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Jean-François BONY, Setsuro FUJIIÉ,
Thierry RAMOND & Maher ZERZERI

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SPECTRAL PROJECTION, RESIDUE OF THE SCATTERING AMPLITUDE AND SCHRÖDINGER GROUP EXPANSION FOR BARRIER-TOP RESONANCES

by Jean-François BONY, Setsuro FUJIIÉ, Thierry RAMOND & Maher ZERZERI

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ABSTRACT. — We study the spectral projection associated to a barrier-top resonance for the semiclassical Schrödinger operator. First, we prove a resolvent estimate for complex energies close to such a resonance. Using that estimate and an explicit representation of the resonant states, we show that the spectral projection has a semiclassical expansion in integer powers of h, and compute its leading term. We use this result to compute the residue of the scattering amplitude at such a resonance. Eventually, we give an expansion for large times of the Schrödinger group in terms of these resonances.

RÉSUMÉ. — On étudie le projecteur spectral associé aux résonances engendrées par le sommet du potentiel d'un opérateur de Schrödinger semiclassique. On démontre d'abord une estimation de la résolvante pour les énergies complexes proches de ces résonances. À l'aide de cette estimation et d'une représentation explicite des états résonants, on prouve que le projecteur spectral admet un développement asymptotique en puissances entières de h, dont on donne le terme principal. Ce résultat nous permet alors de calculer le résidu de l'amplitude de diffusion en ces résonances. Finalement, on décrit le comportement en temps grand du groupe de Schrödinger en fonction des résonances.

1. Introduction

In this paper, we study the behavior of different physical quantities at the resonances generated by the maximum of the potential of a semiclassical Schrödinger operator $P = -h^2\Delta + V$ on $\mathbb{R}^n$. In particular, we show

quantitatively to what extent the presence of these resonances drives the behavior of the scattering amplitude and of the Schrödinger group.

The resonances generated by the maximum point, supposed to be non-degenerate, of the potential (usually called barrier-top resonances) have been studied by Briet, Combes and Duclos [5, 6] and Sjöstrand [37]. These authors have given a precise description of the set $\mathrm{Res}(P) = \{z_\alpha \approx E_0 - ih \sum_{j=1}^n \left(\alpha_j + \frac{1}{2} \right) \lambda_j, \ \alpha \in \mathbb{N}^n \}$ of resonances in any disc of size h centered at the maximum value E_0 of the potential. Here, the λ_j's are the square roots of the eigenvalues of the Hessian of $-2V$ at the maximum point. In particular the resonances lie at distance of order h from the real axis, which is in very strong contrast to the case of shape resonances (the well in the island case), with exponentially small imaginary part (see Helffer and Sjöstrand [22]). The description of resonances in larger discs of size h^δ, $\delta \in]0, 1]$ has been obtained by Kaidi and Kerdelhué [27] under a diophantine condition. For small discs of size one, this question has been treated in the one dimensional case by the third author [36] by means of the complex WKB method. In the two dimensional case, the resonances in discs of size one have also been considered by Hitrik, Sjöstrand and Vũ Ngọc [23] (see also the references in this paper). Here, we consider only the resonances at distance h of the maximum of the potential and we recall their precise localization in Section 2.

Resonances can be defined as the poles of the meromorphic continuation of the cut-off resolvent (see *e.g.* Hunziker [24]). The generalized spectral projection associated to a resonance is defined as the residue of the resolvent at this pole:

$$\Pi_z = -\frac{1}{2i\pi} \oint_{\gamma_z} (P - \zeta)^{-1} d\zeta,$$

as an operator from L^2_{comp} to L^2_{loc}. The multiplicity of a resonance is the rank of this associated spectral projection. In the case of shape resonances, their semiclassical expansion has been computed by Helffer and Sjöstrand in [22]. In Section 4 below, we obtain the semiclassical expansion of the generalized spectral projection for barrier-top resonances. Since the resonances in the present case have a much larger imaginary part, our result is very different from that of the shape resonance case. Using some of the results of [2], we show that, for a simple resonance z_α,

(A) $$\Pi_{z_\alpha} = c_\alpha h^{-|\alpha| - \frac{n}{2}} (\, \cdot \, , \overline{f_\alpha}) f_\alpha,$$

where the resonant state f_α is a Lagrangian distribution, with a WKB expansion near the maximum point of V (that we may suppose to be 0) of

the form

$$f_\alpha(x) \approx e^{i\varphi_+(x)/h} x^\alpha \quad \text{with} \quad \varphi_+(x) \approx \sum_{j=1}^n \frac{\lambda_j}{4} x_j^2.$$

We send the reader to Theorem 4.1 for a more precise statement and the value of c_α.

Resonances appear also in scattering theory (they are called scattering poles in this context). In [30], Lax and Phillips have shown that they coincide with the poles of the meromorphic extension of the scattering amplitude. This result, proved for the wave equation in the exterior of a compact obstacle, was extended by Gérard and Martinez [15] to the long range case for the Schrödinger equation (see also the references in that paper for earlier works). For shape resonances, the residue of the scattering amplitude was calculated in the semiclassical limit by Nakamura [33, 34], Lahmar-Benbernou [28] and Lahmar-Benbernou and Martinez [29]. More generally, upper bounds on the residues of the scattering amplitude have been obtained by Stefanov [40] (in the compact support case) and Michel [32] (in the long range case) for resonances very close to the real axis. In Section 5, we prove a semiclassical expansion of the residues of the scattering amplitude for barrier-top resonances, and we will see in particular that Stefanov's and Michel's upper bounds do not hold in the present setting. Indeed, for long range potentials and under some natural geometric assumptions, we obtain an expansion for the scattering amplitude $\mathcal{A}(\omega, \omega', z, h)$ for the incoming direction ω' and the outgoing direction ω, of the form

(B) Residue $\big(\mathcal{A}(\omega, \omega', z, h), z = z_\alpha\big) \approx h^{-|\alpha|+\frac{1}{2}} e^{i(S^-(\omega')+S^+(\omega))/h} a(\omega, \omega', h),$

where $S^-(\omega')$, $S^+(\omega)$ are classical actions along trajectories tending to the maximum point, and a is a classical symbol in h of order 0. Again, we refer to Theorem 5.1 below for the precise setting and results.

It is commonly believed that resonances play also a crucial role in quantum dynamics. Indeed, it is sometimes possible to describe the long time evolution of the cut-off propagator (for example, the Schrödinger or wave group) in terms of the resonances. Such formulas should generalize the Poisson formula, valid for operators with discrete spectrum. The resonance expansion of the wave group was first obtained by Lax and Phillips [30] in the exterior of a star-shaped obstacle. This result has been generalized, using various techniques, to different non trapping situations (see *e.g.* Vaĭnberg [43] and the references of the second edition of the book [30]). The trapping situations have been treated by Tang and Zworski [42] and

Burq and Zworski [7] for very large times. On the other hand, the time evolution of the quasiresonant states (sorts of quasimodes) has been studied by Gérard and Sigal [16]. A specific study of the Schrödinger group for the shape resonances created by a well in an island has been made by Nakamura, Stefanov and Zworski [35]. There are also some works concerning the situation of a hyperbolic trapped set. We refer to the work of Christiansen and Zworski [9] for the wave equation on the modular surface and on the hyperbolic cylinder, to the work of Häfner and the first author [3] for the wave equation on the de Sitter-Schwarzschild metric, and to the work of Guillarmou and Naud [18] for the wave equation on convex cocompact hyperbolic manifolds. Section 6 is devoted to the computation of the asymptotic behavior for large time of the Schrödinger group localized in energies close to the maximum of the potential. Provided all the resonances taken into account are simple, we obtain in Theorem 6.1 below the expansion, valid for $h > 0$ small enough and all $t > 0$,

(C)
$$\chi e^{-itP/h} \chi \psi(P) = \sum_{z_\alpha \in \mathrm{Res}(P) \cap D(E_0, \mu h)} e^{-itz_\alpha/h} \chi \Pi_{z_\alpha} \chi \psi(P)$$
$$+ \mathcal{O}(h^\infty) + \mathcal{O}(e^{-\mu t} h^{-C(\mu)}),$$

where $\mu > 0$, χ is any function in $C_0^\infty(\mathbb{R}^n)$ and $\psi \in C_0^\infty(\mathbb{R})$ is a cut-off function near the critical energy level E_0. Note that the Π_{z_α}'s appearing in the previous expansion are those given by (A).

For the proof of these different results, we use an estimate on the distorted resolvent $(P_\theta - z)^{-1}$ around the resonances, polynomial with respect to h^{-1}, of the form

(D)
$$\left\| (P_\theta - z)^{-1} \right\| \lesssim h^{-C(\mu)} \prod_{z_\alpha \in \mathrm{Res}(P) \cap D(E_0, 2\mu h)} |z - z_\alpha|^{-1},$$

for all $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-\mu h, \mu h]$. Indeed, such a bound allows to apply the semiclassical microlocal calculus. This estimate is established in Section 3. To prove it, we proceed as in [2] and use the method developed by Martinez [31], Sjöstrand [38] and Tang and Zworski [41]. Similar bounds around the resonances are already known in various situations (see *e.g.* Gérard [14] for two strictly convex obstacles, Michel and the first author [4] in the one dimensional case). Note that, in our setting, a limiting absorption principle has been proved in [1].

2. Settings and resonances

We consider the semiclassical Schrödinger operator on $\mathbb{R}^n$, $n \geqslant 1$,

$$(2.1) \qquad P = -h^2 \Delta + V(x),$$

where V is a smooth real-valued function. We denote by $p(x, \xi) = \xi^2 + V(x)$ the associated classical Hamiltonian. The vector field

$$H_p = \partial_\xi p \cdot \partial_x - \partial_x p \cdot \partial_\xi = 2\xi \cdot \partial_x - \nabla V(x) \cdot \partial_\xi,$$

is the Hamiltonian vector field associated to p. Integral curves $t \mapsto \exp(tH_p)(x, \xi)$ of H_p are called classical trajectories or bicharacteristic curves, and p is constant along such curves. The trapped set at energy E for P is defined as

$$K(E) = \left\{ (x, \xi) \in p^{-1}(E); \ \exp(tH_p)(x, \xi) \not\to \infty \text{ as } t \to \pm\infty \right\},$$

We shall suppose that V satisfies the following assumptions

(H1) $V \in C^\infty(\mathbb{R}^n; \mathbb{R})$ extends holomorphically in the sector

$$\mathcal{S} = \left\{ x \in \mathbb{C}^n; \ |\operatorname{Im} x| \leqslant \delta \langle x \rangle \right\},$$

for some $\delta > 0$. Moreover $V(x) \to 0$ as $x \to \infty$ in $\mathcal{S}$.

(H2) V has a non-degenerate maximum at $x = 0$ and

$$V(x) = E_0 - \sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + \mathcal{O}(x^3),$$

with $E_0 > 0$ and $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$.

(H3) The trapped set at energy E_0 is $K(E_0) = \{(0, 0)\}$.

Notice that (H3) ensures that $x = 0$ is the unique global maximum for V. Moreover, there exists a pointed neighborhood of E_0 in which all the energy levels are non trapping. In the following, $(\mu_k)_{k \geqslant 0}$ denotes the strictly increasing sequence of linear combinations over $\mathbb{N} = \{0, 1, 2, \ldots\}$ of the λ_j's. In particular, $\mu_0 = 0$ and $\mu_1 = \lambda_1$.

The linearization F_p at $(0, 0)$ of the Hamilton vector field H_p is given by

$$F_p = \begin{pmatrix} 0 & 2\operatorname{Id} \\ \frac{1}{2}\operatorname{diag}(\lambda_1^2, \ldots, \lambda_n^2) & 0 \end{pmatrix},$$

and has eigenvalues $-\lambda_n, \ldots, -\lambda_1, \lambda_1, \ldots, \lambda_n$. Thus $(0, 0)$ is a hyperbolic fixed point for H_p and the stable/unstable manifold theorem gives the existence of a stable incoming Lagrangian manifold Λ_- and a stable outgoing Lagrangian manifold Λ_+ characterized by

$$\Lambda_\pm = \left\{ (x, \xi) \in T^*\mathbb{R}^n; \ \exp(tH_p)(x, \xi) \to (0, 0) \text{ as } t \to \mp\infty \right\} \subset p^{-1}(E_0).$$

Moreover, there exist two smooth functions $\varphi_\pm$, defined in a vicinity of 0, satisfying

$$\varphi_\pm(x) = \pm \sum_{j=1}^n \frac{\lambda_j}{4} x_j^2 + \mathcal{O}(x^3),$$

and such that $\Lambda_\pm = \Lambda_{\varphi_\pm} := \{(x,\xi); \ \xi = \nabla\varphi_\pm(x)\}$ near $(0,0)$. Since P is a Schrödinger operator, we have $\varphi_- = -\varphi_+$.

Under the previous assumptions, the operator P is self-adjoint with domain $H^2(\mathbb{R}^n)$, and we define the set $\mathrm{Res}(P)$ of resonances for P as follows (see [24] or [39] for an alternative approach). Let $R_0 > 0$ be a large constant, and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field, such that $F(x) = 0$ for $|x| \leqslant R_0$ and $F(x) = x$ for $|x| \geqslant R_0 + 1$. For $\mu \in \mathbb{R}$ small enough, we denote $U_\mu : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ the unitary operator defined by

$$(2.2) \qquad U_\mu\varphi(x) = \big|\det(1 + \mu dF(x))\big|^{1/2}\varphi(x + \mu F(x)),$$

for $\varphi \in C_0^\infty(\mathbb{R}^n)$. Then the operator $U_\mu P(U_\mu)^{-1}$ is a differential operator with analytic coefficients with respect to μ, and can be analytically continued to small enough complex values of μ. For $\theta \in \mathbb{R}$ small enough, we denote

$$(2.3) \qquad P_\theta = U_{i\theta} P (U_{i\theta})^{-1}.$$

The spectrum of P_θ is discrete in $\mathcal{E}_\theta = \{z \in \mathbb{C}; \ -2\theta < \arg z \leqslant 0\}$, and the resonances of P are by definition the eigenvalues of P_θ in $\mathcal{E}_\theta$. We denote their set by $\mathrm{Res}(P)$. The multiplicity of a resonance is the rank of the spectral projection

$$\Pi_{z,\theta} = -\frac{1}{2i\pi} \oint_\gamma (P_\theta - \zeta)^{-1} d\zeta,$$

where γ is a small enough closed path around the resonance z. The resonances, as well as their multiplicity, do not depend on θ and F. As a matter of fact, the resonances are also the poles of the meromorphic extension from the upper complex half-plane of the resolvent $(P - z)^{-1} : L^2_{\mathrm{comp}}(\mathbb{R}^n) \to L^2_{\mathrm{loc}}(\mathbb{R}^n)$ (see e.g. [20]).

In the present setting, Sjöstrand [37] has given a precise description of the set of resonances in any disc $D(E_0, Ch)$ of center E_0 and radius Ch. This result has also been proved simultaneously by Briet, Combes and Duclos [6] under a slightly stronger hypothesis (a virial assumption).

THEOREM 2.1 (Sjöstrand). — *Assume* (H1)–(H3). *Let* $C > 0$ *be different from* $\sum_{j=1}^n (\alpha_j + \frac{1}{2})\lambda_j$ *for all* $\alpha \in \mathbb{N}^n$. *Then, for* $h > 0$ *small enough,*

there exists a bijection b_h between the sets $\mathrm{Res}_0(P) \cap D(E_0, Ch)$ and $\mathrm{Res}(P) \cap D(E_0, Ch)$ counted with their multiplicity, where

$$\mathrm{Res}_0(P) = \left\{ z_\alpha^0 = E_0 - ih \sum_{j=1}^{n} \left(\alpha_j + \frac{1}{2} \right) \lambda_j; \ \alpha \in \mathbb{N}^n \right\},$$

such that $b_h(z) - z = o(h)$.

In particular, the number of resonances in any disk $D(E_0, Ch)$ is uniformly bounded with respect to h. For $z_\alpha^0 \in \mathrm{Res}_0(P)$, we denote $z_\alpha = b_h(z_\alpha^0)$.

DEFINITION 2.2. — *We shall say that $z_\alpha^0 \in \mathrm{Res}_0(P)$ is simple if $z_\alpha^0 = z_\beta^0$ implies $\alpha = \beta$.*

Remark 2.3. — If $z_\alpha^0 \in \mathrm{Res}_0(P)$ is simple, the corresponding resonance z_α is simple for h small enough and Proposition 0.3 of [37] proves that z_α has a complete asymptotic expansion in powers of h.

Remark 2.4. — The analyticity of V in a full neighborhood of $\mathbb{R}^n$ is used only for the localization of the resonances. Indeed, if the conclusions of Theorem 2.1 and Remark 2.3 hold for V smooth and analytic outside of a compact set, then the results of this paper still apply under this weaker assumption.

The semiclassical pseudodifferential calculus is a tool used throughout this paper, and we fix here some notations. We refer to [13] for more details. For $m(x, \xi, h) \geqslant 0$ an order function and $\delta \geqslant 0$, we say that a function $a(x, \xi, h) \in C^\infty(T^*\mathbb{R}^n)$ is a symbol of class $S_h^\delta(m)$ when, for all $\alpha \in \mathbb{N}^{2n}$,

$$\left| \partial_{x,\xi}^\alpha a(x, \xi, h) \right| \lesssim h^{-\delta|\alpha|} m(x, \xi, h).$$

If $a \in S_h^\delta(m)$, the semiclassical pseudodifferential operator $\mathrm{Op}(a)$ with symbol a is defined by

$$\left(\mathrm{Op}(a)\varphi \right)(x) = \frac{1}{(2\pi h)^n} \iint e^{i(x-y)\cdot\xi/h} a\left(\frac{x+y}{2}, \xi, h \right) \varphi(y) \, dy \, d\xi,$$

for all $\varphi \in C_0^\infty(\mathbb{R}^n)$. We denote by $\Psi_h^\delta(m)$ the space of operators $\mathrm{Op}(S_h^\delta(m))$.

The rest of this paper is organized as follows. In Section 3, we prove a resolvent estimate in the complex plane that we use in all the paper. Then, in Section 4, we compute the spectral projection associated to a resonance. In section 5, we give the asymptotic expansion of the residue of the scattering amplitude at a simple resonance for long range potentials. Section 6 is devoted to the computation of the asymptotic behavior for large t of the Schrödinger group $e^{-itP/h}$, where the spectral projection appears

naturally. At last, we have placed in Appendix A some geometrical considerations about Hamiltonian curves in a neighborhood of the hyperbolic fixed point, that we need in Section 4.

3. Resolvent estimate

In this section, we prove a polynomial estimate for the resolvent of the distorted operator P_θ around the resonances. This estimate is used throughout the paper to control remainder terms. More precisely, we prove the following result.

THEOREM 3.1 (Resolvent estimate). — *Assume* (H1)–(H3). *There exists* $\varepsilon > 0$ *such that, for all* $C > 0$ *and* h *small enough,*

i) *The operator* P *has no resonances in*

$$[E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, 0] \smallsetminus D(E_0, 2Ch).$$

ii) *Assume* $\theta = \nu h |\ln h|$ *with* $\nu > 0$. *Then, there exists* $K > 0$ *such that*

$$(3.1) \qquad \left\| (P_\theta - z)^{-1} \right\| \lesssim h^{-K} \prod_{z_\alpha \in \mathrm{Res}(P) \cap D(E_0, 2Ch)} |z - z_\alpha|^{-1},$$

for all $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, Ch]$.

In particular, the previous theorem states that all the resonances in $[E_0 - \varepsilon, E_0 + \varepsilon] + i[-Ch, 0]$ are those given by Theorem 2.1. The rest of this section is devoted to the proof of Theorem 3.1. We follow the approach of Tang and Zworski [41] and we use the constructions of [2, Section 4] (see also Christianson [10] for hyperbolic orbits), where the propagation of singularities through a hyperbolic fixed point is studied, and of [1, Section 3], where a sharp estimate for the weighted resolvent for real energies is given.

3.1. Definition of a weighted operator Q_z

The distorted operator P_θ defined in (2.3) is a differential operator of order 2 whose symbol $p_\theta \in S_h^0(1)$ satisfies

$$(3.2) \qquad p_\theta(x, \xi, h) = p_{\theta,0}(x, \xi) + h p_{\theta,1}(x, \xi) + h^2 p_{\theta,2}(x, \xi),$$

with $p_{\theta,\bullet} \in S_h^0(\langle \xi \rangle^2)$ and

$$p_{\theta,0}(x, \xi) = p\big(x + i\theta F(x), (1 + i\theta\,{}^t(dF(x)))^{-1}\xi\big).$$

We write the Taylor expansion of $p_{\theta,0}(x,\xi)$ with respect to θ as

(3.3)
$$p_{\theta,0}(x,\xi) = p(x,\xi) - i\theta q(x,\xi) + \theta^2 r(x,\xi,\theta),$$
$$q(x,\xi) = \{p(x,\xi), F(x) \cdot \xi\},$$

for some $r \in S_h^0(\langle \xi \rangle^2)$ which vanishes in $|x| \leqslant R_0$. Notice that

$$q(x,\xi) = 2dF(x)\xi \cdot \xi - \nabla V(x) \cdot F(x),$$

so that for $\varepsilon > 0$ small enough, there exists $R_1 > R_0 + 1$ such that

(3.4)
$$q(x,\xi) \geqslant E_0,$$

for all $(x,\xi) \in p^{-1}([E_0 - 2\varepsilon, E_0 + 2\varepsilon])$ with $|x| \geqslant R_1$.

