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SHUFFLE BIALGEBRAS

by María RONCO (*)

ABSTRACT. — The goal of our work is to study the spaces of primitive elements of some combinatorial Hopf algebras, whose underlying vector spaces admit linear basis labelled by subsets of the set of maps between finite sets. In order to deal with these objects we introduce the notion of shuffle algebras, which are coloured algebras where composition is not always defined. We define bialgebras in this framework and compute the subpaces of primitive elements associated to them. These spaces of primitive elements have natural structure of some type of coloured algebras, which we describe in terms of generators and relations.

RÉSUMÉ. — Le but de ce travail est l'étude des espaces d'éléments primitifs de certaines algèbres de Hopf combinatoires, dont les espaces vectoriels sous-jacents admettent des bases indexées par des sous ensembles de l'ensemble des applications entre ensembles finis. Pour donner une description précise de ces objets nous introduisons la notion d'algèbre shuffle, qui correspond à un type d'algèbre colorée pour laquelle les compositions ne sont pas toujours définies. Nous définissons des bigèbres dans ce contexte et nous calculons leurs sous espaces d'éléments primitifs. Ces espaces d'éléments primitifs peuvent être décrits en terme des générateurs et relations comme des exemples d'autres types d'algèbres colorées.

Introduction

In this paper we study Hopf algebra structures defined on vector spaces spanned by families of maps between finite sets, like the Malvenuto-Reutenauer bialgebra of permutations (see [2]), the bialgebra of surjective maps (see [18]) or the bialgebra of parking functions (see [17]). Our goal is to study these algebras in a similar framework than the one set for algebras spanned by planar rooted trees in [3], [8], [9], [11], [13] and [21]. In order to do that, we need to describe the Malvenuto-Reutenauer algebra, spanned by permutations, as a free object on one generator, and the algebra of surjective maps as a free graded object spanned by one generator in

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each degree, for some algebraic theory. The natural solution of this problem is not properly a type of algebra but of coloured algebra, which we call a shuffle algebra.

Shuffle algebras are a particular case of monoids in the category of $\mathcal{S}$-modules, as described in [19] and [10], where the operations do not preserve the action of the symmetric group. Applying this notion we are able to introduce the notion of shuffle bialgebra, in such a way that the Hopf algebra structures defined on the space spanned by maps between finite sets are induced by shuffle bialgebra structures on these spaces. Hopf algebras associated to shufle bialgebras are, in general, neither commutative nor cocommutative, and this new point of view permit us to compute the subspaces of their primitive elements. Although the primitive elements of the Malvenuto-Reutenauer algebra and of the algebra of planar binary rooted trees have been previously computed in [2], [3] and [8], our description has the advantage of showing them as free objects for some coloured algebraic theories described in terms of generators and relations.

In the general case, there does not exist a standard method to compute the space of primitive elements of a non-cocommutative coalgebra. Shuffle bialgebras present the advantage of being equipped with an associative product $\times$, called the concatenation product, which verifies a nonunital infinitesimal relation with the coproduct. Nonunital infinitesimal bialgebras were introduced in [14], where we proved that any connected nonunital infinitesimal bialgebra is isomorphic to the cofree coalgebra spanned by the space of its primitive elements. This result is the main tool used in the present work to compute the primitive elements.

Afterwards, we extend our results to other algebraic structures: preshuffle algebras and pre-Lie systems, the last ones are given by the underlying spaces of non-symmetric operads. The way we study them is largely inspired by the treatment given by J.-L. Loday to the so-called triples of operads, see [12]. Let us describe briefly the method employed:

(1) Given a linear algebraic theory $\mathcal{T}$, we introduce the notion of $\mathcal{T}$ bialgebra, in such a way that any free $\mathcal{T}$ algebra has a natural structure of $\mathcal{T}$ bialgebra.
(2) In a second step, we identify the theory $\mathcal{T}$ with the space of all the operations of the theory, and compute a basis for a subspace $\mathcal{P}rim_{\mathcal{T}}$ of $\mathcal{T}$, such that the space of primitive elements of any $\mathcal{T}$ bialgebra is closed under the action of the elements of $\mathcal{P}rim_{\mathcal{T}}$.
(3) Afterwards, we prove that any free $\mathcal{T}$ algebra $\mathcal{T}(X)$ is isomorphic, as a coalgebra, to the cofree coalgebra spanned by the space

$\mathcal{P}rim_{\mathcal{T}}(X)$, generated by the operations of $\mathcal{P}rim_{\mathcal{T}}$ on the elements of X. Applying results proved in [14], we get that $\mathcal{P}rim_{\mathcal{T}}(X)$ is the space of primitive elements of $\mathcal{T}(X)$.

(4) Finally, we describe the algebraic theory associated to $\mathcal{P}rim_{\mathcal{T}}$ in terms of generators and relations; and prove that the category of connected $\mathcal{T}$ bialgebras is equivalent to the category of $\mathcal{P}rim_{\mathcal{T}}$ algebras.

It is quite easy to compute the theory $\mathcal{P}rim_{sh}$ when $\mathcal{T}$ is the theory of shuffle algebras. The other examples follow from this case.

The paper is organised as follows:

We recall first elementary definitions of coalgebras as well as some well-known constructions on permutations, maps between finite sets and planar rooted trees, needed in the following sections.

In Section 2 we give the definition of a shuffle algebra, describe the free objects for this theory in terms of spaces spanned by surjective maps, and give the main examples. Shuffle bialgebras are introduced in Section 3, where we show that most of the examples of shuffle algebras given in the previous section have a natural structure of shuffle bialgebras.

In Section 4 we construct functors from the category of shuffle bialgebras to the categories of nonunital infinitesimal bialgebras, of dendriform bialgebras and of 2-associative bialgebras.

In Section 5 we compute the primitive elements of a shuffle bialgebra and we prove a Cartier-Milnor-Moore Theorem in this context.

In Section 6 we introduce the notion of preshuffle bialgebras. From the definition of $\mathcal{P}rim_{sh}$ algebras, we compute the subspaces of primitive elements of preshuffle bialgebras and describe any conilpotent preshuffle bialgebra as an enveloping algebra over its primitive part. Pre-Lie systems are obtained as a particular case of preshuffle algebras. We prove that any pre-Lie system equipped with an admissible coproduct gives rise to a shuffle bialgebra, which shows that if $(A, \circ)$ is a pre-Lie algebra (see [6]) obtained from a pre-Lie system with coproduct then $\circ$ may be extended to an associative product on A. Finally we describe the space of primitive elements of a pre-Lie system with a coproduct.

The last section of the paper contains some applications of the previous results to some good triples of operads (see [12]).

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1. Preliminaries

We introduce here some definitions and notations that are used in the paper.

Let K be a field, $\otimes$ denotes the tensor product of vector spaces over K. In order to simplify redaction, we use sometimes the term *space* instead of K-vector space. Given a graded K-vector space A, A_+ is the space $A \oplus K$ equipped with the canonical maps $K \hookrightarrow A_+ \longrightarrow K$. We denote the degree of a homogeneous element $x \in A_n$ by $|x| = n$.

For any set X, $K[X]$ denotes the vector space spanned by X. For any vector space V, the graded space $T(V) := \bigoplus_{n \geqslant 0} V^{\otimes n}$ is the tensor space over V. The reduced tensor space $\overline{T}(V)$ over V is the subspace $\bigoplus_{n \geqslant 1} V^{\otimes n}$.

The space $\overline{T}(V)$, with the concatenation product given by:

$$(v_1 \otimes \cdots \otimes v_n) \cdot (w_1 \otimes \cdots \otimes w_m) := v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m,$$

for $v_1, \ldots, v_n, w_1, \ldots, w_m \in V$, is the free associative algebra spanned by V.

Coalgebras. A coalgebra C over K is a vector space, equipped with a coproduct $\Delta : C \longrightarrow C \otimes C$, which is coassociative.

We use Sweedler's notation, and denote $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, for $x \in C$. A coalgebra C is counital if there exists a linear map $\epsilon : C \longrightarrow K$ such that $(\epsilon \otimes Id_C) \circ \Delta = Id_C = (Id_C \otimes \epsilon) \circ \Delta$, where we identify $K \otimes C$ and $C \otimes K$ with C, via the canonical isomorphism.

For a counital coalgebra (C, Δ, ϵ), the reduced coproduct is defined on $Ker(\epsilon)$ as the map $\overline{\Delta} := \Delta - Id_C \otimes \epsilon - \epsilon \otimes Id_C$. Note that $\overline{\Delta} : Ker(\epsilon) \longrightarrow Ker(\epsilon) \otimes Ker(\epsilon)$ is coassociative too.

Let $C = \bigoplus_{n \geqslant 0} C_n$ be a graded K-vector space. A graded coassociative coproduct on C is a coassociative coproduct Δ such that $\Delta(C_n) \subseteq \bigoplus_{i=0}^{n} C_i \otimes C_{n-i}$. Given a coassociative coproduct Δ on C and an integer $r \geqslant 1$, Δ^r denotes the homomorphism defined recursively as $\Delta^1 := \Delta$ and $\Delta^{r+1} := (\Delta^r \otimes Id_C) \circ \Delta$, for $r \geqslant 1$.

The coalgebras C we deal with in the paper satisfy that $C_0 = 0$. Given such a coalgebra (C, Δ) we define the coalgebra (C_+, Δ_+), where $C_+ := K \oplus C$ and Δ_+ is the unique coproduct on C_+ such that $\overline{\Delta}_+|_C = \Delta$ and (C_+, Δ_+) is a unital coalgebra.

Let V be a vector space, the deconcatenation coproduct on $\overline{T}(V)$ is given by:

$$\overline{\Delta}^c(v_1 \otimes \cdots \otimes v_n) := \sum_{i=1}^{n-1} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n).$$

Definition 1.1. — Let $C = (\bigoplus_{n\geqslant 1} C_n, \Delta)$ be a graded coalgebra. An element $x \in C$ is called *primitive* if $\Delta(x) = 0$. The subspace of primitive elements of C is denoted by $Prim(C)$.

Definition 1.2. — Let (C, Δ) be a coalgebra. Consider the filtration F_pC on C given by:

$F_1C := \mathrm{Prim}(C)$

$F_pC := \{x \in C \mid \Delta(x) \in F_{p-1}C \otimes F_{p-1}C\}$.

We say that C is *conilpotent* if $C = \bigcup_{p\geqslant 1} F_pC$.

The definition of primitive element for the counital coalgebra C_+ becomes $x \in Prim(C_+)$ if $\Delta_+(x) = x\otimes 1+1\otimes x$. In this case, $Prim(C_+) = Prim(C)$.

The main purpose of this work is to study bialgebra structures on spaces spanned by (coloured) functions between finite sets, permutations and trees. The rest of this section is devoted to introduce definitions and elementary results on these objects.

Permutations and shuffles. Let S_n be the group of permutations on n elements. A permutation σ is denoted by its image $(\sigma(1), \dots, \sigma(n))$. The element 1_n denotes the identity of S_n. The set $S_\infty := \bigcup_{n\geqslant 1} S_n$ is the graded set of all permutations.

Definition 1.3. — Given $1 \leqslant r \leqslant n$, a *composition* $\mathbf{n}$ of n of length r is an ordered family of positive integers $(n_1, \dots, n_r)$ such that $\sum_{i=1}^r n_i = n$. The number r is called the length of the composition $\mathbf{n}$.

For any composition $\mathbf{n} = (n_1, \dots, n_r)$ of n, there exists a homomorphism $S_{n_1} \times \cdots \times S_{n_r} \hookrightarrow S_n$ given by $(\sigma_1, \dots, \sigma_r) \mapsto \sigma_1 \times \cdots \times \sigma_r$, where

$$(\sigma_1 \times \cdots \times \sigma_r)(i) := \sigma_k(i - n_1 - \cdots - n_{k-1}) + n_1 + \cdots + n_{k-1},$$

for $n_1 + \cdots + n_{k-1} < i \leqslant n_1 + \cdots + n_k$. Let $S_{\mathbf{n}}$ or $S_{n_1\times\cdots\times n_r}$ denote indistinctly the subgroup of S_n which is the image of $S_{n_1} \times \cdots \times S_{n_r}$ under this embedding. The operation $\times : S_n \times S_m \longrightarrow S_{n+m}$ defined previously is an associative product on S_∞, called the *concatenation*.

Definition 1.4. — A permutation $\sigma \in S_n$ is *irreducible* if

$$\sigma \notin \bigcup_{i=1}^{n-1} S_i \times S_{n-i}.$$

We denote by Irr_{S_n} the set of irreducible permutations of S_n.

The graded vector space $K[S_\infty] := \bigoplus_{n\geqslant 1} K[S_n]$, equipped with the concatenation product, is the free associative algebra generated by $\bigcup_{n\geqslant 1} Irr_{S_n}$.

Definition 1.5. — Given a composition $\mathbf{n} = (n_1, \dots, n_r)$ of n, a $(n_1, \dots, n_r)$-*shuffle*, or $\mathbf{n}$-*shuffle*, is an element σ of S_n such that:

$$\sigma^{-1}(n_1 + \cdots + n_{k-1} + 1) < \cdots < \sigma^{-1}(n_1 + \cdots + n_k), \text{ for } 1 \leqslant k \leqslant r-1.$$

The set of all $(n_1, \dots, n_r)$-shuffles is denoted either $Sh(n_1, \dots, n_r)$ or $Sh(\mathbf{n})$.

Given positive integers n, m, the permutation $\epsilon_{n,m} := (n+1, \dots, n+m, 1, \dots, n)$ belongs to $Sh(n,m)$.

The following result about Coxeter groups are well-known. For the first assertion see for instance [22], the second one is proved, in a more general context, in [4].

Proposition 1.6. — (1) *Given a permutation $\sigma \in S_n$ and an integer $0 \leqslant i \leqslant n$ there exists unique elements $\sigma^i_{(1)} \in S_i$, $\sigma^{n-i}_{(2)} \in S_{n-i}$ and $\gamma \in Sh(i, n-i)$ such that $\sigma = (\sigma^i_{(1)} \times \sigma^{n-i}_{(2)}) \cdot \gamma$.*
(2) *Given compositions $\mathbf{n}$ of n and $\mathbf{m}$ of m, we have that:*

$$(Sh(\mathbf{n}) \times Sh(\mathbf{m})) \cdot Sh(n,m) = Sh(\mathbf{n} \cup \mathbf{m}),$$

where $\mathbf{n} \cup \mathbf{m} := (n_1, \dots, n_r, m_1, \dots, m_p)$.

The proof of the following lemma is straightforward.

Lemma 1.7. — Let $0 \leqslant r \leqslant n+m$ be an integer and let γ be a *(n,m)*-shuffle. There exist a unique non negative integer $0 \leqslant n_1 \leqslant r$ and permutations $\gamma^r_{(1)} \in S_r$ and $\gamma^{n+m-r}_{(2)} \in S_{n+m-r}$ such that $\gamma = (1_{n_1} \times \epsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma^r_{(1)} \times \gamma^{n+m-r}_{(2)})$, where $n_1 := |\gamma^{-1}(\{1, \dots, n\}) \cap \{1, \dots r\}|$ and $m_1 := r - n_1$. Moreover, $\gamma^r_{(1)}$ belongs to $Sh(n_1, m_1)$ and $\gamma^{n+m-r}_{(2)}$ belongs to $Sh(n-n_1, m-m_1)$.

Remark 1.8. — For any permutation $\gamma \in Sh(n,m)$ there exists unique integers $n_1, \dots, n_r$ and $m_1, \dots, m_r$ such that:

$$\gamma = (1, \dots, n_1, n+1, \dots n+m_1, n_1+1, \dots, n_1 + n_2, \dots, m_1 + \cdots + m_{r-1} + 1, \dots, m),$$

where $\sum_{i=1}^r n_i = n$, $\sum_{j=1}^r m_j = m$, $n_1 \geqslant 0$, $n_i \geqslant 1$ for $i > 2$, $m_j \geqslant 1$ for $j < r$, and $m_r \geqslant 0$.

Functions on finite sets. Given positive integers n and r, let $\mathcal{F}^r_n$ be the set of all maps $f : \{1, \dots, n\} \longrightarrow \{1, \dots, r\}$. An element $f \in \mathcal{F}^r_n$ is denoted by its image $(f(1), \dots, f(n))$. The constant function $(1, \dots, 1) \in \mathcal{F}_n^1$ is denoted by ξ_n. For $n \geqslant 1$, the set $\mathcal{F}_n$ is the disjoint union $\bigcup_{r=1}^n \mathcal{F}_n^r$.

For $1 \leqslant r \leqslant n$, we denote by $\mathcal{ST}_n^r$ the subset of all surjective maps in $\mathcal{F}_n^r$ and by $\mathcal{ST}_n$ the disjoint union $\mathcal{ST}_n := \bigcup_{r=1}^n \mathcal{ST}_n^r \subseteq \mathcal{F}_n$. Clearly, the set S_n of permutations of n elements is equal to $\mathcal{ST}_n^n$.

For any n, m, r and k, the concatenation product $\times : S_n \times S_m \longrightarrow S_{n+m}$ extends to an embedding $\mathcal{F}_n^r \times \mathcal{F}_m^k \longrightarrow \mathcal{F}_{n+m}^{r+k}$, given by

$$f \times g := (f(1), \dots, f(n), g(1) + r, \dots, g(m) + r), \text{ for } f \in \mathcal{F}_n^r \text{ and } g \in \mathcal{F}_m^k.$$

If $\mathbf{n} = (n_1, \dots, n_r)$ is a composition of n, we denote $\xi_{\mathbf{n}}$ the map $\xi_{\mathbf{n}} = (\xi_{n_1} \times \dots \times \xi_{n_r})$.

Remark 1.9. — For any element $f \in \mathcal{F}_n^r$ there exists a unique non-decreasing function $f^{\uparrow} \in \mathcal{F}_n^r$ and a unique permutation $\sigma_f \in Sh(n_1, \dots, n_r)$ such that

$$f = f^{\uparrow} \cdot \sigma_f,$$

where $n_i = |f^{-1}(i)|$ for $1 \leqslant i \leqslant r$, and $\cdot$ denotes the composition of functions.

Let $\mathcal{K}_n$ be the set of all maps $f \in \mathcal{ST}_n$ verifying the following condition:

$$\text{if } f(i) = f(j), \text{ for some } i < j, \text{ then } f(k) \leqslant f(i) \text{ for all } i \leqslant k \leqslant j.$$

It is immediate to check that $f \times g \in \mathcal{K}_{n+m}$, for any $f \in \mathcal{K}_n$ and $g \in \mathcal{K}_m$.

We extend the definition of irreducible permutation to $\bigcup_{n \geqslant 1} \mathcal{F}_n$ as follows:

Definition 1.10. — An element $f \in \mathcal{F}_n$ is called *irreducible* if $f \notin \bigcup_{i=1}^{n-1} \mathcal{F}_i \times \mathcal{F}_{n-i}$. The set of irreducible elements of $\mathcal{F}_n$ is denoted $Irr_{\mathcal{F}_n}$. In a similar way, the set of irreducible elements of $\mathcal{ST}_n$ (respectively, $\mathcal{K}_n$) is the set $Irr_{\mathcal{ST}_n} := \mathcal{ST}_n \cap Irr_{\mathcal{F}_n}$ (respectively, $Irr_{\mathcal{K}_n} := \mathcal{K}_n \cap Irr_{\mathcal{F}_n}$).

The graded space $K[\mathcal{F}_\infty] := \bigoplus_{n \geqslant 1} K[\mathcal{F}_n]$, equipped with the concatenation product, is the free associative algebra spanned by the set $\bigcup_{n \geqslant 1} Irr_{\mathcal{F}_n}$. Analogous results hold for the spaces $K[\mathcal{ST}_\infty] := \bigoplus_{n \geqslant 1} K[\mathcal{ST}_n]$ and $K[\mathcal{K}_\infty] := \bigoplus_{n \geqslant 1} K[\mathcal{K}_n]$

Planar rooted trees

Definition 1.11. — A *planar rooted tree* is a non-empty oriented connected planar graph such that any vertex has at least two input edges and one output edge, equipped with a final vertex called the *root*. For $n \geqslant 2$, a planar n-ary tree is a planar rooted tree such that any vertex has exactly n input edges.

Note that in a planar tree the set of input edges of any vertex is totally ordered. All trees we deal with are reduced planar rooted ones. From now on, we use the term planar tree instead of planar rooted tree.

Notation 1.12. — We denote by T_m the set of all planar trees with $m+1$ leaves and by Y_m the subset of T_m of all planar binary trees with $m+1$ leaves.

Let t be an element of T_m, the leaves of t are numbered from left to right, beginning with 0 up to m. We denote by $\mathfrak{c}_m$ the unique element of T_m, which has $m+1$ leaves and only one vertex (the m-corolla).

Let $X = \bigcup_{n \geqslant 1} X_n$ be a positively graded set. The set $T_m(X)$ is the set of planar binary trees with the internal vertices coloured by the elements of X in such a way that any vertex with k input edges is coloured by an element of X_{k-1}.

Definition 1.13. — Given coloured trees t and w, for any $0 \leqslant i \leqslant |w|$, define $t \circ_i w$ to be the coloured tree obtained by attaching the root of w to the i-th leaf of t.

For instance

$$\begin{array}{ccccc} \searrow & & \swarrow & & \\ & y & & & \\ & \searrow & & \swarrow & \\ & & z & & \\ & & \downarrow & & \end{array} \circ_2 \begin{array}{ccc} \searrow & & \swarrow \\ & x & \\ & \downarrow & \end{array} = \begin{array}{ccccccc} \searrow & & \swarrow & & \searrow & & \swarrow \\ & y & & & & x & \\ & & \searrow & & \swarrow & & \\ & & & z & & & \\ & & & \downarrow & & & \end{array}.$$

Notation 1.14. — Given two coloured trees t and w and $x \in X_1$, we denote by $t \vee_x w$ the tree obtained by joining the roots of t and w to a new root, coloured by x. More generally, we denote by $\bigvee_x(t^0, \ldots, t^r)$ the tree obtained by joining the roots of the trees $t^0, \ldots, t^r$, ordered from left to right, to a new root coloured with an element $x \in X_r$.

