) = G^{\mathrm{mar}}_{\mathcal{R}}(f) \subset G^{\mathrm{mar}}(f) \subset G_\mathbb{Z}(f).$$

Part (b) gives equalities for the singularities f in (a), especially $G^{\mathrm{mar}}(f) = G_\mathbb{Z}(f)$, which is Conjecture 3.2 (a). By Theorem 4.4 (a) M_μ^{mar} is connected.

In [2] for the simple and the exceptional singularities, holomorphic μ-constant families with base spaces $X \cong \mathbb{C}^{\mathrm{mod}(f)}$ are given. The base space is equipped with a good $\mathbb{C}^*$-action (good = positive weights), the point 0 stands for the quasihomogeneous singularity, the other points for semi-quasihomogeneous singularities.

For any singularity $\widetilde{f}_0$, the moduli space $M_\mu^{\mathrm{mar}}(\widetilde{f}_0)$ comes equipped with a $\mathbb{C}$-action, by

$$\mathbb{C} \times M_\mu^{\mathrm{mar}}(\widetilde{f}_0), \quad (c, [(\widetilde{f}, \pm\rho)]) \mapsto [(e^{-c} \cdot \widetilde{f}, \pm\rho \circ \sigma(c)],$$

here $\sigma(c) : ML(e^{-c} \cdot \widetilde{f}) \to ML(\widetilde{f})$ is the canonical isomorphism within the μ-constant family $\{e^{-c} \cdot \widetilde{f} \,|\, c \in \mathbb{C}\}$.

In the case of the quasihomogeneous exceptional singularities f, the germ $(M_\mu^{\mathrm{mar}}(f), [(f, \pm\mathrm{id})])$ and the germ $(X, 0)$ are isomorphic. The $\mathbb{C}$-action on $M_\mu^{\mathrm{mar}}(f)$ factors through to an action of $\mathbb{C}/2\pi i m\mathbb{Z} \cong \mathbb{C}^*$ where $m \in \mathbb{Z}_{\geqslant 1}$ is minimal with $M_h^m = \pm\mathrm{id}$. Any class in $M_\mu^{\mathrm{mar}}(f)$ is obtained by this $\mathbb{C}^*$-action from a class close to $[(f, \pm\mathrm{id})]$. The isomorphism of germs $(M_\mu^{\mathrm{mar}}(f), [(f, \pm\mathrm{id})]) \cong (X, 0)$ is compatible with the $\mathbb{C}^*$-actions. Therefore

$$M_\mu^{\mathrm{mar}}(f) \cong X.$$

In [7] (copied in [8] and [15]) the period map $X \to D_{BL}$ was calculated and shown to be an isomorphism. Therefore Conjecture 5.3 is true (and thus also Conjecture 5.1 and Conjecture 5.4, though Conjecture 5.4 was shown already in [7]).

Finally we come to G^{smar} and M_μ^{smar} and Conjecture 3.2 (b). Assumption (4.1) or (4.2) holds. Suppose that Assumption (4.1) holds, that means that we consider curve or surface singularities, depending on the type. Then by Theorem 4.4 (e), $\{\pm\mathrm{id}\} \subset G_\mathbb{Z}$ acts freely on M_μ^{smar} with quotient M_μ^{mar}, and the quotient map is a double covering. But $M_\mu^{\mathrm{mar}} \cong \mathbb{C}^{\mathrm{mod}(f)}$. The only possible double covering is that which maps two copies of M_μ^{mar} to M_μ^{mar}. By Theorem 4.4 (b) then $G_\mathbb{Z} = G^{\mathrm{smar}}(f) \times \{\pm\mathrm{id}\}$ and

$$G^{\mathrm{smar}}(f) = \mathrm{Stab}_{G_\mathbb{Z}}([(f, \mathrm{id})]) = \{M_h^k \,|\, k \in \mathbb{Z}\}$$

(here M_h is the monodromy of f with $\mathrm{mult}(f) \geqslant 3$; whether this is the surface or curve singularity, depends on the type). □

THEOREM 8.4. — *Also for the simple singularities D_{2k} the Conjectures 3.2 (a) and (b) are true. Therefore here $M_\mu^{\mathrm{mar}} \cong \{pt\}$, and the period map $M_\mu^{\mathrm{mar}} \to D_{BL}$ is an isomorphism (mapping one point to one point). Therefore the Conjectures 5.3 and 5.1 are true (Conjecture 5.4 is trivial for the simple singularities.)*

In the case mult $f \geqslant 3$ *(that is,* mult $f = 3$*, it is the case $n = 1$)*

$$G^{\mathrm{smar}} = \{M_h^a \,|\, a = 0, 1, ..., 2k-2\} \times U,$$
$$G_{\mathbb{Z}} = G^{\mathrm{mar}} = G^{\mathrm{smar}} \times \{\pm \mathrm{id}\}$$

with $U \cong S_3$ for D_4 and $U \cong S_2$ for $k \geqslant 3$. So, $|G_{\mathbb{Z}}| = 36$ for D_4 and $|G_{\mathbb{Z}}| = 4(\mu - 1)$ if $\mu = 2k \geqslant 6$.