We want to gain as much ellipticity as we can near $(0,0)$. As in [2, Section 4], we shall work with a weighted operator, and we start by defining the weights. Let $\widetilde{p}(x,\xi) = p(x,\xi) - E_0$ and $\widetilde{p}_\theta(x,\xi,h) = p_\theta(x,\xi,h) - E_0$. There exists a symplectic map κ defined near $B(0,\varepsilon_2) = \{(x,\xi) \in T^*\mathbb{R}^n; \ |(x,\xi)| \leqslant \varepsilon_2\}$, with $0 < \varepsilon_2 \ll \varepsilon$, such that, setting $(y,\eta) = \kappa(x,\xi)$,

(3.5)
$$\widetilde{p}(x,\xi) = \mathcal{B}(y,\eta)y \cdot \eta.$$

Here $(y,\eta) \mapsto \mathcal{B}(y,\eta)$ is a C^∞ map from $\kappa(B(0,\varepsilon_2))$ to the space $\mathcal{M}_n(\mathbb{R})$ of $n \times n$ matrices with real entries such that

$$\mathcal{B}(0,0) = \operatorname{diag}(\lambda_1, \ldots, \lambda_n).$$

Let U be a unitary Fourier integral operator microlocally defined near $B(0,\varepsilon_2)$ and associated to the canonical transformation κ. Then

(3.6)
$$\widehat{P} = U(P - E_0)U^{-1},$$

is a pseudodifferential operator in $\Psi_h^0(1)$ with a real (modulo $S_h^0(h^\infty)$) symbol $\widehat{p}(y,\eta) = \sum_{j \geqslant 0} \widehat{p}_j(y,\eta)h^j$, such that

$$\widehat{p}_0(y,\eta) = \mathcal{B}(y,\eta)y \cdot \eta.$$

Let $0 < \varepsilon_1 < \varepsilon_2$. Since the trapped set at energy E_0 for p is $\{0\}$, we recall from [17, Appendix] that, for the compact set $\mathcal{K} = B(0, 2R_1) \smallsetminus B(0, \varepsilon_1) \cap p^{-1}([E_0 - 4\varepsilon, E_0 + 4\varepsilon]) \subset T^*\mathbb{R}^n$, there exist $0 < \varepsilon_0 < \varepsilon_1$ and a bounded function $g \in C^\infty(T^*\mathbb{R}^n)$ such that $H_p g$ has compact support and

(3.7)
$$\begin{cases} g(x,\xi) = 0, & \text{if } (x,\xi) \in B(0,\varepsilon_0), \\ H_p g(x,\xi) \geqslant 0, & \text{if } (x,\xi) \in T^*\mathbb{R}^n, \\ H_p g(x,\xi) \geqslant 1, & \text{if } (x,\xi) \in \mathcal{K}. \end{cases}$$

As in [31], we set, for $R \gg R_1$ to be chosen later,

(3.8)
$$g_0(x,\xi) = \chi_0\left(\frac{x}{R}\right)\psi_0(p(x,\xi))g(x,\xi)|\ln h|,$$

where $\chi_0 \in C_0^\infty(\mathbb{R}^n; [0,1])$ with $\chi_0 = 1$ on $B(0,1)$ and $\psi_0 \in C_0^\infty(\mathbb{R}; [0,1])$ with $\operatorname{supp} \psi_0 \subset [E_0 - 4\varepsilon, E_0 + 4\varepsilon]$ and $\psi_0 = 1$ in a neighborhood of $[E_0 - 3\varepsilon, E_0 + 3\varepsilon]$.

We also define functions on the (y, η) side. We set

$$
\begin{cases}
\widehat{g}_1(y, \eta) = (y^2 - \eta^2)\widehat{\phi}_1(y, \eta)|\ln h|, \\
\widehat{g}_2(y, \eta) = \left(\ln \left\langle \dfrac{y}{\sqrt{hM}} \right\rangle - \ln \left\langle \dfrac{\eta}{\sqrt{hM}} \right\rangle \right) \widehat{\phi}_2(y, \eta).
\end{cases}
$$

Here $M > 1$ is a parameter that will be chosen later on. Since we consider the semiclassical regime, we will assume that $hM < 1$. Moreover, $\widehat{\phi}_\bullet = \phi_\bullet \circ \kappa^{-1}$, where $\phi_1 \in C_0^\infty(B(0, \varepsilon_2))$ is such that $\phi_1 = 1$ near $B(0, \varepsilon_1)$ and $\phi_2 \in C_0^\infty(B(0, \varepsilon_0))$ is such that $\phi_2 = 1$ near 0 in $T^*\mathbb{R}^n$. At last, we choose four cut-off functions $\chi_1, \chi_2, \chi_3, \chi_4 \in C_0^\infty(B(0, \varepsilon_2))$ such that, setting again $\widehat{\chi}_\bullet = \chi_\bullet \circ \kappa^{-1}$, we have

$$
\mathbb{1}_{\{0\}} \prec \widehat{\phi}_2 \prec \widehat{\phi}_1 \prec \widehat{\chi}_1 \prec \widehat{\chi}_2 \prec \widehat{\chi}_3 \prec \widehat{\chi}_4.
$$

The notation $f \prec g$ means that $g = 1$ near the support of f. We define the operators

$$
G_{\pm 0} = \operatorname{Op}\left(e^{\pm t_0 g_0}\right), \ G_{\pm j} = \operatorname{Op}\left(e^{\pm t_j \widehat{g}_j}\right) \text{ and } \widetilde{G}_{\pm j} = \operatorname{Op}\left(\widehat{\chi}_j e^{\pm t_j \widehat{g}_j}\right),
$$

for $j = 1, 2$. Notice that $G_{\pm 0}$ is acting on functions of (x, ξ), whereas the other operators are acting on functions of (y, η). The $t_\bullet$'s are real constants that will be fixed below. Then,

$$
G_{\pm 0} \in \Psi_h^0\left(h^{-N_0}\right), \ G_{\pm 1} \in \Psi_h^0\left(h^{-N_1}\right), \ G_{\pm 2} \in \Psi_h^{1/2}\left(h^{-N_2}\right),
$$

$$
(3.9) \qquad \widetilde{G}_{\pm 1} \in \Psi_h^0\left(h^{-N_1}\langle \eta \rangle^{-\infty}\right) \text{ and } \widetilde{G}_{\pm 2} \in \Psi_h^{1/2}\left(h^{-N_2}\langle \eta \rangle^{-\infty}\right),
$$

for some $N_\bullet \in \mathbb{R}$.

We define the operator

$$
Q_z = \left(U^{-1}\left(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1)\right)U + \operatorname{Id} \right) G_{-0}(P_\theta - z)
$$

$$
(3.10) \qquad\qquad G_{+0}\left(U^{-1}\left(\widetilde{G}_{+1}\widetilde{G}_{+2} - \operatorname{Op}(\widehat{\chi}_1)\right)U + \operatorname{Id} \right).
$$

Splitting $P_\theta - z = \operatorname{Op}(\widetilde{p}_\theta \chi_4) + \operatorname{Op}(\widetilde{p}_\theta(1 - \chi_4)) - (z - E_0)$, we write

$$
Q_z = Q_1 + Q_2 - (z - E_0)Q_3,
$$

and we compute the symbols of the operators $Q_\bullet$ separately.

3.2. Computation of Q_z

The goal of this part is to prove the following identity.

LEMMA 3.2. — Let Q_z be the operator defined in (3.10). Then,

$$Q_z = \mathrm{Op}(p_\theta) + \mathrm{Op}(iht_0\{g_0, p_\theta\}) + U^{-1}\,\mathrm{Op}\left(iht_1\{\widehat{g}_1, \widehat{p}_0\}\right.$$
$$\left. + iht_2\{\widehat{g}_2, \widehat{p}_0\}\right)U - z + \mathcal{O}(hM^{-1})$$
$$\text{(3.11)} \qquad\qquad + \mathcal{O}(h^{\frac{3}{2}}M^{-\frac{1}{2}}|\ln h|^2) + \mathcal{O}(|z - E_0|M^{-2}).$$

Remark 3.3. — We will show in the proof of Lemma 3.2 (more precisely in (3.28)) that the operators $(U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id})G_{-0}$ and $G_{+0}(U^{-1}(\widetilde{G}_{+1}\widetilde{G}_{+2} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id})$ are invertible on $L^2(\mathbb{R}^n)$ and $H^2(\mathbb{R}^n)$ for M^{-1} and h small enough. Moreover, their inverses are polynomially bounded in h^{-1}. In particular, the resonances of P are the poles of Q_z^{-1} and to estimate $(P_\theta - z)^{-1}$, it is enough to estimate Q_z^{-1}.

The rest of this section is devoted to the proof of Lemma 3.2. In fact, (3.11) is close to the equation (4.44) of [2] and we will use some identities from [2] when possible.

Proof.

• First we consider Q_1. Since we can assume that $R_0 > \varepsilon_2$, we have

$$\mathrm{Op}(\widetilde{p}_\theta\chi_4)G_{+0} = \mathrm{Op}(\widetilde{p}\chi_4)G_{+0} = \mathrm{Op}(a_1),$$

with $a_1 \in S_h^0(h^{-N_0})$ given, for any $k_0 \in \mathbb{N}$, by

$$a_1(x, \xi) = \sum_{k=0}^{k_0} \frac{1}{k!}\left(\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k \widetilde{p}\chi_4(x, \xi)e^{t_0 g_0(y,\eta)}\right)\bigg|_{y=x,\eta=\xi}$$
$$\text{(3.12)} \qquad\qquad\qquad\qquad\qquad\qquad + h^{k_0 - N_0}S_h^0(1).$$

Then again

$$\text{(3.13)} \qquad G_{-0}\,\mathrm{Op}(\widetilde{p}_\theta\chi_4)G_{+0} = G_{-0}\,\mathrm{Op}(a_1) = \mathrm{Op}(a_2),$$

with $a_2 \in S_h^0(h^{-N_0})$ given, for any $k_1 \in \mathbb{N}$, by

$$a_2(x, \xi) = \sum_{k=0}^{k_1} \frac{1}{k!}\left(\left(\frac{ih}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^k e^{-t_0 g_0(x,\xi)}a_1(y, \eta)\right)\bigg|_{y=x,\eta=\xi}$$
$$\text{(3.14)} \qquad\qquad\qquad\qquad\qquad\qquad + h^{k_1 - N_0}S_h^0(1).$$

The k-th term in (3.14) is easily seen to be $\mathcal{O}(h^k)$, so that choosing k_1 large enough, we conclude that $a_2 \in S_h^0(1)$. Moreover $\mathrm{supp}\, a_2 \subset \mathrm{supp}\,\chi_4$ modulo $S_h^0(h^\infty)$, and

$$\text{(3.15)} \qquad a_2 = \widetilde{p}\chi_4 + iht_0\{g_0, \widetilde{p}\chi_4\} + S_h^0(h^2|\ln h|^2) = \widetilde{p}\chi_4 + a_3,$$

for some $a_3 \in S_h^0(h|\ln h|)$ with $\operatorname{supp} a_3 \subset \operatorname{supp} \chi_4 \cap \operatorname{supp} g_0$ modulo $S_h^0(h^\infty)$.

By Egorov's theorem,

$$(3.16) \qquad U\operatorname{Op}(\widetilde{p}\chi_4)U^{-1} = \operatorname{Op}(\widehat{a}_4) \quad \text{and} \quad U\operatorname{Op}(a_2)U^{-1} = \operatorname{Op}(\widehat{a}_5),$$

where $\widehat{a}_4, \widehat{a}_5 \in S_h^0(1)$ verify $\operatorname{supp} \widehat{a}_4, \operatorname{supp} \widehat{a}_5 \subset \operatorname{supp} \widehat{\chi}_4$ modulo $S_h^0(h^\infty)$. Moreover, from (3.15), we have

$$(3.17) \qquad \widehat{a}_5 = \widehat{a}_4 + iht_0\{\widehat{g}_0, \widehat{p}\widehat{\chi}_4\} + S_h^0(h^2|\ln h|^2) = \widehat{a}_4 + \widehat{a}_6,$$

with $\widehat{g}_0 = g_0 \circ \kappa^{-1}$ and a symbol $\widehat{a}_6 \in S_h^0(h|\ln h|)$ satisfying $\operatorname{supp} \widehat{a}_6 \subset \operatorname{supp} \widehat{\chi}_4 \cap \operatorname{supp} \widehat{g}_0$ modulo $S_h^0(h^\infty)$. Since $\phi_1, \phi_2 \prec \chi_1 \prec \chi_2$, we have $\widehat{g}_1, \widehat{g}_2 \prec \widehat{\chi}_1$ and we get by pseudodifferential calculus

$$(3.18) \qquad \widetilde{G}_{\pm 2}\widetilde{G}_{\pm 1} - \operatorname{Op}(\widehat{\chi}_1) + \operatorname{Id} = G_{\pm 2}G_{\pm 1} + \mathcal{O}(h^\infty).$$

Then, using (3.13), (3.16), (3.17) and (3.18), we obtain

$$Q_1 = U^{-1}\big(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1) + \operatorname{Id}\big)U\operatorname{Op}(a_2)$$
$$U^{-1}\big(\widetilde{G}_{+1}\widetilde{G}_{+2} - \operatorname{Op}(\widehat{\chi}_1) + \operatorname{Id}\big)U + \mathcal{O}(h^\infty)$$
$$= U^{-1}G_{-2}G_{-1}\operatorname{Op}(\widehat{a}_4)G_{+1}G_{+2}U + U^{-1}G_{-2}G_{-1}\operatorname{Op}(\widehat{a}_6)$$
$$(3.19) \qquad G_{+1}G_{+2}U + \mathcal{O}(h^\infty).$$

The first term in the right hand side of (3.19) has already been computed in the equations (4.15)–(4.41) of [2] (the reader should notice however that the symbol p there has to be replaced by $p\chi_4$ here). We have

$$G_{-2}G_{-1}\operatorname{Op}(\widehat{a}_4)G_{+1}G_{+2} = \operatorname{Op}\big(\widehat{a}_4 + iht_1\{\widehat{g}_1, \widehat{p}_0\widehat{\chi}_4\} + iht_2\{\widehat{g}_2, \widehat{p}_0\widehat{\chi}_4\}\big)$$
$$(3.20) \qquad\qquad + \mathcal{O}(hM^{-1}) + \mathcal{O}(h^{\frac{3}{2}}M^{-\frac{1}{2}}|\ln h|^2).$$

On the other hand, since $\operatorname{supp} \phi_2 \subset B(0, \varepsilon_0)$, $\widehat{g}_2 = 0$ near the support of $\widehat{g}_0$ and $\widehat{a}_6$. Thus,

$$G_{-2}G_{-1}\operatorname{Op}(\widehat{a}_6)G_{+1}G_{+2} = G_{-1}\operatorname{Op}(\widehat{a}_6)G_{+1} + \mathcal{O}(h^\infty).$$

And then, working as in (3.12)–(3.15), we obtain

$$(3.21) \quad G_{-2}G_{-1}\operatorname{Op}(\widehat{a}_6)G_{+1}G_{+2} = \operatorname{Op}\big(iht_0\{\widehat{g}_0, \widehat{p}\widehat{\chi}_4\}\big) + \mathcal{O}(h^2|\ln h|^2).$$

Using (3.16) and collecting (3.20) and (3.21), the identity (3.19) gives

$$Q_1 = \operatorname{Op}(\widetilde{p}\chi_4) + \operatorname{Op}(iht_0\{g_0, \widetilde{p}\chi_4\}) + U^{-1}\operatorname{Op}\big(iht_1\{\widehat{g}_1, \widehat{p}_0\}$$
$$(3.22) \qquad + iht_2\{\widehat{g}_2, \widehat{p}_0\}\big)U + \mathcal{O}(hM^{-1}) + \mathcal{O}(h^{\frac{3}{2}}M^{-\frac{1}{2}}|\ln h|^2).$$

• Now we consider Q_2. As in (3.12)–(3.15), we have

$$G_{-0}\operatorname{Op}(\widetilde{p}_\theta(1 - \chi_4))G_{+0} = \operatorname{Op}(b_1),$$

for some $b_1 \in S_h^0(h^{-N_0}\langle\xi\rangle^2)$. Moreover $\operatorname{supp} b_1 \subset \operatorname{supp}(1 - \chi_4)$ modulo $S_h^0(h^\infty)$ and

$$(3.23) \qquad b_1 = \widetilde{p}_\theta(1 - \chi_4) + iht_0\{g_0, \widetilde{p}_\theta(1 - \chi_4)\} + S_h^0(h^2|\ln h|^2).$$

Since $\widehat{\chi}_1 \prec \widehat{\chi}_3$, the pseudodifferential calculus gives $\widetilde{G}_{-1} = \widetilde{G}_{-1}\operatorname{Op}(\widehat{\chi}_3) + \Psi_h^0(h^\infty\langle\eta\rangle^{-\infty})$. Furthermore, using Egorov's theorem, we obtain

$$U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1))U$$
$$= U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1))\operatorname{Op}(\widehat{\chi}_3)U + \Psi_h^0(h^\infty\langle\xi\rangle^{-\infty})$$
$$(3.24) \qquad = U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1))U\operatorname{Op}(b_2) + \Psi_h^0(h^\infty\langle\xi\rangle^{-\infty}),$$

where $b_2 \in S_h^0(\langle\xi\rangle^{-\infty})$ and $\operatorname{supp} b_2 \subset \operatorname{supp}\chi_3$ modulo $S_h^0(h^\infty\langle\xi\rangle^{-\infty})$. Using $\chi_3 \prec \chi_4$, the supports of b_1 and b_2 are disjoint and

$$(3.25) \qquad Q_2 = \operatorname{Op}(b_1) + \mathcal{O}(h^\infty).$$

• It remains to study Q_3. Working as in (3.12)–(3.15), we get $G_{-0}G_{+0} = \operatorname{Id} + \operatorname{Op}(c_1)$ with $c_1 \in S_h^0(h^2|\ln h|^2)$ and $\operatorname{supp} c_1 \subset \operatorname{supp} g_0$ modulo $S_h^0(h^\infty)$. As in (3.16), we have

$$U\operatorname{Op}((1 + c_1)\chi_4)U^{-1} = \operatorname{Op}(\widehat{c}_2),$$

where $\widehat{c}_2 \in S_h^0(1)$. Now (3.18) and (3.24) yield

$$Q_3 = \Big(U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1))U + \operatorname{Id}\Big)\Big(\operatorname{Op}((1 + c_1)\chi_4)$$
$$+ \operatorname{Op}((1 + c_1)(1 - \chi_4))\Big)\Big(U^{-1}(\widetilde{G}_{+1}\widetilde{G}_{+2} - \operatorname{Op}(\widehat{\chi}_1))U + \operatorname{Id}\Big)$$
$$(3.26) \quad = U^{-1}G_{-2}G_{-1}\operatorname{Op}(\widehat{c}_2)G_{+1}G_{+2}U + \operatorname{Op}((1 + c_1)(1 - \chi_4)) + \mathcal{O}(h^\infty),$$

Working as in the equation (4.43) of [2], we get

$$G_{-2}G_{-1}\operatorname{Op}(\widehat{c}_2)G_{+1}G_{+2} = \operatorname{Op}(\widehat{c}_2) + \mathcal{O}(M^{-2}) + \mathcal{O}(h^2|\ln h|^2).$$

Combining (3.26) with the last identity, we finally obtain

$$Q_3 = U^{-1}\operatorname{Op}(\widehat{c}_2)U + \operatorname{Op}((1 + c_1)(1 - \chi_4)) + \mathcal{O}(M^{-2}) + \mathcal{O}(h^2|\ln h|^2)$$
$$(3.27) \quad = \operatorname{Id} + \mathcal{O}(M^{-2}) + \mathcal{O}(h^2|\ln h|^2).$$

• The same way, one can prove

$$\Big(U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \operatorname{Op}(\widehat{\chi}_1))U + \operatorname{Id}\Big)\Big(U^{-1}(\widetilde{G}_{+1}\widetilde{G}_{+2} - \operatorname{Op}(\widehat{\chi}_1))U + \operatorname{Id}\Big)$$
$$= \operatorname{Id} + \mathcal{O}(M^{-2}) + \mathcal{O}(h^2|\ln h|^2),$$

and the same kind of estimate holds for the product the other way round. On the other hand,

$$G_{-0}G_{+0} = \mathrm{Id} + \mathcal{O}(h^2|\ln h|^2) \text{ and } G_{+0}G_{-0} = \mathrm{Id} + \mathcal{O}(h^2|\ln h|^2).$$

Then the two operators $(U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id})G_{-0}$ and $G_{+0}(U^{-1}(\widetilde{G}_{+1}\widetilde{G}_{+2} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id})$ are invertible on $L^2(\mathbb{R}^n)$ for M^{-1} and h small enough and they satisfy

$$(3.28) \quad \begin{aligned} &\left\|\left(\left(U^{-1}(\widetilde{G}_{-2}\widetilde{G}_{-1} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id}\right)G_{-0}\right)^{-1}\right\| = \mathcal{O}(h^{-C}), \\ &\left\|\left(G_{+0}\left(U^{-1}(\widetilde{G}_{+1}\widetilde{G}_{+2} - \mathrm{Op}(\widehat{\chi}_1))U + \mathrm{Id}\right)\right)^{-1}\right\| = \mathcal{O}(h^{-C}), \end{aligned}$$

for some $C > 0$. The same thing can be done on $H^2(\mathbb{R}^n)$ since the operators we consider differ from Id by compactly supported pseudodifferential operators. This shows Remark 3.3.

• Adding (3.22), (3.25) and (3.27), we get Lemma 3.2 □

3.3. Estimates on the inverse of Q_z

Let $\widehat{\varphi} \in C_0^\infty(T^*\mathbb{R}^n; [0,1])$ be such that $\widehat{\varphi} = 1$ near 0. We define

$$(3.29) \qquad \widetilde{K} = U^{-1}\widehat{K}U \text{ with } \widehat{K} = C_1 \mathrm{Op}\left(\widehat{\varphi}\left(\frac{y}{\sqrt{hM}}, \frac{\eta}{\sqrt{hM}}\right)\right),$$

for some large constant $C_1 > 1$ fixed in the following.

LEMMA 3.4. — *Assume that $\delta > 0$, $C_0 > 1$ and $\theta = \nu h|\ln h|$ with $\nu > 0$. Denote $r = \max(|z - E_0|, h)$. Choose $M = \mu\sqrt{\frac{r}{h}}$ and fix $t_2, C_1, t_1, t_0, R, \mu$ large enough in this order. Then, we have, for h small enough,*

i) *For $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-2C_0h, 2C_0h]$ and $\mathrm{Im}\,z \geqslant \delta h$, the operator $Q_z : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible and*

$$(3.30) \qquad \qquad \left\|Q_z^{-1}\right\| = \mathcal{O}(h^{-1}).$$

ii) *For $z \in [E_0 - \varepsilon, E_0 + \varepsilon] + i[-2C_0h, 2C_0h]$, the operator $Q_z - ih\widetilde{K} : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is invertible and*

$$(3.31) \qquad \qquad \left\|(Q_z - ih\widetilde{K})^{-1}\right\| = \mathcal{O}(h^{-1}).$$

This lemma is similar to Proposition 4.1 of [2]. We will only give the proof of part ii) since the first part can be proved the same way (using (3.34) instead of (3.35)).