Any coloured tree t may be written in a unique way as $t = \bigvee_x(t^0, \ldots, t^r)$, with $|t| = \sum_{i=0}^{r} |t^i| + r - 1$ and $x \in X_r$.

2. Shuffle algebras

Our goal is to describe the spaces spanned by coloured permutations and coloured elements of $\mathcal{ST}_\infty$ as free objects for some type of algebraic structure.

Definition 2.1. — A *shuffle* algebra over K is a graded vector space $A = \bigoplus_{n\geqslant 1} A_n$ equipped with linear maps

$$\bullet_\gamma : A_n \otimes_K A_m \to A, \text{ for } \gamma \in Sh(n,m),$$

verifying that:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

whenever $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$ *in* $Sh(n,m,r)$.

Shuffle algebras appear in a natural way as monoids in the category of $\mathcal{S}$-modules, denoted $\mathcal{S}$-*Mod*. Let us describe briefly this category.

The objects in the category $\mathcal{S}$-*Mod* are infinite sequences $M = \{M(n)\}_{n\geqslant 0}$ of K-modules, such that each $M(n)$ is a right $K[S_n]$-module, for $n \geqslant 1$, where $K[S_0] := K$. A homomorphism f from M to N in $\mathcal{S}$-*Mod* is a family of $K[S_n]$-modules homomorphisms $f(n) : M(n) \longrightarrow N(n)$, for $n \geqslant 0$.

The category $\mathcal{S}$-*Mod* is endowed with a symmetric monoidal structure $\otimes_{\mathcal{S}}$ given by:

$$(M \otimes_{\mathcal{S}} N)(n) = \bigoplus_{i=0}^{n} (M(i) \otimes N(n-i)) \otimes_{K[S_i \times S_{n-i}]} K[S_n],$$

where $M(i) \otimes N(n-i)$ has the natural structure of right $K[S_i \times S_{n-i}]$-module.

By Proposition 1.6, the tensor product $(M(i) \otimes N(n-i)) \otimes_{K[S_i \times S_{n-i}]} K[S_n]$ is isomorphic to $M(i) \otimes N(n-i) \otimes K[Sh(i,n-i)]$. Moreover, the associativity and symmetry of $\otimes_{\mathcal{S}}$ are given by the isomorphisms:

(1) $a_{MNR} : (M \otimes_{\mathcal{S}} N) \otimes_{\mathcal{S}} R \longrightarrow M \otimes_{\mathcal{S}} (N \otimes_{\mathcal{S}} R)$, with

$$a_{MNR}((x \otimes y \otimes \sigma) \otimes z \otimes \delta) := x \otimes (y \otimes z \otimes \gamma) \otimes \tau,$$

whenever $(\sigma \times 1_r) \cdot \delta = (1_n \times \gamma) \cdot \tau$ in $Sh(m,n,r)$, for $x \in M(m)$, $y \in N(n)$ and $z \in R(r)$.

(2) $c_{MN} : M \otimes_{\mathcal{S}} N \longrightarrow N \otimes_{\mathcal{S}} M$, with

$$c_{MN}(x \otimes y \otimes \sigma) := y \otimes x \otimes (\epsilon_{n,m} \cdot \sigma),$$

where $\epsilon_{n,m} = (n+1, \dots, n+m, 1, \dots, n) \in Sh(n,m)$, for $x \in M(m)$ and $y \in N(n)$.

Lemma 2.2. — *Let $(M,\circ)$ be a monoid in $(\mathcal{S}\text{-}Mod,\otimes_{\mathcal{S}})$, the space $M = \bigoplus_{n\geqslant 0} M(n)$ has a natural structure of shuffle algebra, given by:*

$$x \bullet_\gamma y := \circ(x\otimes y\otimes \gamma),$$

for $x\in M(n)$ and $y\in M(m)$.

Proof. — The associativity of $\circ$ implies that the products $\bullet_\gamma$ fullfill the conditions of Definition 2.1. □

In [19] and [10], an associative monoid in $(\mathcal{S}\text{-}Mod,\otimes_{\mathcal{S}})$ is called a twisted associative algebra or an *As*-algebra in the category $\mathcal{S}$-*Mod*, respectively.

For any associative graded algebra $(A = \bigoplus_{n\geqslant 0} A_n, \bullet)$, consider $\overline{A} = \{A_n\otimes K[S_n]\}_{n\geqslant 0}$. The $\mathcal{S}$-module $\overline{A}$ has a natural structure of monoid in $(\mathcal{S}\text{-}Mod,\otimes_{\mathcal{S}})$, given by:

$$\circ((x,\sigma)\otimes(y,\tau)\otimes\gamma) := (x\bullet y)\otimes((\sigma\times\tau)\cdot\gamma)\in A_{n+m}\otimes K[S_{n+m}],$$

for $x\in A_n$, $y\in A_m$, $\sigma\in S_n$, $\tau\in S_m$ and $\gamma\in Sh(n,m)$.

Examples 2.3. — **a) The tensor space.** For any vector space V, the space $\overline{T}(V) := \bigoplus_{n\geqslant 1} V^{\otimes n}$ has a natural structure of shuffle algebra, where with the operations $\bullet_\gamma$ are defined by the formula:

$$(v_1\otimes\cdots\otimes v_n)\bullet_\gamma(v_{n+1}\otimes\cdots\otimes v_{n+m}) := v_{\gamma(1)}\otimes\cdots\otimes v_{\gamma(n+m)},$$

for $v_1,\dots,v_{n+m}\in V$.

b) Free shuffle algebras. Consider, on the graded vector space $K[S_\infty] := \bigoplus\limits_{n\geqslant 1} K[S_n]$, the operations $\bullet_\gamma$ given by $\sigma\bullet_\gamma\tau := (\sigma\times\tau)\cdot\gamma$, for $\sigma\in S_n$, $\tau\in S_m$ and $\gamma\in Sh(n,m)$. It is immediate to check that the space $K[S_\infty]$ with the products $\bullet_\gamma$ is a shuffle algebra.

Let $V = \bigoplus_{n\geqslant 1} V_n$ be a graded vector space. The graded space $K[\mathcal{F}_\infty](V)_n$ is the vector space spanned by the elements $f\otimes x_1\otimes\cdots\otimes x_k)\in\mathcal{F}_n^r\otimes X_{n_1}\otimes\cdots\otimes X_{n_r}$ such that the image of f is the subset $\{i_1<\cdots<i_k\}$ of $\{1,\dots,r\}$, with $n_j = |f^{-1}(i_j)|$ for $1\leqslant j\leqslant k$, and $\sum_{j=1}^k n_i = n$. On the graded space $K[\mathcal{F}_\infty](V) := \bigoplus_{n\geqslant 1} K[\mathcal{F}_\infty](V)_n$, we define a structure of shuffle algebra as follows:

$$f\otimes x_1\otimes\cdots\otimes x_r \bullet_\gamma g\otimes y_1\otimes\cdots\otimes y_k := (f\times g)\cdot\gamma\otimes x_1\otimes\cdots\otimes x_r\otimes y_1\otimes\cdots\otimes y_k,$$

for $f\otimes x_1\otimes\cdots\otimes x_r)\in\mathcal{F}_{n,X}$, $g\otimes y_1\otimes\cdots\otimes y_k)\in\mathcal{F}_{m,X}$ and $\gamma\in Sh(n,m)$.

Denote by $\mathbf{Sh}(V)$ the subspace of $K[\mathcal{F}_\infty](V)$ spanned by the elements $f\otimes x_1\otimes\cdots\otimes x_r\in\mathcal{F}_{n,X}$ with $f\in\mathcal{ST}_n$. The space $\mathbf{Sh}(V)$ is closed under the products $\bullet_\gamma$. So, $\mathbf{Sh}(V)$ is a shuffle subalgebra of $K[\mathcal{F}_\infty](V)$.

Proposition 2.3.1. — *For any positively graded vector space V, the algebra $\mathbf{Sh}(V)$ is the free shuffle algebra spanned by V.*

Proof. — From the definition of shuffle algebra and Proposition 1.6, one has that any element in the free shuffle algebra spanned by V is a sum of elements x of type

$$x = x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} (\dots (x_{k-1} \bullet_{\gamma_{k-1}} x_k))),$$

for unique elements $x_i \in V$ and unique shuffles γ_i, $1 \leqslant i \leqslant k$. Let ψ be the unique homomorphism from the free shuffle algebra spanned by V to the space $\mathbf{Sh}(V)$ verifying that:

$$\psi(x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} (\dots (x_{k-1} \bullet_{\gamma_{k-1}} x_k)))) := (\xi_{\mathbf{n}} \cdot \gamma; x_1, \dots, x_k),$$

where

(1) $n_i = |x_i|$, for $1 \leqslant i \leqslant k$,
(2) $\gamma = (1_{n_1+\dots+n_{k-2}} \times \gamma_{k-1}) \cdot \dots \cdot (1_{n_1} \times \gamma_2) \cdot \gamma_1$.

Conversely, let $f : \{1, \dots, n\} \to \{1, \dots, r\}$ be a surjective map and let $n_i := |f^{-1}(i)|$, for $1 \leqslant i \leqslant r$. There exists a unique permutation $\gamma \in Sh(\mathbf{n})$ such that $f = \xi_{\mathbf{n}} \cdot \gamma$. Moreover, there exist unique permutations $\gamma_i \in Sh(n_i, n_{i+1} + \dots + n_k)$ such that:

$$\gamma = (1_{n_1+\dots+n_{k-2}} \times \gamma_k) \cdot \dots \cdot (1_{n_1} \times \gamma_2) \cdot \gamma_1.$$

The inverse of ψ is $\psi^{-1}(f \otimes x_1 \otimes \dots \otimes x_k) = x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} (\dots (x_{k-1} \bullet_{\gamma_{k-1}} x_k)))$. ☐

c) Nonunital infinitesimal bialgebras. Suppose that $(A, \cdot)$ is a graded K-algebra, equipped with a coassociative coproduct $\Delta : A \to A \otimes A$ such that:

$$\Delta(x \cdot y) = \sum x \cdot y_{(1)} \otimes y_{(2)} + \sum x_{(1)} \otimes x_{(2)} \cdot y + x \otimes y, \text{ for } x, y \in A,$$

where $\Delta(z) = \sum z_{(1)} \otimes z_{(2)}$, for $z \in A$. The triple $(A, \cdot, \Delta)$ is called a *nonunital infinitesimal bialgebra* (see [14]).

It is easy to see that the reduced tensor space $\overline{T}(V)$, equipped with the concatenation product and the deconcatenation coproduct, is a graded unital infinitesimal bialgebra which is denoted $\overline{T}^c(V)$.

Let $A = \bigoplus_{n \geqslant 1} A_n$ be a positively graded nonunital infinitesimal bialgebra. The map $\Delta_{n_1,\dots,n_r} : A_n \longrightarrow A_{n_1} \otimes \dots \otimes A_{n_r}$ is given by the composition of Δ^{r-1} with the projection $p_{n_1 \dots n_r} : A^{\otimes n} \longrightarrow A_{n_1} \otimes \dots \otimes A_{n_r}$. For any $x \in A_n$, let $\Delta_{n_1,\dots,n_r}(x) = \sum x_{(1)}^{n_1} \otimes \dots \otimes x_{(r)}^{n_r}$.

The proof of the following result is given, in a more general context, in Theorem 6.7.

LEMMA 2.3.2. — *Let $(\bigoplus_{n\geqslant 1} A_n, \cdot, \Delta)$ be a graded nonunital infinitesimal bialgebra. The graded space A admits a natural structure of shuffle algebra, given by the operations:*

$$x \bullet_\gamma y = \sum_{|y_{(1)}|=i} x_{(1)}^{n_1} \cdot y_{(1)}^{m_1} \cdot x_{(2)}^{n_2} \cdots \cdot y_{(r)}^{m_r}, \text{ for } x \in A_n, y \in A_n, \text{ and } \gamma \in Sh(n,m),$$

where $n_1, \dots, n_r$ and $m_1, \dots, m_r$ are the integers which determine γ, described in Remark 1.8.

For instance, if we consider the unital infinitesimal bialgebra $\overline{T}^c(V)$, the shuffle algebra structure described in point **a)** coincides with the one defined in the previous Lemma.

d) The algebra of parking functions. (see [17] and [16]) Let PF_n be the subset of all functions f in $\mathcal{F}_n^n$ which may be written as a composition $f = f^\uparrow \cdot \sigma$, with $f_\uparrow \in \mathcal{F}_n^n$ such that $f^\uparrow(i) \leqslant i$ for all $1 \leqslant i \leqslant n$ and $\sigma \in S_n$. Such a function is called a *parking* function.

Applying Remark 1.9, we get that for any parking function $f \in PF_n$ there exist unique elements $f^\uparrow \in PF_n$ and $\sigma \in Sh(r_1, \dots, r_n)$ such that $f^\uparrow$ is a non-decreasing parking function and $f = f^\uparrow \cdot \sigma$, where $r_i = |f^{-1}(i)|$.

The concatenation map $\times : PF_n \times PF_m \longrightarrow PF_{n+m}$ is the restriction of the concatenation product $\mathcal{F}_n^n \times \mathcal{F}_m^m \longrightarrow \mathcal{F}_{n+m}^{n+m}$ to $PF_n \times PF_m$. Note that $f \times g$ is also a parking function. Moreover, for any functions $f \in PF_n$, $g \in PF_m$ and $\gamma \in Sh(n,m)$, the product $f \bullet_\gamma g = (f \times g) \cdot \gamma$ belongs to PF_{n+m}. For $n \geqslant 1$, let $\mathbf{PQSym}_n$ denote the K-vector space spanned by the set PF_n. The space spanned by all parking functions $\mathbf{PQSym} := \bigoplus_{n\geqslant 1} \mathbf{PQSym}_n$ is a shuffle subalgebra of $K[\mathcal{F}_\infty]$.

Following **2.2.3** of [16], given a parking function $f \in PF_n$, an integer $b \in \{0, 1, \dots, n\}$ is called a *breakpoint* of f if $|\{i \mid F(i) \leqslant b\}| = b$.

A *Gessel primitive parking function* is an element $f \in PF_n$ such that its unique breakpoints are the trivial ones: 0 and n. Let PPF_n be the subset of prime parking functions of PF_n. It is immediate to check that $f \in PF_n$ if its associated non-decreasing parking function cannot be written a a concatenation of parking functions of smaller degree.

Note that the definition of breakpoint implies that for any parking function $f \in P_n$ and any permutation $\sigma \in S_n$ the sets of breakpoints of f and of $f \cdot \sigma$ are the same. So, the subset PPF_n is invariant under the right action of S_n.

Remark 2.3.3. — (see **2.2.3** of [16]) A element in PPF_n is a parking function which cannot be described as $f \bullet_\gamma g$ for some $f \in PF_k$, $g \in PF_{n-k}$ and $\gamma \in Sh(k, n-k)$.

Remark 2.3.3 implies the following result.

PROPOSITION 2.3.4. — *The shuffle algebra* **PQSym** *is the free shuffle algebra spanned by the set* $PPF := \bigcup_{n\geqslant 1} PPF_n$ *of all prime parking functions.*

The group S_n acts on the right on the set PPF_n, for $n \geqslant 1$. So, $PPF = \{K[PPF_n]\}_{n\geqslant 1}$ is an object in the category $\mathcal{S}$–Mod. Applying Lemma 2.3.4 it is immediate to check that $\mathbf{PQSym} = \overline{T}_{\mathcal{S}}(PPF) = \bigoplus_{n\geqslant 1} PPF^{\otimes_{\mathcal{S}} n}$ in the category $\mathcal{S}$–Mod. The previous assertion means that **PQSym** is the free monoid spanned by PPF in the monoidal category $(\mathcal{S}\text{–Mod}, \otimes_{\mathcal{S}})$.

e) Singular chains of a Lie group. Let

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geqslant 0,\ 0 \leqslant i \leqslant n\}$$

be the standard n-simplex. The degeneracy morphisms $s^i : \Delta^n \longrightarrow \Delta^{n-1}$ are given by the formula

$$s^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-2}, t_{i-1} + t_i, t_{i+1}, \dots, t_n),$$

for $0 \leqslant i \leqslant n-1$. Given an (n,m)-shuffle σ, let $s_\sigma : \Delta^{n+m} \to \Delta^n \times \Delta^m$ be the continuous map:

$$s_\sigma := \big(s_{\sigma^{-1}(n+1)} \circ \cdots \circ s_{\sigma^{-1}(n+m)}\big) \times \big(s_{\sigma^{-1}(1)} \circ \cdots \circ s_{\sigma^{-1}(n)}\big).$$

For any permutation $\beta \in Sh(n,m,r)$ such that $\beta = (1_n \times \delta)\cdot\gamma = (\sigma \times 1_r)\cdot\tau$, with $\delta \in Sh(m,r)$, $\gamma \in Sh(n, m+r)$, $\sigma \in Sh(n,m)$ and $\tau \in Sh(n+m, r)$, it is easy to check that:

$$\begin{aligned}(Id_{\Delta^n} \times s_\delta) \circ s_\gamma &= (s_{i_1} \circ \cdots \circ s_{i_{m+r}}) \times (s_{j_1} \circ \cdots \circ s_{j_{n+r}}) \times (s_{k_1} \circ \cdots \circ s_{k_{n+m}})\\ &= (s_\sigma \times Id_{\Delta^r}) \circ s_\tau,\end{aligned}$$

where

- $\{\beta^{-1}(n+1), \dots, \beta^{-1}(n+m+r)\} = \{i_1 < \cdots < i_{m+r}\}$,
- $\{\beta^{-1}(1), \dots, \beta^{-1}(n), \beta^{-1}(n+m+1), \dots, \beta^{-1}(n+m+r)\} = \{j_1 < \cdots < j_{n+r}\}$,
- $\{\beta^{-1}(1), \dots, \beta^{-1}(n+m)\} = \{k_1 < \cdots < k_{n+m}\}$.

Suppose that $(X, \mu : X \times X \to X)$ is a Lie group, and let $\mathcal{C}(X, K)$ be the space of singular chains on X with coefficients in K. The operations s_σ

with the product μ define homomorphisms $\bullet_\sigma : C_n(X,K)\otimes C_m(X,K)\to C_{n+m}(X,K)$ as follows:

$$f\bullet_\sigma g := \mu\circ(f\times g)\circ s_\sigma,$$

for $f:\Delta^n\to X$ and $g:\Delta^m\to X$ continuous maps.

f) Tensor product of shuffle algebras. Given graded spaces $V=\bigoplus_{n\geqslant 0}V_n$ and $W=\bigoplus_{m\geqslant 0}W_m$ there exist two different products of both spaces:

(1) The *Hadamard product* of V and W, denoted by $V\underset{H}{\otimes}W$, is the graded vector space such that $(V\underset{H}{\otimes}W)_n := V_n\otimes W_n$, for $n\geqslant 0$.

(2) The *tensor product* of V and W, denoted by $V\otimes W$, is the graded vector space such that $(V\otimes W)_n := \bigoplus_{i=0}^n V_i\otimes W_{n-i}$, for $n\geqslant 0$.

We define, for any pair A and B of shuffle algebras, shuffle algebra structures on $A\underset{H}{\otimes}B$ and $A\otimes B$. The proof of the following result is straightforward.

Lemma 2.3.5. — *Let $(A,\bullet_\gamma)$ and $(B,\circ_\delta)$ be two shuffle algebras.*

(1) *The Hadamard product $A\underset{H}{\otimes}B$ has a natural structure of shuffle algebra, given by the operations:*

$$(x\otimes y)\bullet_\gamma(x'\otimes y') := (x\bullet_\gamma x')\otimes(y\circ_\gamma y'),$$

for $x\in A_n, y\in B_n, x'\in A_m, y'\in B_m$ and $\gamma\in Sh(n,m)$.

(2) *The tensor product $A\otimes B$ has a natural structure of shuffle algebra, given by the operations:*

$$(x\otimes y)\bullet_\gamma(x'\otimes y') := \begin{cases}(x\bullet_{\gamma_{(1)}^{n+n'}}x')\otimes(y\circ_{\gamma_{(2)}^{m+m'}}y'), & \text{for}\ \ n=(n+n')_1,\\ 0, & \text{otherwise},\end{cases}$$

where $x\in A_n$, $x'\in A_{n'}$, $y\in A_m$, $y'\in A_{m'}$, $\gamma_{(1)}^{n+n'}\in Sh(n,n')$ and $\gamma_{(2)}^{m+m'}\in Sh(m,m')$ are the permutations defined in Lemma 1.7, and $(n+n')_1 := |\gamma^{-1}(\{1,\dots n+m\}\cap\{1,\dots,n+n'\}|$.

3. Shuffle bialgebras.

Given a shuffle algebra A, the product $*$ on A is given by:

$$x*y := \sum_{\gamma\in Sh(n,m)} x\bullet_\gamma y,$$

for $x\in A_n$ and $y\in A_m$.

It is easy to verify that $*$ is graded and associative applying Proposition 1.6 and the properties of the operations $\bullet_\gamma$. For instance, in 2.3 **a)**, the product $*$ on $\overline{T}(V)$ is the usual shuffle product defined on the tensor space over V.

Given a shuffle algebra $(A, \bullet_\gamma)$, we want to describe coproducts on shuffle algebras which turn the associative algebra $(A, *)$ into a bialgebra. In particular, all the examples of shuffle algebras given in 2.3 may be equipped in a natural way with a shuffle bialgebra structure.

Definition 3.1. — Let $(A, \bullet_\gamma)$ be a positively graded shuffle algebra, such that A is equipped with a graded coassociative coproduct Δ. We say that $(A, \bullet_\gamma, \Delta)$ is a *shuffle bialgebra* if Δ verifies the following relations:

$$\Delta(x \bullet_\gamma y) = \sum_{r=1}^{n+m-1} \big(\sum (x_{(1)} \bullet_{\gamma^r_{(1)}} y_{(1)}) \otimes (x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} y_{(2)})\big),$$

where $\gamma^r_{(1)}$ and $\gamma^{n+m-r}_{(2)}$ are the permutations defined in Lemma 1.7, the second sum is taken over all $|x_{(1)}| = n_1$ and $|y_{(1)}| = m_1$, and

$$x_{(1)} \bullet_{\gamma^r_{(1)}} y_{(1)} := \begin{cases} x, & \text{for } n_1 = n \\ y, & \text{for } n_1 = 0, \end{cases}$$

$$x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} y_{(2)} := \begin{cases} x, & \text{for } n_1 = 0 \\ y, & \text{for } n_1 = n, \end{cases}$$

Proposition 3.2. — *Let $(A, \bullet_\gamma, \Delta_A)$ and $(B, \circ_\delta, \Delta_B)$ be shuffle bialgebras. The Hadamard product $A \underset{H}{\otimes} B$ with the operations $\bullet_\gamma$ given in Lemma 2.3.5 and the coproduct given by:*

$$\Delta_{A \underset{H}{\otimes} B}(x \otimes y) = \sum_{|x_{(1)}| = |y_{(1)}|} (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}),$$

is a shuffle bialgebra.

