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IMBALANCES IN ARNOUX-RAUZY SEQUENCES

by J. CASSAIGNE, S. FERENCZI & L.Q. ZAMBONI

1. Introduction.

The regular continued fraction algorithm provides a formidable link between the arithmetic/diophantine properties of an irrational number α, the ergodic/dynamical properties of a rotation by angle α on the 1-dimensional torus, and the symbolic/combinatorial properties of a class of binary sequences known as the Sturmian infinite words (see below). A fundamental problem is to generalize and extend this rich interaction to dimension two or greater, starting either from a dynamical system or a specified class of sequences. A primary motivation is that such a generalization could yield a satisfying algorithm of simultaneous rational approximation in $\mathbb{R}^n$. In case $n = 2$ there are a number of different dynamical systems all of which are natural candidates to play the role of a rotation in dimension one: for instance, promising results have been obtained by considering the dynamics of three-interval exchange transformations on the unit interval [15], [16], or of two rotations on the circle [2], [5]. However, ever since the early work of Rauzy in [27], the most natural generalization was thought to be the one stemming from a rotation on the 2-torus; in this case, the associated symbolic counterpart is given by a class of sequences of complexity $2n+1$ introduced by Arnoux and Rauzy in [4] which are a natural generalization of the Sturmian sequences. Though the resulting arithmetic/ergodic/combinatorial interaction is very satisfying in the special case of the so-called Tribonacci system, a more general canonical

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equivalence (through what is called a natural coding, see definition below) between two-dimensional rotations and Arnoux-Rauzy sequences was only a conjecture. The aim of this paper is to provide a counter-example to this conjecture, through hitherto unsuspected combinatorial properties of some very special Arnoux-Rauzy sequences.

Let ω be a sequence with values in a finite alphabet $\mathcal{A}$. A *factor* of ω of length n is a string $\omega_i \dots \omega_{i+n-1}$ for some i in $\mathbb{N}$. The *(block) complexity function* $p_\omega : \mathbb{N} \to \mathbb{N}$ assigns to each n the number of distinct factors of ω of length n. A fundamental result due to Morse and Hedlund [23] states that a sequence ω is ultimately periodic if and only if the complexity satisfies $p_\omega(n) \leqslant n$ for some n. Sequences of complexity $p(n) = n + 1$ are called *Sturmian words.* The most well known Sturmian word is the so-called Fibonacci sequence

$$121121211211212112121121121211211212112121121121211 \dots$$

fixed by the morphism $1 \mapsto 12$ and $2 \mapsto 1$. In [24] Morse and Hedlund showed that each Sturmian word may be realized geometrically by an irrational rotation on the circle. More precisely, every Sturmian word is obtained by coding the symbolic orbit of a point x on the circle (of circumference one) under a rotation by an irrational angle α where the circle is partitioned into two complementary intervals, one of length α and the other of length $1 - \alpha$. And conversely each such coding gives rise to a Sturmian word. The irrational α is called the *slope* of the Sturmian word. An alternative characterization using continued fractions was given by Rauzy (though it may have been known earlier) in [25] and [26], and later by Arnoux and Rauzy in [4]. Sturmian words admit various other types of characterizations of geometric and combinatorial nature (see for instance [20]). For example they are characterized by the following balance property: a sequence ω is Sturmian if and only if ω is a binary non-ultimately periodic sequence with the property that for any two factors u and v of ω of equal length, we have $-1 \leqslant |u|_i - |v|_i \leqslant 1$ for each letter i. Here $|u|_i$ denotes the number of occurrences of i in u.

The condition $p_\omega(n) = n + 1$ implies the existence of exactly one right special and one left special factor of each length. A factor u of ω is called right special (respectively left special) if for some distinct letters $a, b \in \mathcal{A}$ the words ua and ub (respectively au and bu) are each factors of ω. In [4] Arnoux and Rauzy introduced a class of uniformly recurrent (minimal) sequences ω of complexity $p_\omega(n) = 2n + 1$ characterized by the following combinatorial criterion known as the $\star$ condition: ω admits exactly one right special and one left special factor of each length. We call them *Arnoux-Rauzy sequences* or *A-R sequences* for short. This condition distinguishes them from other sequences of complexity $2n+1$ such as those obtained by coding trajectories of 3-interval exchange transformations [15], [16] or those of *Chacon type*, i.e., topologically isomorphic to the subshift generated by the Chacon sequence [8], [14]. A-R sequences have seen a recent surge of interest: [3], [7], [9], [10], [11], [17], [21], [22], [29], [30], [31]. In [1] Arnoux showed that the Tribonacci sequence may be geometrically realized by an exchange of six intervals on the circle. The result was later generalized by Arnoux and Rauzy in [4] to all A-R sequences.