Proof. — Let $\omega_1, \ldots, \omega_5 \in C_0^\infty(T^*\mathbb{R}^n; [0,1])$ be such that

$$(3.32) \quad \mathbb{1}_{\{0\}} \prec \omega_1 \prec \omega_2 \prec \phi_2 \prec \mathbb{1}_{B(0,\varepsilon_1)} \prec \omega_3 \prec \omega_4 \prec \phi_1 \prec \omega_5 \prec \mathbb{1}_{B(0,\varepsilon_2)}.$$

As usual, we denote $\widehat{\omega}_\bullet = \omega_\bullet \circ \kappa^{-1}$. We now recall some ellipticity estimates proved in [2] by means of Gårding's inequality and Calderòn-Vaillancourt's theorem. From the equations (4.50), (4.51), (4.54), (4.55) and (4.64) of [2], we have

$$\left(\mathrm{Op}\big(-h\{\widehat{g}_2, \widehat{p}_0\}(1-\widehat{\omega}_2^2)\big)u, u\right) \geqslant -Ch|\ln h|\big\|\mathrm{Op}(\widehat{\omega}_4 - \widehat{\omega}_1)u\big\|^2$$
$$(3.33) \qquad\qquad\qquad\qquad + \mathcal{O}(h^\infty)\|u\|^2,$$

$$(3.34) \qquad \left(\mathrm{Op}\big(-h\{\widehat{g}_2, \widehat{p}_0\}\widehat{\omega}_2^2\big)u, u\right) \geqslant -ChM^{-1}\|u\|^2,$$

$$\left(\mathrm{Op}\big(-ht_2\{\widehat{g}_2, \widehat{p}_0\}\widehat{\omega}_2^2 + C_1 h\widehat{\varphi}\big)u, u\right) \geqslant \delta \min(t_2, C_1)h\big\|\mathrm{Op}(\widehat{\omega}_2)u\big\|^2$$
$$(3.35) \qquad\qquad\qquad\qquad + \mathcal{O}(hM^{-1})\|u\|^2,$$

$$\left(\mathrm{Op}\big(-h\{\widehat{g}_1, \widehat{p}_0\}(1-\widehat{\omega}_4^2)\big)u, u\right) \geqslant -Ch|\ln h|\big\|\mathrm{Op}(\widehat{\omega}_5 - \widehat{\omega}_3)u\big\|^2$$
$$(3.36) \qquad\qquad\qquad\qquad + \mathcal{O}(h^\infty)\|u\|^2,$$

$$\left(\mathrm{Op}\big(-h\{\widehat{g}_1, \widehat{p}_0\}\widehat{\omega}_4^2\big)u, u\right) \geqslant \delta h|\ln h|\big\|\mathrm{Op}(\widehat{\omega}_4 - \widehat{\omega}_1)u\big\|^2$$
$$(3.37) \qquad\qquad\qquad\qquad + \mathcal{O}(h^2|\ln h|)\|u\|^2,$$

for some $\delta, C > 0$ which do not depend on h, M and the $t_\bullet$'s.

From (3.3) and since $\theta = \nu h|\ln h|$,

$$\mathrm{Op}(p_\theta) + \mathrm{Op}\big(iht_0\{g_0, p_\theta\}\big) = \mathrm{Op}(p - i\theta q + iht_0\{g_0, p\})$$
$$(3.38) \qquad\qquad\qquad\qquad + \Psi_h^0(h^2|\ln h|^2\langle\xi\rangle^2).$$

Let $\omega_6 \in C_0^\infty(T^*\mathbb{R}^n; [0,1])$ be such that

$$(3.39) \quad \mathbb{1}_{B(0,R_1) \cap p^{-1}([E_0 - 2\varepsilon, E_0 + 2\varepsilon])} \prec \omega_6 \prec \mathbb{1}_{B(0,2R_1) \cap p^{-1}([E_0 - 3\varepsilon, E_0 + 3\varepsilon])}.$$

From the definition (3.8) of g_0, we have

$$-\{g_0, p\} = \chi_0\Big(\frac{x}{R}\Big)\psi_0(p)H_p g|\ln h| + \frac{2}{R}\xi \cdot (\partial_x \chi_0)\Big(\frac{x}{R}\Big)\psi_0(p(x,\xi))g|\ln h|.$$

Using Gårding's inequality, (3.7) implies

$$\left(\mathrm{Op}(-ht_0\{g_0, p\}\omega_6^2)u, u\right) \geqslant t_0 h|\ln h|\big\|\mathrm{Op}(\omega_6 - \omega_3)u\big\|^2$$
$$(3.40) \qquad\qquad - C\frac{t_0}{R}h|\ln h|\big\|\mathrm{Op}(1 - \omega_2)u\big\|^2 + \mathcal{O}(h^2|\ln h|)\|u\|^2.$$

Let $\psi \in C_0^\infty([E_0 - 2\varepsilon, E_0 + 2\varepsilon]; [0, 1])$ with $\psi = 1$ near $[E_0 - \varepsilon, E_0 + \varepsilon]$. Using the functional calculus for pseudodifferential operators, we can write

$$\big(\operatorname{Op}(q)u, u \big) = \big(\operatorname{Op}(q)\psi(P)u, u \big) + \big(\operatorname{Op}(q)(1 - \psi(P))u, u \big)$$
$$= \big(\operatorname{Op}(q\psi(p))u, u \big) + \big(\operatorname{Op}(q)(P + i)^{-1}(P + i)(1 - \psi(P))u, u \big)$$
$$+ \mathcal{O}(h)\|u\|^2.$$

Note that the operator $\operatorname{Op}(q)(P + i)^{-1}$ is uniformly bounded on $L^2(\mathbb{R}^n)$. Gårding's inequality together with (3.4) give

$$\big(\operatorname{Op}(q)u, u \big) \geqslant \delta \big\| \operatorname{Op}(\psi(p)(1 - \omega_6))u \big\|^2 - C \big\| (P + i)(1 - \psi(P))u \big\| \|u\|$$
$$\text{(3.41)} \qquad\qquad - C \big\| \operatorname{Op}(\omega_6 - \omega_4)u \big\|^2 + \mathcal{O}(h)\|u\|^2.$$

Adding (3.33), (3.35), (3.36) and (3.37) and using Gårding's inequality, we obtain

$$- \operatorname{Im}\big((U^{-1} \operatorname{Op}(iht_1\{\widehat{g}_1, \widehat{p}_0\} + iht_2\{\widehat{g}_2, \widehat{p}_0\})U - ih\widetilde{K})u, u \big)$$
$$\geqslant \delta t_1 h |\ln h| \big\| \operatorname{Op}(\omega_4 - \omega_1)u \big\|^2 + \delta \min(t_2, C_1) h \big\| \operatorname{Op}(\omega_2)u \big\|^2$$
$$- C t_1 h |\ln h| \big\| \operatorname{Op}(\omega_5 - \omega_3)u \big\|^2 - C t_2 h |\ln h| \big\| \operatorname{Op}(\omega_4 - \omega_1)u \big\|^2$$
$$\text{(3.42)} \qquad + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}(h^2 |\ln h|)\|u\|^2.$$

Combining the formulas (3.11) and (3.38) and the estimates (3.40), (3.41) and (3.42), we get

$$- \operatorname{Im}\big((Q_z - ih\widetilde{K})u, u \big)$$
$$\geqslant \delta \min(t_2, C_1) h \big\| \operatorname{Op}(\omega_2)u \big\|^2 + \delta t_1 h |\ln h| \big\| \operatorname{Op}(\omega_4 - \omega_1)u \big\|^2$$
$$+ t_0 h |\ln h| \big\| \operatorname{Op}(\omega_6 - \omega_3)u \big\|^2 + \delta \nu h |\ln h| \big\| \operatorname{Op}(\psi(p)(1 - \omega_6))u \big\|^2$$
$$- C t_2 h |\ln h| \big\| \operatorname{Op}(\omega_4 - \omega_1)u \big\|^2 - C t_1 h |\ln h| \big\| \operatorname{Op}(\omega_5 - \omega_3)u \big\|^2$$
$$- C \frac{t_0}{R} h |\ln h| \big\| \operatorname{Op}(1 - \omega_2)u \big\|^2 - C \nu h |\ln h| \big\| \operatorname{Op}(\omega_6 - \omega_4)u \big\|^2$$
$$- C \nu h |\ln h| \big\| (P + i)(1 - \psi(P))u \big\| \|u\| + \operatorname{Im} z \|u\|^2$$
$$\text{(3.43)} \qquad + \mathcal{O}(h^{\frac{3}{2}} M^{-\frac{1}{2}} |\ln h|^2)\|u\|^2 + \mathcal{O}(hM^{-1})\|u\|^2 + \mathcal{O}(|z - E_0| M^{-2})\|u\|^2.$$

Now, assume that $\operatorname{Im} z \in [-2C_0 h, 2C_0 h]$ and $\operatorname{Re} z - E_0$ is small. We choose the parameters, in this order, $\min(t_2, C_1) \gg C_0$, $t_1 \gg t_2$, $t_0 \gg \max(t_1, \nu)$ then $R \gg 1$ and finally $M = \mu\sqrt{\frac{\tau}{h}}$ with $\mu \gg 1$. Then, for h small enough, Gårding's inequality implies

$$\big\| (Q_z - ih\widetilde{K})u \big\| \|u\| \geqslant - \operatorname{Im}\big((Q_z - ih\widetilde{K})u, u \big)$$
$$\text{(3.44)} \qquad\qquad \geqslant h \|\psi(P)u\|^2 + \mathcal{O}(h |\ln h|) \big\| (P + i)(1 - \psi(P))u \big\|^2.$$

On the other hand, from (3.11), we have

$$Q_z - ih\widetilde{K} = P - z + \Psi_h^0\big(h|\ln h|\langle\xi\rangle^2\big) + \mathcal{O}(h|\ln h|).$$

Then,

$$
\begin{aligned}
\big\|(Q_z - ih\widetilde{K})u\big\| &\geqslant \big\|(1 - \psi(P))(Q_z - ih\widetilde{K})u\big\| \\
&\geqslant \big\|(1 - \psi(P))(P - z)u\big\| + \mathcal{O}(h|\ln h|)\big\|(P + i)u\big\| \\
&\gtrsim \big\|(P + i)(1 - \psi(P))u\big\| + \mathcal{O}(h|\ln h|)\big\|(P + i)u\big\| \\
&\gtrsim \big\|(P + i)(1 - \psi(P))u\big\| + \mathcal{O}(h|\ln h|)\big\|\psi(P)u\big\|,
\end{aligned}
$$

(3.45)

for all h small enough.

Adding (3.44) and $C_2 h|\ln h|$ times the square of (3.45), we obtain

$$\big\|(Q_z - ih\widetilde{K})u\big\|\|u\| + C_2 h|\ln h|\big\|(Q_z - ih\widetilde{K})u\big\|^2 \gtrsim h\|(P + i)u\|^2,$$

for C_2 fixed large enough. Then, using $\|(Q_z - ih\widetilde{K})u\|\|u\| \leqslant \delta h\|u\|^2 + \frac{1}{\delta h}\|(Q_z - ih\widetilde{K})u\|^2$ with $0 < \delta \ll 1$, we finally obtain

$$\big\|(Q_z - ih\widetilde{K})u\big\| \gtrsim h\|(P + i)u\|. \tag{3.46}$$

Since we can obtain the same way the same estimate for the adjoint $(Q_z - ih\widetilde{K})^*$, we get the lemma. $\qquad\square$

To prove the part i) of Theorem 3.1 (the resonance free zone), we will use in addition the following lemma.

LEMMA 3.5. — *Assume $|z - E_0| \geqslant h$. Under the assumptions of Lemma 3.4, we have*

$$\big\|\widetilde{K}Q_z u\big\| = |z - E_0|\big\|\widetilde{K}u\big\| + \mathcal{O}(h^{\frac{1}{2}}|z - E_0|^{\frac{1}{2}})\|u\|.$$

Proof. — Since $\|\widetilde{K}\| \lesssim 1$, (3.11) gives

$$
\begin{aligned}
\widetilde{K}Q_z = {}&\widetilde{K}\operatorname{Op}(\widetilde{p}_\theta) + \widetilde{K}\operatorname{Op}(iht_0\{g_0, p_\theta\}) + \widetilde{K}U^{-1}\operatorname{Op}\big(iht_1\{\widehat{g}_1, \widehat{p}_0\} \\
&+ iht_2\{\widehat{g}_2, \widehat{p}_0\}\big)U - (z - E_0)\widetilde{K} + \mathcal{O}(hM^{-1}) \\
&+ \mathcal{O}(h^{\frac{3}{2}}M^{-\frac{1}{2}}|\ln h|^2) + \mathcal{O}(|z - E_0|M^{-2}).
\end{aligned}
$$

(3.47)

Since the support of $\widehat{g}_0$ does not intersect the support of the symbol of $\widehat{K}$, we obtain

$$\widetilde{K}\operatorname{Op}(iht_0\{g_0, p_\theta\}) = \mathcal{O}(h^\infty). \tag{3.48}$$

Moreover, working as in (3.24),

$$
\begin{aligned}
\widetilde{K}\operatorname{Op}(\widetilde{p}_\theta) &= U^{-1}\widehat{K}U\operatorname{Op}(\widetilde{p}\chi_4) + \mathcal{O}(h^\infty) \\
&= U^{-1}\widehat{K}\operatorname{Op}(\widehat{p})U + \mathcal{O}(h^\infty).
\end{aligned}
$$

We now rescale the variables as in [8] and in the equation (4.18) of [2]. We define a unitary transformation V on $L^2(\mathbb{R}^n)$ by

$$(Vf)(y) = (hM)^{-\frac{n}{4}} f\big((hM)^{-\frac{1}{2}} y\big).$$

If $a(y, \eta)$ is a symbol, then

$$V^{-1} \operatorname{Op}_h(a(y, \eta))V = \operatorname{Op}_{\frac{1}{M}}\big(a((hM)^{\frac{1}{2}} Y, (hM)^{\frac{1}{2}} H)\big).$$

If possible, we will identify in the following an operator with its conjugation by V. As in [2, (4.24)], we define the class of symbols $a \in \widetilde{\mathcal{S}}_{\frac{1}{M}}(m)$, for an order function $m(Y, H)$, by

$$\big|\partial_x^\alpha \partial_H^\beta a(Y, H)\big| \lesssim \langle Y\rangle^{-\frac{|\alpha|}{2}} \langle H\rangle^{-\frac{|\beta|}{2}} m(Y, H).$$

We refer to the appendix of [2] for the pseudodifferential calculus in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$. From [2, (4.23)], we have that $\widehat{p} \in \widetilde{\mathcal{S}}_{\frac{1}{M}}(hM\langle(Y, H)\rangle^2)$. Since $\varphi \in C_0^\infty(T^*\mathbb{R}^n)$, we also have $\varphi(Y, H) \in \widetilde{\mathcal{S}}_{\frac{1}{M}}(\langle(Y, H)\rangle^{-\infty})$. Then, the pseudodifferential calculus in $\widetilde{\mathcal{S}}_{\frac{1}{M}}$ implies

(3.49) $$\widetilde{K} \operatorname{Op}(\widetilde{p}_\theta) = \mathcal{O}(hM).$$

The same way, [2, Equation (4.38)] gives

$$iht_1\{\widehat{g}_1, \widehat{p}_0\} \in \widetilde{\mathcal{S}}_{\frac{1}{M}}(h^{\frac{3}{2}} M^{\frac{1}{2}} |\ln h| \langle(Y, H)\rangle).$$

So,

$$\widetilde{K}U^{-1} \operatorname{Op}(iht_1\{\widehat{g}_1, \widehat{p}_0\})U = U^{-1}\widehat{K} \operatorname{Op}(iht_1\{\widehat{g}_1, \widehat{p}_0\})U + \mathcal{O}(h^\infty)$$

(3.50) $$= \mathcal{O}(h^{\frac{3}{2}} M^{\frac{1}{2}} |\ln h|).$$

Working in $S_h^{1/2}$, we get

$$\widetilde{K}U^{-1} \operatorname{Op}(iht_2\{\widehat{g}_2, \widehat{p}_0\})U = U^{-1}\widehat{K} \operatorname{Op}(iht_2\{\widehat{g}_2, \widehat{p}_0\}\widehat{\omega}_2)U + \mathcal{O}(h^\infty).$$

Since $\widehat{\omega}_2 \prec \widehat{\phi}_2$, [2, Equation (4.48)] yields that $iht_2\{\widehat{g}_2, \widehat{p}_0\}\widehat{\omega}_2 \in S_{\frac{1}{M}}^0(h)$. Using Calderòn-Vaillancourt's theorem for this operator, we finally obtain

(3.51) $$\widetilde{K}U^{-1} \operatorname{Op}(iht_2\{\widehat{g}_2, \widehat{p}_0\})U = \mathcal{O}(h).$$

The lemma follows from (3.47), the choice of M in Lemma 3.4 and the estimates (3.48), (3.49), (3.50) and (3.51). $\qquad\square$

3.4. Proof of Theorem 3.1

We first prove that (3.1) holds for

$$z \in [E_0 - Ah, E_0 + Ah] + i[-C_0 h, C_0 h].$$

Here, $A > 0$ is any fixed constant. We use a method due to Tang and Zworski [41]. For $z \in [E_0 - 2Ah, E_0 + 2Ah] + i[-2C_0 h, 2C_0 h]$, the quantity M can always be replaced by $\mu \gg 1$ in Lemma 3.4 (see (3.43)–(3.44)). Then, $z \mapsto Q_z$ is holomorphic in this set and $\|\widetilde{K}\|_{\text{tr}} = \mathcal{O}(1)$. As usual (see Section 4 of [2] for instance), we can find an operator K such that $\|K\| \lesssim 1$, rank $K = \mathcal{O}(1)$ and such that (3.31) holds with $\widetilde{K}$ replaced by K. Furthermore, thanks to Remark 3.3, the resonances coincide with the poles of Q_z^{-1} (with the same multiplicity). Mimicking the proof of Proposition 4.2 of [2] or Lemma 6.5 of [4] (which are adaptations of Lemma 1 of [41]), the estimates (3.30) and (3.31) imply

$$\left\|Q_z^{-1}\right\| \lesssim h^{-K_1} \prod_{z_\alpha \in \text{Res}(P) \cap D(E_0, 2C_0 h)} |z - z_\alpha|^{-1},$$

for some $K_1 > 0$ and any $z \in [E_0 - Ah, E_0 + Ah] + i[-C_0 h, C_0 h]$. On the other hand, Remark 3.3 gives

$$\left\|(P_\theta - z)^{-1}\right\| \lesssim h^{-K_2} \|Q_z^{-1}\|,$$

for some $K_2 > 0$. This proves (3.1) for $z \in [E_0 - Ah, E_0 + Ah] + i[-C_0 h, C_0 h]$.

Thanks to Theorem 2.1 which describes all the resonances in any neighborhood of size h of E_0, it remains to prove that P has no resonance in

$$(3.52) \qquad \left([E_0 - \varepsilon, E_0 + \varepsilon] \smallsetminus [E_0 - Ah, E_0 + Ah]\right) + i[-C_0 h, C_0 h],$$

for one $A > 0$ and that the resolvent satisfies in this region an upper bound polynomial with respect to h^{-1}. In particular, we can assume that $|z - E_0| \geqslant h$. Using Lemma 3.4, Lemma 3.5 and $\|\widetilde{K}Q_z u\| \lesssim \|Q_z u\|$, we get

$$\|Q_z u\| \geqslant \delta h \|(P + i)u\| - h \|\widetilde{K}u\|,$$

$$\|Q_z u\| \geqslant \delta |z - E_0| \|\widetilde{K}u\| + \mathcal{O}(h^{\frac{1}{2}} |z - E_0|^{\frac{1}{2}}) \|u\|,$$

for some $\delta > 0$. Then, summing the first identity with $h\delta^{-1} |z - E_0|^{-1}$ times the second one, we obtain

$$\|Q_z u\| \gtrsim h \|(P + i)u\| + \mathcal{O}(h^{\frac{3}{2}} |z - E_0|^{-\frac{1}{2}}) \|u\|,$$

since $h\delta^{-1} |z - E_0|^{-1} \lesssim 1$. If now we assume that $|z - E_0| \geqslant Ah$, we get

$$\|Q_z u\| \gtrsim h \|(P + i)u\| + \mathcal{O}(hA^{-\frac{1}{2}}) \|u\| \gtrsim h \|(P + i)u\|,$$

for A large enough. Thanks to Remark 3.3, this implies that P has no resonance in the region given in (3.52) and that (3.1) holds in this set.

4. Spectral projection

The purpose of this part is to give the asymptotic expansion of the generalized spectral projection Π_{z_α} associated to an isolated resonance z_α in some $D(E_0, Ch)$. We recall that Π_{z_α} is the operator from $L^2_{\text{comp}}(\mathbb{R}^n)$ to $L^2_{\text{loc}}(\mathbb{R}^n)$ defined by

$$\Pi_{z_\alpha} = -\frac{1}{2i\pi} \oint_\gamma (P - z)^{-1} dz,$$

where γ is a simple loop in the complex plane, oriented counterclockwise, such that z_α is the only resonance in the bounded domain delimited by γ.

THEOREM 4.1 (Asymptotic expansion for the spectral projection). — *Assume* (H1)–(H3). *Let* $\alpha \in \mathbb{N}^n$ *be such that* z_α^0 *is simple. Then, as operators from* $L^2_{\text{comp}}(\mathbb{R}^n)$ *to* $L^2_{\text{loc}}(\mathbb{R}^n)$,

$$(4.1) \qquad \Pi_{z_\alpha} = c(\,\cdot\,, \overline{f})f,$$

where

$$(4.2) \qquad c(h) = h^{-|\alpha|-\frac{n}{2}} \frac{e^{-i\frac{\pi}{2}(|\alpha|+\frac{n}{2})}}{\alpha!(2\pi)^{\frac{n}{2}}} \prod_{j=1}^n \lambda_j^{\alpha_j+\frac{1}{2}},$$

and the function $f(x, h)$ *satisfies the following properties:*

i) *It is locally uniformly in* $L^2(\mathbb{R}^n)$: *for all* $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\|\varphi f\|_{L^2(\mathbb{R}^n)} \lesssim 1.$$

ii) *It satisfies the Schrödinger equation:*

$$(P - z_\alpha)f = 0.$$

iii) *It is outgoing: there exists* $R > 0$ *such that*

$f = 0$ *microlocally near each* (x, ξ) *with* $|x| > R,\ \cos(x, \xi) < -1/2.$

iv) *Finally, locally near* 0, *we have*

$$f = d(x, h)e^{i\varphi_+(x)/h},$$

where $d(x, h) \in S_h^0(1)$ *is a classical symbol satisfying*

$$d(x, h) \sim \sum_{j=0}^{+\infty} d_j(x)h^j \quad \text{and} \quad d_0(x) = x^\alpha + \mathcal{O}(x^{|\alpha|+1}).$$

We prove this result the following way. Using [2], we compute $(P - z)^{-1}v$ for some well prepared WKB function v and z on a loop around the resonance z_α. Integrating with respect to z, we get $\Pi_{z_\alpha} v$ and thus the resonant state f. To finish the proof, we obtain the constant c computing $(v, \overline{f})$ by a stationary phase argument.