Proof. — Let $x \in A_n$, $y \in B_n$, $z \in A_m$, $w \in B_m$ and $\gamma \in Sh(n, m)$. We have that

$$\Delta((x \otimes y) \bullet_\gamma (z \otimes w)) =$$
$$\sum_{r,s} \big(\sum (x_{(1)} \bullet_{\gamma^r_{(1)}} z_{(1)} \otimes y_{(1)} \bullet_{\gamma^s_{(1)}} w_{(1)}) \otimes (x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} z_{(2)} \otimes y_{(2)} \bullet_{\gamma^{n+m-s}_{(2)}} w_{(2)})\big),$$

where the second sum is taken over all elements such that $|x_{(1)} \bullet_{\gamma^r_{(1)}} z_{(1)}| = |y_{(1)} \bullet_{\gamma^s_{(1)}} w_{(1)}|$.

For $1 \leqslant r, s \leqslant n+m$, we have that $|x_{(1)} \bullet_{\gamma^r_{(1)}} z_{(2)}| = r$ and $|y_{(1)} \bullet_{\gamma^s_{(1)}} w_{(2)}| = s$. So, $|x_{(1)} \bullet_{\gamma^r_{(1)}} z_{(2)}| = |y_{(1)} \bullet_{\gamma^s_{(1)}} w_{(2)}|$ if, and only if $r = s$, where

$\gamma^r_{(1)} \in Sh(n_1, r-n_1)$. In this case, we get that $|x_{(1)}| = n_1 = |y_{(1)}|$ and $|z_{(2)}| = r - n_1 = |w_{(2)}|$, which implies the result. $\square$

Examples 3.3. — **a) The free shuffle algebra.** Let (C, Θ) be a positively graded coalgebra. Since $\mathbf{Sh}(C)$ is the free shuffle algebra spanned by the graded vector space C and $\mathbf{Sh}(C) \otimes \mathbf{Sh}(C)$ is a shuffle algebra, there exists a unique coproduct Δ_θ on $\mathbf{Sh}(C)$ which extends Θ.

For $f = \xi_{\mathbf{n}} \cdot \sigma$, with $\mathbf{n} = (n_1, \dots, n_r)$, $\sigma \in Sh(\mathbf{n})$ and elements $x_1, \dots, x_r \in V$, with $x_i \in V_{n_i}$, it is not difficult to check that Δ_θ on $\mathbf{Sh}(C)$ is given by the following formula:

$$\Delta_\theta(f \otimes x_1 \otimes \cdots \otimes x_r) :=$$
$$\sum_{i=0}^{n} \Big(\sum_{|x_{j(1)}| = m^i_j} (\xi_{\mathbf{m}^i} \cdot \sigma^i_{(1)} \otimes x_{1(1)} \otimes \cdots \otimes x_{r(1)}) \otimes (\xi_{\mathbf{n}-\mathbf{m}^i} \cdot \sigma^{n-i}_{(2)} \otimes x_{1(2)} \otimes \cdots \otimes x_{r(2)}) \Big),$$

where

(1) $\sigma = \delta_i \cdot (\sigma^i_{(1)} \otimes \sigma^{n-i}_{(2)})$, with $\delta_i \in Sh(i, n-i)$,
(2) for each $1 \leqslant i \leqslant n-1$,
$$m^i_j := |\delta_i^{-1}\{1, \dots, i\} \cap \{n_1 + \cdots + n_{j-1} + 1, \dots, n_1 + \cdots + n_j\}|,$$
(3) $\mathbf{m}^i := (m^i_1, \dots, m^i_r)$ and $\mathbf{n} - \mathbf{m}^i := (n_1 - m^i_1, \dots, n_r - m^i_r)$,
(4) $\Theta(x_j) = \sum x_{j(1)} \otimes x_{j(2)}$, for $1 \leqslant j \leqslant r$.

For example, suppose that $C = \bigoplus_{n \geqslant 1} K\xi_n$ is the vector space spanned by one element in each degree, equipped with the coproduct $\Theta(\xi_n) = \sum_{i=0}^n \xi_i \otimes \xi_{n-i}$. For $f = (2,3,3,5,4,1,4,3)$, we have that:

$$\begin{aligned}\Delta_\theta(f) = (1) \otimes (2,2,5,3,1,3,2) + (1,2) \otimes (2,4,3,1,3,2) + \\ (1,2,2) \otimes (4,3,1,3,2) + (1,2,2,3) \otimes (3,1,3,2) + (1,2,2,4,3) \otimes (1,3,2) + \\ (2,3,3,5,4,1) \otimes (2,1) + (2,3,3,5,4,1,4) \otimes (1).\end{aligned}$$

For instance, if $C_0 = Kx_0$, with $|x_0| = 1$, the underlying vector space of $\mathbf{Sh}(C_0)$ is $K[S_\infty]$, and Δ is the unique coproduct such that:

$$\Delta(\sigma) := \sum_{r=1}^{n-1} \sigma^r_{(1)} \otimes \sigma^{n-r}_{(2)},$$

for $\sigma \in S_n$, where $\sigma = \delta_r \cdot (\sigma^r_{(1)} \times \sigma^{n-r}_{(2)})$, with $\delta_r^{-1} \in Sh(r, n-r)$, for $1 \leqslant r \leqslant n-1$.

b) Nonunital infinitesimal bialgebras. Let $(A, \cdot, \Delta)$ be a graded nonunital infinitesimal bialgebra.

Lemma 3.3.1. — *The associated shuffle algebra $(A, \bullet_\gamma)$, equipped with the coproduct Δ, is a shuffle bialgebra.*

Proof. — Let $\gamma \in Sh(n,m)$ be the permutation given by the sequences $n_1, \dots, n_s$ and $m_1, \dots, m_s$, as described in Remark 1.8. For any $1 \leqslant r \leqslant n+m-1$, there exists $1 \leqslant k \leqslant s$ such that

$$\sum_{i=1}^{k-1}(n_i + m_i) + n'_k + m'_k = r,$$

where either $0 < n'_k < n_k$ and $m'_k = 0$, or $n'_k = n_k$ and $0 \leqslant m'_k \leqslant m_k$.

If $\gamma = (1_{n_1} \times \epsilon_{n-n_1,m_1} \times 1_{m-m_1}) \cdot (\gamma^r_{(1)} \times \gamma^{n+m-r}_{(2)})$, then $\gamma^r_{(1)}$ is given by the sequences $n_1, \dots, n_{k-1}, n'_k$ and $m_1, \dots, m_{k-1}, m'_k$, and $\gamma^{n+m-r}_{(2)}$ is given by the sequences $n_k - n'_k, n_{k+1}, \dots, n_s$ and $m_k - m'_k, m_{k+1}, \dots, m_s$.

Given elements $x \in A_n$, $y \in A_m$, the coassociativity of Δ and the relation between $\cdot$ and Δ state that:

$$\begin{aligned}
&\Delta(x \bullet_\gamma y) = \\
&\sum\Big(\sum_{1\leqslant k\leqslant r}\Big(\sum_{1\leqslant j_k\leqslant n_k}(x^{n_1}_{(1)} \cdot y^{m_1}_{(1)} \cdot \dots \cdot x^{j_k}_{(k)}) \otimes (x^{n_k-j_k}_{(k+1)} \cdot y^{n_k}_{(k)} \cdot \dots \cdot y^{m_r}_{(r)})\Big) + \\
&\qquad \Big(\sum_{1\leqslant l_k\leqslant m_k}(x^{n_1}_{(1)} \cdot \dots \cdot x^{n_k}_{(k)} \cdot y^{l_k}_{k}) \otimes (y^{m_k-l_k}_{(k+1)} \cdot x^{n_{k+1}}_{(k+1)} \cdot \dots \cdot y^{m_r}_{(r)})\Big)\Big) = \\
&\qquad\qquad \sum_{r=0}^{n+m}(x_{(1)} \bullet_{\gamma^r_{(1)}} y_{(1)}) \otimes (x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} y_{(2)}),
\end{aligned}$$

which implies the result. □

c) Monoids in $(\mathcal{S}\text{–}Mod, \otimes_{\mathcal{S}})$. For an $\mathcal{S}$-module M, a coproduct on M is a family of homomorphisms of $K[S_n]$-modules $\Omega_n : M(n) \longrightarrow \bigoplus_{i=0}^{n} M(i) \otimes M(n-i) \otimes K[Sh(i, n-i)]$, for each $n \geqslant 0$. For $x \in M(n)$, we have that

$$\Omega(x) = \sum_{i=0}^{n}\Big(\sum_{\sigma\in Sh(i,n-i)} x^\sigma_{(1)} \otimes x^\sigma_{(2)} \otimes \sigma\Big).$$

Given permutations $\sigma \in Sh(n, m+r)$, $\tau \in Sh(m,r)$, $\delta \in Sh(n+m, r)$ and $\omega \in Sh(n,m)$, such that $(1_n \times \tau) \cdot \sigma = (\omega \times 1_r) \cdot \delta$, the coassociativity of Ω implies that:

$$\sum x^\sigma_{(1)} \otimes (x^\sigma_{(2)})^\tau_{(1)} \otimes (x^\sigma_{(2)})^\tau_{(2)} = \sum (x^\delta_{(1)})^\omega_{(1)} \otimes (x^\delta_{(1)})^\omega_{(2)} \otimes x^\delta_{(2)}.$$

A monoid $(M,\circ)$ in the category $(\mathcal{S}\text{–}Mod,\otimes_{\mathcal{S}})$ is a bialgebra if it is equipped with a coassociative coproduct verifying the condition:

$$\Omega(\circ(x\otimes y\otimes\gamma))=\sum(x^{\delta}_{(1)}\otimes y^{\tau}_{(1)}\otimes\alpha_1)\otimes(x^{\delta}_{(2)}\otimes y^{\tau}_{(2)}\otimes\alpha_2)\otimes\rho,\qquad(*)$$

for $x\in M(n)$, $y\in M(m)$ and $\gamma\in Sh(n,m)$, where

$$(1_{n_1}\times\epsilon_{m_1,n_2}\times 1_{m_2})\cdot(\delta\times\tau)\cdot\gamma=(\alpha_1\times\alpha_2)\cdot\rho \text{ in } Sh(n_1,m_1,n_2,m_2),$$

with $\alpha_1\in Sh(n_1,m_1)$, $\alpha_2\in Sh(n_2,m_2)$ and $\rho\in Sh(r,n+m-r)$, where $r=n_1+m_1$.

We have seen that an algebra M in $(\mathcal{S}\text{–}Mod,\otimes_{\mathcal{S}})$ is a shuffle algebra. However, even if $(M,\circ,\Omega)$ is a bialgebra in $(\mathcal{S}-Mod,\otimes_{\mathcal{S}})$, it is not always a shuffle bialgebra. But it is possible to obtain two shuffle bialgebras from it, as we describe above.

Proposition 3.3.2. — *Let $(M,\circ,\Omega)$ be a bialgebra in $(\mathcal{S}\text{–}Mod,\otimes_{\mathcal{S}})$.*

(1) *Let Ω_0 be the coproduct on M defined as follows :*

$$\Omega_0(x):=\sum_{i=0}^{n}x^{1_n}_{(1)}\otimes x^{1_n}_{(2)},$$

where 1_n is considered as a $(i,n-i)$-shuffle for $0\leqslant i\leqslant n$. The shuffle algebra $(M=\bigoplus_{n\geqslant 0}M(n),\bullet_\gamma)$, equipped with this coproduct , is a shuffle bialgebra.

(2) *Let Ω_{top} be the coproduct on M given by:*

$$\Omega_{top}(x):=\sum_{i=0}^{n}x^{\epsilon_{i,n-i}}_{(2)}\otimes x^{\epsilon_{i,n-i}}_{(1)},$$

where $\epsilon_{i,n-i}$ is considered as a $(i,n-i)$-shuffle, for $0\leqslant i\leqslant n$. The data $(M=\bigoplus_{n\geqslant 0}M(n),\bullet_\gamma,\Omega_{top})$ is a shuffle bialgebra.

Proof. — For elements $x\in M(n)$ and $y\in M(m)$, a shuffle $\gamma\in Sh(n,m)$ and an integer $0\leqslant r\leqslant n+m$, we have that:

$$\gamma=(1_{n_1}\times\epsilon_{n-n_1,m_1}\times 1_{m-m_1})\cdot(\gamma^{r}_{(1)}\times\gamma^{n+m-r}_{(2)}),$$

for $n_1=|\gamma^{-1}(\{1,\dots,r\})\cap\{1,\dots,n\}|$ and $m_1=r-n_1$.

If $\delta=1_n$ and $\tau=1_m$, then:

$$\begin{aligned}&(1_{n_1}\times\epsilon_{m_1,n-n_1}\times 1_{m-m_1})\cdot\gamma=\\&(1_{n_1}\times\epsilon_{m_1,n-n_1}\times 1_{m-m_1})\cdot(1_{n_1}\times\epsilon_{n-n_1,m_1}\times 1_{m-m_1})\cdot(\gamma^{r}_{(1)}\times\gamma^{n+m-r}_{(2)})=\\&\qquad(\gamma^{r}_{(1)}\times\gamma^{n+m-r}_{(2)})\cdot 1_{n+m},\end{aligned}$$

which implies the first statement.

To prove the second one, suppose that $\delta = \epsilon_{n_1(n-n_1)}$ and $\tau = \epsilon_{m_1(m-m_1)}$. We have that $\epsilon_{r(n+m-r)} =$

$$(1_{n_1}\times\epsilon_{m_1(n-n_1)}\times 1_{m-m_1})\cdot(\epsilon_{n_1(n-n_1)}\times\epsilon_{m_1(m-m_1)})\cdot(1_{n-n_1}\times\epsilon_{n_1(m-m_1)}\times 1_{m_1}).$$

Moreover, if $\gamma = (1_{n-n_1} \times \epsilon_{n_1,m-m_1} \times 1_{m_1}) \cdot (\gamma_{(1)}^{n+m-r} \times \gamma_{(2)}^{r})$, then

$$\epsilon_{r,n+m-r} \cdot (\gamma_{(1)}^{n+m-r} \times \gamma_{(2)}^{r}) = (\gamma_{(2)}^{r} \times \gamma_{(1)}^{n+m-r}) \cdot \epsilon_{n+m-r,r}.$$

So, we get that $(x \bullet_\gamma y)_{(1)}^{\epsilon_{r,n+m-r}} = x_{(1)}^{\epsilon_{n_1,n-n_1}} \bullet_{\gamma_{(2)}^{r}} y_{(1)}^{\epsilon_{m_1,m-m_1}}$ and $(x \bullet_\gamma y)_{(2)}^{\epsilon_{r,n+m-r}} = x_{(2)}^{\epsilon_{n_1,n-n_1}} \bullet_{\gamma_{(1)}^{n+m-r}} y_{(2)}^{\epsilon_{m_1,m-m_1}}$.

We conclude that:

$$\Omega_{top}(x \bullet_\gamma y) = \sum_{r=0}^{n+m} (x \bullet_\gamma y)_{(2)}^{\epsilon_{r,n+m-r}} \otimes (x \bullet_\gamma y)_{(1)}^{\epsilon_{r,n+m-r}} =$$
$$\sum_{r=0}^{n+m} x_{(2)}^{\epsilon_{n_1,n-n_1}} \bullet_{\gamma_{(1)}^{n+m-r}} y_{(2)}^{\epsilon_{m_1,m-m_1}} \otimes x_{(1)}^{\epsilon_{n_1,n-n_1}} \bullet_{\gamma_{(2)}^{r}} y_{(1)}^{\epsilon_{m_1,m-m_1}},$$

which ends the proof. □

d) The bialgebra of parking functions. (see [16]) Given a function $f \in \mathcal{F}_n^k$, recall that there exist a unique non-decreasing function $f^\uparrow \in \mathcal{F}_n^k$, and a unique permutation $\sigma \in Sh(\mathbf{n})$ such that $f = f^\uparrow \cdot \sigma$, where $\mathbf{n} = (n_1, \dots, n_k)$ for $n_i = |(f^\uparrow)^{-1}(i)|$.

Following [16], there exists a graded map $Park : \bigcup_{n\geqslant 1} \mathcal{F}_n \longrightarrow \bigcup_{n\geqslant 1} PF_n$ defined as follows.

Let $f^\uparrow \in \mathcal{F}_n$ be a non-decreasing function, the parking function $Park(f^\uparrow)$ is given by:

$$Park(f^\uparrow)(j) := \begin{cases} 1, & \text{for } j = 1, \\ \mathrm{Min}\{Park(f^\uparrow)(j-1)) + f^\uparrow(j) - f^\uparrow(j-1),\ j\}, & \text{for } j > 1. \end{cases}$$

If $f = f^\uparrow \cdot \sigma$, then *Park(f)* $:= Park(f^\uparrow) \cdot \sigma$.

Remark 3.3.3. — (1) If $f \in \mathcal{F}_n^k$ be a function, then $Park(f)$ is the unique parking function such that $f(i) < f(j)$ (respectively, $f(i) = f(j)$) if, and only if, $Park(f)(i) < Park(f)(j)$ (respectively, $Park(f)(i) = Park(f)(j)$) for $1 \leqslant i, j \leqslant n$.

(2) For any pair of parking functions f, g, $Park(f \times g) = Park(f) \times Park(g)$.

(3) If $f \in PF_n$ is a parking function and $\gamma \in S_n$ is a permutation, then

$$Park(f \cdot \gamma) = Park(f) \cdot \gamma.$$

Let $\mathbf{PQSym} = \bigoplus_{n \geqslant 1} PF_n$ be the graded space of all parking functions. The coproduct on $\mathbf{PQSym}$ is defined as follows (see [16]):

For $f \in PF_n$ and $0 \leqslant r \leqslant n$,

$$\Delta_{PQSym}(f) := \sum_{r=1}^{n-1} Park(f_1^r) \otimes Park(f_2^{n-r}),$$

for $f_1^r := (f(1), \dots, f(r))$ and $f_2^{n-r} := (f(r+1), \dots, f(n))$.

Proposition 3.3.4. — *The shuffle algebra* $(\mathbf{PQSym}, \bullet_\gamma)$*, equipped with the coproduct* Δ_{PQSym}*, is a shuffle bialgebra.*

Proof. — Let $f \in PF_n$, $g \in PF_m$ be parking functions, and let γ be a *(n,m)*-shuffle. For $1 \leqslant r \leqslant n-1$, we want to check that:

$$Park((f \bullet_\gamma g)_1^r) \otimes Park((f \bullet_\gamma g)_2^{n+m-r}) = \\ (Park(f_1^{n_1}) \bullet_{\gamma_{(1)}^r} Park(g_1^{m_1})) \otimes (Park(f_2^{n-n_1}) \bullet_{\gamma_{(2)}^{n+m-r}} Park(g_2^{m-m_1})),$$

where $\gamma = (1_{n_1} \times \epsilon_{m_1, n-n_1} \times 1_{m-m_1}) \cdot (\gamma_{(1)}^r \times \gamma_{(2)}^{n+m-r})$.

Computing $(f \times g) \cdot (1_{n_1} \times \epsilon_{m_1, n-n_1} \times 1_{m-m_1})$, we get that:

$$(f \bullet_\gamma g)_1^r = (f_1^{n_1} \times g_1^{m_1}[n-n_1]) \cdot \gamma_{(1)}^r$$

and

$$(f \bullet_\gamma g)_2^{n-r} = (f_2^{n-n_1} \times g_2^{m-m_1}[n_1]) \cdot \gamma_{(2)}^{n+m-r},$$

where $f_1^{n_1} \times g_1^{m_1}[n-n_1] = (f(1), \dots, f(n_1), g(1)+n, \dots, g(m_1)+n)$.

By Remark 3.3.3, we get that

$$\begin{aligned} Park((f \bullet_\gamma g)_1^r) &= Park(f_1^{n_1}) \bullet_{\gamma_{(1)}^r} Park(g_1^{m_1})), \\ Park((f \bullet_\gamma g)_2^{n+m-r}) &= Park(f_2^{n-n_1}) \bullet_{\gamma_{(2)}^{n+m-r}} Park(g_2^{m-m_1})), \end{aligned}$$

which ends the proof. □

4. Relations with dendriform and 2-associative algebras

We want to relate shuffle bialgebras to other algebraic structures. In [1], M. Aguiar constructs functors relating infinitesimal bialgebras to dendriform algebras (see [11]) and brace algebras. We want to include shuffle bialgebras in his framework.

The following results are immediate to verify applying the definition of shuffle bialgebra.

Lemma 4.1. — *Given a shuffle algebra $(A, \bullet_\gamma)$ there exist three associative algebra structures on A, defined as follows:*

(1) *the products $x \bullet_0 y$ and $x \bullet_{top} y$ defined by:*

$$x \bullet_0 y := x \bullet_{1_{n+m}} y,$$
$$x \bullet_{top} y := y \bullet_{\epsilon_{m,n}} x,$$

(2) *the product $x * y := \sum_{\gamma \in Sh(n,m)} x \bullet_\gamma y$,*

where $x \in A_n$ and $y \in A_m$.

Lemma 4.2. — *Let $(A, \bullet_i, \Delta)$ be a shuffle bialgebra.*

(1) *The relationship between Δ and the associative products $\bullet_0$ and $\bullet_{top}$ is given by the following equalities:*

$$\Delta(x \bullet_0 y) = \sum (x \bullet_0 y_{(1)}) \otimes y_{(2)} + \sum x_{(1)} \otimes (x_{(2)} \bullet_0 y) + x \otimes y,$$
$$\Delta(x \bullet_{top} y) = \sum (x \bullet_{top} y_{(1)}) \otimes y_{(2)} + \sum x_{(1)} \otimes (x_{(2)} \bullet_{top} y) + x \otimes y,$$

for $x, y \in A$. So, $(A, \bullet_0, \Delta)$ and $(A, \bullet_{top}, \Delta)$ are nonunital infinitesimal bialgebras.