Proof. — Consider the curve singularity $f = x^{2k-1} - xy^2$ with weights $(w_x, w_y) = (\frac{1}{2k-1}, \frac{k-1}{2k-1})$. Because of $\mathrm{mod}(f) = 0$ and Theorem 4.4 (b), M_μ^{smar} consists of $|G_{\mathbb{Z}}/G^{\mathrm{smar}}|$ many points, and $G^{\mathrm{smar}} = \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \mathrm{id})])$. By Theorem 6.2 and Theorem 6.1 (g), this group can be calculated: The restriction of the homomorphism $()_{\mathrm{hom}} : \mathcal{R}^f \to \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \mathrm{id})])$ to the finite group $\mathrm{Stab}_{G_w}(f) \subset \mathcal{R}^f$ with

$$\mathrm{Stab}_{G_w}(f) := \{\varphi \in \mathcal{R}^f \,|\, \varphi \text{ is a quasihomogeneous coordinate change}\}$$

is an isomorphism $()_{\mathrm{hom}} : \mathrm{Stab}_{G_w}(f) \to \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \mathrm{id})])$.

It is easy to see that $\mathrm{Stab}_{G_w}(f)$ is generated by the coordinate changes

$$\varphi_1 : (x, y) \mapsto (e(w_x)x, e(w_y)y) \qquad \text{with } (\varphi_1)_{\mathrm{hom}} = M_h,$$
$$\varphi_2 : (x, y) \mapsto (x, -y),$$
$$\varphi_3 : (x, y) \mapsto (-\frac{1}{2}x + \frac{1}{2}y, \frac{3}{2}x + \frac{1}{2}y) \quad \text{only for } k = 2.$$

The element φ_2 [and φ_3 if $k = 2$] generates a subgroup of G_w isomorphic to S_2 [respectively S_3 if $k = 2$]. The image under $()_{\mathrm{hom}}$ is called U. Thus

$$G^{\mathrm{smar}} = \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \mathrm{id})]) = \{M_h^k \,|\, k = 0, 1, ..., 2k-2\} \times U.$$

By Theorem 3.3 (g) $-\mathrm{id} \notin \mathrm{Stab}_{G_{\mathbb{Z}}}([(f, \mathrm{id})])$, therefore $G^{\mathrm{mar}} = G^{\mathrm{smar}} \times \{\pm \mathrm{id}\}$ and $|G^{\mathrm{mar}}| = 36$ for D_4 and $|G^{\mathrm{mar}}| = 4(\mu - 1)$ for $k \geqslant 3$. It rests to see $|G_{\mathbb{Z}}| = |G^{\mathrm{mar}}|$. Then $G_{\mathbb{Z}} = G^{\mathrm{mar}}$, and everything is proved.

For the calculation of $|G_{\mathbb{Z}}|$ we go over to $\widetilde{f} = f(x, y) + x_2^2 + x_3^2 + x_4^2$ with $n = 4$. It is well known that then $(Ml(\widetilde{f}), I)$ is the root lattice of type D_{2k} and M_h is a Coxeter element. Then $G_{\mathbb{Z}} = \mathrm{Aut}(Ml(f), M_h, I)$ because I is nondegenerate.

Choose a basis $e_1, ..., e_\mu$ of $Ml(\widetilde{f})$ which corresponds to the standard Dynkin diagram,

$$I(e_i, e_i) = 2, \quad I(e_i, e_{i+1}) = -1 \text{ for } 1 \leqslant i \leqslant \mu - 2, \quad I(e_{\mu-2}, e_\mu) = -1.$$
$$I(e_i, e_j) = 0 \quad \text{for all other } i \text{ and } j \text{ with } i < j.$$

Then

$$\begin{aligned} M_h &= s_{e_1} \circ s_{e_2} \circ ... \circ s_{e_\mu}, \\ B_1 &:= \ker(M_h + \mathrm{id}) \cap Ml(\widetilde{f}) = \mathbb{Z} \cdot b_1 + \mathbb{Z} \cdot b_2, \quad \textit{where} \\ b_1 &= e_{\mu-1} - e_\mu, \quad b_2 = e_{\mu-1} + e_{\mu-3} + ... + e_1, \\ & I(b_1, b_1) = 4, \quad I(b_1, b_2) = 2, \quad I(b_2, b_2) = 2k. \end{aligned}$$

If $k = 2$ this is an A_2-lattice. If $k \geqslant 3$, $I(\beta, \beta) = 4 \iff \beta = \pm b_1$ for $\beta \in B_1$. One sees easily $|\operatorname{Aut}(B_1, I)| = 12$ if $k = 2$ and $|\operatorname{Aut}(B_1, I)| = 4$ if $k \geqslant 3$.