Remark 4.2.

i) Since f is not necessarily in $\mathcal{S}'(\mathbb{R}^n)$, saying "$f = 0$ microlocally near ρ_0" means that there exists $\phi \in C_0^\infty(\mathbb{R}^{2n})$ with $\phi(\rho_0) \neq 0$ such that, for every $\chi \in C_0^\infty(\mathbb{R}^n)$, $\mathrm{Op}(\phi)(\chi f) = \mathcal{O}(h^\infty)$ in $L^2(\mathbb{R}^n)$.

ii) The properties i)–iv) of Theorem 4.1 characterize uniquely the resonant state $f(x, h)$ modulo $\mathcal{O}(h^\infty)$. In particular, the usual propagation of singularities implies that this function is a classical Lagrangian distribution of order 0 with Lagrangian manifold Λ_+.

For the punctual well in the island situation, the generalized spectral projection has been computed by Helffer and Sjöstrand [22]. In particular, they have proved that this operator is almost orthogonal. Indeed, if the resonance z is isolated and the cut-off $\chi \in C_0^\infty(\mathbb{R}^n)$ is equal to 1 near the well, then $\chi\Pi_z\chi$ is exponentially close to the spectral projection associated to the Dirichlet problem in the well and $\|\chi\Pi_z\chi\| = 1 + \mathcal{O}(e^{-\delta/h})$ for some $\delta > 0$. The situation is very different in the present setting since, for $\chi \neq 0$, $\|\chi\Pi_{z_\alpha}\chi\|$ is of order $h^{-|\alpha|-\frac{n}{2}}$.

From the previous discussion, the polynomial upper bound on the resolvent proved in Theorem 3.1 occurs effectively. More precisely, in every disc $D(z_\alpha, \varepsilon h)$, with $\varepsilon > 0$, the cut-off resolvent can not be bounded by anything smaller than $\frac{h^{-C_\alpha}}{|z-z_\alpha|}$ for some $C_\alpha > 0$. Moreover, since $C_\alpha \geqslant |\alpha| + \frac{n}{2}$, this constant can not be taken uniformly with respect to z_α.

One may perhaps prove Theorem 4.1 with other methods than the one we use here. In the one dimensional case, the resolvent can be written in term of a basis of solutions of $(P - z)u = 0$ and of their Wronskian. Thus, it must be possible to use the results of [36] in which the scattering amplitude, which can be expressed through the Wronskians of the Jost solutions, has been computed. In any dimension, another approach is perhaps also possible. One may first try to calculate the resonant state f with various methods (using, for example, the works of Briet, Combes and Duclos [5], Sjöstrand [37] or Hassell, Melrose and Vasy [19]). It then remains to calculate the constant c. This question is equivalent to the calculation of the scalar product $(f, \overline{f}) = \int f^2$. If we neglect the problems of integration at infinity, this calculation is reduced to a problem of stationary phase at point 0. But, since f^2 vanishes to order $2|\alpha|$, the knowledge of d_0 is not

enough and we must explicitly know the $|\alpha|$ first terms in the expansion of f in powers of h. In this computation, the situation becomes, in a sense, similar to that of the eigenvectors of the harmonic oscillator for which the "good variable" is $\frac{x}{\sqrt{h}}$. However, this is not the case in Theorem 4.1 since the factor $e^{i\varphi_+(x)/h}$ in f has modulus 1.

It may be possible to obtain some results when z_α^0 is not simple. In that case, various situations may occur: several resonances can be very close to each other, the resonances can have a non-trivial multiplicity and they can be multiple poles of the resolvent. We refer to [37, Section 4] where such phenomena are shown. In the remainder of this discussion, we consider the simplest case where a double resonance can appear. We assume that $\lambda_1 = \lambda_2 < \lambda_3$ and that $\widetilde{z} = b_h(z_{(1,0,\dots)}) = b_h(z_{(0,1,0,\dots)})$ is a double resonance. Then, near $\widetilde{z}$, the resolvent can be written

$$(z - P)^{-1} = \frac{\Pi_2}{(z - \widetilde{z})^2} + \frac{\Pi_1}{z - \widetilde{z}} + H(z),$$

where H is holomorphic near $\widetilde{z}$. In that case, rank $\Pi_2 \leqslant 1$ and rank $\Pi_1 = 2$. It seems possible to calculate Π_1 with a proof similar to that of Theorem 4.1. Using Proposition A.3, we can construct two initial data v_1, v_2 such that the microsupport of v_j and Λ_- intersect along a Hamiltonian curve which goes to 0 along the j-th vector basis. Then, computing the residue of $(z-P)^{-1}v_j$, we obtain that $\Pi_1(v_j + (P - \widetilde{z})\partial_z v_j)$ is of the form $f_j = x_j e^{i\varphi_+(x)/h}$ modulo a constant. In the following, we can neglect $(P - \widetilde{z})\partial_z v_j$ as it gives lower order terms. Since f_1 and f_2 can not be collinear, $\{f_1, f_2\}$ (resp. $\{\overline{f_1}, \overline{f_2}\}$) forms a basis of Im Π_1 (resp. Im Π_1^*). To finish the computation of Π_1, it is sufficient to calculate $(v_j, \overline{f_k})$. The scalar products $(v_j, \overline{f_j})$ can be calculated as in the proof of Theorem 4.1. But, according to the choice of the v_j's and to the form of the f_k's, $(v_j, \overline{f_k})$ appears to be smaller when $j \neq k$. Eventually, in the $\{f_1, f_2\}$ and $\{\overline{f_1}, \overline{f_2}\}$ bases, the operator Π_1 seems to be a 2×2-matrix whose diagonal coefficients are given by (4.2) at the first order and whose off-diagonal coefficients are of lower order. One can probably also say something about Π_2. But, one may need to calculate several lower order terms in the semiclassical expansions (for the resonance for example). This operator seems to have a smaller norm.

4.1. Construction of "test functions"

To prove the theorem, it is enough to show that

$$(4.3) \qquad\qquad \chi \Pi_{z_\alpha} \chi = c(\,\cdot\,, \overline{\chi f}) \chi f,$$

for $\chi \in C_0^\infty(\mathbb{R}^n)$. Let $\Pi_{z_\alpha,\theta}$ be the spectral projection of P_θ at the resonance z_α. It is the operator on $L^2(\mathbb{R}^n)$ defined by

$$(4.4) \qquad \Pi_{z_\alpha,\theta} = -\frac{1}{2i\pi} \oint_\gamma (P_\theta - z)^{-1} dz,$$

We now assume that the distortion occurs outside of the support of χ. In particular, $\chi \Pi_{z_\alpha} \chi = \chi \Pi_{z_\alpha,\theta} \chi$. Let J be the anti-linear operator on $L^2(\mathbb{R}^n)$ defined by

$$J : \begin{cases} L^2(\mathbb{R}^n) & \longrightarrow & L^2(\mathbb{R}^n) \\ u & & \overline{u}. \end{cases}$$

Since P is a Schrödinger operator with a real potential, $JP = PJ$ and a direct calculation gives $(P_\theta - z)^{-1} = J\big((P_\theta - z)^{-1}\big)^* J$. Thus, $\Pi_{z_\alpha,\theta}$ can be written $\Pi_{z_\alpha,\theta} = (\,\cdot\,, \overline{g_\theta}) g_\theta$ with $g_\theta \in L^2(\mathbb{R}^n)$. The same way, $\Pi_{z_\alpha} = (\,\cdot\,, \overline{g}) g$ for some $g \in L^2_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover, from [39, Proof of Lemma 3.5], we can always assume that $g_\theta = U_\theta g$. In particular, $\chi g_\theta = \chi g$.

Since z_α^0 is simple, for all $j \in \{1\} \cup \mathrm{supp}\,\alpha$ (where $\mathrm{supp}\,\alpha = \{j \in \mathbb{N}; \, \alpha_j \neq 0\}$), $\lambda_j = \lambda \cdot \beta$ with $\beta \in \mathbb{N}^n$ implies $|\beta| = 1$. Then, from Lemma A.1 and Proposition A.3, there exists a Hamiltonian curve $\gamma^- = (x(t), \xi(t)) \subset \Lambda_-$ such that, for all $j \in \{1\} \cup \mathrm{supp}\,\alpha$, we have $\gamma_{\lambda_j}^- = \gamma_{\lambda_j,0}^- \neq 0$.

We now construct the "test functions", supported microlocally near the "test curve" γ^-, on which we will evaluate the spectral projection. Let $u(x, z, h)$ be a function defined in a vicinity of 0 but not at 0. We assume that u is a WKB solution of $(P - z)u = 0$. More precisely, near the x-projection of $\gamma^- \smallsetminus \{0\}$, we have

$$(4.5) \qquad u(x, z, h) = b(x, z, h) e^{i\psi(x)/h}.$$

Here ψ is a C^∞ function solving the eikonal equation $|\nabla \psi|^2 + V(x) = E_0$. We assume that $\Lambda_\psi = \{(x, \nabla\psi(x))\}$ intersects transversely Λ_- along γ^-. Note that the construction of such a phase, whose associated Lagrangian manifold projects nicely on the x-space in a vicinity of γ^-, can always be done thanks to [1, Proposition C.1]. The symbol $b(x, z, h)$ is classical: for all $N \in \mathbb{N}$,

$$b(x, z, h) = \sum_{j=0}^N b_j(x, z) h^j + S_h^0(h^{N+1}),$$

uniformly for $z \in D(E_0, C_0 h)$. Moreover, b and the b_j's are C^∞ with respect to x and analytic with respect to $z \in D(E_0, C_0 h)$. Finally, we assume that u satisfies

$$(P - z)u = \mathcal{O}(h^\infty),$$

and $b_0(x, z) \neq 0$ near the x-projection of γ^-. For that, it is enough to solve the usual transport equations. Finally, we suppose that $u = 0$ outside a neighborhood of the spacial projection of γ^-. Then, we set

$$(4.6) \qquad v = [P, \tau]u,$$

where $\tau \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\tau$ close to 0 and $\tau = 1$ near 0. We consider

$$(4.7) \qquad w = (P_\theta - z)^{-1}v.$$

In all the proof of Theorem 4.1, we will work with z in a ring $\mathcal{R}_h = D(z_\alpha^0, C_2 h) \smallsetminus D(z_\alpha^0, C_1 h)$ such that z_α^0 is the unique element of $\operatorname{Res}_0(P)$ in $D(z_\alpha^0, C_2 h)$. Note that Theorem 3.1 implies that $\|w\|_{H^2(\mathbb{R}^n)} \lesssim h^{-C}$ uniformly for $z \in \mathcal{R}_h$, for some $C > 0$.

4.2. Calculation of w before the critical point

We begin the proof by showing that w is 0 in the incoming region. More precisely, we have

LEMMA 4.3. — *Let $\rho \in \mathbb{R}^{2n}$ be such that $\rho \notin \Lambda_+$ and $\exp(]-\infty, 0]H_p)(\rho)$ does not meet the microsupport of v. Then, $w = 0$ microlocally near ρ, uniformly in $z \in \mathcal{R}_h$.*

Proof. — This lemma can be proved as Theorem 2 of [4]. First, assume $\rho \notin p^{-1}(E_0)$. Using the elliptic equation $(P_\theta - z)w = v$, the norm estimates $\|v\|, \|w\| \lesssim h^{-C}$ and the condition $\rho \notin \operatorname{MS}(v)$, the standard pseudodifferential calculus implies that $\rho \notin \operatorname{MS}(w)$. More precisely, for all $f \in C_0^\infty(\mathbb{R})$ with $f = 1$ near E_0, we have

$$(4.8) \qquad (1 - f(P))w = \mathcal{O}(h^\infty),$$

uniformly in $z \in \mathcal{R}_h$.

Assume now that $\rho \in p^{-1}(E_0)$. From the hypotheses, the half-curve $\exp(tH_p)(\rho)$, $t \leqslant 0$, does not meet $\operatorname{MS}(v)$ and goes to ∞ as $t \to -\infty$. Then, one can find a symbol $\omega \in S_h^0(1)$ such that $\omega = 1$ near ρ, $H_p\omega \leqslant 0$, $\exp(]-\infty, 0]H_p)(\operatorname{supp}\omega)$ does not meet $\operatorname{MS}(v)$ and $\exp(-TH_p)(\operatorname{supp}\omega) \subset \Gamma^-(R, d, \sigma)$ for some $T, R \gg 1$, $d > 0$ and $\sigma < 0$. Here, $\Gamma^-(R, d, \sigma) = \{(x, \xi) \in T^*\mathbb{R}^n; |x| > R, d^{-1} < |\xi| < d \text{ and } \cos(x, \xi) \leqslant \sigma\}$. Then, mimicking the proof of [4, Theorem 2], we get $\operatorname{Op}(\omega)w = \mathcal{O}(h^\infty)$, uniformly in $z \in \mathcal{R}_h$. The unique difference with its proof is that the 0 in the left hand side of [4, (3.4)] is replaced by $\mathcal{O}(h^\infty)$ (here, we use that $\operatorname{supp}\omega \cap \operatorname{MS}(v) = \emptyset$). $\qquad \square$

We will now calculate w on Λ_- near 0. First, using $\mathrm{MS}(v) \cap \Lambda_- \subset \gamma^-$, the previous lemma implies the following consequence.

Remark 4.4. — We have $w = 0$ microlocally near each point of $\Lambda_- \setminus \gamma^-$.

On the other hand, near γ^-, we have the following lemma. Note that the results of this lemma and of Remark 4.4 are uniform for $z \in \mathcal{R}_h$.

LEMMA 4.5. — *Let $\rho \in \gamma^-$ be a point close enough to 0. Then, $w = u$ microlocally near ρ.*
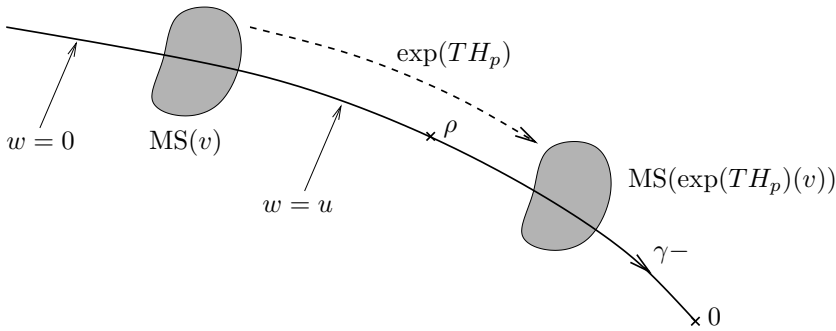
Figure 4.1. *The geometrical setting of Lemma 4.5.*

Proof. — We define

(4.9)
$$\widetilde{w} = \frac{i}{h} \int_0^T e^{-it(P-z)/h}[P, \tau]u \, dt,$$

where $T > 0$ is chosen such that $\exp(TH_p)(\mathrm{MS}(v)) \cap \exp(]-\infty, 0]H_p)(\rho) = \emptyset$ (see Figure 4.1). Then, microlocally near each point of γ^-, we have

$$(P_\theta - z)\widetilde{w} = (P - z)\widetilde{w}$$
$$= [P, \tau]u - e^{-iT(P-z)/h}[P, \tau]u.$$

For the first equality, we have used that $P = P_\theta$ near the spacial projection of $\mathrm{MS}(\widetilde{w}) \cap \gamma^- \subset \exp([0, +\infty[H_p)(\mathrm{MS}(v))$. Thus, microlocally near γ^-, we have

$$(P_\theta - z)(w - \widetilde{w}) = e^{-iT(P-z)/h}[P, \tau]u.$$

In particular, the choice of T and the Egorov theorem imply $(P_\theta - z)(w - \widetilde{w}) = 0$ microlocally near $\exp(]-\infty, 0]H_p)(\rho)$. On the other hand, combining Lemma 4.3 (for w) and the Egorov theorem (for $\widetilde{w}$), we obtain $w - \widetilde{w} = 0$

microlocally near $\exp(-SH_p)(\rho)$, for all S large enough. Using moreover that $\|w - \widetilde{w}\| \leqslant h^{-C}$, the propagation of singularities implies that

$$(4.10) \qquad w = \widetilde{w} \text{ microlocally near } \rho.$$

Then, microlocally near ρ, we have

$$\widetilde{w} = \frac{i}{h} \int_0^T e^{-it(P-z)/h}[P,\tau]u \, dt$$

$$= \frac{i}{h} \int_0^T e^{-it(P-z)/h}\big((P-z)\tau u - \tau(P-z)u\big) dt$$

$$(4.11) \qquad = \frac{i}{h} \int_0^T e^{-it(P-z)/h}(P-z)\tau u \, dt = -e^{-iT(P-z)/h}\tau u + \tau u = u,$$

which proves the lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In fact, one can prove more directly Lemma 4.3 and Lemma 4.5 by applying the proof of Theorem 2 of [4] to the function $w - \widetilde{w}$.

4.3. Representation of w at the critical point

We will use the variable $\sigma = (z - E_0)/h$, the notation $\sigma_\alpha^0 = (z_\alpha^0 - E_0)/h$ and the set $\mathcal{R} = D(\sigma_\alpha^0, C_2) \smallsetminus D(\sigma_\alpha^0, C_1)$. Note that σ_α^0 and $\mathcal{R}$ does not depend on h and $\mathcal{R}_h = E_0 + h\mathcal{R}$. Since $\tau = 1$ near 0, we have

$$(4.12) \qquad (P-z)w = (P_\theta - z)w = [P,\tau]u = 0,$$

in a neighborhood of 0. On the other hand, let $\rho \in \Lambda_- \cap \{|x| = \delta\}$ with $\delta > 0$ small enough. From Remark 4.4 and Lemma 4.5, we have

$$(4.13) \qquad w = \begin{cases} be^{i\psi/h} & \text{if } \rho \in \gamma^- \\ 0 & \text{if } \rho \notin \gamma^-, \end{cases}$$

microlocally near ρ. Moreover $\|w\| \lesssim h^{-C}$. Then, we are in position to apply Theorem 2.1 and Theorem 2.5 of [2] which give a representation of w microlocally near $(0,0)$. More precisely, Theorem 5.1 of [2] states that, microlocally near $(0,0)$,

$$w = \frac{1}{(2\pi h)^{\frac{1}{2}}} e^{i\varphi_+(x)/h} e^{i\psi(0)/h} A_-(x, \sigma, h)$$

$$(4.14) \qquad\qquad\qquad + \frac{1}{(2\pi h)^{\frac{1}{2}}} \int_{-1}^{+\infty} e^{i\varphi(t,x)/h} A_+(t, x, \sigma, h) \, dt.$$

Concerning the symbol A_+, we will only use that $\sigma \mapsto A_+$ is a holomorphic function of $\sigma \in D(0, C_0)$ which decays uniformly exponentially in t (see [2, Proposition 5.11]). The constant $\psi(0)$ is defined by

$$(4.15) \qquad \psi(0) := \lim_{t \to +\infty} \psi(x(t)) = \psi(x(s)) - \varphi_-(x(s)),$$

for all $s \geqslant 0$.

The symbol $A_-(x, \sigma, h) \in S_h^0(h^{-C})$, holomorphic for $\sigma \in \mathcal{R}$, is constructed the following way. There exists an expandible symbol $a(t, x, \sigma, h) \in S_h^0(1)$ of the form

$$a(t, x, \sigma, h) \sim \sum_{j=0}^{+\infty} a_j(t, x, \sigma) h^j,$$

where the a_j's satisfy

$$a_j(t, x, \sigma) \sim \sum_{k=0}^{+\infty} a_{j, \mu_k}(t, x, \sigma) e^{-(S+\mu_k)t}$$

and

$$a_{j, \mu_k}(t, x, \sigma) = \sum_{\ell=0}^{M_{j, \mu_k}} a_{j, \mu_k, \ell}(x, \sigma) t^\ell.$$

We refer to Helffer and Sjöstrand [21] for the definition of expandible functions. Here, S is defined by

$$S = S(\sigma) := \sum_{j=1}^{n} \frac{\lambda_j}{2} - i\sigma.$$

The symbols $a_j, a_{j, \mu_k}, a_{j, \mu_k, \ell}$ are holomorphic for $\sigma \in D(0, C_0)$. Moreover, as in [2, (6.26)], $a_{0,0}$ does not depend on t (and σ) and

$$(4.16) \qquad a_{0,0}(0) = |g_{\lambda_1}^-| \lambda_1^{\frac{3}{2}} e^{-i\frac{\pi}{4}} e^{-\int_0^{+\infty} \Delta\psi(x(s)) - (\sum \lambda_j/2 - \lambda_1)\, ds} b_0(x(0)),$$

with $g_{\lambda_j}^- = \pi_x(\gamma_{\lambda_j}^-)$, π_x being the spatial projection. Let $\varphi_\star(t, x) = \varphi(t, x) - (\varphi_+(x) + \psi(0))$ be the expandible function

$$\varphi_\star(t, x) \sim \sum_{k=1}^{+\infty} \varphi_{\mu_k}(t, x) e^{-\mu_k t} \quad \text{and} \quad \varphi_{\mu_k}(t, x) = \sum_{\ell=0}^{N_{\mu_k}} \varphi_{\mu_k, \ell}(x) t^\ell,$$

constructed in [2, Section 5]. Recall that $\varphi(t, x)$ satisfies the eikonal equation

$$(4.17) \qquad \partial_t \varphi + (\partial_x \varphi)^2 + V(x) = E_0.$$

We consider the expandible symbol (see (5.77) of [2])

$$(4.18) \qquad \widetilde{a} = \sum_{q < Q_1} \frac{a}{q!} \left(\frac{i\varphi_\star}{h} \right)^q \sim \sum_{j=1-Q_1}^{+\infty} \widetilde{a}_j(t, x, \sigma) h^j,$$

for some $Q_1 \in \mathbb{N}$ fixed large enough,

$$\widetilde{a}_j(t, x, \sigma) \sim \sum_{k=0}^{+\infty} \widetilde{a}_{j,\mu_k}(t, x, \sigma) e^{-(S+\mu_k)t}$$

and

$$\widetilde{a}_{j,\mu_k}(t, x, \sigma) = \sum_{\ell=0}^{\widetilde{M}_{j,\mu_k}} \widetilde{a}_{j,\mu_k,\ell}(x, \sigma) t^\ell.$$

Then, $A_-(x, \sigma, h)$ is a symbol, holomorphic with respect to $\sigma \in \mathcal{R}$, such that

$$(4.19) \qquad A_-(x, \sigma, h) \sim \sum_{j=1-Q_1}^{+\infty} h^j \sum_{k=0}^{K_1} \sum_{\ell=0}^{\widetilde{M}_{j,\mu_k}} \frac{\ell!}{(S+\mu_k)^{\ell+1}} \widetilde{a}_{j,\mu_k,\ell}(x, \sigma),$$

for some $K_1 \in \mathbb{N}$ large enough.