(2) *The product $*$ may be extended to $A_+ = A \oplus K$in a unique way such that $1_K * x = x = x * 1_K$ for $x \in A_+$. The triple $(A_+, *, \Delta_+)$ is a bialgebra, which means that:*

$$\Delta_+(x * y) = \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} * y_{(2)}), \text{ for } x, y \in A_+.$$

The previous result implies that there exists two functors, H_0 and H_{top}, from the category of shuffle bialgebras to the category of graded nonunital infinitesimal bialgebras. Let $G : Gr\epsilon \longrightarrow Sh$ be the functor which assigns to any graded nonunital infinitesimal bialgebra $(A, \cdot, \Delta)$ the shuffle bialgebra $(A, \bullet_\gamma, \Delta)$. It is easy to check that the compositions $H_0 \circ G$ and $H_{top} \circ G$ are equal to the identity functor.

Definition 4.3. — (see [14]) Let B be a K-vector space equipped with two associative products $*$ and $\cdot$, and a coassociative coproduct Δ, such that:

(1) $(B, *, \Delta)$ is a bialgebra over K,
(2) $(B, \cdot, \Delta)$ is an infinitesimal unital bialgebra.

Then $(B, *, \cdot, \Delta)$ is called a 2-*associative bialgebra.*

Lemma 3.3.1 implies that if $(A, \bullet_\gamma, \Delta)$ is a shuffle bialgebra, then $(A_+ = K \oplus A, *, \bullet_0, \Delta)$ is a 2-associative bialgebra.

Definition 4.4. — A *dendriform algebra* (see [11]) over K is a vector space D equipped with two bilinear maps $\succ, \prec : D \otimes D \longrightarrow D$ which verify the following relations:

(1) $x \succ (y \succ z) = (x \succ y + x \prec y) \succ z$,
(2) $x \succ (y \prec z) = (x \succ y) \prec z$,
(3) $x \prec (y \succ z + y \prec z) = (x \prec y) \prec z$,

for $x, y, z \in D$.

Note that any dendriform algebra D has a natural structure of associative algebra with the product $*$, defined by: $x * y = x \succ y + x \prec y$.

For nonnegative integers n, m, let $Sh^{\succ}(n,m)$ and $Sh^{\prec}(n,m)$ be the following subsets of $Sh(n,m)$:
a) $Sh^{\succ}(n,m) := \{\sigma \in Sh(n,m) \mid \sigma(n+m) = n+m\}$,
b) $Sh^{\prec}(n,m) := \{\sigma \in Sh(n,m) \mid \sigma(n+m) = n\}$.

It is immediate to check that $Sh(n,m)$ is the disjoint union of $Sh^{\succ}(n,m)$ and $Sh^{\prec}(n,m)$.

Let $(A, \bullet_\gamma)$ be a shuffle algebra. Define on A the operations $\succ$ and $\prec$ as follows:

(1) $x \succ y := \displaystyle\sum_{\gamma \in Sh^{\succ}(n,m)} x \bullet_\gamma y$,
(2) $x \prec y := \displaystyle\sum_{\gamma \in Sh^{\prec}(n,m)} x \bullet_\gamma y$,

for $x \in A_n$ and $y \in A_m$. Note that the associative product $*$ defined at the beginning of Section 3 is the sum of $\succ$ and $\prec$. The proof of the following Lemma may be obtained by straightforward calculation.

Lemma 4.5. — *Let $(A, \bullet_\gamma)$ be a shuffle algebra, then $(A, \succ, \prec)$ is a dendriform bialgebra. Moreover, if $(A, \bullet_\gamma, \Delta)$ is a shuffle bialgebra, then $\succ$, $\prec$ and Δ verify the following equalities:*

$$\Delta(x \succ y) = \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \succ y_{(2)}) + \sum x_{(1)} \otimes (x_{(2)} \succ y) + \sum y_{(1)} \otimes (x \succ y_{(2)}) + \sum (x * y_{(1)}) \otimes y_{(2)} + x \otimes y,$$

$$\Delta(x \prec y) = \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \prec y_{(2)}) + \sum y_{(1)} \otimes (x \prec y_{(2)}) + \sum (x * y_{(1)}) \otimes y_{(2)} + \sum x_{(1)} \otimes (x_{(2)} \prec y) + y \otimes x,$$

for all $x, y \in A$.

The Lemma above states that any shuffle bialgebra has a natural structure of dendriform bialgebra.

5. Primitive elements of shuffle bialgebras

We recall some results proved in [14] that we need in order to study primitive elements in shuffle algebras.

Following [14], let $(H,\cdot,\Delta)$ be a triple such that (H,Δ) is a conilpotent coassociative coalgebra and $(H,\cdot)$ is an associative algebra. Define the linear map $e\in End_K(H)$ as follows:

$$e(x):=x-\sum x_{(1)}\cdot x_{(2)}+\cdots+(-1)^{r+1}\sum x_{(1)}\cdot x_{(2)}\cdot\cdots\cdot x_{(r)}+\cdots=\sum_{r\geqslant 1}(-1)^{r+1}\cdot^{r}\circ\Delta^{r}(x),$$

where $\Delta^r(x)=\sum x_{(1)}\otimes x_{(2)}\otimes\cdots\otimes x_{(r)}$.

Proposition 5.1. — *(see Proposition 2.5 of* [14]*) Any conilpotent non-unital infinitesimal bialgebra $(H,\cdot,\Delta)$ verifies that:*

1. *the image of e belongs to $Prim(H)$,*
2. *the restriction $e\mid_{Prim(H)}=Id_{Prim(H)}$, and*
3. *$e(x\cdot y)=0$ for all $x,y\in Ker(\epsilon)$.*
4. *any element x of $Ker(\epsilon)$ verifies that*

$$x=e(x)+\sum e(x_{(1)})\cdot e(x_{(2)})+\cdots+\sum e(x_{(1)})\cdot\cdots\cdot e(x_{(n)})+\ldots,$$

where $\Delta^n(x)=\sum x_{(1)}\otimes\cdots\otimes x_{(n)}$.

Theorem 5.2. — *(see Theorem 2.6 of* [14] *) Any conilpotent infinitesimal bialgebra H is isomorphic to*

$$(\overline{T}^c(Prim(H)):=(\bigoplus_{n\geqslant 1}Prim(H)^{\otimes n},\nu,\Delta),$$

where ν is the concatenation product and Δ is the deconcatenation coproduct.

We want to prove a Cartier-Milnor-Moore theorem in the context of shuffle bialgebras. In order to do it, we introduce the notion of $\mathcal{P}rim_{sh}$ algebra.

Definition 5.3. — A *$\mathcal{P}rim_{sh}$ algebra* is a graded vector space V equipped with operations $B^\gamma : V_n\otimes V_m\longrightarrow V$, for $\gamma\in Sh(n,m)\setminus\{1_{n+m},\epsilon_{nm}\}$, and a binary operation $\{-,-\}$ which satisfy the following relations:

1. $\{x,\{y,z\}\}=\{\{x,y\},z\}+B^{1_n\times\epsilon_{rm}}(\{x,z\};y)-B^{\epsilon_{mn}\times 1_r}(y;\{x,z\})$;
2. $\{x;B^\gamma(y;z)\}=B^{\overline{\gamma}}(y;\{x,z\})+B^{\tilde{\gamma}}(\{x,y\};z)$, where $\overline{\gamma}:=(\epsilon_{mn}\times 1_r)\cdot(1_n\times\gamma)$ and $\tilde{\gamma}=1_n\times\gamma$;

(3) $\{B^{\gamma}(x;y),z\} = B^{\underline{\gamma}}(\{x;z\};y) + B^{\gamma\times 1_r}(x;\{y,z\})$, where $\underline{\gamma} := (1_m \times \epsilon_{rn})\cdot(\gamma\times 1_r)$;
(4) $B^{\gamma}(B^{\delta}(x;y);z) = B^{\tau}(x;B^{\sigma}(y;z))$, whenever $(\delta\times 1_r)\cdot\gamma = (1_n\times\sigma)\cdot\tau$, with $\sigma\neq 1_{m+r}$;

for $x\in V_n$, $y\in V_m$ and $z\in V_r$.

Note that relation (4) of Definition 5.3 makes no sense when the permutations δ and γ are such that $(\delta\times 1_r)\cdot\gamma = \tau\in Sh(n,m+r)$. In this case, the permutation τ verifies that $\tau(1)<\tau(n+m)$ and $\tau(n+m)+1<\tau(n)$. Conversely, if a $(n,m+r)$-shuffle τ verifies these conditions, then there exist unique permutations $\delta\in Sh(n,m)\setminus\{1_{n+m},\epsilon_{nm}\}$ and $\gamma\in Sh(n+m,r)\setminus\{1_{n+m+r},\epsilon_{(n+m)r}\}$ such that $\tau=(\delta\times 1_r)\cdot\gamma$. The following Lemma generalizes this situation, its proof is immediate applying Proposition 1.6.

LEMMA 5.4. — *Let $(m_1,\dots,m_q)$ be a composition of m. If a permutation $\tau\in Sh(n,m)$ is such that $\tau(1)<\tau(n+m_1)$ and $\tau(n+m_1+\dots+m_{q-1})+1<\tau(n)$, then there exist unique families of permutations $\sigma_i\in Sh(n+m_1+\dots+m_{i-1},m_i)$, for $1\leqslant i\leqslant q$, such that:*

(1) $\sigma_i\neq 1_{n+m_1+\dots m_{i-1}}$ *and* $\sigma_i\neq\epsilon_{(n+m_1+\dots+m_{i-1})m_i}$,
(2) $\tau=(\sigma_1\times 1_{m-m_1})\cdot(\sigma_2\times 1_{m-m_1-m_2})\cdot\dots\cdot\sigma_q$,
(3) *for* $2\leqslant i\leqslant q-1$, *let* $\tau_i := (\sigma_1\times 1_{m_2+\dots+m_i})\cdot(\sigma_2\times 1_{m_3+\dots+m_i})\cdots\cdot\sigma_i$. *The element* $\tau_i\in Sh(n,m_1+\dots+m_i)$ *verifies that* $\tau_i(1)<\tau_i(n+m_1)$ *and* $\tau_i(n+m_1+\dots+m_{i-1})+1<\tau_i(n)$.

Notation 5.5. — Let V be a $\mathcal{P}rim_{sh}$ algebra. Given elements $x\in V_n$ and $y_i\in V_{m_i}$, for $1\leqslant i\leqslant q$, and a permutation $\tau\in Sh(n,m_1+\dots+m_q)$ such that $\tau(1)<\tau(n+m_1)$ and $\tau(n+m_1+\dots+m_{q-1})+1<\tau(n)$, we denote by $B^{\tau}_q(x;y_1,\dots,y_q)$ the following element of $V_{n+m_1+\dots+m_q}$:

$$B^{\tau}_q(x;y_1,\dots,y_q) := (B^{\sigma_q}(\dots((B^{\sigma_2}(B^{\sigma_1}(x;y_1);y_2)\dots);y_q),$$

where $\sigma_1,\dots,\sigma_q$ are the permutations described in Lemma 5.4.

Let $(A,\bullet_{\gamma})$ be a shuffle algebra over K. In order to simplify notation, given elements $x\in A_n$ and $y\in A_m$, we shall keep the notations $x\bullet_0 y$ for the element $x\bullet_{1_{n+m}} y$, and $y\bullet_{top} x$ for the element $x\bullet_{\epsilon_{nm}} y$. Recall that both operations are associative.

Define binary operations $\{-,-\}$ and B^{γ} on A by the formulas:

(1) $\{x,y\} := x\bullet_{top} y - x\bullet_0 y$,
(2) $B^{\gamma}(x;y) := x\bullet_{\gamma} y$,

for $x\in A_n$, $y\in A_m$ and $\gamma\in Sh(n,m)\setminus\{1_{n+m},\epsilon_{nm}\}$.

To see that there exists a functor from the category *Sh-alg* of shuffle algebras to the category of $\mathcal{P}rim_{sh}$ algebras, we need the following result.

Proposition 5.6. — *Let $(A, \bullet_\gamma)$ be a shuffle algebra. The space A with the operations $\{-,-\}$ and B^γ is a $\mathcal{P}rim_{sh}$ algebra.*

Proof. — We need to check that the operations defined on A verify the relations of Definition 5.3. For the first equality note that $\epsilon_{(m+r)n} = (1_m \times \epsilon_{rn}) \cdot (\epsilon_{nm} \times 1_r)$, which implies that:

$$\begin{aligned}\{x, \{y, z\}\} = x \bullet_{top} y \bullet_{top} z + x \bullet_0 y \bullet_0 z - x \bullet_{top} (y \bullet_0 z) - x \bullet_0 (y \bullet_{top} z) = \\ \{\{\{x, y\}, z\} + (x \bullet_0 y) \bullet_{top} z + (x \bullet_{top} y) \bullet_0 z - x \bullet_{top} (y \bullet_0 z) - x \bullet_0 (y \bullet_{top} z) = \\ \{\{\{x, y\}, z\} + \{x, z\} \bullet_{1_n \times \epsilon_{rm}} y - y \bullet_{\epsilon_{mn} \times 1_r} \{x, z\}.\end{aligned}$$

The second and the third equalities may be verified in an analogous way, using the properties of the $\bullet_\gamma$'s. The fourth one is immediate from the definition of shuffle algebra. □

Remark 5.7. — For a shuffle algebra A, given elements $x \in A_n, y_1, \dots y_q$, with $y_i \in A_{m_i}$, and a permutation $\tau \in Sh(n, m)$ such that $\tau(n+m_1) < \tau(1)$ and $\tau(n + m - m_q) + 1 < \tau(n)$, the element $B^\tau_q(x; y_1, \dots, y_q)$ defined in Notation 5.5 verifies:

$$B^\tau_q(x; y_1, \dots, y_q) = x \bullet_\tau (y_1 \bullet_0 \cdots \bullet_0 y_q),$$

where $m = m_1 + \cdots + m_q$.

The following Proposition states that the subspace of primitive elements of a shuffle bialgebra is closed under the operations $\{-,-\}$ and $\bullet_\gamma$.

Proposition 5.8. — *Let $(A, \bullet_\gamma, \Delta)$ be a shuffle bialgebra, the subspace of primitive elements $Prim(A)$ is a $\mathcal{P}rim_{sh}$ subalgebra of A.*

Proof. — It is immediate to check that $\Delta(x \bullet_{top} y) = x \otimes y = \Delta(x \bullet_0 y)$, for $x, y \in Prim(A)$, which implies that $\Delta(\{x, y\}) = 0$ whenever $x, y \in \mathrm{Prim}(A)$.

For any permutation $\gamma \in Sh(n, m) \setminus \{1_{n+m}, \epsilon_{nm}\}$, the coproduct verifies that:

$$\Delta(x \bullet_\gamma y) = \sum_r \Big(\sum (x_{(1)} \bullet_{\gamma^r_{(1)}} y_{(1)}) \otimes (x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} y_{(2)})\Big),$$

where $\gamma = (\gamma^r_{(1)} \times \gamma^{n+m-r}_{(2)}) \cdot (1_{n_1} \times \epsilon_{(n-n_1)m_1} \times 1_{m-m_1})$.

Since $\sum x_{(1)} \otimes x_{(2)} = 0$ and $\sum y_{(1)} \otimes y_{(2)} = 0$, we have that $(x_{(1)} \bullet_{\gamma^r_{(1)}} y_{(1)}) \otimes (x_{(2)} \bullet_{\gamma^{n+m-r}_{(2)}} y_{(2)}) \neq 0$ if, and only if, either $r = n_1 = n$, and therefore $\gamma = 1_{n+m}$, or $r = m_1 = m$, and $\gamma = \epsilon_{nm}$. As $\gamma \notin \{1_{n+m}, \epsilon_{nm}\}$, we get that $\Delta(x \bullet_\gamma y) = 0$. □

Given a positively graded vector conilpotent coalgebra on (C, Θ) we may describe, applying Theorem 5.2, the free shuffle algebra $\mathbf{Sh}(C)$ in terms of its primitive elements.

Let $e : \mathbf{Sh}(C) \longrightarrow \mathit{Prim}(\mathbf{Sh}(C))$ be the linear map defined in Proposition 5.1. The restriction of the map e to the elements $\xi_n \otimes x$, with $n \geqslant 1$ and $x \in C_n$ gives a linear isomorphism between$C$ and the subspace $e(C)$ of $\mathit{Prim}(\mathbf{Sh}(C))$. In order to simplify notation, we denote by $\overline{x}$ the image under e of the element $\xi_n \otimes x$.

Let $\mathcal{P}(\mathbf{Sh}(C))$ be the subspace of $\mathbf{Sh}(C)$ spanned by the set $e(C)$ with the operations B^γ and $\{-,-\}$, and let $\mathcal{P}(\mathbf{Sh}(C))^{\bullet_0 n} \subseteq \mathbf{Sh}(C)$ be the subspace spanned by all the elements of the form $z_1 \bullet_0 z_2 \bullet_0 \cdots \bullet_0 z_n$, with $z_j \in \mathcal{P}(\mathbf{Sh}(C))$, for $1 \leqslant j \leqslant n$.

Proposition 5.9. — *The space* $T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C))) := \bigoplus_{n\geqslant 1} \mathcal{P}(\mathbf{Sh}(C))^{\bullet_0 n}$ *coincides with* $\mathbf{Sh}(C)$.

Proof. — We prove first that $C \subseteq T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$. If $x \in C_1$, then $\overline{x} = \xi_1 \otimes x \in \mathcal{P}(\mathbf{Sh}(C))$. Suppose that $x \in C_n$, for $n \geqslant 2$, the definition of e implies that

$$\xi_n \otimes x = \overline{x} + \sum (\xi_{n_1}; x_{(1)}) \bullet_0 \overline{x}_{(2)},$$

where $\Theta(x) = \sum x_{(1)} \otimes x_{(2)}$. The elements $\overline{x}$ and $\overline{x}_{(2)}$ belong to $e(C) \subseteq \mathcal{P}rim_{sh}(C)$. Applying a recursive argument on the degree of x, we get that:

$$\xi_{n_1} \otimes x_{(1)} = \sum_k \overline{y}_1^k \bullet_0 \overline{y}_2^k \bullet_0 \cdots \bullet_0 \overline{y}_{r_k}^k,$$

with $y_l^k \in C_{m_l^k}$, for $1 \leqslant l \leqslant r_k$. So, $\xi_n \otimes x$ belongs to $T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$.

Since $(\mathbf{Sh}(C), \bullet_\gamma)$ is the free shuffle algebra spanned by the vector space C, any homogeneous element $y \in \mathbf{Sh}(C)_n$ may be written in a unique way as $y = \sum_l (\xi_{n_l} \otimes x_l) \bullet_{\gamma_l} y'_l$, with $x_l \in C_{n_l}$, $y'_l \in \mathbf{Sh}(C)$ such that $|y'_l| < n$ and $\gamma_l \in \mathit{Sh}(n_l, n - n_l)$.

We have proved yet that $\xi_n \otimes x = \sum_l \overline{x}_1^l \bullet_0 \cdots \bullet_0 \overline{x}_{r-l}^l$. To end the proof we are going to see that for any collection of elements y and $\{z_i\}_{1\leqslant i\leqslant q}$ in $\mathcal{P}(\mathbf{Sh}(C))$, the element $w = y \bullet_\gamma (z_1 \bullet_0 z_2 \bullet_0 \cdots \bullet_0 z_q)$, belongs to $\oplus_{n\geqslant 1} \mathcal{P}(\mathbf{Sh}(C))^{\bullet_0 n}$. Let $|y| = m$,$|z_i| = r_i$, for $1 \leqslant i \leqslant q$, and $r = \sum_{i=1}^q r_i$. The result is obvious for $\gamma = 1_{m+r}$.

For $\gamma \neq 1_{m+r}$, we proceed by induction on q. If $q = 1$, then

- $w = B^\gamma(y; z_1)$, for $\gamma \neq \epsilon_{mr_1}$, and
- $w = \{z_1, y\} + z_1 \bullet_0 y$, for $\gamma = \epsilon_{mr_1}$.

If $q > 1$ and $\gamma(m) < \gamma(m + r_1 + \cdots + r_{q-1}) + 1$, then there exists $\tilde{\gamma} \in \mathit{Sh}(m, \overline{r})$ such that $\gamma = \tilde{\gamma} \times 1_{r_q}$, where $\overline{r} = r_1 + \cdots + r_{q-1}$. We have that $w = (y \bullet_{\tilde{\gamma}} (z_1 \bullet_0 \cdots \bullet_0 z_{q-1})) \bullet_0 z_q$. By recursive hypothesis, the element $y \bullet_{\tilde{\gamma}} (z_1 \bullet_0 \cdots \bullet_0 z_{q-1})$ belongs to $T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$, which implies that $w \in T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$.