Any $\psi \in G_\mathbb{Z}$ maps B_1 to itself because $\psi \circ M_h = M_h \circ \psi$. There is an exact sequence

$$1 \mapsto \{\psi \in G_\mathbb{Z} \,|\, \psi = \mathrm{id} \text{ on } B_1\} \to G_\mathbb{Z} \to \operatorname{Aut}(B_1, I).$$

Orlik's conjecture holds also for D_{2k} (see the proof of Theorem 8.3 (b)), with $p_{ch} = (t^{\mu-1} + 1)\Phi_2 = p_1 \cdot p_2$, $p_1 = t^{\mu-1} + 1$, $p_2 = \Phi_2$: There are $a_1 \in Ml(\widetilde{f})$, $a_2 \in B_1$ with

$$Ml(\widetilde{f}) = \left(\bigoplus_{i=0}^{\mu-2} \mathbb{Z} \cdot M_h^i(a_1) \right) \oplus \mathbb{Z} \cdot a_2 =: B_2 \oplus \mathbb{Z} \cdot a_2.$$

Any $\psi \in G_\mathbb{Z}$ with $\psi = \mathrm{id}$ on B_1 restricts to an automorphism of B_2. Lemma 8.2 applies and shows $\psi_{|B_2} = \pm(M_{h|B_2})^k$ for some k. Now $\psi_{|B_1} = \mathrm{id}$ forces $\psi = (-M_h)^k$ for some $k \in \mathbb{Z}$ (here M_h is the mondromy of $\widetilde{f}$, so minus the monodromy of f). Now

$$\begin{aligned} |G_\mathbb{Z}| &= |\{(-M_h)^k \,|\, k \in \mathbb{Z}\}| \cdot |\text{image of } G_\mathbb{Z} \text{ in } \operatorname{Aut}(B_1, I)| \\ &\leqslant \begin{cases} 3 \times 12 = |G^{\mathrm{mar}}| & \text{if } k = 2, \\ (2k-1) \cdot 4 = |G^{\mathrm{mar}}| & \text{if } k \geqslant 3. \end{cases} \end{aligned}$$

Therefore $G_\mathbb{Z} = G^{\mathrm{mar}}$. And $G_\mathbb{Z} \to \operatorname{Aut}(B_1, I)$ is surjective. □

Remarks 8.5. — (i) Let f be any quasihomogeneous singularity such that its Milnor lattice and its monodromy satisfy the properties (i) – (iii) in Lemma 8.2. Property (i) implies that there are no nontrivial μ-constant deformations of weight 0. Therefore a holomorphic μ-constant family with base space $X \cong \mathbb{C}^{\mathrm{mod}(f)}$ and good $\mathbb{C}^*$-action on it exists, where the point 0

stands for the quasihomogeneous singularity and all other points for semi-quasihomogeneous singularities, and it contains representatives of any right equivalence class in the μ-homotopy class of f [2]. Then the proof of Theorem 8.3 (c) goes through without changes. Then $M_\mu^{\mathrm{mar}} \cong X \cong \mathbb{C}^{\mathrm{mod}(f)}$, and the period map $M_\mu^{\mathrm{mar}} \to D_{BL}$ is an isomorphism.

In [9, ch. 6] most of this (only not Conjecture 3.2 (b) for G^{smar}) was carried out for the Brieskorn-Pham singularities with pairwise coprime exponents.

(ii) In Theorem 8.3 the quasihomogeneous singularities Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16} are missing. Also they have holomorphic μ-constant families with base spaces $X \cong \mathbb{C}^{\mathrm{mod}(f)}$ as in (i). A part of the proof of Theorem 8.3 applies and shows that each component of M_μ^{mar} is isomorphic to X, and it shows that in the case $\mathrm{mult}(f) \geqslant 3$ Conjecture 3.2 (b) is true. The only missing part is a proof of Conjecture 3.2 (a): $G^{\mathrm{mar}} = G_{\mathbb{Z}}$. This will be shown in another paper. Then the proof of Theorem 8.3 goes through and gives everything else. Then the period map $M_\mu^{\mathrm{mar}} \to D_{BL}$ is an isomorphism.

(iii) For the simple singularities $G_{\mathbb{Z}}$ had also been calculated in [30][31]. There specific properties of the ADE root lattices were used. Also $G_{\mathbb{Z}} = G^{\mathrm{mar}}$ is contained implicitly in [30][31].

(iv) In the case of the simple and the exceptional singularities, M_μ^{mar} [respectively any component of M_μ^{mar} for the cases Z_{12}, Q_{12}, U_{12}, Z_{18}, Q_{16}, U_{16}] is simply connected. A discussion in another paper will show that this holds also for the simple-elliptic singularities $\widetilde{E}_k$ ($k = 6, 7, 8$) and the hyperbolic singularities T_{pqr} ($\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$).

But I expect that it does not hold for the six bimodal quadrangle singularities. In those cases D_{BL} is a line bundle over the hyperbolic plane $\mathbb{H}$, and the image of the period map $M_\mu^{\mathrm{mar}} \to D_{BL}$ is the restriction of the line bundle to the complement in $\mathbb{H}$ of the countably many elliptic fixed points of a certain triangle group with angles > 0 [7].

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