In the following, we will need some informations on the φ_{μ_k}. Let $j \in \{1, \ldots, n\}$ be such that $\alpha_j \neq 0$ or $j = 1$. Since z_α^0 is simple, λ_j can not be written as a non-trivial combination of the λ_k's (i.e. $\lambda_j = \lambda \cdot \beta$ implies $\beta_k = \delta_{j,k}$). Therefore, calculating the term in $e^{-\lambda_j t}$ of (4.17), we obtain

$$-\lambda_j \varphi_{\lambda_j} + \partial_t \varphi_{\lambda_j} + 2\partial_x \varphi_+ \cdot \partial_x \varphi_{\lambda_j} = 0.$$

Working as in Section 6.1 of [1] (see also (5.59) of [2] for $j = 1$), one can prove that φ_{λ_j} does not depend on t (i.e. $N_{\lambda_j} = 0$), that

$$(4.20) \qquad 2\partial_x \varphi_+ \cdot \partial_x \varphi_{\lambda_j} - \lambda_j \varphi_{\lambda_j} = 0,$$

and that

$$(4.21) \qquad \varphi_{\lambda_j}(x) = -\lambda_j g_{\lambda_j}^- x_j + \mathcal{O}(x^2).$$

Since $g_{\lambda_j}^-$ is collinear to the j-th vector of basis, we also denote this j-th component of the vector $g_{\lambda_j}^-$ by $g_{\lambda_j}^-$.

4.4. Integration with respect to z

Let γ be a fixed simple loop in $\mathcal{R}$ around 0 oriented counterclockwise and $\gamma_h = E_0 + h\gamma \subset \mathcal{R}_h$. We integrate w on the loop γ_h. First, since z_α is a simple resonance for h small and since v is a holomorphic function with respect to $z \in D(E_0, C_0 h)$, the equations (4.4) and (4.7) give

$$(4.22) \qquad \Pi_{z_\alpha, \theta} v(x, z_\alpha) = -\frac{1}{2i\pi} \oint_{\gamma_h} w(x, z) \, dz = -\frac{h}{2i\pi} \oint_{\gamma} w(x, \sigma) \, d\sigma.$$

On the other hand, we can also calculate this quantity microlocally near $(0, 0)$ with the help of (4.14). Since $\sigma \mapsto A_+(\sigma)$ is holomorphic in $D(0, C_0)$,

the second term in the right hand side of (4.14) gives no contribution to this integral. Moreover, for $\mu_k \neq \lambda \cdot \alpha$, the function $(S + \mu_k)^{-1}$ is holomorphic for $\sigma \in D(\sigma_\alpha^0, C_2)$. This implies that only the terms of (4.19) with $\mu_k = \lambda \cdot \alpha$ give a non-zero contribution to the integral over σ.

We now look for the terms with $\mu_k = \lambda \cdot \alpha$ in (4.18). Among these terms, the one which gives the higher possible power of h^{-1}, is given by $q = |\alpha|$ and is equal to

$$\frac{a_{0,0}(x)}{|\alpha|!} \left(\frac{i}{h}\right)^{|\alpha|} \frac{|\alpha|!}{\alpha!} \prod_{j=1}^{n} \left(\varphi_{\lambda_j}(x)\right)^{\alpha_j}.$$

Here, we have used the fact that z_α^0 is simple. Note that, since $a_{0,0}$ and φ_{λ_j}, with $\alpha_j \neq 0$, does not depend on t (see the discussion before (4.20)), this term does not depend on t. Then, A_- satisfies, as $h \to 0$,

$$A_-(x, \sigma, h) \sim \sum_{j=0}^{+\infty} a_j^-(x, \sigma) h^{-|\alpha|+j} + \mathcal{H}(x, \sigma, h),$$

where the a_j^-'s are holomorphic with respect to $\sigma \in \mathcal{R}$ and C^∞ with respect to x near 0. The function $\sigma \mapsto \mathcal{H}$ is holomorphic in $D(\sigma_\alpha^0, C_2)$. Moreover,

$$a_0^-(x, \sigma) = \frac{i^{|\alpha|} a_{0,0}(x)}{(\sum \lambda_j/2 + \lambda \cdot \alpha - i\sigma) \alpha!} \prod_{j=1}^{n} \left(\varphi_{\lambda_j}(x)\right)^{\alpha_j}.$$

Using the previous discussion, together with (4.14) and (4.22), we obtain that

$$\Pi_{z_\alpha, \theta} v(x, z_\alpha) = -\frac{1}{2i\pi} \oint_\gamma w(x, z) \, dz$$

(4.23)
$$\sim e^{i\varphi_+(x)/h} e^{i\psi(0)/h} \sum_{j=0}^{+\infty} \widehat{a}_j(x) h^{\frac{1}{2} - |\alpha| + j},$$

microlocally near $(0,0)$. Moreover,

(4.24)
$$\widehat{a}_0(x) = -\frac{i^{|\alpha|+1}}{(2\pi)^{\frac{1}{2}} \alpha!} a_{0,0}(x) \prod_{j=1}^{n} \left(\varphi_{\lambda_j}(x)\right)^{\alpha_j}.$$

To be more precise, in the C^∞ case, Theorem 2.1 of [2] gives only uniqueness for z outside of a set $\Gamma(h)$, which is finite uniformly with respect to h. Then, to prove (4.22), we integrate first on a loop $\widetilde{\gamma}_h \in \mathcal{R}_h \setminus (\Gamma(h) + D(0, \varepsilon h))$ of length of order h and which may depend on h in a non trivial way. But, since the function w is holomorphic in $\mathcal{R}_h$, we can deform the contour $\widetilde{\gamma}_h$ to γ_h and thus justify (4.22).

4.5. Construction and properties of $\widetilde{f}$

We define the functions $\widetilde{f}$ and $\widetilde{f}_\theta$ by

$$(4.25) \quad \widetilde{f}(x,h) := \widehat{c}^{-1}\Pi_{z_\alpha}v(x,z_\alpha) \quad \text{and} \quad \widetilde{f}_\theta(x,h) := \widehat{c}^{-1}\Pi_{z_\alpha,\theta}v(x,z_\alpha),$$

where, using the notation $(g^-)^\alpha = \prod_{j=1}^n (g_{\lambda_j}^-)^{\alpha_j}$,

$$(4.26) \qquad \widehat{c}(h) := -\frac{i^{|\alpha|+1}}{(2\pi)^{\frac{1}{2}}\alpha!}a_{0,0}(0)(-\lambda g^-)^\alpha h^{\frac{1}{2}-|\alpha|}e^{i\psi(0)/h}.$$

As usual, we have $\chi\widetilde{f} = \chi\widetilde{f}_\theta$ if the distortion holds outside of the support of $\chi \in C_0^\infty$. From (4.25), $\widetilde{f}$ (resp. $\widetilde{f}_\theta$) is in the image of Π_{z_α} (resp. $\Pi_{z_\alpha,\theta}$). Moreover, using (4.16) (which gives that $a_{0,0}(0) \neq 0$), (4.21), (4.23), (4.24), (4.25) and (4.26), we have, microlocally near $(0,0)$,

$$\widetilde{f} = \widetilde{d}(x,h)e^{i\varphi_+(x)/h},$$

where $\widetilde{d}(x,h) \in S_h^0(1)$ is a classical symbol satisfying

$$(4.27) \qquad \widetilde{d}(x,h) \sim \sum_{j=0}^{+\infty} \widetilde{d}_j(x)h^j \quad \text{and} \quad \widetilde{d}_0(x) = x^\alpha + \mathcal{O}(x^{|\alpha|+1}).$$

In particular, $\widetilde{f}$ is not identically zero. Then, Π_{z_α} can be written as

$$(4.28) \qquad\qquad \Pi_{z_\alpha} = \widetilde{c}(\,\cdot\,,\overline{\widetilde{f}}\,)\widetilde{f},$$

and $\widetilde{f}$ satisfies iv) of Theorem 4.1. Furthermore, using Lemma 4.3, integrating over z and coming back to the definition of $\widetilde{f}$ (see (4.25)), we immediately obtain the point iii) of Theorem 4.1. Since $\widetilde{f}$ is in the image of Π_{z_α} which is the spectral projection at a simple resonance, the point ii) of Theorem 4.1 is clear. Combining iii), iv), (4.8), which gives a uniform bound outside of the energy level, together with "the transport equation" ii), we get the point i) by a standard argument of propagation of singularities.

4.6. Calculation of $\left(v(z_\alpha), \overline{\widetilde{f}}\,\right)$

Here we calculate the scalar product between $v(x,z_\alpha)$ and $\overline{\widetilde{f}}\,(x)$. From (4.5) and (4.6), the function v is supported near $\operatorname{supp}\partial_x\tau$ and microsupported near $\{(x,\xi) \in \mathbb{R}^{2n};\ x \in \operatorname{supp}\partial_x\tau \text{ and } (x,\xi) \in \Lambda_\psi\}$. Then, if $\operatorname{supp}\tau$ is close enough to 0, the previous section and (4.5) imply that

$$\left(v(z_\alpha), \overline{\widetilde{f}}\,\right) = \left([P,\tau]be^{i\psi/h}, \overline{\widetilde{d}e^{i\varphi_+/h}}\right) + \mathcal{O}(h^\infty).$$

A direct calculus gives

$$[P, \tau]\big(be^{i\psi/h}\big) = \widetilde{b}(x, h)e^{i\psi(x)/h}, \tag{4.29}$$

with

$$\widetilde{b}(x, h) \sim \sum_{j=0}^{+\infty} \widetilde{b}_j(x)h^{1+j} \quad \text{and} \quad \widetilde{b}_0(x) = -2i\partial_x\tau \cdot \partial_x\psi b_0(x). \tag{4.30}$$

Then, using that $\varphi_+ = -\varphi_-$, we get

$$\big(v(z_\alpha), \overline{\widetilde{f}}\,\big) = \int \widetilde{b}(x, h)\widetilde{d}(x, h)e^{i(\psi(x)-\varphi_-(x))/h}dx + \mathcal{O}(h^\infty). \tag{4.31}$$

The critical points of the phase $\psi - \varphi_-$ (i.e. the points x such that $\nabla\psi(x) = \nabla\varphi_-(x)$) are the points in the spatial projection of $\Lambda_\psi \cap \Lambda_- = \gamma^-$. Moreover, since this intersection is transversal, the phase function $\psi - \varphi_-$ is non degenerate in the directions that are transverse to $\pi_x\gamma^-$ (π_x being the spatial projection). Then, applying the method of the stationary phase in the orthogonal directions of $\pi_x\gamma^-$ (written $(\pi_x\gamma^-)^\perp$) and parameterizing the curve $\pi_x\gamma^-$ by $x(t)$, (4.31) gives

$$\big(v(z_\alpha), \overline{\widetilde{f}}\,\big) = \int r(t, h)e^{i(\psi(x(t))-\varphi_-(x(t)))/h}dt + \mathcal{O}(h^\infty), \tag{4.32}$$

with $r(t, h) \sim \sum_{j=0}^{+\infty} r_j(t)h^{\frac{n+1}{2}+j}$ and

$$r_0(t) = (2\pi)^{\frac{n-1}{2}} \frac{e^{i\frac{\pi}{4}\operatorname{sgn}(\psi-\varphi_-)''_{|(\pi_x\gamma^-)^\perp}}}{\big|\det(\psi-\varphi_-)''_{|(\pi_x\gamma^-)^\perp}\big|^{\frac{1}{2}}}|\partial_t x(t)|\widetilde{b}_0(x(t))\widetilde{d}_0(x(t)). \tag{4.33}$$

From (4.15), we have $\psi(x(t)) - \varphi_-(x(t)) = \psi(0)$ for all $t \in \mathbb{R}$. In particular, (4.32) can be written

$$\big(v(z_\alpha), \overline{\widetilde{f}}\,\big) = e^{i\psi(0)/h}s(h), \tag{4.34}$$

with

$$s(h) \sim \sum_{j=0}^{+\infty} s_j h^{\frac{n+1}{2}+j} \quad \text{and} \quad s_0 = \int r_0(t)\, dt. \tag{4.35}$$

From (4.24) and (4.26), we have

$$\widetilde{d}_0(x) = \frac{a_{0,0}(x)}{a_{0,0}(0)} \prod_{j=1}^{n} \Big(\frac{\varphi_{\lambda_j}(x)}{-\lambda_j g_{\lambda_j}^-}\Big)^{\alpha_j}. \tag{4.36}$$

Using (4.20), we have the transport equation

$$\partial_t \varphi_{\lambda_j}(x(t)) = \partial_t x(t) \cdot \partial_x \varphi_{\lambda_j}(x(t)) = 2\xi(t) \cdot \partial_x \varphi_{\lambda_j}(x(t))$$
$$= -2\partial_x\varphi_+(x(t)) \cdot \varphi_{\lambda_j}(x(t)) = -\lambda_j\varphi_{\lambda_j}(x(t)),$$

which gives

$$(4.37) \qquad \varphi_{\lambda_j}(x(t)) = e^{-\lambda_j t}\varphi_{\lambda_j}(x(0)).$$

On the other hand, since $\varphi_{\lambda_j}(x)$ is C^∞ and $x(t)$ is expandible, the function $t \mapsto \varphi_{\lambda_j}(x(t))$ is expandible. Moreover, since λ_j can not be written as a nontrivial combination of the λ_k's, the Taylor expansion (4.21) of φ_{λ_j} shows that the term in $e^{-\lambda_j t}$ in the expansion of $\varphi_{\lambda_j}(x(t))$ is $-\lambda_j(g_{\lambda_j}^-)^2 e^{-\lambda_j t}$. Since (4.37) gives another asymptotic expansion, the uniqueness of the expansion implies that

$$\varphi_{\lambda_j}(x(t)) = -\lambda_j(g_{\lambda_j}^-)^2 e^{-\lambda_j t}.$$

Then, combining with (4.36), we obtain

$$(4.38) \qquad \widetilde{d}_0(x(t)) = (g^-)^\alpha e^{-\lambda \cdot \alpha t}\big(1 + \mathcal{O}(e^{-\varepsilon t})\big).$$

Note here that the curve γ^- has been chosen in Section 4.1 such that $(g^-)^\alpha \neq 0$.

From the construction of u in (4.5) and since z_α is a classical symbol (see Remark 2.3) with $z_\alpha = z_\alpha^0 + \mathcal{O}(h^2) = E_0 - ih(\lambda \cdot \alpha + \sum \lambda_j/2) + \mathcal{O}(h^2)$, the function b_0 satisfies the usual transport equation

$$2\partial_x\psi \cdot \partial_x b_0 + \Big(\Delta\psi - \lambda \cdot \alpha - \sum \lambda_j/2\Big)b_0 = 0.$$

Mimicking the proof of (4.37), we get

$$(4.39) \qquad b_0(x(t)) = e^{-\int_0^t \Delta\psi(x(s)) - \left(\sum \lambda_j/2 + \lambda \cdot \alpha\right) ds}b_0(x(0)).$$

Therefore, (4.30) gives

$$\widetilde{b}_0(x(t)) = -ib_0(x(t))\partial_t\tau(x(t))$$
$$(4.40) \qquad = -ie^{-\int_0^t \Delta\psi(x(s)) - \left(\sum \lambda_j/2 + \lambda \cdot \alpha\right) ds}b_0(x(0))\partial_t\tau(x(t)).$$

From Proposition C.1 of [1] and since $g_1^- \neq 0$, we have

$$(\psi - \varphi_-)''(x(t)) = \begin{pmatrix} 0 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} + \mathcal{O}(e^{-\varepsilon t}),$$

Using $x(t) = g_1^- e^{-\lambda_1 t} + \mathcal{O}(e^{-(\lambda_1 + \varepsilon)t})$, we get

$$(4.41) \qquad \left|\det(\psi - \varphi_-)''\big|_{(\pi_x\gamma^-)^\perp}(x(t))\right|^{\frac{1}{2}} = \left(\prod_{j=2}^n \lambda_j\right)^{\frac{1}{2}} + \mathcal{O}(e^{-\varepsilon t}).$$

and

$$\text{(4.42)} \qquad \operatorname{sgn}(\psi - \varphi_-)''\big|_{(\pi_x \gamma^-)^\perp}(x(t)) = n - 1,$$

for t large enough.

Finally, using the expansion of $x(t)$, we have

$$\text{(4.43)} \qquad |\partial_t x(t)| = |g^-_{\lambda_1}|\lambda_1 e^{-\lambda_1 t}\big(1 + \mathcal{O}(e^{-\varepsilon t})\big).$$

Combining the definitions of s_0 (4.35) and of r_0 (4.33) with the relations (4.38), (4.40), (4.41), (4.42) and (4.43), the constant s_0 does not vanish if $\partial_t \tau(x(t)) \geqslant 0$ and the support of $\partial_t \tau(x(t))$ is sufficiently small near T large enough.

4.7. End of the proof of Theorem 4.1

From (4.25) and (4.28), we have

$$\widehat{c}\widehat{f} = \widetilde{c}\big(v(z_\alpha), \overline{\widetilde{f}}\,\big)\widetilde{f}.$$

In particular, using (4.26) and (4.34), we get

$$\widetilde{c} = \frac{\widehat{c}}{\big(v(z_\alpha), \overline{\widetilde{f}}\,\big)} = -\frac{i^{|\alpha|+1}}{(2\pi)^{\frac{1}{2}}\alpha! s(h)} a_{0,0}(0)(-\lambda g^-)^\alpha h^{\frac{1}{2}-|\alpha|}$$

$$\text{(4.44)} \qquad \sim \sum_{j=0}^{+\infty} \widetilde{c}_j h^{-\frac{n}{2}-|\alpha|+j},$$

with

$$\text{(4.45)} \qquad \widetilde{c}_0 = -\frac{i^{|\alpha|+1} a_{0,0}(0)(-\lambda g^-)^\alpha}{(2\pi)^{\frac{1}{2}}\alpha! s_0}.$$

At this point, the function $\widetilde{f}$ and the constant $\widetilde{c}$ may depend on v. Nevertheless, since $\Pi_{z_\alpha} = \widetilde{c}(\,\cdot\,, \overline{\widetilde{f}}\,)\widetilde{f}$ and $\widetilde{d}_0$ (the first term in the expansion of $\widetilde{f}$ given in (4.27)) do not depend on v, the constant $\widetilde{c}_0$ also does not depend on v.

We choose a sequence of functions τ (say τ_N), with $\partial_t \tau_N(x(t)) \geqslant 0$, such that $\partial_t \tau_N(x(t))$ converges to the Dirac mass δ_t for some fixed $t > 0$. Then, from the definition of s_0 (4.35) and of $\widetilde{b}_0$ (4.40), we get

$$\widetilde{c}_0 = \frac{i^{|\alpha|+1} a_{0,0}(0)(-\lambda g^-)^\alpha}{i(2\pi)^{\frac{n}{2}}|\partial_t x(t)|b_0(x(t))\widetilde{d}_0(x(t))\alpha!} \frac{\big|\det(\psi - \varphi_-)''\big|_{(\pi_x \gamma^-)^\perp}\big|^{\frac{1}{2}}}{e^{i\frac{\pi}{4}\operatorname{sgn}(\psi-\varphi_-)''\big|_{(\pi_x \gamma^-)^\perp}}}.$$

Combining (4.16), (4.38), (4.39), (4.41), (4.42) and (4.43), we obtain

$$
\widetilde{c}_0 = \frac{i^{|\alpha|} e^{-i\frac{\pi}{4}} e^{-\int_0^{+\infty} \Delta\psi(x(s)) - (\sum \lambda_j/2 - \lambda_1)\, ds} (-\lambda)^\alpha}{(2\pi)^{\frac{n}{2}} e^{-\int_0^t \Delta\psi(x(s)) - (\sum \lambda_j/2 - \lambda_1)\, ds} \alpha!}
$$

$$
\times \frac{\left(\prod_{j=1}^n \lambda_j\right)^{\frac{1}{2}}}{e^{i(n-1)\frac{\pi}{4}}} \left(1 + \mathcal{O}(e^{-\varepsilon t})\right).
$$

Then, letting t going to $+\infty$ and using that $\widetilde{c}_0$ does not depend on t, it follows

$$
(4.46) \qquad\qquad \widetilde{c}_0 = \frac{i^{|\alpha|}(-\lambda)^\alpha \left(\prod_{j=1}^n \lambda_j\right)^{\frac{1}{2}}}{(2\pi)^{\frac{n}{2}} e^{in\frac{\pi}{4}} \alpha!}.
$$

We now consider a fixed v as in the beginning of this subsection. With $c(h)$ as in (4.2), (4.46) gives that $\widetilde{c} = c\check{c}$ where

$$
\check{c} \sim \sum_{j=0}^{+\infty} \check{c}_j h^j \quad \text{and} \quad \check{c}_0 = 1.
$$

Now, we define $f := \check{c}^{\frac{1}{2}} \widetilde{f}$. Then, (4.28) gives (4.1) and the properties of f given in Theorem 4.1 follow from the properties of $\widetilde{f}$ given in Section 4.5 and $\check{c}_0 = 1$.

5. Residue of the scattering amplitude

In this section, we give the semiclassical expansion of the residue of the scattering amplitude at an isolated resonance. To define the scattering matrix, we assume that the potential is long range:

(H4) For some $\rho > 0$, we have $|V(x)| \lesssim \langle x \rangle^{-\rho}$ for all $x \in \mathcal{S}$.