If $q > 1$ and $\gamma(m) \geqslant \gamma(m + r_1 + \cdots + r_{q-1}) + 1$, then we have to consider three situations:

(1) If $\gamma(1) < \gamma(m + r_1)$ then $w = B_q^{\gamma}(y; z_1, \ldots, z_q) \in \mathcal{P}(\mathbf{Sh}(C))$,

(2) if $\gamma(1) \geqslant \gamma(m + r_1)$ then, then $\gamma = (\epsilon_{mr_1} \times 1_{r-r_1}) \cdot (1_{r_1} \times \tilde{\gamma})$, for some $\tilde{\gamma} \in Sh(m, r - r_1)$. We get that:

$$w = \{z_1, y\} \bullet_{1_{r_1} \times \tilde{\gamma}} (z_2 \bullet_0 \cdots \bullet_0 z_q) + z_1 \bullet_0 (y \bullet_{\tilde{\gamma}} (z_2 \bullet_0 \cdots \bullet_0 z_q)).$$

But, by a recursive argument, the elements $\{z_1, y\} \bullet_{1_{r_1} \times \tilde{\gamma}} (z_2 \bullet_0 \cdots \bullet_0 z_q)$ and $y \bullet_{\tilde{\gamma}} (z_2 \bullet_0 \cdots \bullet_0 z_q)$ belong to $T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$, which implies that $w \in T^{\bullet_0}(\mathcal{P}(\mathbf{Sh}(C)))$. □

For any graded coalgebra C, let $\mathcal{P}rim_{sh}(C)$ denotes the free $\mathcal{P}rim_{sh}$ algebra spanned by the underlying space of C. Lemma 5.4 implies that any homogeneous element of degree n in $\mathcal{P}rim_{sh}(C)$ is a sum of elements of type $B_q^{\gamma}(x; y_1, \ldots, y_q)$, with $x = \{\{\{\{x_1, x_2\}, x_3\}, \ldots\}, x_n\}$, for $x_1, \ldots, x_n \in C$, and $y_1, \ldots, y_q \in \mathcal{P}rim_{sh}(C)$, with $|x| + \sum_{i=1}^{q} |y_i| = n$. Let $\{C\}_0$ be the vector space spanned by the elements $\{\{\{\{x_1, x_2\}, x_3\}, \ldots\}, x_n\}$, with $x_i \in C$, we have that $\{C\}_0$ is a subspace of $\mathcal{P}rim_{sh}(C)$.

Define recursively

$$\{C\}_n := \{C\}_{n-1} \bigoplus (\bigoplus_{m \geqslant 1} B_m),$$

where B_m is the space spanned by the elements $B_m^{\gamma}(x; y_1, \ldots, y_m)$, with $x \in \{C\}_0$, $y_j \in \{C\}_{n-1}$ for $1 \leqslant j \leqslant m$ Note that $\{C\}_n \subseteq \{C\}_{n+1}$. It is clear that $\mathcal{P}rim_{sh}(C) = \sum_{n \geqslant 0} \{C\}_n$ as a vector space.

Proposition 5.10. — *Let C be a positively graded coalgebra. The subspace $\mathcal{P}(\mathbf{Sh}(C))$ is the subspace of primitive elements of $\mathbf{Sh}(C)$. Moreover, it is the free $\mathcal{P}rim_{sh}$ algebra $\mathcal{P}rim_{sh}(C)$.*

Proof. — Note first that it suffices to prove the result for the case where $C = \bigoplus_{n \geqslant 1} C_n$ with C_n a space of finite dimension, for all $n \geqslant 1$. For the general case, C is a limit of graded vector spaces which verify this condition and the result follows.

Proposition 5.8 states that $\mathcal{P}(\mathbf{Sh}(C)) \subseteq \mathrm{Prim}(\mathbf{Sh}(C))$, while Proposition 5.9 implies that $\mathbf{Sh}(C) = \overline{T}(\mathcal{P}(\mathbf{Sh}(C)))$ as a vector space. From Theorem 5.2, one has that $\mathbf{Sh}(C) = \overline{T}(Prim(\mathbf{Sh}(C))$, so $\mathcal{P}(\mathbf{Sh}(C)) = Prim(\mathbf{Sh}(C))$.

For the second assertion note that, since $\mathbf{Sh}(C)$ is the free associative algebra over the space $[Irr_{\mathcal{ST}}(C) = \bigoplus_{n\geqslant 1} Irr_{\mathcal{ST}_n}(C)$ spanned by the irreducible surjective maps coloured with elements of C, the previous assertion states that $\mathcal{P}(\mathbf{Sh}(C))$ is isomorphic as a a vector space to $Irr_{\mathcal{ST}}(C)$. From Proposition 5.8 we know that $\mathcal{P}(\mathbf{Sh}(C))$ is a $\mathcal{P}rim_{sh}$ algebra which contains $e(C)$. To see that it is free, it suffices to define a linear isomorphism $\beta : Irr_{\mathcal{ST}}(C) \longrightarrow \mathcal{P}rim_{sh}(C)$. On $C \subset Irr_{\mathcal{P}}(C)$, β coincides with the identity map. Let $y = (\xi_n \otimes x) \bullet_\gamma y_1 \in Irr_{\mathcal{P}}(C)$, with $x \in C_n$ and $n \geqslant 1$. We define $\beta(y)$ as follows:

If $y_1 \in Irr_{\mathcal{P}}(C)$ and $\gamma \neq \epsilon_{nm_1}$, then $\beta(y) := B_1^\gamma(x; \beta(y_1))$.

If $y_1 \in Irr_{\mathcal{P}}(C)$ and $\gamma = \epsilon_{nm_1}$, then

$$\beta(y) := \begin{cases} \{\beta(y_1), x\}, & \text{for } \beta(y_1) \in \{C\}_0, \\ B_q^{\overline{\tau}}(\{w, x\}; z_1, \dots, z_q), & \text{for } \beta(y_1) = B_q^\tau(w; z_1, \dots, z_q), \end{cases}$$

where $|y_1| = m_1$, $w \in \{C\}_0$, $|w| = s$, $z_j \in \mathcal{P}rim_{sh}(C)$ for $1 \leqslant j \leqslant q$, $\sum_j |z_j| = r$ and $\overline{\tau} := (\tau \times 1_n) \cdot (1_s \times \epsilon_{nr})$.

Suppose that $y_1 = t_1 \bullet_0 \cdots \bullet_0 t_p$, with $t_i \in Irr_{\mathcal{P}}(C)$ and $p > 1$. The fact that y is irreducible implies that $\gamma(n + h_1 + \cdots + h_{p-1} + 1) < \gamma(n)$, for $|t_i| = h_i$.

If $\gamma(1) < \gamma(n + h_1)$, then $\beta(y) := B_p^\gamma(x; \beta(t_1), \dots, \beta(t_p))$.

If $\gamma(n + h_1) < \gamma(1) - 1$, then

$$\beta(y) := \begin{cases} B_{p-1}^{\overline{\gamma}}(\{\beta(t_1), x\}; \beta(t_2), \dots, \beta(t_p)), & \text{for } \beta(t_1) \in \{C\}_0 \\ B_{q+p-1}^{\overline{\tau}}(\{w, x\}; z_1, \dots, z_q, \beta(t_2), \dots, \beta(t_p)), & \text{for } \beta(t_1) = B_q^\tau(w; z_1, \dots, z_q), \end{cases}$$

where $\overline{\gamma} = \gamma \cdot (\epsilon_{h_1 n} \times 1_{h_2 + \cdots + h_p})$, $\overline{\tau} := \gamma \cdot (1_n \times \tau \times 1_{h_2 + \cdots + h_r}) \cdot (\epsilon_{sr} \times 1_{h-s})$, and $h := \sum_{i=1}^p h_i$.

It is not difficult to check that β is bijective, which implies that $\mathcal{P}(\mathbf{Sh}(X))$ is isomorphic to $\mathcal{P}rim_{sh}(X)$. □

The following result is a straightforward consequence of Theorem 5.2 and the previous results.

Proposition 5.11. — *Let (C, Θ) be a positively graded coalgebra. The nonunital infinitesimal bialgebra $\mathbf{Sh}(C)$ is isomorphic to $\overline{T}^c(\mathcal{P}rim_{sh}(C))$, where $\mathcal{P}rim_{sh}(C)$ is the free $\mathcal{P}rim_{sh}$ algebra spanned by the vector space C.*

Example 5.12. — **Primitive elements of the $K[\mathcal{ST}_\infty]$.** As a shuffle algebra $K[\mathcal{ST}_\infty]$ is freely generated by the set $\{\xi_n\}_{n\geqslant 1}$, while it is the free associative algebra spanned by the irreducible elements of $\mathcal{ST}_\infty$.

Let Θ be the unique coproduct on $\bigoplus_{n\geqslant 1} K[\xi_n]$ such that $\Theta(\xi_n) = \sum_{i=1}^{n-1} \xi_i \otimes \xi_{n-i}$, for $n \geqslant 1$. Given an irreducible element $f \in Irr_{\mathcal{ST}_n}$, with $n_1 = |f^{-1}(1)|$, we associate to it a primitive element $E_\theta(f)$ as follows:

If $n_1 = n$, then $f = \xi_n$ and

$$E(\xi_n) := e(\xi_n) = \sum_{r=1}^{n} (-1)^{r-1} \Big(\sum_{n_1+\cdots+n_r=n} \xi_{n_1} \bullet_0 \cdots \bullet_0 \xi_{n_r} \Big).$$

If $n_1 < n$, then $f = \xi_{n_1} \bullet_\gamma f_1$, with $\gamma \neq 1_n$. There exists a unique family $g_1, \dots, g_r$ of irreducible elements, such that $f_1 = g_1 \bullet_0 \cdots \bullet_0 g_r$. Let $|g_j| = m_j$, for $1 \leqslant j \leqslant r$. Since $f \in Irr_{\mathcal{ST}_n}$, we get that $\gamma(n_1) > \gamma(n_1 + m_1 + \cdots + m_{r-1} + 1)$. The element $E(f)$ is given by:

(1) $E(f) := E(\xi_{n_1}) \bullet_\gamma (E(g_1) \bullet_0 \cdots \bullet_0 E(g_r))$, for $\gamma(1) < \gamma(n_1 + m_1)$,

(2) $E(f) := \{E(g_1), E(\xi_n)\} \bullet_{\tilde{\gamma}} (E(g_2) \bullet_0 \cdots \bullet_0 E(g_r)$, for $\gamma(1) \geqslant \gamma(n_1 + m_1)$,

where $\gamma = (\epsilon_{n_1,m_1} \times 1_{m_2+\cdots+m_r}) \cdot \tilde{\gamma}$.

Note that $E(f) = f + \sum_i f_i$, such that the f_i are not irreducible, which implies that the family $\{E(f)\}_{f \text{ irreducible}}$ is linearly independent. Propositions 5.9 and 5.10 imply that the family $\{E(f)\}_{f\in Irr_{\mathcal{ST}_\infty}}$ is a basis of the space of primitive elements of $K[\mathcal{ST}_\infty]$.

For example, let $f = (3,2,4,1,6,4,1,5,5) = \xi_2 \bullet_{(3,4,5,1,6,7,2,8,9)} ((2,1) \bullet_0 (1,3,1,2,2))$. We get that

$$\begin{aligned}
&E(\xi_2) = (1,1) - (1,2), \qquad\qquad E(2,1) = (2,1) - (1,2),\\
&E(2,1,1) = \{E(\xi_1), E(\xi_2)\} = (2,1,1) - (3,1,2) - (1,2,2) + (1,2,3),\\
&E(1,3,1,2,2) = E(\xi_2) \bullet_{(1,3,2,4,5)} E(2,1,1) = (1,3,1,2,2) - (1,4,1,2,3) -\\
&\qquad\qquad (1,2,1,3,3) + (1,2,1,3,4) - (1,4,2,3,3) + (1,5,2,3,4) +\\
&\qquad\qquad\qquad\qquad (1,3,2,4,4) - (1,3,2,4,5),\\
&E(3,2,4,1,6,4,1,5,5) = \{E(2,1), E(\xi_2)\} \bullet_{(1,2,5,3,6,7,4,8,9)} E(1,3,1,2,2).
\end{aligned}$$

We want to prove the equivalence between the categories of conilpotent shuffle bialgebras and $\mathcal{P}rim_{sh}$ algebras. More precisely, given a $\mathcal{P}rim_{sh}$ algebra $(V, B^\gamma, [-,-])$, let $\mathcal{U}_{Sh}(V)$ be the shuffle bialgebra obtained by taking the quotient of the free shuffle algebra $\mathbf{Sh}(V)$ by the ideal (as a shuffle algebra) spanned by the set:

$$\{B^\gamma(x;y) - \overline{B}^\gamma(x;y),\ \{x,z\} - [x,z]\},$$

with $x \in V_n$, $y \in V_m$, $z \in V_r$ and $\gamma \in Sh(n,m)$ such that $\gamma(1) < \gamma(n+m_1)$ and $\gamma(n + m_1 + \cdots + m_{n-1} + 1) < \gamma(n)$. Here B^γ and $\{-,-\}$ denote the operations associated to the shuffle algebra $\mathbf{Sh}(V)$.

Theorem 5.13. — **a)** *Let $(H, \circ_\gamma, \Delta)$ be a conilpotent shuffle bialgebra. As a shuffle bialgebra H is isomorphic to $\mathcal{U}_{Sh}(Prim(H))$, where $Prim(H)$ is the $\mathcal{P}rim_{sh}$ algebra of primitive elements of H.*
b) *Let $(V, \overline{B}_n^\gamma, \{-,-\})$ be a $\mathcal{P}rim_{sh}$ algebra. As $\mathcal{P}rim_{sh}$ algebra V is isomorphic to* $\mathrm{Prim}(\mathcal{U}_{Sh}(V))$.

Proof. — **a)** .Define $\varphi : \mathbf{Sh}(Prim(H)) \longrightarrow H$ as follows:

$$\varphi(x_1 \bullet_{\gamma_1} (x_2 \bullet_{\gamma_2} (\dots (x_{n-1} \bullet_{\gamma_{n-1}} x_n)))) := x_1 \circ_{\gamma_1} (x_2 \circ_{\gamma_2} (\dots (x_{n-1} \circ_{\gamma_{n-1}} x_n))),$$

where $x_i \in Prim(H)$ for $1 \leqslant i \leqslant n$. Note that $\varphi(B^\gamma(x;y)) = \varphi(x \bullet_\gamma y) = x \circ_\gamma y = \overline{B}^\gamma(x;y)$, and $\varphi(\{x,y\}) = \varphi(x \bullet_{top} y - x \bullet_0 y) = [x;y]$, so φ factorizes through $\mathcal{U}_{Sh}(\mathrm{Prim}(H))$. Moreover, it is immediate to check that φ is a bialgebra homomorphism. Applying Theorem 5.2, the inverse morphism is given by

$$\varphi^{-1}(x) = cl(e(x)) + \sum cl(x_{(1)} \bullet_0 x_2)) + \dots + \sum cl(x_1 \bullet_0 \dots \bullet_0 x_n),$$

where cl denotes the class of the element in $\mathcal{U}_{Sh}(Prim(H))$.

b) It is clear that $V \subseteq \mathrm{Prim}(\mathcal{U}_{Sh}(V))$. Proposition 5.11 implies that $\mathrm{Prim}(\mathbf{Sh}(V)) = \mathcal{P}rim_{sh}(V)$. So, the primitive elements of $\mathcal{U}_{Sh}(V)$ are generated by V under the operations B^γ and $\{,\}$, which are precisely the elements of V in the quotient. □

6. Preshuffle bialgebras and pre-Lie systems

We want to apply our description of primitive elements of shuffle bialgebras to other structures. We introduce the notion of preshuffle algebras, related to leveled trees, and study pre-Lie systems (see [6]) as a particular type of preshuffle algebras.

Shuffle algebras are related to a weak version of monoids in the category $(\mathcal{S}\text{–}Mod, \otimes_{\mathcal{S}})$, when we do no ask for a compatibility relation between the operations $\bullet_\gamma$ and the action of the symmetric group. In the present section, we consider the family of *block-shuffles*, which is closed under compositions, and define a preshuffle algebra as a graded space equipped with operations $\bullet_\gamma$ only when γ is a block-shuffle. For any non-symmetric operad $\mathbb{P} = \{\mathbb{P}_n\}_{n\geqslant 0}$ the space $A = \bigoplus_{n\geqslant 0} \mathbb{P}_{n+1}$ has a natural structure of preshuffle algebra. More precisely, non-symmetric operads are, via this identification, preshuffle algebras satisfying some extra conditions.

DEFINITION 6.1. — (1) A *preshuffle* algebra over K is a graded vector space $A = \bigoplus_{n\geqslant 1} A_n$ equipped with linear maps

$$\bullet_i : A\otimes A_m \to A,\ \text{for } 0\leqslant i\leqslant m,$$

satisfying:

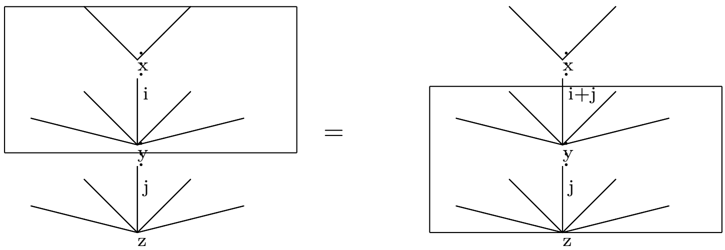
$$(x\bullet_i y)\bullet_j z = x\bullet_{i+j}(y\bullet_j z),\ \text{for } 0\leqslant i\leqslant |y| \text{ and } 0\leqslant j\leqslant |z|.$$

(2) A *pre-Lie system* is a preshuffle algebra $(A, \bullet_i)$ such that the operations $\bullet_i$ satisfy the following additional conditions:

$$x\bullet_i (y\bullet_j z) = y\bullet_{j+|x|}(x\bullet_i z),\ \text{for } 0\leqslant i< j,$$

for any elements $x, y, z\in A$.

The relations verified by a preshuffle algebra may be pictured as follows:



For non-negative integers n and $0\leqslant i\leqslant m$, let $\omega_i^{n,m}$ be the permutation $\omega_i^{n,m} := \epsilon_{n,i}\times 1_{m-i}\in Sh(n,m)$. The element $\omega_m^{n,m}$ is equal to $\epsilon_{n,m}$.

It is immediate to verify that:

$$(1_n\times\omega_j^{m,r})\cdot\omega_i^{n,m+r} = (\omega_{i-j}^{n,m}\times 1_r)\cdot\omega_j^{n+m,r},$$

for $0\leqslant j\leqslant r$ and $0\leqslant i\leqslant m+r$, which implies that any shuffle algebra is a preshuffle algebra with the product $\bullet_i$ given by:

$$x\bullet_i y := x\bullet_{\omega_i^{n,m}} y,\ \text{for } x\in A_n,\ y\in A_m \text{ and } 0\leqslant i\leqslant m.$$

DEFINITION 6.2. — (see [6]) A *pre-Lie algebra* is a vector space V equipped with a bilinear map $\circ : V\otimes V\longrightarrow V$ satisfying that:

$$x\circ(y\circ z) - (x\circ y)\circ z = x\circ(z\circ y) - (x\circ z)\circ y,$$

for any elements $x, y, z\in V$.

Associative algebras are particular examples of pre-Lie algebras.

Any pre-Lie system $(A, \bullet_i)$ induces a pre-Lie algebra structure on the space A with the product $\circ$ given by:

$$x \circ y = \sum_{0 \leqslant i \leqslant m} x \bullet_i y,$$

for $x \in A$ and $y \in A_m$.

Definition 6.3. — Let $(A, \bullet_i)$ be a positively graded preshuffle algebra, such that A is equipped with a graded coassociative coproduct Δ. We say that $(A, \bullet_i, \Delta)$ is a *preshuffle bialgebra* if it verifies:

(1) $\Delta(x \bullet_0 y) = \sum x_{(1)} \otimes (x_{(2)} \bullet_0 y) + x \otimes y + \sum (x \bullet_0 y_{(1)}) \otimes y_{(2)}$.

(2) $$\Delta(x \bullet_i y) = \sum_{|y_{(1)}| \leqslant i} y_{(1)} \otimes (x \bullet_{i - |y_{(1)}|} y_{(2)}) + \sum_{|y_{(1)}| = i} (x_{(1)} \bullet_i y_{(1)}) \otimes (x_{(2)} \bullet_0 y_{(2)}) + \sum_{|y_{(1)}| \geqslant i} (x \bullet_i y_{(1)}) \otimes y_{(2)},$$

for $1 \leqslant i \leqslant |y|$.

(3) $\Delta(x \bullet_{|y|} y) = \sum y_{(1)} \otimes (x \bullet_{|y_{(2)}|} y_{(2)}) + y \otimes x + \sum (x_{(1)} \bullet_{|y|} y) \otimes x_{(2)}$.

We shall use indistinctly the terms *pre-Lie system with coproduct* or *coalgebra structure on a pre-Lie system* to design a preshuffle bialgebra $(A, \bullet_i, \Delta)$ such that $(A, \bullet_i)$ is a pre-Lie system.

Clearly any shuffle bialgebra is a preshuffle bialgebra.

Examples 6.4. — **a) The free preshuffle algebra.** Let V be a positively graded vector space. In **b)** of Examples 2.3 we introduce a shuffle algebra structure on the space $K[\mathcal{F}_\infty](V)$, so $K[\mathcal{F}_\infty](V)$ is a preshuffle algebra with the operations $\bullet_i := \bullet_{\omega_i}$. Moreover, the subspaces $K[\mathcal{K}_\infty(V)] := \bigoplus_{n \geqslant 1} K[\mathcal{K}_n(V)]$ and $K[\mathcal{ST}_\infty(V)] := \bigoplus_{n \geqslant 1} K[\mathcal{ST}_n(V)]$ are closed under the operations $\bullet_i$, so they are sub-preshuffle algebras of $K[\mathcal{F}_\infty(V)]$.

Theorem 6.4.1. — *For any positively graded vector space V, the space $K[\mathcal{K}_\infty(V)]$, with the operations $\bullet_i$, is the free preshuffle algebra spanned by V. We denote it by* $\mathbf{Psh}(V)$.

Proof. — Any element of $x \in V_n$ is identified with the pair $\xi_n \otimes x$. The result follows easily using that any element z in the free preshuffle algebra spanned by V is of the form $x \bullet_i y$, with $x \in V_r$ for some $1 \leqslant r$ and y an element of the free preshuffle algebra such that $|y| < |z|$. □

Given a coalgebra (C, Θ), the coproduct $\Delta_\theta : \mathbf{Sh}(C) \longrightarrow \mathbf{Sh}(C) \otimes \mathbf{Sh}(C)$, defined in Example **c)** of 3.3, restricts to $\mathbf{Psh}(C)$. So, any free preshuffle algebra is a preshuffle bialgebra.