Using the constructions of Isozaki and Kitada (see [25] and [26]), the assumption (H4) allows to define the scattering matrix $S(z,h)$, $z \in]0, +\infty[$ related to the pair $P_0 = -h^2\Delta$ and P as a unitary operator

$$
S(z,h) : L^2(\mathbb{S}^{n-1}) \longrightarrow L^2(\mathbb{S}^{n-1}).
$$

In the short range case (*i.e.* $\rho > 1$), this operator coincides with the usual scattering matrix. Next, introduce the operator $\mathcal{T}(z,h)$ defined by

$$
S(z,h) = \mathrm{Id} - 2i\pi\mathcal{T}(z,h).
$$

Its kernel $\mathcal{T}(\omega, \omega', z, h)$ is smooth away from the diagonal of $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ (see [26]). Here, ω (resp. ω') is called the outgoing (resp. incoming) direction. Finally, the scattering amplitude is defined for $\omega \neq \omega'$ by

$$\mathcal{A}(\omega, \omega', z, h) = c(z, h)\mathcal{T}(\omega, \omega', z, h),$$

with

$$c(z, h) = -(2\pi)z^{-\frac{n-1}{4}}(2\pi h)^{\frac{n-1}{2}}e^{-i\frac{(n-3)\pi}{4}}.$$

In [15], Gérard and Martinez have shown that for $\omega \neq \omega'$ fixed, the scattering amplitude has a meromorphic continuation to a neighborhood of $]0, +\infty[$, whose poles are the resonances of P. Moreover, the multiplicity of each pole is less or equal to the multiplicity of the resonance. Notice that, since the kernel of the residue of the scattering matrix is not singular at $\omega = \omega'$ (see Theorem 1.1 (iii) of [15]), we drop the assumption $\omega \neq \omega'$ in the sequel.

We will now make some hypotheses on the behavior of the classical curves. Let $(x(t), \xi(t)) = \exp(tH_p)(x, \xi)$ be a Hamiltonian curve in $p^{-1}(E_0)$. Under the hypotheses (H1)–(H4), there are only two possible behaviors for $x(t)$ as $t \to \pm\infty$: either it escapes to ∞, or it goes to 0. From the long range assumption (H4), if $x(t)$ escapes to ∞, then $\xi(t)$ has a limit in $\sqrt{E_0}\mathbb{S}^{n-1}$. Moreover the set of points with asymptotic direction ω and ω',

$$\Lambda_{\omega'}^- = \left\{(x, \xi) \in p^{-1}(E_0); \ \xi(t) \longrightarrow \sqrt{E_0}\omega' \text{ as } t \to -\infty\right\},$$

$$\Lambda_\omega^+ = \left\{(x, \xi) \in p^{-1}(E_0); \ \xi(t) \longrightarrow \sqrt{E_0}\omega \text{ as } t \to +\infty\right\},$$

are Lagrangian submanifolds of $T^*\mathbb{R}^n$ (see [12]). We suppose that

(H5) $\Lambda_{\omega'}^-$ and Λ_- (resp. Λ_ω^+ and Λ_+) intersect in a finite number N_- (resp. N_+) of bicharacteristic curves, with each intersection transverse.

We denote these curves, respectively,

$$\gamma_k^- : t \mapsto \gamma_k^-(t) = (x_k^-(t), \xi_k^-(t)), \quad 1 \leqslant k \leqslant N_-,$$

and

$$\gamma_\ell^+ : t \mapsto \gamma_\ell^+(t) = (x_\ell^+(t), \xi_\ell^+(t)), \quad 1 \leqslant \ell \leqslant N_+.$$

Note that, from Proposition 2.5 of [1], the intersections $\Lambda_{\omega'}^- \cap \Lambda_-$ and $\Lambda_\omega^+ \cap \Lambda_+$ are never empty (i.e. $N_- \geqslant 1$ and $N_+ \geqslant 1$). From [21, Equation (2.7)], the curve $\gamma_\star^\pm$ with $\star = k, \ell$ satisfies

$$x_\star^\pm(t) \sim \sum_{j=1}^{+\infty} g_{\mu_j}^{\star,\pm}(t)e^{\pm\mu_j t} \quad \text{with} \quad g_{\mu_j}^{\star,\pm}(t) = \sum_{m=0}^{M_{\mu_j}^{\star,\pm}} g_{\mu_j,m}^{\star,\pm}t^m \quad \text{as } t \to \mp\infty.$$

From Lemma A.1, if λ_j satisfies $\lambda \cdot \alpha = \lambda_j \implies |\alpha| = 1$, then $M_{\lambda_j}^{\star,\pm} = 0$. Moreover, there always exists a μ_j such that $g_{\mu_j}^{\star,\pm} \neq 0$. We define

$$\lambda_\star^\pm = \min\{\mu_j;\ g_{\mu_j}^{\star,\pm} \neq 0\}.$$

We know that $\lambda_\star^\pm$ is one of the λ_j's and that $M_{\lambda_j}^{\star,\pm} = 0$ (see [1, (2.18)]). We shall denote

$$S_k^- = \int_{-T_k^-}^{+\infty} x_k^-(s)\partial_x V(x_k^-(s))\, ds \quad \text{and} \quad S_\ell^+ = \int_{-\infty}^{T_\ell^+} x_\ell^+(s)\partial_x V(x_\ell^+(s))\, ds,$$

for some $T_\star^\pm$ large enough which is equal to $+\infty$ in the short range case $\rho > 1$.

Moreover, in the short range case, the bicharacteristic curves in $\Lambda_\alpha^\pm$, $\alpha \in \mathbb{S}^{n-1}$, are the bicharacteristic curves

$$\gamma_\pm(t,z,\alpha) = \big(x_\pm(t,z,\alpha), \xi_\pm(t,z,\alpha)\big)$$

for which there exists a $z \in \alpha^\perp \sim \mathbb{R}^{n-1}$ such that

$$\lim_{t\to\pm\infty} \big|x_\pm(t,z,\alpha) - 2\sqrt{E_0}\alpha t - z\big| = 0,$$

$$\lim_{t\to\pm\infty} \big|\xi_\pm(t,z,\alpha) - \sqrt{E_0}\alpha\big| = 0.$$

These trajectories are smooth with respect to t, z, α. We denote by $z_\star^\pm$ the impact parameter of the curve $\gamma_\star^\pm$. Let

$$D_k^- = \lim_{t\to+\infty} \left| \det \frac{\partial x_-(t,z,\omega')}{\partial(t,z)} \Big|_{z=z_k^-} \right| e^{-(\Sigma_j \lambda_j - 2\lambda_k^-)t},$$

$$D_\ell^+ = \lim_{t\to-\infty} \left| \det \frac{\partial x_+(t,z,\omega)}{\partial(t,z)} \Big|_{z=z_\ell^+} \right| e^{(\Sigma_j \lambda_j - 2\lambda_\ell^+)t},$$

be the Maslov determinants for $\gamma_\star^\pm$ which exist and satisfy $0 < D_\star^\pm < +\infty$ (see [1]). We shall also denote by $\nu_\star^\pm$ the Maslov index of the curve $\gamma_\star^\pm$.

THEOREM 5.1 (Residue of the scattering amplitude). — *Assume* (H1)–(H5). *Let* $\alpha \in \mathbb{N}^n$ *be such that* z_α^0 *is simple. Then, the residue of the scattering amplitude satisfies*

$$\text{Residue}\,\big(\mathcal{A}(\omega,\omega',z,h), z = z_\alpha\big) = \sum_{k=1}^{N_-}\sum_{\ell=1}^{N_+} a_{k,\ell} h^{-|\alpha|+\frac{1}{2}} e^{i(S_k^- + S_\ell^+)/h} + \mathcal{O}(h^\infty),$$

where

$$a_{k,\ell}(h) = b_k^-(h)b_\ell^+(h) \quad \text{and} \quad b_\star^\pm(h) \sim \sum_{j=0}^{+\infty} b_{\star,j}^\pm h^j.$$

Moreover, $b^{\pm}_{\star,0} = 0$ if and only if $(g^{\star,\pm})^{\alpha} = 0$. Finally, in the short range case $\rho > 1$, we have

$$b^{-}_{k,0} b^{+}_{\ell,0} = \frac{e^{-i\frac{\pi}{2}(|\alpha|-\frac{1}{2})}}{\sqrt{2\pi}\alpha!} E_0^{\frac{n-1}{4}} (\lambda^{-}_k \lambda^{+}_\ell)^{\frac{3}{2}} \prod_{j=1}^{n} \lambda_j^{\alpha_j - \frac{1}{2}}$$

$$\times e^{-i\nu^{-}_k \pi/2} e^{-i\nu^{+}_\ell \pi/2} (D^{-}_k D^{+}_\ell)^{-\frac{1}{2}} (g^{k,-})^{\alpha} (g^{\ell,+})^{\alpha} |g^{k,-}_{\lambda^{-}_k}| |g^{\ell,+}_{\lambda^{+}_\ell}|.$$

In the last formula $(g^{\star,\pm})^{\alpha}$ is a shorthand for $\prod_{j=1}^{n}(g^{\star,\pm}_{\lambda_j})^{\alpha_j}$ where $g^{\star,\pm}_{\lambda_j}$ is identified with its j-th coordinate. To prove the theorem, we first obtain a representation formula for the scattering amplitude involving the resolvent. Then we apply Theorem 4.1 to express the residue of the scattering amplitude with the help of the resonant state f. Finally, the result follows from the computation of two scalar products which are done with the stationary phase method.

Remark 5.2. — Stefanov [40] (in the compact support case) and Michel [32] (in the long range case) have given a priori estimates for the residue of the scattering amplitude. For the resonances z_0 very close to the real axis (more precisely $|\operatorname{Im} z_0| \lesssim h^{\frac{3n+5}{2}}$) and under a separation condition, they have proved that the residue satisfies

$$\big| \operatorname{Residue} \big(\mathcal{A}(\omega, \omega', z, h), z = z_0 \big) \big| \lesssim h^{-\frac{n-1}{2}} |\operatorname{Im} z_0|.$$

In the present situation, these results do not apply since the resonances are "too far" from the real axis. Furthermore, the previous estimate does not hold. Indeed, the imaginary part of z_α behaves like $-|\alpha|h$ but the residue is typically of order $h^{-|\alpha|+\frac{1}{2}}$.

In the one dimensional case, Theorem 5.1 can probably be deduced from the computation of the scattering amplitude obtained by the third author in [36].

For a point-well in the island case and under some geometrical assumptions, the asymptotic of the residue of the scattering amplitude has been computed by Nakamura [33, 34], Lahmar-Benbernou [28] and Lahmar-Benbernou and Martinez [29].

It is possible to compare Theorem 5.1 with the semiclassical expansion of the scattering amplitude for real energy obtained in [1]. Assume for simplicity that the λ_j's are non-resonant ($\mathbb{Z}$-independent for example), $\lambda_n < 2\lambda_1$, $N_- = N_+ = 1$, $N_\infty = 0$ and $g^{1,-}_{\lambda_j} \neq 0$ for all $j \in \{1, \ldots, n\}$. In particular, we have $k = \ell = 1$. Let $J \in \{1, \ldots, n\}$ be the first j with $g^{1,+}_{\lambda_j} \neq 0$ (thus,

$\lambda_J = \lambda_1^+$). In that case, Theorem 2.6 (a) of [1] gives

$$(5.1) \qquad \mathcal{A}(\omega, \omega', E, h) = \left(f(E)\Gamma\left(\frac{\Sigma(E)}{\lambda_J}\right) + o(1) \right) h^{\frac{\Sigma(E)}{\lambda_J} - \frac{1}{2}} e^{i(S_1^- + S_1^+)/h},$$

for E real with $E - E_0 = \mathcal{O}(h)$. Here,

$$\Sigma(E) = \sum_{j=1}^{n} \frac{\lambda_j}{2} - i\frac{E - E_0}{h},$$

and $f(E)$ is an explicit function, analytic near E_0. Thus, the main term in (5.1), defined in [1] for E real, has a meromorphic extension in a fix neighborhood of E_0. Moreover, its poles are exactly the pseudo-resonances $z_\alpha^0 \in \text{Res}_0(P)$ with $\alpha = (0, \ldots, 0, \alpha_J, 0, \ldots, 0)$ and the corresponding residue coincides with that given in Theorem 5.1. In particular, this principal term does not contribute to the residue at the other (pseudo)-resonances. The cases (b) and (c) in Theorem 2.6 of [1] only appear for resonant λ_j's and the corresponding main terms in the semiclassical expansion of the scattering amplitude have poles at some $z_\alpha^0 \in \text{Res}_0(P)$ which are not simple.

5.1. Representation formula for the scattering amplitude

In this section, we recall a representation formula of the scattering amplitude for complex energies due to Gérard and Martinez [15]. Their approach consists in extending the formula of Isozaki and Kitada [26] to complex energies. For this purpose, they show that the phases and the symbols involved in that formula can be chosen to be analytic in a suitable complex neighborhood of $\mathbb{R}^{2n}$. We only recall what will be useful in the following and refer to [15] for the details.

For $R > 0$ large enough, $d > 0$, $\varepsilon > 0$ and $\sigma \in]0,1[$, we denote

$$\Gamma_\mathbb{C}^\pm(R, d, \varepsilon, \sigma) = \Big\{ (x, \xi) \in \mathbb{C}^{2n}; \ |\operatorname{Re} x| > R, \ d^{-1} < |\operatorname{Re} \xi| < d,$$

$$|\operatorname{Im} x| \leqslant \varepsilon \langle \operatorname{Re} x \rangle, \ |\operatorname{Im} \xi| \leqslant \varepsilon \langle \operatorname{Re} \xi \rangle$$

$$\text{and } \pm \cos(\operatorname{Re} x, \operatorname{Re} \xi) \geqslant \pm\sigma \Big\},$$

$$\Gamma^\pm(R, d, \sigma) = \Gamma_\mathbb{C}^\pm(R, d, \varepsilon, \sigma) \cap \mathbb{R}^{2n}.$$

Let $\varepsilon > 0$, $d \gg 1$, $-1 < \sigma_1^- < \sigma_1^+ < 0 < \sigma_2^- < \sigma_2^+ < 1$ and $R_1 > 0$ be sufficiently large. For $k = 1, 2$, we denote $\Gamma^k = \Gamma_\mathbb{C}^+(R_1, d, \varepsilon, \sigma_k^+) \cup \Gamma_\mathbb{C}^-(R_1, d, \varepsilon, \sigma_k^-)$. In [15], Gérard and Martinez construct some phases $\varphi_k \in C^\infty(\mathbb{R}^{2n}; \mathbb{R})$ and some symbols $t_k \in C^\infty(\mathbb{R}^{2n}) \cap S_h^0(1)$ satisfying the general assumptions of Isozaki and Kitada [25] and the following properties.

The phases φ_k have a holomorphic extension to Γ^k and satisfy

(5.2)
$$\begin{cases} (\nabla_x \varphi_k(x,\xi))^2 + V(x) = \xi^2, \\ \partial_x^\alpha \partial_\xi^\beta \big(\varphi_k(x,\xi) - x \cdot \xi\big) = \mathcal{O}\big(\langle x \rangle^{1-\rho-|\alpha|}\big), \end{cases}$$

uniformly in Γ^k. Moreover, $\Lambda_{\varphi_k(\cdot,\sqrt{E_0}\omega)} = \{(x, \partial_x \varphi_k(x, \sqrt{E_0}\omega))\} \subset \Lambda_\omega^- \cup \Lambda_\omega^+$.

There exist two symbols $a_k(x,\xi,h) \in C^\infty(\mathbb{R}^{2n}, \mathbb{C})$ supported inside $\Gamma^k \cap \mathbb{R}^{2n}$, with

$$a_k(x,\xi,h) \sim \sum_{j=0}^{+\infty} a_{k,j}(x,\xi) h^j,$$

such that

$$\big|\partial_x^\alpha \partial_\xi^\beta a_k(x,\xi,h)\big| \lesssim \langle x \rangle^{-|\alpha|} \quad \text{and} \quad \big|\partial_x^\alpha \partial_\xi^\beta a_{k,j}(x,\xi)\big| \lesssim \langle x \rangle^{-j-|\alpha|}.$$

Moreover, for some $\delta > 0$ with $-1 < \sigma_k^- - \delta < \sigma_k^+ + \delta < 1$, we have

(5.3)
$$\big|\partial_x^\alpha \partial_\xi^\beta (a_{k,0}(x,\xi) - 1)\big| \lesssim \langle x \rangle^{-\rho-|\alpha|},$$

for $(x,\xi) \in \Gamma^+(2R_1, d/2, \sigma_k^+ + \delta) \cup \Gamma^-(2R_1, d/2, \sigma_k^- - \delta)$. Finally, they extend holomorphically with respect to $X = |x|$ and $\Xi = |\xi|$ for X in $\{\mathrm{Re}\, X > 3R_1,\ |\mathrm{Im}\, X| < \varepsilon \langle \mathrm{Re}\, X \rangle\}$ and Ξ in a complex neighborhood of $\sqrt{E_0}$. Furthermore, their extensions continue to satisfy estimates analogous to the previous ones.

The symbols t_k are then defined by

(5.4)
$$t_k(x,\xi,h) = e^{-i\varphi_k(x,\xi)/h}(P - \xi^2)\big(a_k(\cdot,\xi,h) e^{i\varphi_k(\cdot,\xi)/h}\big),$$

and satisfy, for some $\widetilde{\varepsilon} > 0$,

(5.5)
$$\big|\partial_x^\alpha \partial_\xi^\beta t_k(x,\xi,h)\big| = \mathcal{O}\big(e^{-\widetilde{\varepsilon}\langle x \rangle/h}\big),$$

uniformly with respect to h and

$$(x,\xi) \in \Gamma_{\mathbb{C}}^+\big(2R_1, d/2, \varepsilon, \sigma_k^+ + \delta\big) \cup \Gamma_{\mathbb{C}}^-\big(2R_1, d/2, \varepsilon, \sigma_k^- - \delta\big).$$

Under the assumption (H4), Gérard and Martinez [15] have proved that the scattering amplitude can be written

(5.6)
$$\mathcal{A}(\omega, \omega', z, h) = \widetilde{c}(z,h) g(\omega, \omega', z, h) + f(\omega, \omega', z, h),$$

where $f(\omega, \omega', z, h)$ has a holomorphic extension in a (fixed) neighborhood of E_0,

$$g(\omega, \omega', z, h) = \Big((P_\theta - z)^{-1} U_{i\theta}\big(e^{i\varphi_2(x,\sqrt{z}\omega')/h} t_2(x, \sqrt{z}\omega', h)\big),$$
$$U_{\overline{i\theta}}\big(e^{i\varphi_1(x,\sqrt{z}\omega)/h} t_1(x, \sqrt{z}\omega, h)\big)\Big),$$

and

$$(5.7) \qquad \widetilde{c}(z,h) = \pi(2\pi h)^{-\frac{n+1}{2}} z^{\frac{n-3}{4}} e^{-i\frac{(n-3)\pi}{4}}.$$

By assumption, the resonance z_α is simple for h small enough. Moreover, Theorem 3.1 implies that $\Pi_{z_\alpha,\theta} = \mathcal{O}(h^{-M})$ for $\theta = \nu h|\ln h|$ and some $M > 0$. Then Lemma 5.4 of [4] (see also Proposition 5.1 of [28] in the case of a well in the island) states that

$$\mathcal{R} := \mathrm{Residue}\left(\mathcal{A}(\omega, \omega', z, h), z = z_\alpha\right)$$
$$= -\widetilde{c}(z_\alpha, h)\Big(\Pi_{z_\alpha,\theta}\chi U_{i\theta}\big(e^{i\varphi_2(x,\sqrt{z_\alpha}\omega')/h}t_2(x,\sqrt{z_\alpha}\omega',h)\big),$$
$$\chi U_{\overline{i\theta}}\big(e^{i\varphi_1(x,\sqrt{\overline{z_\alpha}}\omega)/h}t_1(x,\sqrt{\overline{z_\alpha}}\omega,h)\big)\Big) + \mathcal{O}(h^\infty),$$

where $\chi \in C_0^\infty(\mathbb{R}^n)$ satisfies $\mathbb{1}_{|x|\leqslant 2R_1} \prec \chi \prec \mathbb{1}_{|x|\leqslant 3R_1}$ with $R_0 \gg R_1$. In particular, there is no distortion (*i.e.* $F = 0$) on the support of χ and Theorem 4.1 implies

$$\mathcal{R} = \widehat{c}\Big(f, e^{i\varphi_1(x,\sqrt{\overline{z_\alpha}}\omega)/h}\chi t_1(x,\sqrt{\overline{z_\alpha}}\omega,h)\Big)$$
$$(5.8) \qquad\qquad \times \Big(e^{i\varphi_2(x,\sqrt{z_\alpha}\omega')/h}\chi t_2(x,\sqrt{z_\alpha}\omega',h), \overline{f}\Big) + \mathcal{O}(h^\infty),$$

where $\widehat{c} = -\widetilde{c}(z_\alpha, h)c(h)$ with $c(h)$ given by (4.2).

5.2. Computation of $(f, e^{i\varphi_1/h}\chi t_1)$

We will calculate the scalar product $(f, e^{i\varphi_1/h}\chi t_1)$ by the stationary phase method. First, we will prove that this quantity has an asymptotic expansion in power of h and then calculate the first term using a limit at the origin. We will use arguments close to the ones developed in Section 4.6 or [1, Section 7].

Denote

$$u = e^{i\varphi_1(x,\sqrt{\overline{z_\alpha}}\omega)/h}a_1(x,\sqrt{\overline{z_\alpha}}\omega,h) \quad \text{and} \quad v = e^{i\varphi_1(x,\sqrt{\overline{z_\alpha}}\omega)/h}t_1(x,\sqrt{\overline{z_\alpha}}\omega,h).$$

From Theorem 4.1 ii) and (5.4), we have

$$(f, \chi v) = (f, \chi(P - \overline{z_\alpha})u) = ((P - z_\alpha)f, \chi u) + (f, [\chi, P]u)$$
$$(5.9) \qquad\qquad\qquad = -(f, [P, \chi]u).$$

From (5.5) and the choice of χ, $(P - \overline{z_\alpha})u = 0$ microlocally near $\Gamma^+(2R_1, d/2, \sigma_1^+ + \delta)$. Moreover, since z_α has an asymptotic expansion in power of h, we can write, microlocally near $\Gamma^+(2R_1, d/2, \sigma_1^+ + \delta)$,

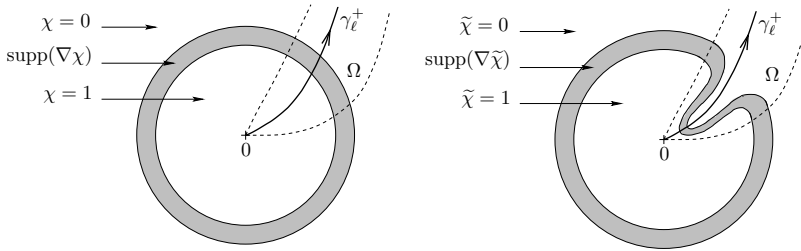$$u = \widetilde{a}(x,h)e^{i\varphi_1(x,\sqrt{E_0}\omega)/h},$$

Figure 5.1. The functions χ and $\widetilde{\chi}$.

where $\widetilde{a}$ has an asymptotic expansion in power of h. Note that $(\mathrm{supp}\,\nabla\chi \times \mathbb{R}^n)\cap\bigcup_\ell \gamma_\ell^+ \subset \Gamma^+(2R_1, d/2, \sigma_1^+ + \delta)$. Using Maslov's theory, we can extend the function u near Ω, a small neighborhood of $\bigcup_\ell \gamma_\ell^+ \cap (B(0, 3R_0) \times \mathbb{R}^n)$, such that u is still a solution of $(P - \overline{z_\alpha})u = 0$ microlocally in Ω. Let $\widetilde{\chi}(x, \xi) \in C^\infty(T^*\mathbb{R}^n)$ be such that $\widetilde{\chi}(x, \xi) = \chi(x)$ out of Ω (see Figure 5.1). In particular, $(P - \overline{z_\alpha})u = 0$ microlocally near the support of $\chi - \widetilde{\chi}$. So, we have

$$(f, \chi v) = - \left(f, [P, \mathrm{Op}(\widetilde{\chi})]u\right) - \left((P - z_\alpha)f, \mathrm{Op}(\chi - \widetilde{\chi})u\right)$$
$$+ \left(f, \mathrm{Op}(\chi - \widetilde{\chi})(P - \overline{z_\alpha})u\right)$$

$$(5.10) \qquad = - \left(f, [P, \mathrm{Op}(\widetilde{\chi})]u\right) + \mathcal{O}(h^\infty).$$

On the other hand, since $\overline{z_\alpha} = E_0 + \mathcal{O}(h)$, the microsupport of $[P, \mathrm{Op}(\widetilde{\chi})]u$ satisfies

$$\mathrm{MS}\left([P, \mathrm{Op}(\widetilde{\chi})]u\right) \subset \Lambda_{\varphi_1(\cdot, \sqrt{E_0}\omega)} \cap \mathrm{supp}\,\nabla\widetilde{\chi}$$
$$\subset \left(\Lambda_\omega^- \cap \Gamma^-(R_1, d, \sigma_1^-/2)\right) \cup \left(\Lambda_\omega^+ \cap \Omega\right).$$

Moreover, Theorem 4.1 gives $\mathrm{MS}(f) \subset \Lambda_+$. Then, modulo $\mathcal{O}(h^\infty)$, the non-zero contributions to $(f, [P, \mathrm{Op}(\widetilde{\chi})]u)$ comes from the values of the functions f and $[P, \mathrm{Op}(\widetilde{\chi})]u$ microlocally on the set $\bigcup_\ell \gamma_\ell^+$ (which is the intersection of the two microsupports). Let g_ℓ^+ be $C_0^\infty(T^*\mathbb{R}^n)$ functions with support in a small enough neighborhood of $\gamma_\ell^+ \cap (B(0, 3R_0) \times \mathbb{R}^n)$ such that $g_\ell^+ = 1$ in a similar neighborhood. Then, (5.10) becomes

$$(5.11) \qquad (f, \chi v) = - \sum_{\ell=1}^{N_+} \left(f, \mathrm{Op}(g_\ell^+)[P, \mathrm{Op}(\widetilde{\chi})]u\right) + \mathcal{O}(h^\infty).$$

We now compute $\operatorname{Op}(g_\ell^+)[P, \operatorname{Op}(\widetilde{\chi})]u$. From Proposition C.1 of [1], the Lagrangian manifold Λ_ω^+ has a nice projection with respect to x in a neighborhood of any point of γ_ℓ^+ close to $(0,0)$. Then, Maslov's theory implies that u can be written as

$$u(x) = a_\ell^+(x, h)e^{i\psi_\ell^+(x)/h},$$

microlocally in such a neighborhood. From the construction of [25] and [15], we see that

$$(5.12) \qquad \psi_\ell^+(x_\ell^+(t)) = x_\ell^+(t)\xi_\ell^+(t) - \int_t^{T_\ell^+} x_\ell^+(s)\partial_x V(x_\ell^+(s))\, ds,$$

for some $T_\ell^+ > 0$ large enough (equal to $+\infty$ in the short range case). The symbol a_ℓ^+ has an asymptotic expansion $a_\ell^+(x, h) \sim \sum_j a_{\ell,j}^+(x)h^j$ with $a_{\ell,0}^+(x_\ell^+(t)) \neq 0$. Moreover, in the short range case, Equation (7.12) of [1] gives

$$(5.13) \qquad a_{\ell,0}^+(x_\ell^+(t)) = e^{i\nu_\ell^+\pi/2}2^{\frac{1}{2}}E_0^{\frac{1}{4}}(D_\ell^+(t))^{-\frac{1}{2}}e^{-t(\sum \lambda_k/2 + \lambda \cdot \alpha)},$$

where ν_ℓ^+ is the Maslov index of the curve γ_ℓ^+ and $D_\ell^+(t)$ is the Maslov's determinant

$$D_\ell^+(t) = \left| \det \frac{\partial x_+(t, z, \omega)}{\partial(t, z)}\big|_{z=z_\ell^+} \right|.$$

Moreover, from Section 6 of [1], we know that

$$(5.14) \qquad D_\ell^+ = \lim_{t \to -\infty} D_\ell^+(t)e^{t\left(\sum \lambda_k - 2\lambda_\ell^+\right)},$$

exists and satisfies $0 < D_\ell^+ < +\infty$. So,

$$(5.15) \qquad \operatorname{Op}(g_\ell^+)[P, \operatorname{Op}(\widetilde{\chi})]u = \widetilde{a}_\ell^+(x, h)e^{i\psi_\ell^+(x)/h},$$

with

$$\widetilde{a}_\ell^+(x, h) \sim \sum_{j=0}^{+\infty} \widetilde{a}_{\ell,j}^+(x)h^{1+j},$$

and

$$(5.16) \qquad \widetilde{a}_{\ell,0}^+(x) = -i(\{p, \widetilde{\chi}\}g_\ell^+)(x, \partial_x\psi_\ell^+(x))a_{\ell,0}^+(x).$$

Since the support of $g_\ell^+(x, \xi)\partial_{x,\xi}\widetilde{\chi}(x, \xi)$ is close enough to $(0,0)$, Theorem 4.1 iv) and (5.11) imply that

$$(5.17) \quad (f, \chi v) = -\sum_{\ell=1}^{N_+} \int d(x, h)\overline{\widetilde{a}_\ell^+(x, h)}e^{i(\varphi_+(x) - \psi_\ell^+(x))/h}dx + \mathcal{O}(h^\infty).$$

We proceed now as in (4.32). In the support of the symbol $\widetilde{a}_\ell^+$, the critical points of the phase function $\varphi_+ - \psi_\ell^+$ (i.e. the points x such that

$\partial_x \varphi_+(x) = \partial_x \psi_\ell^+(x))$ are the points in the spacial projection of γ_ℓ^+. Since this intersection $\Lambda_+ \cap \Lambda_\omega^+ = \gamma_\ell^+$ is transverse from the assumption (H5), the phase $\varphi_+ - \psi_\ell^+$ is non degenerate in the directions transverse to $\pi_x \gamma_\ell^+$. Therefore, performing the method of the stationary phase in the orthogonal directions of $\pi_x \gamma_\ell^+$ (as in (4.32)) and parameterizing the curve $\pi_x \gamma_\ell^+$ by $x_\ell^+(t)$, (5.17) gives

$$(5.18) \quad (f, \chi v) = -\sum_{\ell=1}^{N_+} \int b_\ell^+(t, h) e^{i(\varphi_+(x_\ell^+(t)) - \psi_\ell^+(x_\ell^+(t)))/h} dt + \mathcal{O}(h^\infty).$$

with $b_\ell^+(t, h) \sim \sum_{j \geqslant 0} b_{\ell,j}^+(t) h^{\frac{n+1}{2}+j}$ and

$$b_{\ell,0}^+(t) = (2\pi)^{\frac{n-1}{2}} \frac{e^{i\frac{\pi}{4} \operatorname{sgn}(\varphi_+ - \psi_\ell^+)''|_{(\pi_x \gamma_\ell^+)^\perp}}}{\left| \det(\varphi_+ - \psi_\ell^+)|_{(\pi_x \gamma_\ell^+)^\perp} \right|^{\frac{1}{2}}} |\partial_t x_\ell^+(t)| d_0(x_\ell^+(t)) \overline{\tilde{a}_{\ell,0}^+(x_\ell^+(t))}.$$

Since $\gamma_\ell^+ \in \Lambda_+ \cap \Lambda_\omega^+$, $\varphi_+(x_\ell^+(t))$ and $\psi_\ell^+(x_\ell^+(t))$ have the same derivative (with respect to t), and (5.12) gives

$$\varphi_+(x_\ell^+(t)) - \psi_\ell^+(x_\ell^+(t)) = \int_{-\infty}^T x_\ell^+(s) \partial_x V(x_\ell^+(s)) ds = S_\ell^+,$$

for all $t \geqslant 0$. Then, combining with (5.16), we get

$$(5.19) \quad (f, \chi v) = \sum_{\ell=1}^{N_+} e^{iS_\ell^+/h} h^{\frac{n+1}{2}} \tilde{b}_\ell^+(h),$$

with $\tilde{b}_\ell^+(h) \sim \sum_{j \geqslant 0} \tilde{b}_{\ell,j}^+ h^j$ and

$$\tilde{b}_{\ell,0}^+ = -i(2\pi)^{\frac{n-1}{2}} \int \frac{e^{i\frac{\pi}{4} \operatorname{sgn}(\varphi_+ - \psi_\ell^+)''|_{(\pi_x \gamma_\ell^+)^\perp}}}{\left| \det(\varphi_+ - \psi_\ell^+)|_{(\pi_x \gamma_\ell^+)^\perp} \right|^{\frac{1}{2}}} |\partial_t x_\ell^+(t)| d_0(x_\ell^+(t))$$

$$(5.20) \qquad\qquad\qquad \times \overline{a_{\ell,0}^+(x_\ell^+(t))} \partial_t \tilde{\chi}(x_\ell^+(t), \xi_\ell^+(t)) dt.$$

From (5.10), $(f, \chi v)$ does not depend on $\tilde{\chi}$, modulo $\mathcal{O}(h^\infty)$. In particular, changing $\tilde{\chi}$ in a neighborhood of a fixed curve γ_ℓ^+, we obtain that each $b_{\ell,0}^+$ does not depend on $\tilde{\chi}$. From Proposition C.1 of [1], we have, up to a linear change of variables in $\mathbb{R}^n$,

$$(\varphi_+ - \psi_\ell^+)''(x(t)) = \operatorname{diag}(\lambda_1, \ldots, \lambda_{j(\ell)-1}, 0, \lambda_{j(\ell)+1}, \ldots, \lambda_n) + \mathcal{O}(e^{-\varepsilon t}),$$

where $j(\ell)$ is such that $\lambda_{j(\ell)} = \lambda_\ell^+$. Since $x_\ell^+(t) = g_{\lambda_\ell^+}^{\ell,+} e^{\lambda_\ell^+ t} + \mathcal{O}(e^{(\lambda_\ell^+ + \varepsilon)t})$ as expandable symbol (see [2, Definition 5.2]), this implies

$$(5.21) \qquad \left| \det(\varphi_+ - \psi_\ell^+)_{|_{(\pi_x \gamma_\ell^+)^\perp}} \right|^{\frac{1}{2}} (x_\ell^+(t)) = \left(\prod_{j \neq j(\ell)} \lambda_j \right)^{\frac{1}{2}} + \mathcal{O}(e^{\varepsilon t}),$$

$$(5.22) \qquad \operatorname{sgn}(\varphi_+ - \psi_\ell^+)''_{|_{(\pi_x \gamma_\ell^+)^\perp}} (x_\ell^+(t)) = n - 1,$$

$$(5.23) \qquad |\partial_t x_\ell^+(t)| = |g_{\lambda_\ell^+}^{\ell,+}| \lambda_\ell^+ e^{\lambda_\ell^+ t} \big(1 + \mathcal{O}(e^{\varepsilon t})\big),$$

as $t \to -\infty$. On the other hand, (4.38) gives

$$(5.24) \qquad d_0(x_\ell^+(t)) = (g^{\ell,+})^\alpha e^{\lambda \cdot \alpha t} \big(1 + \mathcal{O}(e^{\varepsilon t})\big).$$

We first consider the long range case ($\rho > 0$). If $(g^{\ell,+})^\alpha = 0$, then (5.20) and (5.24) imply that $b_{\ell,0}^+ = 0$. We will now prove that $b_{\ell,0}^+ \neq 0$ if $(g^{\ell,+})^\alpha \neq 0$. Let $T > 0$ be sufficiently large such that the quantities in (5.21), (5.22) and (5.24) do not vanish and (5.23) holds at $t = -T$. Then, if $\widetilde{\chi}(x_\ell^+(t))$ satisfies $\partial_t \widetilde{\chi}(x_\ell^+(t)) \leqslant 0$ and has its support close enough to T, the previous discussion, $a_{\ell,0}^+(x_\ell^+(T)) \neq 0$ and (5.20) imply that $b_{\ell,0}^+ \neq 0$.

Let us now consider the short range case ($\rho > 1$). Assume that the support of $\partial_t \widetilde{\chi}(x_\ell^+(t))$ is sufficiently negative. Then, the formula (5.20) and the estimates (5.13), (5.14), (5.21), (5.22), (5.23) and (5.24) give

$$\widetilde{b}_{\ell,0}^+ = -i(2\pi)^{\frac{n-1}{2}} \frac{2^{\frac{1}{2}} e^{i(n-1)\frac{\pi}{4}}}{\left(\prod_{j \neq j(\ell)} \lambda_j \right)^{\frac{1}{2}}} |g_{\lambda_\ell^+}^{\ell,+}| \lambda_\ell^+ (g^{\ell,+})^\alpha e^{-i\nu_\ell^+ \pi/2} E_0^{\frac{1}{4}} (D_\ell^+)^{-\frac{1}{2}}$$
$$\times \int \partial_t \widetilde{\chi}(x_\ell^+(t))(1 + o(1)) \, dt,$$

where the $o(1)$ does not depend on $\widetilde{\chi}$. Now, we take a sequence of functions $\widetilde{\chi}$ such that the support of $\partial_t \widetilde{\chi}(x_\ell^+(t))$ goes to $-\infty$ and $\partial_t \widetilde{\chi}(x_\ell^+(t)) \leqslant 0$ (see Figure 5.1). Since $b_{\ell,0}^+$ does not depend on $\widetilde{\chi}$, the previous expression gives

$$(5.25) \quad \widetilde{b}_{\ell,0}^+ = i(2\pi)^{\frac{n-1}{2}} \frac{2^{\frac{1}{2}} e^{i(n-1)\frac{\pi}{4}}}{\left(\prod_{j \neq j(\ell)} \lambda_j \right)^{\frac{1}{2}}} |g_{\lambda_\ell^+}^{\ell,+}| \lambda_\ell^+ (g^{\ell,+})^\alpha e^{-i\nu_\ell^+ \pi/2} E_0^{\frac{1}{4}} (D_\ell^+)^{-\frac{1}{2}}.$$

5.3. End of the proof of Theorem 5.1

Following the approach of Section 5.2, one can prove that

$$(5.26) \qquad \left(e^{i\varphi_2(x, \sqrt{z_\alpha}\omega')/h} \chi t_2(x, \sqrt{z_\alpha}\omega', h), \overline{f} \right) = \sum_{k=1}^{N_-} e^{iS_k^-/h} h^{\frac{n+1}{2}} \widetilde{d}_k^-(h),$$

with $\widetilde{d}_k^-(h) \sim \sum_{j \geqslant 0} \widetilde{d}_{k,j}^- h^j$ and $\widetilde{d}_{k,j}^- = 0$ if and only if $(g^{k,-})^\alpha = 0$. Moreover, in the short range case, we have

$$(5.27) \quad \widetilde{d}_{k,0}^- = i(2\pi)^{\frac{n-1}{2}} \frac{2^{\frac{1}{2}} e^{i(n-1)\frac{\pi}{4}}}{\left(\prod_{j \neq j(k)} \lambda_j\right)^{\frac{1}{2}}} |g_{\lambda_k^-}^{k,-}| \lambda_k^-(g^{k,-})^\alpha e^{-i\nu_k^- \pi/2} E_0^{\frac{1}{4}} (D_k^-)^{-\frac{1}{2}}.$$

Then, combining the representation of the residue given in (5.8) with the constants given in (4.2) and (5.7), and the scalar products (5.19) and (5.26), we obtain

$$\text{Residue}\left(\mathcal{A}(\omega, \omega', z, h), z = z_\alpha\right) = h^{-|\alpha|+\frac{1}{2}} \sum_{k=1}^{N_-} \sum_{\ell=1}^{N_+} a_{k,\ell}(h) e^{i(S_k^- + S_\ell^+)/h},$$

with $a_{k,\ell}(h) \sim \sum_{j \geqslant 0} a_{k,\ell}^j h^j$ and $a_{k,\ell}^0 = 0$ if and only if $(g^{k,-})^\alpha (g^{\ell,+})^\alpha = 0$. Moreover, in the short range case, (5.25) and (5.27) imply

$$a_{k,\ell}^0 = \frac{e^{-i\frac{\pi}{2}(|\alpha|-\frac{1}{2})}}{\sqrt{2\pi}\alpha!} E_0^{\frac{n-1}{4}} (\lambda_k^- \lambda_\ell^+)^{\frac{3}{2}} \prod_{j=1}^n \lambda_j^{\alpha_j - \frac{1}{2}}$$

$$(5.28) \qquad \times e^{-i\nu_k^- \pi/2} e^{-i\nu_\ell^+ \pi/2} (D_k^- D_\ell^+)^{-\frac{1}{2}} (g^{k,-})^\alpha (g^{\ell,+})^\alpha |g_{\lambda_k^-}^{k,-}| |g_{\lambda_\ell^+}^{\ell,+}|.$$

6. Large time behavior of the Schrödinger group

In this section, we prove a resonance expansion for the cut-off Schrödinger propagator. The proof relies on the resolvent estimate in Theorem 3.1 and on standard arguments.

THEOREM 6.1 (Schrödinger group expansion). — *Assume* (H1)–(H3). *Let* $\mu > 0$ *be different from* $\sum_{j=1}^n (\alpha_j + \frac{1}{2})\lambda_j$ *for all* $\alpha \in \mathbb{N}^n$. *Let* $\chi \in C_0^\infty(\mathbb{R}^n)$ *and* $\psi \in C_0^\infty([E_0 - \varepsilon, E_0 + \varepsilon])$ *for some* $\varepsilon > 0$ *small enough. Then, there exists* $K = K(\mu) > 0$ *such that*

$$\chi e^{-itP/h} \chi \psi(P)$$
$$= \sum_{z_\alpha \in \text{Res}(P) \cap D(E_0, \mu h)} -\chi \, \text{Residue}\left(e^{-itz/h}(P-z)^{-1}, z = z_\alpha\right)\chi \psi(P)$$
$$+ \mathcal{O}(h^\infty) + \mathcal{O}(e^{-\mu t} h^{-K}),$$

for all $t \geqslant 0$. *In particular, if all the* z_α^0 *in* $D(E_0, \mu h)$ *are simple, we have*

$$\chi e^{-itP/h} \chi \psi(P) = \sum_{z_\alpha \in \text{Res}(P) \cap D(E_0, \mu h)} e^{-itz_\alpha/h} \chi \Pi_{z_\alpha} \chi \psi(P)$$
$$+ \mathcal{O}(h^\infty) + \mathcal{O}(e^{-\mu t} h^{-K}),$$

for all $t \geqslant 0$. *Here,* Π_{z_α} *is the generalized spectral projection associated to* z_α *and described in Theorem 4.1.*

Remark 6.2. — Note that the previous expansions make sense only for $t > \frac{K}{\mu}|\ln h|$.

One might think that the resonance expansion holds for shorter times. But, in fact, it is not possible to do much better. This follows from the paper of De Bièvre and Robert [11] which is stated with slightly different hypotheses. In the one dimensional case, they have proved that the coherent states propagate through a maximum of the potential for times of order $\frac{1}{\lambda_1}|\ln h|$ and that they stay at $(0,0)$ before. On the other hand, the sum of the generalized spectral projections over the resonances appearing in Theorem 6.1 can not be microlocalized only at $(0,0)$ thanks to Theorem 4.1. Thus, if the resonance expansion with a small error holds at time $t \geqslant 0$, we have necessarily $t \geqslant \frac{1}{\lambda_1}|\ln h|$ in the one dimensional case. If we only want to prove that $t \to +\infty$ as $h \to 0$, we can more simply apply the standard propagation of singularities with an initial data microlocalized in $\Lambda_- \smallsetminus \{(0,0)\}$.

There is also a simplest way to justify this phenomena. Let μ be such that $\sum \lambda_j/2 < \mu < \lambda_1 + \sum \lambda_j/2$. Then, z_0 is the unique resonance in $D(E_0, \mu h)$ for h small enough and z_0^0 is always simple. Assume that, for some $t \geqslant 0$, we can write

$$(6.1) \qquad \chi e^{-itP/h}\chi\psi(P) = e^{-itz_0/h}\chi\Pi_{z_0}\chi\psi(P) + R,$$

where R is small. The left hand side is of order 1 since the propagator is unitary. On the other hand, from Theorem 2.1 and Theorem 4.1, the right hand side is of order $e^{-t\sum \lambda_j/2}h^{-\frac{n}{2}}$. Then, (6.1) implies $t \geqslant \frac{n}{\sum \lambda_j}|\ln h|$. Remark that this critical time coincides with the one obtained by De Bièvre and Robert in the one dimensional case.

The situation is different for the well in the island case which was treated by Nakamura, Stefanov and Zworski [35]. In that setting, the cut-off Schrödinger group is well approximated by the resonance expansion after a fix time. This is in agreement with the geometrical interpretation since a fixed amount of time is enough to dispel the part of the initial data which is not localized in the well.

Nevertheless, Gérard and Sigal [16] have proved that the Schrödinger group when acting on a quasiresonant state (sort of a quasimode) associated to a quasiresonance z_h is well approximated by $e^{-itz_h/h}$ for all time $t \geqslant 0$.

Remark 6.3. — When $t/|\ln h| \to +\infty$ as $h \to 0$, the sum over the resonances is negligible and Theorem 6.1 simply yields $\chi e^{-itP/h}\chi\psi(P) = \mathcal{O}(h^\infty)$.

The remainder terms $\mathcal{O}(h^\infty)$ in Theorem 6.1 come from the C^∞ pseudo-differential calculus. Thus, if the cut-off functions χ, ψ are in some Gevrey

class, it is perhaps possible to replace these remainder terms by $\mathcal{O}(e^{-h^{-\delta}})$ for some $\delta > 0$. In that case, the sum over the resonances will dominate the remainders until t is of order $h^{-\delta}$.