The free preshuffle algebra spanned by the one dimensional space Kx_0 of degree one is just the space $K[S_\infty] := \bigoplus_{n\geqslant 1} K[S_n]$, with the operations $\bullet_{\omega_i}$.

b) Infinitesimal bialgebras Given a graded nonunital infinitesimal bialgebra $(A, \cdot, \Delta)$, example **b)** of 3.3 shows that there exists a natural way to define a shuffle bialgebra structure on A, where the coproduct is Δ and the operations $\bullet_\gamma$ are constructed using $\cdot$ and Δ. It is easy to see that the preshuffle algebra structure on A, given by $\bullet_i = \bullet_{\omega_i^{n,m}}$ is in fact a pre-Lie system. So, any graded nonunital infinitesimal bialgebra gives rise to a coalgebra structure on a pre-Lie system.

c) The free pre-Lie system. The graded vector space $K[T_\infty]$ spanned by the set of planar trees $T_\infty := \bigcup_{n\geqslant 1} T_n$, with the products $\circ_i$ described in Definition 1.13 is a pre-Lie system. Moreover, the subspace $K[Y_\infty]$, spanned by the set of planar binary trees, is a sub-pre-Lie system of $K[T_\infty]$.

For a graded vector space $V = \bigoplus_{n\geqslant 1} V_n$, let $V[-1]$ be the graded space such that $V[-1]_0 = 0$, $V[-1]_1 = Kid$ and $V[-1]_n = V_{n-1}$. The free nonsymmetric operad spanned by $V[-1]$ (see for instance [15]) is the vector space $\bigoplus_{n\geqslant 1} K[T_{n-1}(V)]$ spanned by the set of planar rooted trees with the internal vertices coloured by the elements of V, in such a way that a vertex with $r+1$ inputs is coloured by an element x of V_r, with the maps $\circ_i$ induced by the pre-Lie system structure of the space $K[T_\infty]$ in an obvious way. So, the free pre-Lie system $\mathbf{PLie}(V)$ on V is just the space of coloured trees $K[T_\infty(V)] = \bigoplus_{n\geqslant 1} K[T_n(V)]$ with the compositions $\circ_i$.

d) The space of Hochschild cochains. (see [6]) Let A be a unital K-algebra, and let $C^*(A) := \bigoplus_{n\geqslant 0} \mathrm{Hom}_K(A^{\otimes n}, A)$ be the space of Hochschild cochains on A.

The space $C^*(A)[1] := \bigoplus_{n\geqslant 0} \mathrm{Hom}_K(A^{\otimes(n+1)}, A)$ is a pre-Lie system with the operations $\bullet_i$ defined as follows:

$$g \bullet_i f := f \circ (id_A^{\otimes(i-1)} \times g \times id_A^{\otimes(n-i)}),$$

for $g \in C^m(A, A)$ and $f \in C^n(A, A)$.

Consider on $C^*(A)[1]$ the following coproduct:

$$\Delta(f) := \sum_{i=1}^{n-1} f_{(1)}^i \otimes f_{(2)}^{n-i+1}, \text{ for } f \in C^n(A, A),$$

where

(1) $f_{(1)}^i(x_1, \dots, x_i) := f(x_1, \dots, x_i, 1_A, \dots, 1_A) \in C^i(A, A)$

(2) $f_{(2)}^{n-i+1}(x_1,\dots,x_{n+1-i}) := f(1_A,\dots,1_A,x_1,\dots,x_{n+1-i}) \in C^{n+1-i}(A,A).$

It is easy to see that Δ defines a coproduct on the pre-Lie system. This example motivated M. Gerstenhaber's definition of pre-Lie systems.

e) The underlying space of an algebraic operad. Let K be a field of characteristic 0, and let $\mathbb{P}$ be a K-linear operad as described in [7]. Consider the graded K-vector space $\mathbb{P}[1] := \bigoplus_{n\geqslant 0}\mathbb{P}(n+1)$ equipped with the maps:

$$\lambda \bullet_i \nu := \gamma_{1,\dots,1,n,1,\dots,1}(\nu\otimes 1\otimes\cdots\otimes 1\otimes\lambda\otimes 1\otimes\cdots\otimes 1),$$

where $1\in\mathbb{P}(1)=\mathbb{P}[1]_0$ is the identity operation, and $\lambda\in\mathbb{P}(m)$ is at the $i+1$-th place. It is easy to check that $\mathbb{P}[1]$ with these products is a pre-Lie system over K.

As an example of coproduct on a pre-Lie system given by a non-symmetric operad consider the operad As.

The pre-Lie system structure of $As[1]=\bigoplus_{n\geqslant 0}K[S_{n+1}]$ is given by the operations:

$$(\sigma\bullet_i\tau) = (\tau_{(1)}^{i}\times\sigma\times\tau_{(2)}^{m-i-1})\cdot\delta_i^n,$$

where $\tau=(\tilde\tau_{(1)}^{i}\times 1_1\times\tilde\tau_{(2)}^{m-i-1})\cdot\delta$ with $\delta\in \mathit{Sh}(i,1.m-i-1)$, $\tilde\tau_{(1)}^{i}\in S_i$, $\tilde\tau_{(2)}^{m-i-1}\in S_{m-i-1}$, and

$$\delta_i^n(k) := \begin{cases} \delta(k), & \text{for } \delta(k)\leqslant i \text{ and } k<\delta^{-1}(i+1),\\ \delta(k)+n-1, & \text{for } \delta(k)> i \text{ and } k<\delta^{-1}(i+1),\\ i+r+1, & \text{for } k=\delta^{-1}(i+1)+r \text{ and } 0\leqslant r<n,\\ \delta(k-n+1), & \text{for } \delta(k)\leqslant i \text{ and } k>\delta^{-1}(i+1)+n-1,\\ \delta(k-n+1)+n-1, & \text{for } \delta(k)> i \text{ and } k>\delta^{-1}(i+1)+n-1.\end{cases}$$

In fact, $\sigma\bullet_i\tau$ is obtained by replacing $i+1$ in the image of τ by $(\sigma(1)+i,\dots,\sigma(n)+i)$. For instance,

$$(2,4,1,3)\bullet_1(1,3,2,5,4) = (1,6,3,5,2,4,8,7).$$

To define a coproduct on $As[1]$, let $\gamma\in S_{m+1}$ be a permutation, for an integer $0\leqslant i\leqslant m$, there exists unique decompositions:

$$\gamma = (\tilde\gamma_{(1)}^{i+1}\times\tilde\gamma_{(2)}^{m-i})\cdot\delta = (\tilde\gamma_{(1)}^{i}\times\tilde\gamma_{(2)}^{m+1-i})\cdot\epsilon,$$

where $\tilde\gamma_{(i)}^{j}\in S_j$, for $i=1,2$, $\delta^{-1}\in \mathit{Sh}(i+1,m-i)$and $\epsilon\in \mathit{Sh}(i,m-i+1)$. Define

$$\Delta_{As}(\gamma) := \sum_{i=0}^{m}\tilde\gamma_{(1)}^{i+1}\otimes\tilde\gamma_{(2)}^{m+i-i}.$$

Proposition 6.4.2. — *The space* $As[1] = \bigoplus_{n\geqslant 1} K[S_{n+1}]$*, equipped with the operations* $\bullet_i$ *and the coproduct* Δ_{As} *is a coalgebra structure on a pre-Lie system.*

Proof. — We know that $(As[1], \bullet_i)$ is a pre-Lie system. To check that Δ is coassociative, it suffices to note that, for $\gamma \in S_{m+1}$, we have that:

$$\begin{aligned}(\Delta_{As} \otimes id_{As[1]}) \circ \Delta_{As}(\gamma) &= \sum_{i+j+k=m} \tilde{\gamma}^{i+1}_{(1)} \otimes \tilde{\gamma}^{j+1}_{(2)} \otimes \tilde{\gamma}^{k+1}_{(3)} \\ &= (id_{As[1]} \otimes \Delta_{As}) \circ \Delta_{As}(\gamma),\end{aligned}$$

where, for each compositions (i, j, k) of m, the following equalities hold:

$$\gamma = (\tilde{\gamma}^{i+1}_{(1)} \times \tilde{\gamma}^{j}_{(2)} \times \tilde{\gamma}^{k}_{(3)}) \cdot \delta_1 = (\tilde{\gamma}^{i}_{(1)} \times \tilde{\gamma}^{j+1}_{(2)} \times \tilde{\gamma}^{k}_{(3)}) \cdot \delta_2 = (\tilde{\gamma}^{i}_{(1)} \times \tilde{\gamma}^{j}_{(2)} \times \tilde{\gamma}^{k+1}_{(3)}) \cdot \delta_3,$$

with $\tilde{\gamma}^{p}_{(l)} \in S_p$, for $l = 1, 2, 3$, $\delta_1 \in Sh(i+1, j, k)$, $\delta_2 \in Sh(i, j+1, k)$ and $\delta_3 \in Sh(i, j, k+1)$.

To prove the relationship between Δ_{As} and the operations $\bullet_i$, note that for any $\gamma \in S_n$ and any $0 \leqslant i \leqslant n$, there exist unique order preserving bijections $\varphi^{i}_{(1)} : \{1, \dots, i\} \longrightarrow \gamma^{-1}(\{1, \dots, i\})$ and $\varphi^{n-i}_{(2)} : \{1, \dots, n-i\} \longrightarrow \gamma^{-1}(\{i+1, \dots, n\})$. The permutations $\tilde{\gamma}^{i}_{(1)}$ and $\tilde{\gamma}^{n-i}_{(2)}$ are given by the formulas:

$$\begin{aligned}&\tilde{\gamma}^{i}_{(1)} = (\gamma(\varphi^{i}_{(1)}(1)), \dots, \gamma(\varphi^{i}_{(1)}(i)))\\ &\tilde{\gamma}^{n-i}_{(2)} = (\gamma(\varphi^{n-i}_{(2)}(1)), \dots, \gamma(\varphi^{n-i}_{(2)}(n-i))).\end{aligned}$$

Using the formulas above, it is easily seen that, for $\sigma \in S_{n+1}$, $\tau \in S_{m+1}$ and $0 \leqslant j \leqslant n+m$, we have that:

$$(\sigma \tilde{\bullet}_i \tau)^{j+1}_{(1)} = \begin{cases} \tilde{\tau}^{j+1}_{(1)}, & \text{for } 0 \leqslant j < i \\ \tilde{\sigma}^{j-i+1}_{(1)} \bullet_i \tilde{\tau}^{i+1}_{(1)}, & \text{for } i \leqslant j \leqslant i+n \\ \sigma \bullet_i \tilde{\tau}^{j-n+1}_{(1)}, & \text{for } i+n < j \leqslant n+m. \end{cases}$$

$$(\sigma \tilde{\bullet}_i \tau)^{n+m-j+1}_{(2)} = \begin{cases} \sigma \bullet_{i-j} \tilde{\tau}^{m-j+1}_{(2)}, & \text{for } 0 \leqslant j < i \\ \tilde{\sigma}^{n+i-j+1}_{(2)} \bullet_0 \tilde{\tau}^{m-i+1}_{(2)}, & \text{for } i \leqslant j \leqslant i+n \\ \tilde{\tau}^{m+n-j+1}_{(2)}, & \text{for } i+n < j \leqslant n+m, \end{cases}$$

which ends the proof. □

For any graded vector space V there exists homomorphisms of preshuffle bialgebras

$$\mathbf{Sh}(V) \hookleftarrow \mathbf{Psh}(V) \twoheadrightarrow \mathbf{PLie}(V).$$

The following results extend Lemma 2.3.5 and Proposition 3.2 to preshuffle algebras, their proof is straightforward.

Lemma 6.5. — *Let $(A, \bullet_i)$ and $(B, \circ_j)$ be preshuffle algebras (respectively pre-Lie systems).*

(1) *The Hadamard product $A \underset{H}{\otimes} B$ has a natural structure of preshuffle algebra (respectively pre-Lie system), given by the operations:*

$$(x \otimes y) \bullet_i (x' \otimes y') := (x \bullet_i x') \otimes (y \circ_i y'),$$

for $x \in A_n, y \in B_n, x' \in A_m, y' \in B_m$ and $0 \leqslant i \leqslant m$.

(2) *The tensor product $A \otimes B$ has a natural structure of preshuffle algebra (respectively pre-Lie system), given by the operations:*

$$(x \otimes y) \bullet_i (x' \otimes y') := \begin{cases} (x \bullet_i x') \otimes (y \circ_{i-|x'|} y'), & \text{for } \ i = |x'|, \\ 0, & \text{otherwise.} \end{cases}$$

Proposition 6.6. — *Let $(A, \bullet_i, \Delta_A)$ and $(B, \circ_j, \Delta_B)$ be two preshuffle bialgebras. The Hadamard product $A \underset{H}{\otimes} B$ with the operations $\bullet_i$ given in Definition 6.5 and the coproduct given by:*

$$\Delta_{A \underset{H}{\otimes} B}(x \otimes y) = \sum_{|x_{(1)}| = |y_{(1)}|} (x_{(1)} \otimes y_{(1)}) \otimes (x_{(2)} \otimes y_{(2)}),$$

where $\Delta_A(x) = \sum x_{(1)} \otimes x_{(2)}$ and $\Delta_B(y) = \sum y_{(1)} \otimes y_{(2)}$, is a preshuffle bialgebra.

Let $(A, \bullet_i, \Delta)$ be a coalgebra structure on a pre-Lie system, we want to show that there exist a natural way of defining operations $\bullet_\gamma$ on A is such a way that $(A, \bullet_\gamma, \Delta)$ is a shuffle bialgebra.

(1) Given a composition $(n_1, \dots, n_p)$ of n, we denote by $\Delta_{n_1,\dots,n_p}$ the composition $\pi_{n_1,\dots,n_p} \circ \Delta^p$, where $\pi_{n_1,\dots,n_p}$ is the projection from $A^{\otimes p}$ to
$A_{n_1} \otimes \dots \otimes A_{n_p}$.

(2) Let γ be an *(n,m)*-shuffle. There exist unique compositions $(n_1, \dots, n_r)$ of n and $(m_1, \dots, m_{r+1})$ of m such that $m_1 \geqslant 0$, $m_{r+1} \geqslant 0$, $m_i \geqslant 1$ for $2 \leqslant i \leqslant r$, and $n_j \geqslant 1$ for $1 \leqslant j \leqslant r$, such that

$$\gamma = (n+1, \dots n+m_1, 1, \dots, n_1, n+m_1+1, \dots, n+m_1+m_2, \\ n_1+1, \dots, n_1+n_2, \dots, n+m),$$

that is

$$\gamma(j)=\begin{cases} j+n-\sum_{i=1}^{k} n_i, & \text{for } 0<j-\sum_{i=1}^{k} n_i+m_i \leqslant m_{k+1}, \text{ with } 0\leqslant k\leqslant r\\ j-\sum_{i=1}^{k} m_i, & \text{for } m_{k+1}<j-\sum_{i=1}^{k} n_i+m_i \leqslant m_{k+1}+n_{k+1}, \text{ with } 1\leqslant k\leqslant r.\end{cases}$$

For instance, if $\gamma=(1,3,4,2,5,6)\in \mathit{Sh}(2,4)$, then $(m_1,m_2,m_3)=(0,2,2)$, and $(n_1,n_2)=(1,1)$.

Given elements $x\in A_n$ and $y\in A_m$, define the element $x\bullet_\gamma y\in A_{n+m}$ as follows:

$$x\bullet_\gamma y := \sum x_{(1)}^{n_1}\bullet_{m_1}(\dots(x_{(r-1)}^{n_{r-1}}\bullet_{m_1+\dots+m_{r-1}}(x_{(r)}^{n_r}\bullet_{m_1+\dots+m_r} y)))),$$

where $\Delta_{n_1,\dots,n_p}(x)=\sum x_{(1)}^{n_1}\otimes\dots\otimes x_{(p)}^{n_p}$.

Theorem 6.7. — *Let $(A,\bullet_i,\Delta)$ be a coalgebra structure on a pre-Lie system. The graded space A equipped with the operations $\bullet_\gamma$ defined above for any shuffle γ, is a shuffle bialgebra.*

Proof. — Let $x\in A_n$, $y\in A_m$ and $z\in A_r$ be homogeneous elements of A, and let $\gamma\in \mathit{Sh}(n,m+r)$, $\delta\in \mathit{Sh}(m,r)$, $\lambda\in \mathit{Sh}(n+m,r)$ and $\sigma\in \mathit{Sh}(n,m)$ be such that

$$(1_n\times\delta)\cdot\gamma=(\sigma\times 1_r)\cdot\lambda.$$

We want to verify that $x\bullet_\gamma(y\bullet_\delta z)=(x\bullet_\sigma y)\bullet_\lambda z$.

Let γ be the permutation given by the integers $(n_1,\dots,n_p)\vdash n$ and $(h_1,\dots,h_{p+1})\vdash m+r$. We proceed by a recursive argument on p.

If $p=1$, then $\gamma=\omega_h^{n,m+r}$.

Suppose that $\delta\in \mathit{Sh}(m,r)$ is given by integers $(m_1,\dots,m_q)\vdash m$ and $(r_1,\dots,r_{q+1})\vdash r$, we have to consider two different cases.

a) If there exists $0\leqslant k\leqslant q$ such that $0<h-\sum_{i=1}^{k} r_i+m_i<r_{k+1}$, then $(1_n\times\delta)\cdot\gamma=(\sigma\times 1_r)\cdot\lambda$, where $\sigma=\omega_{m_1+\dots+m_k}^{n,m}$ and λ is the $(n+m,r)$ shuffle associated to the compositions $(m_1,\dots,m_k,n,m_{k+1},\dots,m_q)$ of $n+m$ and $(r_1,\dots,r_k,\underline{h},r_{k+1}-\underline{h},r_{k+2},\dots,r_{q+1})$ of r, with $\underline{h}:=h-\sum_{i=1}^{k} r_i+m_i$.

Applying the properties of a pre-Lie system, we get that:

$$x\bullet_\gamma(y\bullet_\delta z)=$$

$$x\bullet_h(y_{(1)}^{m_1}\bullet_{r_1}(\dots(y_{(q)}^{m_q}\bullet_{r_1+\dots+r_q} z)))=$$

$$y_{(1)}^{m_1}\bullet_{r_1}(\dots y_{(k)}^{m_k}\bullet_{r_1+\dots+r_k}(x\bullet_{h-\sum_{1\leqslant i\leqslant k} m_i}(y_{(k+1)}^{m_{k+1}}\bullet_{r_1+\dots+r_{k+1}}(\dots(y_{(q)}^{m_q}\bullet_{r_1+\dots+r_q} z))))).$$

Since $(x \bullet_{m_1+\cdots+m_k} y)^{m_j}_{(j)} = \begin{cases} y^{m_j}_{(j)}, & \text{for } 1 \leqslant j \leqslant k, \\ x, & \text{for } j = k+1 \\ y^{m_{j-1}}_{(j-1)}, & \text{for } j < k+1, \end{cases}$, we get the result.

b) If there exists $0 \leqslant k \leqslant q$ such that $0 < h - \sum_{i=1}^{k-1}(r_i + m_i) + r_k < m_k$, then $\sigma = \omega^{n,m}_{m_1+\cdots+m_{k-1}+\underline{h}}$, with $\underline{h} := h - \sum_{i=1}^{k-1}(r_i+m_i) - r_k$, and λ is the $(n+m, r)$-shuffle associated to the compositions $(m_1, \ldots, m_{k-1}, m_k + n, \ldots, m_p)$ of $n+m$ and $(r_1, \ldots, r_{p+1})$ of r.

We have that

(1) $(x \bullet_{m_1+\cdots+m_{k-1}+\underline{h}} y)^{m_j}_{(j)} = y^{m_j}_{(j)}$, for $j \neq k$,
(2) $(x \bullet_{m_1+\cdots+m_{k-1}+\underline{h}} y)^{n+m_k}_{(k)} = x \bullet_{\underline{h}} y^{m_k}_{(k)}$.

In this case, using the properties of pre-Lie systems, it is immediate to check that:

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_{m_1+\cdots+m_{k-1}+\underline{h}} y) \bullet_\lambda z.$$

For $p > 1$, note that if γ is the $(n, m+r)$-shuffle associated to the compositions $(n_1, \ldots, n_p) \vdash n$ and $(h_1, \ldots, h_{p+1})$ of $m+r$, then

$$x \bullet_\gamma (y \bullet_\delta z) = x^{n_1}_{(1)} \bullet_{h_1} (x^{n-n_1}_{(2)} \bullet_{\tilde{\gamma}} (y \bullet_\delta z),$$

where $\tilde{\gamma}$ is the $(n - n_1, m+r)$-shuffle associated to the compositions $(n_2, \ldots, n_p)$ of $n - n_1$ and $(h_1 + h_2, \ldots, h_{p+1})$ of $m+r$.

We get that:

$$x \bullet_\gamma (y \bullet_\delta z) = x^{n_1}_{(1)} \bullet_{h_1} ((x^{n-n_1}_{(2)} \bullet_{\tilde{\sigma}} y) \bullet_{\tilde{\lambda}} z) = (x^{n_1}_{(1)} \bullet_{k_1} (x^{n-n_1}_{(2)} \bullet_{\tilde{\sigma}} y)) \bullet_\lambda z,$$

where $(1_{n-n_1} \times \delta) \cdot \tilde{\gamma} = (\tilde{\sigma} \times 1_r) \cdot \tilde{\lambda}$, and
$(1_{n+m-n_1} \times \tilde{\lambda}) \cdot \omega^{n_1, n+m+r-n_1}_{h_1} = (\omega^{n_1, n+m-n_1}_{k_1} \times 1_r) \cdot \lambda$.

So, we have that

$$x \bullet_\gamma (y \bullet_\delta z) = (x \bullet_\sigma y) \bullet_\lambda z,$$

with $\sigma := (1_{n_1} \times \tilde{\sigma}) \cdot \omega^{n_1, n+m-n_1}_{k_1}$ and $(1_n \times \delta) \cdot \gamma = (\sigma \times 1_r) \cdot \lambda$, which ends the proof. □

Note that Theorem 6.7 implies that for any pre-Lie algebra $(A, \circ)$ defined from a pre-Lie system with coproduct, it is possible to define an associative structure on A just by constructing the shuffle algebra structure $(A, \bullet_\gamma)$ and taking the associated product $* = \sum \bullet_\gamma$.

Primitive elements of preshuffle bialgebras.

Since any shuffle bialgebra is a preshuffle algebra, we look for the operations obtained by compositions and linear combinations of the primitive operations $\{-,-\}$ and B^{γ}, introduced in Section 4, which can be defined in terms of the multiplications $\bullet_i$ of a preshuffle algebra.

Let $(A,\bullet_i)$ be a preshuffle algebra, and let $x\in A_n$, $y\in A_m$ and $z\in A_r$ be elements of A. Note that $\{x,y\}=x\bullet_{top}y-x\bullet_0 y$ and $B^{\omega_i^{n,m}}(x;y)=x\bullet_i y$ are defined for all $1\leqslant i\leqslant m-1$. But also the element

$$B^{1_n\times\omega_i^{m,r}}(\{x,z\};y)=z\bullet_{i+n}(x\bullet_0 y)-x\bullet_0(z\bullet_i y)$$

may be defined in A for $1\leqslant i\leqslant m$. In a similar way, the element

$$B_q^{\omega}(\{x_1,z\};x_2,\dots,x_q;y)=$$
$$z\bullet_{n+i}(x_1\bullet_0 x_2\bullet_0\dots x_q\bullet_0 y)-x_1\bullet_0(z\bullet_{n_{\geqslant 2}+i}(x_2\bullet_0\cdots\bullet_0 x_q\bullet_0 y)),$$

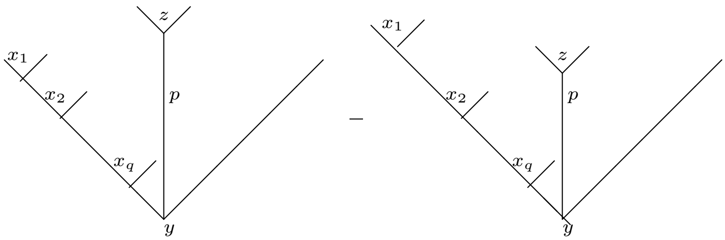
where $\omega=1_{n_1}\times\omega_{n_2+\dots+n_q+i}^{m,n_2+\dots+n_q+r}$, for $|x_i|=n_i$, may be defined on A.

Definition 6.8. — Let $(A,\bullet_i)$ be a preshuffle algebra over K. For $q\geqslant 0$ and $1\leqslant p\leqslant n_r$, let L_q^p the $q+2$-ary operation defined by:

$$L_0^p(y;z):=z\bullet_p y,\quad 0<p<|y|,$$
$$L_0^{|y|}(y;z):=\{y,z\}=y\bullet_{top}z-y\bullet_0 z,$$
$$L_q^p(x_1,\dots,x_q;y;z):=z\bullet_{p+n}(x_1\bullet_0\cdots\bullet_0 x_q\bullet_0 y)$$
$$-x_1\bullet_0(z\bullet_{p+n_{\geqslant 2}}(x_2\bullet_0\cdots\bullet_0 x_q\bullet_0 y)),\quad q\geqslant 1,$$

where $(x_1,\dots,x_q;y;z)$ denotes the element $x_1\otimes\cdots\otimes x_q\otimes y\otimes z\in A^{\otimes(q+2)}$, $n_k:=|x_k|$, $n_{\geqslant k}:=\sum_{i=k}^q n_i$ and $n=n_{\geqslant 1}$, for $1\leqslant k\leqslant q$.

In the set of planar leveled rooted trees, the element $L_q^p(x_1,\dots,x_q;y;z)$ may be represented by the element



Since the relations between the products $\bullet_i$ and the coproduct Δ are the same that the ones between the operations $\bullet_{\omega_i}$ and the coproduct in a shuffle bialgebra, we get the following result. Its proof is a consequence of Proposition 5.8.

Proposition 6.9. — *Let $(A = \bigoplus_{k\geqslant 1} A_k, \bullet_i, \Delta)$ be a preshuffle bialgebra. If the elements $x_1, \dots x_q, y, z$ belong to Prim(A), then $L_q^p(x_1, \dots, x_q; y; z)$ belongs to Prim(A), for any $1 \leqslant p \leqslant |x_q|$.*

Note that, $\Delta(L_0^i(x; y)) = 0$ for all $y \in A$ and $x \in Prim(A)$ and $0 < i < |x|$.

We want to prove analogous results for preshuffle algebras that the ones proved in Propositions 5.9 and 5.10 for shuffle algebras. Note that, since $\mathbf{Psh}(V) \hookrightarrow \mathbf{Sh}(V)$ is a homomorphism of preshuffle bialgebras, then $Prim(\mathbf{Psh}(V)) = \mathbf{Psh}(V) \bigcap Prim(\mathbf{Sh}(V)$. So, the first two relations and the last one of Definition 5.3 are satisfied by the operations $\bullet_0$ and L_0^j.

Lemma 6.10. — *Let $x_1, \dots, x_q, y, z$ be elements of a preshuffle algebra A. With the same notations that in Definition 6.8, the product $\bullet_0$ and the operations L_q^j defined above verify the following equalities:*
1) *For $j < |y|$,*

$$L_q^j(x_1, \dots, x_q; y; z \bullet_0 w) = \sum_{k=0}^{q} L_k^{j+n_{\geqslant k}}(x_1, \dots, x_k; L_{q-k}^j(x_{k+1}, \dots, x_q; y; w); z),$$

and

$$L_q^{|y|}(x_1, \dots, x_q; y; z \bullet_0 w) = \sum_{k=0}^{q} L_k^{|y|+n_{\geqslant k+1}}(x_1, \dots, x_k, L_{q-k}^{|y|}(x_{k+1}, \dots, x_q; y; w); z) + L_q^{|y|}(x_1, \dots, x_q; y; z) \bullet_0 w,$$

2) *For $1 \leqslant j \leqslant |z|$, $L_q^j(x_1, \dots, x_q, z \bullet_0 y; w) = L_q^j(x_1, \dots, x_q; z; w) \bullet_0 y$, and*

$$L_q^{j+|z|}(x_1, \dots, x_q; z \bullet_0 y; w) = \begin{cases} L_1^j(z, y; w) + z \bullet_0 L_0^j(y; z), & \text{for } q = 0 \\ L_{q+1}^j(x_1, \dots, x_q, z; y; w), & \text{for } q \geqslant 1, \end{cases}$$

3)

$$L_q^j(x_1, \dots, x_{q-1}, z \bullet_0 x_q; y; w) = \begin{cases} L_{q+1}^j(x_1, \dots, x_{q-1}, z, x_q; y; w), & \text{for } q \geqslant 2, \\ L_2^j(z, x_1; y; w) + z \bullet_0 L_1^j(x_1; y; w), & \text{for } q = 1. \end{cases}$$

Proof. — The formulas are straightforward to check. We prove for instance the last one, the other ones may be obtained similarly.

For $q \geqslant 2$, the result is obvious.

For $q = 1$, we have that:

$$L_1^j(z \bullet_0 x; y; w) = w \bullet_{j+n+r} (z \bullet_0 x \bullet_0 y) - (z \bullet_0 x) \bullet_0 (w \bullet_j y) =$$
$$(w \bullet_{j+n+r} (z \bullet_0 x \bullet_0 y) - z \bullet_0 (w \bullet_{j+n} (x \bullet_0 y)))$$
$$+(z \bullet_0 (w \bullet_{j+n} (x \bullet_0 y)) - (z \bullet_0 x) \bullet_0 (w \bullet_j y)) =$$
$$L_2^j(z, x; y; w) + z \bullet_0 L_1^j(x; y; w).$$

□

We introduce some notation, in order to prove the relations satisfied by the operations L_q^p.

Notation 6.11. — Let $(A, \bullet_i)$ be a preshuffle algebra and let $x_1, \dots, x_n$ be elements of A. Given a partition $\underline{p} = \{p_1, \dots, p_m\}$ of n, with $p_i \geqslant 0$ for $1 \leqslant i \leqslant m$, and $1 \leqslant j \leqslant n-1$, we denote by $\mathbf{x}$ the element $x_1 \otimes \cdots \otimes x_n \in A^{\otimes n}$, by $\mathbf{x}^{p_1}$ the element $x_{p_1+1} \otimes \cdots \otimes x_n \in A^{\otimes n-p_1}$ and by $\mathbf{x}^{\leqslant j}$ the element $(x_1, \dots, x_j) \in A^{\otimes j}$. The degree of $\mathbf{x}$ is $|\mathbf{x}| := \sum_{i=1}^n |x_i|$. For any $1 \leqslant j \leqslant m-1$, let $\underline{p}_j := (p_{j+1}, \dots, p_m)$ be the partition of $n - p_1 - \cdots - p_j$.

Let $\mathbf{x} = (x_1, \dots, x_n)$, y, $\mathbf{z} = (z_1, \dots, z_m)$, t and w be a collection of elements in A. Given nonnegative integers $0 \leqslant j \leqslant |y|$, $0 \leqslant k \leqslant |w|$ and $1 \leqslant l \leqslant m$ we define:

(1) for a partition $\underline{p} = \{p_1, \dots, p_m\}$ of n,

$$L_{\underline{p}}^j(\mathbf{x}, y, z) := L_n^j(x_1, \dots, x_n; y; z), \quad \text{for } m = 1,$$
$$L_{\underline{p}}^j(\mathbf{x}, y, \mathbf{z}) := L_{p_1}^{j+n_{>p_1}}(x_1, \dots, x_{p_1}; L_{\underline{p}_1}^j(\mathbf{x}^{p_1}; y; (z_2, \dots, z_m); z_1),$$

where $n_{>j} := \displaystyle\sum_{i=j+1}^{q} |x_i|$.

(2) for a partition $\underline{p} = (p_1, \dots, p_{m+1})$ of n,

$$L_{\underline{p}}^{jk}(\mathbf{x}, y, \mathbf{z}, t) := L_{p_1}^{j+k+n_{>p_1}+|\mathbf{z}|}(x_1, \dots, x_{p_1}; L_{\underline{p}_1}^j(\mathbf{x}^{p_1}, y, \mathbf{z}); t).$$

(3) for a partition $\underline{p} = (p_1, \dots, p_{l+1})$ of n,

$$L_{l\underline{p}}^k(\mathbf{x}, y, \mathbf{z}, w, t)$$
$$:= L_{p_1+m-l+1}^k(x_1, \dots, x_{r_1}, L_{\underline{r}_1}^{|y|}(\mathbf{x}^{p_1}, y, \mathbf{z}^{\leqslant l}), z_{l+1}, \dots, z_m; w; t).$$

Theorem 6.12. — *Let $(A, \bullet_i)$ be a preshuffle algebra over K. Given elements $\mathbf{x} = (x_1, \dots, x_n)$, y, $\mathbf{z} = (z_1, \dots, z_m)$, w and t , the operations L_n^j*

verify the following relations:

$$\mathbf{a})\ L_n^j(\mathbf{x};y;L_0^k(w;t)) = \sum_{r=0}^{n} L_r^{j+k+n_{>r}}(\mathbf{x}^{\leqslant r};L_{n-r}^j(x_{r+1},\dots,x_n;y;w);t)+$$
$$\delta_{j|y|}L_{n+1}^k(\mathbf{x},y;w;t)-$$
$$\delta_{k|w|}\sum_{r=0}^{n} L_r^{j+n_{>r}}(L_r^{j+k+n_{>r}}(\mathbf{x}^{\leqslant r};L_{n-r}^j(x_{r+1},\dots,x_n;y;t);w),$$

where $\delta_{pq} := \begin{cases} 1, & \text{for } p=q,\\ 0, & \text{otherwise.}\end{cases}$

b) *For* $m \geqslant 1$,

$$L_n^j(\mathbf{x};y;L_m^k(\mathbf{z};w;t)) =$$
$$\sum_{\underline{p}}\big(L_{\underline{p}}^{jk}(\mathbf{x},y,\mathbf{z}',t) - L_{p_1}^{j+n_{>p_1}}(x_1,\dots,x_{p_1};L_{\underline{p}_1}^{jk}(\mathbf{x}^{p_1},y,\mathbf{z}'^1,t);z_1)\big)+$$
$$\delta_{j|y|}\Big(\sum_{l=1}^{m}\sum_{\underline{q}} L_{l\underline{q}}^k(\mathbf{x},y,\mathbf{z},w,t) - \sum_{l=2}^{m}(\sum_{\underline{r}} L_{r_1}^{|y|+n_{>r_1}}(x_1,\dots,x_{r_1};L_{(l-1)\underline{r}_1}^k(\mathbf{x}^{r_1},y,\mathbf{z}^1,w,t);z_1)+$$
$$L_{n+m+1}^k(\mathbf{x},y,\mathbf{z};w;t) - \sum_{s=0}^{n} L_s^{|y|+n_{>s}}(\mathbf{x}^{\leqslant s};L_{n+m-s}^k(x_{s+1},\dots,x_n,y,z_2,\dots,z_m;w;t);z_1)\Big),$$

where the first sum is taken over all partitions $\underline{p} = (p_1,\dots,p_{m+1})$*, the second one is taken over all partitions* $\underline{q} = (q_1,\dots,q_{l+1})$*, and the third one over all* $\underline{r} = (r_1,\dots,r_{l+1})$ *of* n *and* $\mathbf{z}' := (z_1,\dots,z_m,w)$.

Proof. — For $n,m = 0,1$ the formulas may be checked by a straightforward calculation. The other cases are obtained by recursive arguments on n and m, applying Lemma 6.10 and the formula:

$$L_{n+1}^i(x_1,\dots,x_{n+1};y;w) = L_n^{i+|x_{n+1}|}(x_1,\dots,x_n;x_{n+1}\bullet_0 y;w),$$

for $q \geqslant 1$. □

Definition 6.13. — A $\mathcal{P}rim_{psh}$ *algebra* is a graded vector space V equipped with operations $L_n^j : V^{\otimes n}\otimes V_m\otimes V \longrightarrow V$, for $n \geqslant 0$ and $1 \leqslant j \leqslant m$, which satisfies the relations of the Theorem 6.12.

Note that the Theorem 6.12 implies that the free $\mathcal{P}rim_{psh}$ algebra over a vector space V is linearly spanned by elements $x = L_n^p(x_1,\dots,x_n;y;z)$, where $x_1,\dots,x_n,y$ are elements in the free algebra, and $z \in V$.

Theorem 6.12 states that there exists a functor from the category *Presh* of preshuffle algebras to the category of $\mathcal{P}rim_{psh}$ algebras. Given a preshuffle bialgebra H, the subspace of primitive elements $Prim(H)$ is a $\mathcal{P}rim_{psh}$ subalgebra of H.

We prove for preshuffle algebras analogous results that those obtained for shuffle algebras in Section 4.

Let (C,Θ) be a positively graded coalgebra, since the triple $(\mathbf{Psh}(C),\bullet_0,\Delta_\Theta)$ is a conilpotent nonunital infinitesimal bialgebra, the map

$$e(\xi_n,x)\mapsto(\xi_n;x)-\sum(\xi_{n_1,n_2};x_{(1)},x_{(2)})+\dots\\ +(-1)^{r+1}\sum(\xi_{n_1,\dots,n_r};x_{(1)},\dots,x_{(r)})+\dots$$

gives a linear isomorphism between C and the subset $e(C)$ of $Prim(\mathbf{Psh}(C))$.

We denote by $\mathcal{P}(\mathbf{Psh}(C))$ the subspace of $\mathbf{Psh}(C)$ spanned by the space $e(C)$ with the operations L_n^i, and by $\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 n}$ the space spanned by all the elements of the form $z_1\bullet_0 z_2\bullet_0\cdots\bullet_0 z_n$, with each $z_j\in\mathcal{P}(\mathbf{Psh}(C))$, for $1\leqslant j\leqslant n$. Note that Theorem 6.12 states that any element w in $\mathcal{P}(\mathbf{Psh}(C))$ is a sum of elements of type $L_i^n(x_1,\dots,x_n;y;t)$, with $x_1,\dots,x_n,y\in\mathcal{P}(\mathbf{Psh}(C))$ and $t\in e(C)$.

Proposition 6.14. — *Let (C,Θ) be a positively graded coalgebra. Any element z in $\mathbf{Psh}(C)$ belongs to $\bigoplus_{n\geqslant 1}\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 n}$.*

Proof. — We only need to check that an element

$$z=e(\xi_n\otimes x)\bullet_j(z_1\bullet_0 z_2\bullet_0\cdots\bullet_0 z_r),$$

with $x\in C_n$ and $z_i\in\mathcal{P}(\mathbf{Psh}(C))$, belongs to $\bigoplus_{n\geqslant 1}\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 n}$.

We show it applying a recursive argument on r.

If $r=0$, then $z=e(\xi_n\otimes x)$ belongs to $e(C)$, and the result is obvious.

If $r=1$ and $0<j<|z_1|$, then $z=L_0^j(z_1;e(\xi_n\otimes x))$ belongs to $\mathcal{P}(\mathbf{Psh}(C))$.

If $r=1$ and $j=|z_1|$, then $z=L_0^{|z_1|}(z_1;e(\xi_n\otimes x))+z_1\bullet_0 e(\xi_n\otimes x)$ belongs to $\mathcal{P}(\mathbf{Psh}(C))\bigoplus\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 2}$.

Suppose that $r\geqslant 2$. If $0<j\leqslant|z_1|+\cdots+|z_r|$, then there exists $1\leqslant k\leqslant r$ such that $|z_1|+\cdots+|z_{k-1}|<j\leqslant|z_1|+\cdots+|z_k|$, and

$$z=(e(\xi_n\otimes x)\bullet_j(z_1\bullet_0\cdots\bullet_0 z_k))\bullet_0 z_{k+1}\bullet_0\cdots\bullet_0 z_r.$$

Clearly, if $k<r$ the result follows immediately by recursive hypothesis.

If $k=r$, then

$$z=L_{r-1}^{j-m}(z_1,\dots,z_{r-1};z_r;e(\xi_n\otimes x))+z_1\bullet_0(e(\xi_n\otimes x)\bullet_{i-|z_1|}(z_2\bullet_0\cdots\bullet_0 z_r)),$$

where $m=|z_1|+\cdots+|z_{r-1}|$. But $L_{r-1}^{j-m}(z_1,\dots,z_{r-1};z_r;e(\xi_n\otimes x))\in\mathcal{P}(\mathbf{P}sh(C))$ and, by recursive hypothesis, $e(\xi_n\otimes x)\bullet_{j-|z_1|}(z_2\bullet_0\cdots\bullet_0 z_r)\in\bigoplus_{n\geqslant 1}\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 n}$. So, $z\in\bigoplus_{n\geqslant 1}\mathcal{P}(\mathbf{Psh}(C))^{\bullet_0 n}$. □

For a vector space V, let $\mathcal{P}rim_{psh}(V)$ denotes the free $\mathcal{P}rim_{psh}$ algebra spanned by V.

Proposition 6.15. — *Let (C,Θ) be a positively graded coalgebra. The subspace $\mathcal{P}(\mathbf{Psh}(C))$ is the subspace of primitive elements of $\mathbf{Psh}(C)$. Moreover, it is the free $\mathcal{P}rim_{psh}$ algebra spanned by C.*

Proof. — As for shuffle algebras, it suffices to prove the result for the case where C_n is finite dimensional, for all $n\geqslant 1$.

Proposition 6.9 states that $\mathcal{P}(\mathbf{Psh}(C))\subseteq \mathit{Prim}(\mathbf{Psh}(C))$, while Proposition 6.14 implies that $\mathbf{Psh}(C)=\overline{T}(\mathcal{P}(\mathbf{Psh}(C)))$ as a vector space. From Theorem 5.2 one has that $\mathbf{Psh}(C)=\overline{T}(\mathit{Prim}(\mathbf{Psh}(C)))$, so $\mathcal{P}(\mathbf{Psh}(C))=\mathit{Prim}(\mathbf{Psh}(C))$.

For the second point, we know that the dimension of the subspace $\mathcal{P}(\mathbf{Psh}(C))_n$ of homogeneous elements of degree n of $\mathcal{P}(\mathbf{Psh}(C))$ is $|\mathit{Irr}_{\mathcal{K}_n}(C)|$.

We need to check that the dimension of the homogeneous subspace of degree n of $\mathcal{P}rim_{psh}(C)$ is precisely $|\mathit{Irr}_{\mathcal{K}_n}(C)|$.

Let $\mathcal{P}rim_{psh}(C)_n$ denotes the subspace of degree n of $\mathcal{P}rim_{psh}(C)$. Let $X=\bigcup_{n\geqslant 1}X_n$ be an homogeneous basis of C, a basis of $\mathcal{P}rim_{psh}(C)_n$ is given by the set $\mathbb{B}_n$, where:

$$\mathbb{B}_n:=X_n\bigcup$$
$$\{L_q^j(\mathbf{x};y;z)\ |1\leqslant j\leqslant |y|,\ x_1,\dots,x_q,y\in\bigcup_{k=1}^{n-1}\mathbb{B}_k\ ,\ z\in X\ ,\ q\geqslant 0 \text{ and}\sum_{i=1}^{q}|x_i|+|y|+|z|=n\}.$$

On the other hand, note that if $f\otimes x_1\otimes\cdots\otimes x_n\in \mathit{Irr}_{\mathcal{K}_n}(C)$, then $f=\xi_{|f^{-1}(1)|}\bullet_j f'$, for $1\leqslant j\leqslant |f'|$. Moreover, if $f'=f_1\bullet_0\cdots\bullet_0 f_r$, with $f_i\in \mathit{Irr}_{\mathcal{K}_{n_i}}$ for $1\leqslant i\leqslant r$, then $f=\xi_{|f^{-1}(1)|}\bullet_j f'$ with $n_1+\cdots+n_{r-1}<j\leqslant |f'|$.

Define a map $\alpha:\bigcup_{n\geqslant 1}\mathbb{B}_n\longrightarrow\bigcup_{n\geqslant 1}\mathit{Irr}_{\mathcal{K}_n}(C)$ as follows:

(1) $\alpha(x):=(\xi_n\otimes x)$, for $x\in X_n$,

(2) $\alpha(L_0^j(y;z)) := (\xi_{|z|} \otimes z) \bullet_j \alpha(y)$, for $1 \leqslant j \leqslant |y|$,

(3) $\alpha(L_q^j(\mathbf{x};y;z)) :=$
$$(\xi_{|z|} \otimes z) \bullet_{j+|\mathbf{x}|} (\alpha(x_1) \bullet_0 \cdots \bullet_0 \alpha(x_q) \bullet_0 \alpha(y)),$$
for $1 \leqslant j \leqslant |y|$.

The previous argument states that α is well-defined.