Burq and Zworski [7] (see also Tang and Zworski [42]) have obtained a long time expansion of semiclassical propagators in terms of resonances close to the real axis. Their result in the present situation gives $\chi e^{-itP/h}\chi\psi(P) = \mathcal{O}(h^{\infty})$ for all $t > h^{-L}$ for some $L > 0$.

Proof. — Let $f \in C_0^{\infty}([E_0 - 3\varepsilon, E_0 + 3\varepsilon])$ be such that $f = 1$ near $[E_0 - 2\varepsilon, E_0 + 2\varepsilon]$. Then, from the pseudodifferential calculus, we get

$$I := \chi e^{-itP/h}\chi\psi(P) = \chi e^{-itP/h}f(P)\chi\psi(P) + \mathcal{O}(h^{\infty})$$

$$= \int_{\mathbb{R}} e^{-itz/h}f(z)\chi dE_z\chi\psi(P) + \mathcal{O}(h^{\infty}),$$

where dE_z, the spectral projection, is given by the Stone formula

$$dE_z = \frac{1}{2\pi i}\big(R_+(z) - R_-(z)\big)\,dz,$$

and $R_{\pm}(z) = (P - z)^{-1}$ is analytic for $\pm\,\mathrm{Im}\,z > 0$. Then,

$$I = \frac{1}{2\pi i}\int_{\mathbb{R}} e^{-itz/h}f(z)\chi\big(R_+(z) - R_-(z)\big)\chi\psi(P)\,dz + \mathcal{O}(h^{\infty}),$$

Making a change of contour, we obtain

$$I = \sum_{z_\alpha \in \mathrm{Res}(P) \cap D(E_0, \mu h)} -\chi\,\mathrm{Residue}\,\big(e^{-itz/h}R_+(z), z = z_\alpha\big)\chi\psi(P)$$

(6.2)
$$+ I_1 + I_2 + I_3 + I_4 + I_5 + \mathcal{O}(h^{\infty}),$$

where

(6.3)
$$I_j = \frac{1}{2\pi i}\int_{\Gamma_j} e^{-itz/h}f(z)\chi\big(R_+(z)-R_-(z)\big)\chi\psi(P)\,dz \quad \text{for } j = 1, 5,$$

$$I_j = \frac{1}{2\pi i}\int_{\Gamma_j} e^{-itz/h}\chi\big(R_+(z)-R_-(z)\big)\chi\psi(P)\,dz \qquad \text{for } j = 2, 3, 4,$$

and $\Gamma_1 =]-\infty, E_0 - 2\varepsilon]$, $\Gamma_2 = E_0 - 2\varepsilon + i[0, -\mu h]$, $\Gamma_3 = [E_0 - 2\varepsilon, E_0 + 2\varepsilon] - i\mu h$, $\Gamma_4 = E_0 + 2\varepsilon + i[-\mu h, 0]$ and $\Gamma_5 = [E_0 + 2\varepsilon, +\infty[$ (see Figure 6.1). The theorem will follow from the estimates on the I_j's given below.

• Estimates on I_1 and I_5. Using that $\Gamma_1 \cap \mathrm{supp}\,\psi = \emptyset$, there exists $g \in C_0^{\infty}(\mathbb{R})$ such that $g = 1$ near $\mathrm{supp}\,f \cap \Gamma_1$ and $g = 0$ near $\mathrm{supp}\,\psi$. Then, by pseudodifferential calculus, $g(P)\chi\psi(P) = \mathcal{O}(h^{\infty})$. Therefore, (6.3) yields

$$I_1 = \chi e^{-itP/h}\mathbb{1}_{\Gamma_1}(P)f(P)\chi\psi(P)$$

$$= \chi e^{-itP/h}\mathbb{1}_{\Gamma_1}(P)f(P)g(P)\chi\psi(P)$$

(6.4)
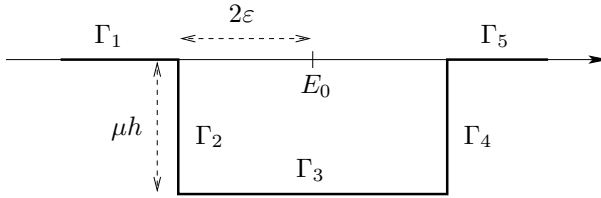$$= \mathcal{O}(h^{\infty}).$$

Figure 6.1. The contours Γ_j.

The same way, we get $I_5 = \mathcal{O}(h^\infty)$.

• Estimates on I_3. Using Theorem 3.1 for $R_\pm$, we obtain

$$(6.5) \qquad \|I_3\| \lesssim \int_{\Gamma_3} \left|e^{-itz/h}\right| \left\|\chi\big(R_+(z) - R_-(z)\big)\chi\right\| dz = \mathcal{O}(e^{-\mu t}h^{-K}).$$

• Estimates on I_2 and I_4. Let $\theta = \nu h|\ln h|$ be as in Theorem 3.1 and assume that the distortion occurs outside of the support of χ. Then, $\chi R_+(z)\chi = \chi(P_\theta - z)^{-1}\chi$ and $\chi R_-(z)\chi = \chi(P_{-\theta} - z)^{-1}\chi$. In particular, we can write

$$(6.6) \qquad I_2 = \frac{1}{2\pi i}\int_{\Gamma_2} e^{-itz/h}\chi\big((P_\theta - z)^{-1} - (P_{-\theta} - z)^{-1}\big)\chi\psi(P)\,dz.$$

Let $k \in C_0^\infty(\mathbb{R})$ be such that $k = 1$ near $E_0 - 2\varepsilon$ and $k = 0$ near $\operatorname{supp}\psi$ (see Figure 6.2). Then, for $z \in \Gamma_2$,

$$
\begin{aligned}
(P_{\pm\theta} - z)^{-1} &= (P_{\pm\theta} - z)^{-1}k(P) + (P_{\pm\theta} - z)^{-1}(1-k)(P)\\
&= (P_{\pm\theta} - z)^{-1}k(P) + (P - z)^{-1}(1-k)(P)\\
&\quad + (P_{\pm\theta} - z)^{-1}(P - P_{\pm\theta})(P - z)^{-1}(1-k)(P).
\end{aligned}
$$

Therefore (6.6) becomes

$$(6.7) \qquad I_2 = J_1^+ - J_1^- + J_2^+ - J_2^-,$$

where

$$
\begin{aligned}
J_1^\pm &= \frac{1}{2\pi i}\int_{\Gamma_2} e^{-itz/h}\chi(P_{\pm\theta} - z)^{-1}k(P)\chi\psi(P)\,dz\\
J_2^\pm &= \frac{1}{2\pi i}\int_{\Gamma_2} e^{-itz/h}\chi(P_{\pm\theta} - z)^{-1}(P - P_{\pm\theta})\\
&\qquad\qquad (P - z)^{-1}(1-k)(P)\chi\psi(P)\,dz.
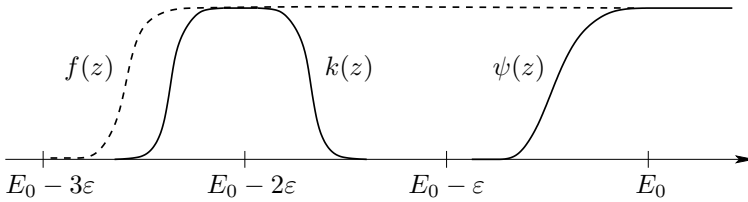\end{aligned}
$$

Figure 6.2. The cut-off functions f, k and ψ.

Since k and ψ have disjoint supports, the pseudodifferential calculus gives $k(P)\chi\psi(P) = \mathcal{O}(h^\infty)$. Thus, Theorem 3.1 implies

$$(6.8) \qquad \|J_1^\pm\| \lesssim \int_{\Gamma_2} \left\|(P_{\pm\theta} - z)^{-1}\right\| \left\|k(P)\chi\psi(P)\right\| |dz| = \mathcal{O}(h^\infty),$$

since $\operatorname{Im} z \leqslant 0$ for $z \in \Gamma_2$.

On the other hand, $P - P_{\pm\theta} \in \Psi_h^0(\theta\langle\xi\rangle^2)$ and $(P - z)^{-1}(1 - k)(P) \in \Psi_h^0(\langle\xi\rangle^{-2})$ uniformly in $z \in \Gamma_2$. Moreover, $P - P_{\pm\theta}$ is a differential operator whose coefficients are supported outside of the support of χ. Then, the microlocal analysis gives

$$\left\|(P - P_{\pm\theta})(P - z)^{-1}(1 - k)(P)\chi\right\| = \mathcal{O}(h^\infty),$$

uniformly in $z \in \Gamma_2$. Combining this estimate with Theorem 3.1, we get

$$(6.9) \qquad\qquad\qquad J_2^\pm = \mathcal{O}(h^\infty).$$

Using (6.8) and (6.9) to estimate (6.7), we conclude $I_2 = \mathcal{O}(h^\infty)$. The same way, we have $I_4 = \mathcal{O}(h^\infty)$. $\qquad\qquad\qquad\square$

Appendix A. Construction of test curves

In this section, we construct Hamiltonian curves in Λ_- with a prescribed asymptotic expansion at infinity. They are used in Section 4, where test functions for the projection are built in a microlocal neighborhood of these curves. We will work on Λ_-, but the same work can be done in Λ_+.

Let $\gamma^-(t)$ be a Hamiltonian curve in Λ_-. From [21, Section 3], the curve γ^- satisfies, in the sense of expandible functions,

$$(A.1) \qquad \gamma^-(t) \sim \sum_{k=1}^{+\infty} \gamma_{\mu_k}^-(t)e^{-\mu_k t} \quad \text{with} \quad \gamma_{\mu_k}^-(t) = \sum_{m=0}^{M_{\mu_k}} \gamma_{\mu_k, m}^- t^m.$$

The spectrum of F_p is $\sigma(F_p) = \{-\lambda_n, \ldots, -\lambda_1, \lambda_1, \ldots, \lambda_n\}$. We denote by Π_μ the spectral projection on the eigenspace of F_p associated to $-\mu$. Remark that

$$(A.2) \qquad \operatorname{Ker}(F_p + \mu) \oplus \operatorname{Im}(F_p + \mu) = \mathbb{R}^{2n}.$$

LEMMA A.1. — Let $\gamma^-(t)$ be a Hamiltonian curve in Λ_-. Assume that λ_j is such that $\lambda_j = \alpha \cdot \lambda$, $\alpha \in \mathbb{N}^n$, implies $|\alpha| = 1$. Then, $M_{\lambda_j} = 0$ and $\gamma^-_{\lambda_j, 0} \in \operatorname{Ker}(F_p + \lambda_j)$.

Proof. — We have $\partial_t \gamma^-(t) = H_p(\gamma^-(t))$. Taking the Taylor expansion of H_p at 0, we get

$$(A.3) \quad \partial_t \gamma^-(t) = F_p(\gamma^-(t)) + G_2(\gamma^-(t)) + \cdots + G_K(\gamma^-(t)) + \mathcal{O}(e^{-(\lambda_j + \varepsilon)t}),$$

where G_k is a polynomial of order k and $K > \lambda_j / \lambda_1$. Since λ_j can not be written as the sum of at least two terms μ_ℓ, the cross products G_k in the previous formula provide no term of the form $e^{-\lambda_j t}$. Then,

$$(A.4) \qquad \sum_{m=0}^{M_{\lambda_j}} -\lambda_j \gamma^-_{\lambda_j, m} t^m + m \gamma^-_{\lambda_j, m} t^{m-1} = \sum_{m=0}^{M_{\lambda_j}} F_p(\gamma^-_{\lambda_j, m}) t^m,$$

which can be written

$$(A.5) \qquad \begin{cases} (F_p + \lambda_j)\gamma^-_{\lambda_j, m} = 0 & \text{for } m = M_{\lambda_j} \\ (F_p + \lambda_j)\gamma^-_{\lambda_j, m} = (m+1)\gamma^-_{\lambda_j, m+1} & \text{for } 0 \leqslant m < M_{\lambda_j}. \end{cases}$$

If $M_{\lambda_j} \geqslant 1$, the previous equation, together with (A.2), gives a contradiction. Thus, $M_{\lambda_j} = 0$ and $\gamma^-_{\lambda_j, 0} \in \operatorname{Ker}(F_p + \lambda_j)$ from (A.5). $\qquad \square$

We begin the construction with the following formal result.

LEMMA A.2. — If $\widetilde{\gamma}^-_{\lambda_j, 0} \in \operatorname{Ker}(F_p + \lambda_j)$ for all $j \in \{1, \ldots, n\}$, then there exists a formal Hamiltonian curve γ^- of the form (A.1) such that

$$(A.6) \qquad \forall j \in \{1, \ldots, n\}, \quad \Pi_{\lambda_j}(\gamma^-_{\lambda_j, 0}) = \widetilde{\gamma}^-_{\lambda_j, 0}.$$

Proof. — We construct the coefficients $\gamma^-_{\mu_k}$ inductively. Using the Taylor expansion of H_p at 0 as in (A.3), one can see that it is enough to find $\gamma^-_{\mu_k}$, $k \geqslant 0$, such that

$$(A.7) \quad \sum_{m=0}^{M_{\mu_k}} -\mu_k \gamma^-_{\mu_k, m} t^m + m \gamma^-_{\mu_k, m} t^{m-1} = \sum_{m=0}^{M_{\mu_k}} F_p(\gamma^-_{\mu_k, m}) t^m + \sum_{m=0}^{N_{\mu_k}} R_{\mu_k, m} t^m,$$

where the $R_{\mu_k, m}$ depend only on the $\gamma^-_{\mu_\ell}$ for $\ell < k$. Assume that the $\gamma^-_{\mu_\ell}$ have been chosen to satisfy (A.7) for all $\mu_\ell < \mu_k$ and (A.6) for all $\lambda_j < \mu_k$.

If $\mu_k \notin \{\lambda_1, \ldots, \lambda_n\}$, then it is enough to take $M_{\mu_k} = N_{\mu_k}$,

$$\gamma^-_{\mu_k, M_{\mu_k}} = -(F_p + \mu_k)^{-1} R_{\mu_k, M_{\mu_k}},$$

and, for $0 \leqslant m < M_{\mu_k}$,

$$\gamma^-_{\mu_k, m} = (F_p + \mu_k)^{-1} \left((m+1) \gamma^-_{\mu_k, m+1} - R_{\mu_k, m} \right).$$

If $\mu_k = \lambda_j$ for some j, then we take $M_{\mu_k} = N_{\mu_k} + 1$ and

$$\gamma^-_{\mu_k, M_{\mu_k}} = M^{-1}_{\mu_k} \Pi_{\lambda_j} R_{\mu_k, M_{\mu_k}-1}$$

$$\gamma^-_{\mu_k, M_{\mu_k}-1} = (M_{\mu_k} - 1)^{-1} \Pi_{\lambda_j} R_{\mu_k, M_{\mu_k}-2} - K_{\lambda_j}(1 - \Pi_{\lambda_j}) R_{\mu_k, M_{\mu_k}-1}$$

$$\vdots \quad = \quad \vdots$$

$$\gamma^-_{\mu_k, 0} = \widetilde{\gamma}^-_{\lambda_j, 0} - K_{\lambda_j}(1 - \Pi_{\lambda_j}) R_{\mu_k, 0}.$$

Here, K_{λ_j} is the inverse of the map $F_p + \lambda_j : \text{Im}(F_p + \lambda_j) \longrightarrow \text{Im}(F_p + \lambda_j)$. With these choices, (A.6) and (A.7) are always verified. $\qquad \square$

PROPOSITION A.3. — If $\widetilde{\gamma}^-_{\lambda_j, 0} \in \text{Ker}(F_p + \lambda_j)$ for all $j \in \{1, \ldots, n\}$, then there exists a Hamiltonian curve $\gamma^- \in \Lambda_-$ such that

$$\forall j \in \{1, \ldots, n\}, \quad \Pi_{\lambda_j}(\gamma^-_{\lambda_j, 0}) = \widetilde{\gamma}^-_{\lambda_j, 0}.$$

Proof. — Let

$$\rho(t) = \sum_{\mu_k \leqslant N} \sum_{m=0}^{M_{\mu_k}} \gamma^-_{\mu_k, m} t^m e^{-\mu_k t},$$

where the $\gamma^-_{\mu_k, m}$ are given by Lemma A.2 and N will be fixed below. Since (A.7) is verified for all $\mu_k \leqslant N$, we have

$$\partial_t \rho(t) = H_p(\rho(t)) + R(t),$$

with $R(t) = \mathcal{O}(e^{-(N+\varepsilon)t})$. We seek a solution of the form $\gamma^-(t) = \rho(t) + r(t)$. Then, r must satisfies

(A.8) $$\partial_t r = H_p(\rho + r) - H_p(\rho) - R.$$

Let $T_N \geqslant 0$ be such that $|R(t)| \leqslant e^{-Nt}$ and $|\rho(t)| \leqslant 1$ for all $t \geqslant T_N$. We define $r_j(t)$ by induction as

(A.9) $$\begin{cases} r_0(t) = 0 \\ r_{j+1}(t) = -\int_t^{+\infty} \left(H_p(\rho + r_j) - H_p(\rho) - R \right)(s)\, ds. \end{cases}$$

LEMMA A.4. — *For N large enough, the functions $(r_j)_{j \geqslant 0}$ exist on $[T_N, +\infty[$ and*

$$(A.10) \qquad |r_j(t)| \leqslant e^{-Nt},$$

for all $t \geqslant T_N$.

Proof of Lemma A.4. — Define

$$(A.11) \qquad C_1 = \sup_{|u| \leqslant 2} |dH_p(u)|.$$

We will prove the lemma inductively. First, r_0 satisfies (A.10). Assume now that r_{j-1} exists on $[T_N, +\infty[$ and verifies (A.10). In particular, $|r_{j-1}(t)| \leqslant 1$ for $t \geqslant T_N$. Then, (A.9) gives

$$|r_j(t)| \leqslant \int_t^{+\infty} \big(\big|H_p(\rho + r_{j-1}) - H_p(\rho)\big| + |R|\big) ds$$

$$\leqslant \int_t^{+\infty} \big(C_1|r_{j-1}| + |R|\big) ds \leqslant \int_t^{+\infty} (C_1 + 1)e^{-Ns} ds$$

$$(A.12) \qquad \leqslant \frac{C_1 + 1}{N} e^{-Nt},$$

for $t \geqslant T_N$. Therefore, if $N \geqslant C_1 + 1$, r_j satisfies (A.10) and the lemma follows. $\qquad \square$

LEMMA A.5. — *For N large enough, we have*

$$(A.13) \qquad \big|r_{j+1}(t) - r_j(t)\big| \leqslant \frac{e^{-Nt}}{2^j},$$

for $j \geqslant 0$ and $t \geqslant T_N$.

Proof of Lemma A.5. — For $j = 0$ and N large enough, Lemma A.4 gives $|r_1(t) - r_0(t)| = |r_1(t)| \leqslant e^{-Nt}$. Assume that (A.13) holds for some $j - 1 \geqslant 0$. Using (A.9), we get

$$|r_{j+1}(t) - r_j(t)| \leqslant \int_t^{+\infty} \big|H_p(\rho + r_j) - H_p(\rho + r_{j-1})\big| ds$$

$$\leqslant C_1 \int_t^{+\infty} \big|r_j - r_{j-1}\big| ds \leqslant C_1 \int_t^{+\infty} \frac{e^{-Ns}}{2^{j-1}} ds$$

$$(A.14) \qquad \leqslant \frac{2C_1}{N} \frac{e^{-Nt}}{2^j}.$$

Then, for $N \geqslant 2C_1$, (A.13) holds and the lemma follows. $\qquad \square$

LEMMA A.6. — *For N large enough, there exists $r \in C^\infty([T_N, +\infty[)$ such that*

i) *for $t \geqslant T_N$, we have $|r(t)| \leqslant e^{-Nt}$,*

ii) *for all $j \geqslant 0$,*

$$\left\| e^{Nt}(r_j - r) \right\|_{L^\infty([T_N, +\infty[)} \leqslant 2^{1-j},$$

iii) *the curve $\gamma^- = \rho + r$ satisfies $\partial_t \gamma^- = H_p(\gamma^-)$.*

Proof of Lemma A.6. — Using standard arguments, Lemma A.5 provides us with a function $r \in C^0([T_N, +\infty[)$ satisfying ii). Then, part i) follows directly from Lemma A.4. On the other hand,

$$\left| \int_t^{+\infty} \big(H_p(\rho + r_j) - H_p(\rho) - R \big) - \big(H_p(\rho + r) - H_p(\rho) - R \big) ds \right|$$

$$\leqslant \int_t^{+\infty} \big| H_p(\rho + r_j) - H_p(\rho + r) \big| ds$$

$$\leqslant \int_t^{+\infty} C_1 |r_j - r| ds$$

$$\leqslant C_1 2^{1-j} \int_t^{+\infty} e^{-Ns} ds \leqslant \frac{C_1}{N} 2^{1-j} \longrightarrow 0,$$

as $j \to +\infty$. Then, taking the limit $j \to +\infty$ in (A.9), we obtain

$$r(t) = - \int_t^{+\infty} \big(H_p(\rho + r) - H_p(\rho) - R \big)(s) \, ds.$$

Thus, $r \in C^\infty([T_N, +\infty[)$ and $\gamma^- = \rho + r$ satisfies $\partial_t \gamma^- = H_p(\gamma^-)$. □

To finish the proof of Proposition A.3, we impose in addition that $\lambda_n < N$. Then, the function γ^- of Lemma A.6 is a Hamiltonian curve in Λ_- and, since $r(t) = o(e^{-\lambda_n t})$, Lemma A.2 assures that $\Pi_{\lambda_j}(\gamma^-_{\lambda_j,0}) = \widetilde{\gamma}^-_{\lambda_j,0}$ for all $j \in \{1, \ldots, n\}$. □

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Jean-François BONY
Université Bordeaux 1
IMB (UMR CNRS 5251)
33405 Talence (France)
Jean-Francois.Bony@math.u-bordeaux1.fr

Setsuro FUJIIÉ
University of Hyogo
Graduate School of Material Science (Japan)
fujiie@sci.u-hyogo.ac.jp

Thierry RAMOND
Université Paris Sud 11
LMO (UMR CNRS 8628)
91405 Orsay (France)
thierry.ramond@math.u-psud.fr

Maher ZERZERI
Université Paris 13
LAGA (UMR CNRS 7539)
93430 Villetaneuse (France)
zerzeri@math.univ-paris13.fr