Conversely, let $f \otimes x_1 \otimes \cdots \otimes x_n \in Irr_{\mathcal{K}_n}(C)$, with $x_1, \dots, x_n \in X$, be such that $f = \xi_{|f^{-1}(1)|} \bullet_j f'$, where $f' = f_1 \bullet_0 \cdots \bullet_0 f_r$, with $f_i \in Irr_{\mathcal{K}_{n_i}}$ for $1 \leqslant i \leqslant r$, and $n_1 + \cdots + n_{r-1} < j \leqslant |f'|$. If $r = 0$, then $(f; x_1, \dots, x_n) = (\xi_n \otimes x_1)$ and $\beta(\xi_n \otimes x_1) = x_1 \in X$.

If $r = 1$, then $\beta(f \otimes x_1 \otimes \cdots \otimes x_n)) = L_0^j(\beta(f' \otimes x_2 \otimes \cdots \otimes x_n); x_1)$.

If $r > 1$, then there exists a composition $(j_1, \dots, j_r)$ of $n-1$ such that $\displaystyle\sum_{l=j_1+\cdots+j_{i-1}+2}^{j_1+\cdots+j_i+1} |x_l| = n_i$, for $1 \leqslant i \leqslant r$. In this case, we have that:

$$\beta(f \otimes x_1 \otimes \cdots \otimes x_n) =$$
$$L_{r-1}^j(\beta(f_1 \otimes x_2 \otimes \cdots \otimes x_{j_1+1}), \dots, \beta(f_{r-1} \otimes \cdots \otimes x_{j_1+\cdots+j_{r-1}+1}); \beta(f_r \otimes \cdots \otimes x_n); x_1).$$

Clearly, β is the inverse map of α, which implies that the dimension of $\mathcal{P}rim_{psh}(C)_n$ is $|Irr_{\mathcal{K}_n}(C)|$, for $n \geqslant 1$. So, $\mathcal{P}rim_{psh}(C) = Prim(\mathbf{Psh}(C))$. $\square$

The following result is a straightforward consequence of Theorem 5.2 and Proposition 6.15.

Proposition 6.16. — *Let (C, Θ) be a positively graded coalgebra. There exists a coalgebra isomorphism between $\mathbf{Psh}(C)$ and $\overline{T}^c(\mathcal{P}rim_{psh}(C))$.*

Example 6.17. — **Basis of primitive elements for the Malvenuto-Reutenauer bialgebra.** In [5] and [2] the authors describe different basis for the subspace of primitive elements of the Malvenuto-Reutenauer bialgebra. We construct another one using our description of primitive elements of a preshuffle bialgebra.

The bialgebra associated to the free preshuffle algebra $\mathbf{Psh}(Kx_0)$, spanned by one element in degree one, is the Malvenuto-Reutenauer bialgebra $K[S_\infty]$. The dimension of the subspace of its primitive elements of degree n is the number of irreducible permutations of S_n. Using Proposition 5.10, we associate to any $\sigma \in Irr_{S_n}$ a primitive element $E(\sigma)$ in the following way:

(1) $E(1) := (1)$, for $(1) \in S_1$.

(2) For $\sigma \in Irr_{S_n}$ with $n > 1$, there exists a unique family of irreducible permutations $\sigma_1, \dots, \sigma_r$ and $\delta \in Sh(1, n-1)$ such that $\sigma = ((1) \times \sigma_1 \times \cdots \times \sigma_r) \cdot \delta$. Since σ is irreducible, we get that $\sigma^{-1}(1) > n_1 + \cdots + n_{r-1} + 1$ for $|\sigma_i| = n_i$. We define

$$E(\sigma) := L_{r-1}^{\sigma^{-1}(1)-1-n_1-\cdots-n_{r-1}}(E(\sigma_1), \dots, E(\sigma_{r-1}); E(\sigma_r); (1)),$$

where the operations L_i^j are the operations introduced of Definition 6.8.

Propositions 6.14 and 6.15 imply that the set $\{E(\sigma)\}_{\sigma \in \bigcup Irr_{S_n}}$ is a basis of the subspace of primitive elements of the Malvenuto-Reutenauer bialgebra. Forexample, we have that:

$$E_{(2,1)} = (2,1) - (1,2), \qquad E_{(3,1,2)} = L_0^1(E_{(2,1)}; (1)) = (3,1,2) - (2,1,3),$$
$$E_{(3,4,2,5,7,1,6)} = L_2^1(E_{(2,3,1)}, E_{(1)}; E_{(2,1)}; (1)) =$$
$$(3,4,2,5,7,1,6) - (2,3,1,5,7,4,6) - (3,4,2,5,6,1,7) + (2,3,1,5,6,4,7) -$$
$$(2,4,3,5,7,1,6) + (1,3,2,5,7,4,6) + (2,4,3,5,6,1,7) - (1,3,2,5,6,4,7).$$

Given a $\mathcal{P}rim_{psh}$ algebra $(V, \overline{L}_n^i)$, let $\mathcal{U}_{Psh}(V)$ be the preshuffle bialgebra obtained by making the quotient of the free preshuffle algebra $\mathbf{Psh}(V)$ by the ideal (as a preshuffle algebra) spanned all the elements:

$$L_q^i(x_1, \dots, x_q; y; z) - \overline{L}_q^i(x_1, \dots, x_q; y; z),$$

with $x_1, \dots, x_q, y, z \in V$, $q \geqslant 0$ and $1 \leqslant i \leqslant |\, y \,|$, where L_q^i denotes the operations associated to the preshuffle algebra $K[\mathcal{K}(V)]$.

The proof of the following result is obtained applying the same steps and arguments that those used in the proof of Theorem 5.13.

Theorem 6.18. — **a)** *Let $(H, \circ_i, \Delta)$ be a conilpotent preshuffle bialgebra, then H is isomorphic to $\mathcal{U}_{Psh}(Prim(H))$, where $Prim(H)$ is the $\mathcal{P}rim_{psh}$ algebra of primitive elements of H.*
b) *Let $(V, \overline{L}_q^i)$ be a $\mathcal{P}rim_{psh}$ algebra, then V is isomorphic to $Prim(\mathcal{U}_{Psh}(V))$.*

Primitive elements of pre-Lie systems.

Let $(A, \bullet_i, \Delta)$ be a coalgebra structure on a pre-Lie system, we want to compute its primitive elements.
By Proposition 6.9, the elements $L_n^p(x_1, \dots, x_n; y; z)$ are primitive in A, for $1 \leqslant p < |y|$, whenever the elements $x_1, \dots, x_n, y, z$ belong to $Prim(A)$. However, an easy calculation shows that $L_n^p(x_1, \dots, x_n; y; z) = 0$ for any $x_1, \dots, x_n, y, z \in A$ and $n \geqslant 1$, which motivates the following definition.

Definition 6.19. — A $\mathcal{P}rim_{PLie}$ algebra over K is a graded vector space V equipped with a family of binary operations $\{-,-\} : V\otimes V \longrightarrow V$ and $\bullet_p : V\otimes V_n \longrightarrow V$, for $1\leqslant p\leqslant n-1$, such that:

(1) $\{\{x,y\},z\} = \{x,\{y,z\}\} + y\bullet_{|x|}\{x,z\}$, *for* $x,y,z\in V$.
(2) $\{x\bullet_p y, z\} = x\bullet_p \{y,z\}$,
(3) $\{x, y\bullet_p z\} = y\bullet_{|x|+p}\{x,z\}$,
(4) $\{x,y\}\bullet_p z = y\bullet_{|x|+p}(x\bullet_p z) - x\bullet_p(y\bullet_p z)$, *for* $1\leqslant p<|z|$,
(5) $(x\bullet_p y)\bullet_q z = x\bullet_{p+q}(y\bullet_q z)$,
(6) $x\bullet_p(y\bullet_q z) = y\bullet_{|x|+q}(x\bullet_p z)$, *if* $1\leqslant p<q<|z|$,

for $x,y,z\in V$.

Clearly, any grafting bialgebra $(A,\bullet_i,\Delta)$ has a natural structure of $\mathcal{P}rim_{PLie}$ algebra, such that $Prim(A)$ is a $\mathcal{P}rim_{PLie}$ subalgebra of A.

For any positively graded coalgebra (C,Θ), there exists a natural extension of the coproduct to a coassociative coproduct Δ_Θ such that $(\mathbf{PLie}(C),\bullet_i,\Delta_\Theta)$ is a pre-Lie system with coproduct. Moreover, the vector space $\mathbf{PLie}(C)$ equipped with the associative product $\circ_0$ and Δ_Θ is a nonunital infinitesimal conilpotent bialgebra, so it is isomorphic to $\overline{T}^c(Prim(\mathbf{PLie}(C)))$.

Let $\mathcal{P}(\mathbf{PLie}(C))$ the subspace of $\mathbf{PLie}(C)$ spanned by $e(C)$ with the operations $\{-,-\}$ and $\circ_p$.

Proposition 6.20. — *Let (C,Θ) be a positively graded coalgebra. Any element z in $\mathbf{PLie}(C)$ may be written as a sum $z=\sum_k z_1^k\circ_0 z_2^k\circ_0\cdots\circ_0 z_{r_k}^k$, with $z_i^k\in\mathcal{P}(\mathbf{PLie}(C))$.*

Proof. — The space $\mathbf{PLie}(C)$ is a quotient of $\mathbf{Psh}(C)$, the projection is denoted by Π. Let e_{Psh} (respectively, e_{Gr}) denotes the projection from $\mathbf{Psh}(C)$ (respectively, $\mathbf{PLie}(C)$) into its primitive part.

The set $\Pi^{-1}(e_{PLie}(x))$ has a unique element, for any $x\in C$; which implies that the restriction of Π to $e_{Psh}(C)$ is a monomorphism, whose image is $e_{PLie}(C)$.

Moreover, since Π sends the product $\bullet_p$ to $\circ_p$, for $p\geqslant 0$, we have that $\Pi(\mathcal{P}(\mathbf{Psh}(C)))\subseteq\mathcal{P}(\mathbf{PLie}(C)))$.

Let $z\in\mathbf{PLie}(C)$, there exist at least one element $\tilde z\in\mathbf{Psh}(C)$ such that $\Pi(\tilde z)=z$.

We know that $\tilde z=\sum_k \tilde z_1^k\bullet_0\tilde z_2^k\bullet_0\cdots\bullet_0\tilde z_{r_k}^k$, with $\tilde z_i^j\in\mathcal{P}(\mathbf{Psh}(C))$.

So, $z=\sum_k\Pi(\tilde z_1^k)\circ_0\Pi(\tilde z_2^k)\circ_0\cdots\circ_0\Pi(\tilde z_{r_k}^k)$, with $\Pi(\tilde z_1^k)\in\mathcal{P}(\mathbf{PLie}(C))$. □

Proposition 6.21. — *Let (C,Θ) be a positively graded coalgebra. The subspace $\mathcal{P}(\mathbf{PLie}(C))$ is the subspace of primitive elements of $\mathbf{PLie}(C)$. Moreover, it is the free $\mathcal{P}rim_{PLie}$ algebra $\mathcal{P}rim_{PLie}(C)$, spanned by C.*

Proof. — The proof of the first assertion is identical to the ones given for preshuffle and shuffle algebras. To prove that $\mathcal{P}(\mathbf{PLie}(C))$ is the free $\mathcal{P}rim_{PLie}$ algebra $\mathcal{P}rim_{PLie}(C)$, we may suppose that the space C_n is finite dimensional, for all $n \geqslant 1$.

Let X be an homogeneous basis of C. Since the associative algebra $((K[T_\infty(C)]), \bullet_0)$ is free on the set $\{t = \bigvee_x(|, t^1, \dots, t^r)\}$, the dimension of the subspace of homogeneous elements of degree n of $Prim(\mathbf{PLie}(C))$ is the number of trees of the form $t = \bigvee_x(|, t^1, \dots, t^r)$, where $x \in X_r$ and $|t| = \sum_{1 \leqslant j \leqslant r} |t^j| + r - 1$.

Let $\{X\}$ be the set of all elements of the form $z = \{x_1, \{\dots, \{x_{k-1}, x_k\}\}\}$, with $k \geqslant 1$ and $x_i \in X$. From Definition 6.19 we have that the elements of X and the elements of type:

$$z = x_1 \bullet_{i_1} (\cdots \bullet_{i_{r-2}} (x_{r-1} \bullet_{i_{r-1}} (x_r \bullet_{i_r} w))),$$

with $i_1 > \cdots > i_r$, $1 \leqslant i_j < |x_{j+1}| + \cdots + |x_r| + |w|$, $x_j \in X$, and $w \in \{X\}$, are a basis of $\mathcal{P}rim_{PLie}(C)$ as a vector space.

The map γ from the basis described above to the set $\{t = \bigvee_x(|, t^1, \dots, t^r) \mid$ with $x \in X_r$ and $|t| = \sum_{1 \leqslant j \leqslant r} |t^j| + r - 1\}$, is defined as follows:

$$\gamma(x) := (\mathfrak{c}_n, x), \text{ for } x \in X$$

$$\gamma(\{x_1, \{\dots, \{x_{k-1}, x_k\}\}\}) := \bigvee_{x_1}(|, \gamma(\{x_2, \{\dots, \{x_{k-1}, x_k\}\}\})),$$

for $x_1, \dots, x_k \in X$

$$\gamma(x_1 \bullet_{i_1} (\cdots \bullet_{i_{r-2}} (x_{r-1} \bullet_{i_{r-1}} (x_r \bullet_{i_r} w)))) := (\mathfrak{c}_{n_1}, x_1) \circ_{i_1} (\dots ((\mathfrak{c}_{n_r}, x_r) \circ_{i_r} \gamma(w))),$$

for $|x_i| = n_i$.

Clearly, γ is a graded bijection, which sends elements of degree n of the basis to trees of the same degree. So, $\mathcal{P}(\mathbf{PLie}(C))$ is a quotient of the free $\mathcal{P}rim_{PLie}$ algebra over C such that both spaces have the same dimension on each degree, which implies they are isomorphic. □

There exists a natural equivalence between the categories of conilpotent coalgebra structures on a pre-Lie system and $\mathcal{P}rim_{PLie}$ algebras. As in previous cases we define, for any $\mathcal{P}rim_{PLie}$ algebra $(V, [-,-], \circ_p)$, the universal envelopping pre-Lie system $\mathcal{U}_{PLie}(V)$ as the quotient of the free pre-Lie system $\mathbf{PLie}(V)$ by the ideal spanned by the elements:

$\{x, y\} - [x, y]$ and $x \bullet_p y - x \circ_p y$, for $x, y \in V$ and $1 \leqslant p < |y|$, where $\{-,-\}$ and $\bullet_p$ denote the operations associated to the pre-Lie system $\mathbf{PLie}(V)$.

The proof of the following result is similar to the proof given for shuffle and preshuffle bialgebras.

THEOREM 6.22. — **a)** *Let $(H, \circ_i, \Delta)$ be a conilpotent coalgebra structure on a pre-Lie system, then H is isomorphic to $\mathcal{U}_{PLie}(Prim(H))$, where $Prim(H)$ is the $\mathcal{P}rim_{PLie}$ algebra of primitive elements of H.*
b) *Let $(V, \{-,-\}, \bullet_p)$ be a $\mathcal{P}rim_{PLie}$ algebra, then V is isomorphic to $Prim(\mathcal{U}_{PLie}(V))$.*

7. Some triples of operads

Note that preshuffle algebras, shuffle algebras and pre-Lie systems algebras are not described by classical linear operads, but by coloured operads. In this section we give an easy way to compute primitive elements of infinitesimal bialgebras having two associative products applying the results obtained in Section 6. The examples that we study describe two *good triples* of K-linear operads, in the sense of [12], where the co-operad is always the associative co-operad.

Recall that a 2-associative algebra is simply a vector space equipped with two associative products $\cdot$ and $\circ$. In [14] we give a description of the free 2-associative algebra on a vector space V which we denote by **2-ass**(V).

DEFINITION 7.1. — A 2-*infinitesimal nonunital bialgebra* is a 2-associative algebra $(A, \cdot, \circ)$ equipped with a coassociative coproduct Δ, such that the triples $(A, \cdot, \Delta)$ and $(A, \circ, \Delta)$ are infinitesimal nonunital bialgebras.

For any preshuffle bialgebra $(A, \bullet_i, \Delta)$ the triple $(A, \bullet_0, \bullet_{top}, \Delta)$ is a 2-infinitesimal nonunital bialgebra. If we look at the structure of $\mathcal{P}rim_{psh}$ algebra of A described in Definition 6.8, then the unique operations which are defined using the products $\bullet_0$ and $\bullet_{top}$ are the $n+2$-ary products $L_n^{|y|}(x_1, \dots, x_n; y; z)$, for $n \geqslant 1$. Note that they do not verify any relationship.

DEFINITION 7.2. — A $Mag(\infty)$ algebra over K is a vector space M, equipped with n-linear maps $\mu_n : M^{\otimes n} \longrightarrow M$, for $n \geqslant 2$.

Let $(A, \cdot, \circ)$ be a 2-associative algebra, define $\mu_n : A^{\otimes n} \longrightarrow A$ be the n-ary operation:

$$\mu_2(x_1, x_2) := x_1 \cdot x_2 - x_1 \circ x_2,$$
$$\mu_n(x_1, \dots, x_n)$$
$$:= (x_1 \cdot (\cdots (x_{n-2} \cdot x_{n-1}))) \circ x_n - x_1 \cdot ((x_2 \cdot (\cdots (x_{n-2} \cdot x_{n-1}))) \circ x_n),$$

for $x_1, \ldots, x_n \in A$ and $n \geqslant 2$. Clearly, (A, μ_n) is a $Mag(\infty)$ algebra.

Proposition 7.3. — *Let $(A, \cdot, \circ, \Delta)$ be a 2-infinitesimal nonunital bialgebra, the subspace $Prim(A)$ of primitive elements of A is closed under the products μ_n, for $n \geqslant 2$.*

Proof. — The result is a straigthforward consequence of Proposition 6.9, applying that $\mu_n(x_1, \ldots, x_n)$ coincides with $L_{n-2}^{|x_{n-1}|}(x_1, \ldots, x_{n-2}; x_{n-1}; x_n)$, for $x \bullet_0 y := x \cdot y$ and $x \bullet_{|y|} y = y \circ x$. □

Applying similar arguments that the ones we use in Section 6, we get that

Proposition 7.4. — (1) *For any vector space V there exists a unique coassociative coproduct Δ on $\mathbf{2\text{-}ass}(V)$ such that all the elements of V are primitive and $(\mathbf{2\text{-}ass}(V).\cdot, \circ, \Delta)$ is a 2-infinitesimal nonunital bialgebra. Moreover, the space of primitive elements $Prim(\mathbf{2\text{-}ass}(V))$ is the free $Mag(\infty)$ algebra on V.*

(2) *If we denote by $\mathbf{Mag}_\infty(V)$ the free $Mag(\infty)$ algebra on V, then $\mathbf{2\text{-}ass}(V)$ is isomorphic, as a coalgebra, to $\overline{T}^c(\mathbf{Mag}_\infty(V))$.*

(3) *The functor which sends any 2-infinitesimal nonunital bialgebra H to the $Mag(\infty)$ algebra $Prim(H)$ gives an equivalence between the category of conilpotent 2-infinitesimal nonunital bialgebras and the category of $Mag(\infty)$ algebras.*

The following example is studied in [12].

Definition 7.5. — A duplicial algebra $(A, /, \backslash)$ is a 2-associative algebra over K such that the associative products $/$ and $\backslash$ satisfy the following relation:

$$x/(y\backslash z) = (x/y)\backslash z,$$

for $x, y, z \in A$.

Note that for any pre-Lie system $(A, \bullet_i)$, the space A with the products:

$$x/y := x \bullet_0 y \text{ and } x\backslash y := x \bullet_{top} y$$

is a duplicial algebra.

It is not difficult to verify (see [20] or [12]) that the free duplicial algebra spanned by a vector space V is the space of planar binary rooted trees $K[Y_\infty(V)]$, with the vertices coloured by the elements of V. We denote it by $\mathbf{Dup}(V)$.

Let $(A, /, \backslash)$ be a duplicial algebra, a duplicial bialgebra structure on A is given by a coassociative coproduct Δ such that $(A, /, \backslash, \Delta)$ is a 2-associative bialgebra. In particular, for any vector space V, the free duplicial algebra

Dup$V$) is a duplicial bialgebra. The unique operation of $\mathcal{P}rim_{PLie}$ which may be defined in any duplicial algebra is the product $\{-,-\}$, which does not verify any relation.

Definition 7.6. — *A magmatic algebra over K is a vector space M equipped with a bilinear map $M \otimes M \longrightarrow M$.*

Consider the functor from the category of duplicial algebras to the category of magmatic algebras, which maps $(A, /, \backslash) \mapsto (A, \{-,-\})$. If $(A, /, \backslash, \Delta)$ is a duplicial bialgebra, then $(Prim(A), \{-,-\})$ is a magmatic subalgebra of $(A, \{-,-\})$.

In this case again, we get a structure theorem relating conilpotent duplicial bialgebras and magmatic algebras. For any vector space V, let $\{V, V\}$ denote the subspace of the free duplicial algebra $\mathbf{Dup}(V)$ spanned by the elements of V under the operation $\{-,-\}$. The proof of the following results may be obtained following the same arguments that in Proposition 6.21.

Proposition 7.7. — *The coalgebra $\overline{T}^c\{V, V\}$ is isomorphic to $\mathbf{Dup}(V)$. Moreover, $\{V, V\}$ is the free magmatic algebra spanned by V.*

Consider the functor from the category of duplicial algebras to the category of magmatic algebras, given by $(A, /, \backslash) \mapsto (A, \{-,-\})$, and let $\mathcal{U}_{dup}$ be its left adjoint. A Cartier-Milnor-Moore type theorem for conilpotent duplicial bialgebras follows applying the general results obtained in [12] for triples of operads, we just state it.

Theorem 7.8. — **a)** *Any conilpotent duplicial bialgebra A is isomorphic to $\mathcal{U}_{dup}(Prim(A))$, where $Prim(A)$ is the magmatic algebra of primitive elements of A.*
b) *Any magmatic algebra M is isomorphic to $Prim(\mathcal{U}_{dup}(M))$.*

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