For each $i \in \{1,2,3\}$ define morphims σ_i by $\sigma_i(i) = i$ and $\sigma_i(j) = ij$ for $i \neq j \in \{1,2,3\}$. We call the σ_i *standard A-R morphisms.* A non-trivial morphism $\sigma : \{1,2,3\} \to \{1,2,3\}^*$ is called an *A-R morphism* if it fixes an A-R sequence (this is indeed the case for the σ_i). It is readily verified that any composition of the standard A-R morphims in which each σ_i occurs at least once is a primitive A-R morphism. Arnoux and Rauzy proved that each A-R sequence is in the shift orbit closure of a unique sequence of the form

$$\sigma_{n_1} \circ \sigma_{n_2} \circ \sigma_{n_3} \circ \cdots (1)$$

where the sequence $(n_k) \in \{1,2,3\}^{\mathbb{N}}$ takes on each value in $\{1,2,3\}$ an infinite number of times (see [4]; see also [12] and [29] for some generalizations of this result).

The most famous example of an A-R morphism is the so-called Tribonacci morphism $1 \mapsto 12$, $2 \mapsto 13$, $3 \mapsto 1$. This example was made famous by Rauzy in [27] where he showed that the symbolic subshift generated by this morphism is measure-theoretically conjugate to an exchange of three fractal domains on a compact set in $\mathbb{R}^2$. This example was also studied in great detail by Messaoudi in [21], [22] and Ito-Kimura in [18]. Let $\mathbb{T}^n$ denote the n-torus $\mathbb{R}^n/L$ where L is a lattice (discrete maximal subgroup) in $\mathbb{R}^n$. A *rotation* on $\mathbb{T}^n$ is a pair (R, α) where $\alpha \in \mathbb{R}^n$, and $R : \mathbb{T}^n \to \mathbb{T}^n$ is defined by $R(x) = x + \alpha \pmod{L}$. We say a sequence $\omega = \omega_1\omega_2\omega_3\ldots \in \{1,2,\ldots,n\}$ is a *natural coding* of a rotation (R, α) of $\mathbb{T}^n$ if there exists a fundamental domain Ω for R together with a partition $\Omega = \Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_n$ such that on each Ω_i the map R is a translation by a vector α_i and the sequence ω is the symbolic coding of the R-orbit of a point $x \in \Omega$ with respect to the Ω_i, i.e., $R^k(x) \in \Omega_i$ whenever $\omega_k = i$. In [28] Rauzy proved that the Tribonacci sequence is a natural coding of a rotation on the 2-torus. More generally, it was believed that each A-R sequence is a natural coding of a rotation on $\mathbb{T}^2$. This conjecture was formulated, though not written, by Arnoux and

Rauzy at the time of [4], and since then there have been several efforts made towards its resolution. Most recently, Arnoux and Ito [3] obtained some partial results in support of this conjecture in case the sequence is fixed by an A-R morphism. See also [6] for related results in this direction.

In this paper we exhibit a counterexample to the conjecture; we first construct an A-R sequence ω_* which is unbalanced in the following sense: for each positive integer N there exist factors u and v of ω_* of equal length and $i \in \{1,2,3\}$ such that $|u|_i - |v|_i > N$. In itself this sequence provides a strong counterexample to a conjecture of Droubay, Justin and Pirillo (see 4.5 in [12]) which states that for each Episturmian sequence ω on $\{1,2,3\}$ and all factors u and v of ω of equal length, the inequality $-2 \leqslant |u|_i - |v|_i \leqslant 2$ holds for all $i \in \{1,2,3\}$. We then use a result of Rauzy on bounded remainder sets in [28], later generalized by Ferenczi in [13], to establish the existence of an A-R sequence which is not a natural coding of a rotation on the 2-torus.

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2. Imbalances in A-R sequences.

Let ω be a sequence with values in a finite alphabet $\mathcal{A}$. If u is a factor of ω we denote its length by $|u|$. Let N be a positive integer. We will say that ω is N-balanced if for all factors u and v of equal length we have $-N \leqslant |u|_i - |v|_i \leqslant N$ for all $i \in \mathcal{A}$. As an immediate consequence of the definition we have

LEMMA 2.1. — *If a sequence $\omega \in \mathcal{A}^{\mathbb{N}}$ contains two factors u and v such that $|u|_i - |v|_i > N$ and $|v|_j - |u|_j > N$ for some $i, j \in \mathcal{A}$, then ω is not N-balanced.*

In [12], Droubay, Justin and Pirillo study various combinatorial properties of so-called *Episturmian sequences.* A sequence is called Episturmian if it is closed under reversal and has at most one right special factor of each length. It is readily verified that Sturmian sequences and A-R sequences are Episturmian [12]. In 4.5 in [12] the authors conjecture that an Episturmian sequence on N letters is $(N-1)$-balanced. By taking $N \geqslant 2$ the following proposition provides a counterexample to the belief that A-R sequences are 2-balanced (see [12]):

PROPOSITION 2.2. — *For each positive integer N there exists a primitive A-R morphism σ such that for every A-R sequence ω the A-R sequence $\sigma(\omega)$ is not N-balanced. In particular the fixed point of σ is not N-balanced.*

Proof. — We show by induction that for each positive integer n there exist nonnegative integers a_n, b_n, c_n and a primitive A-R morphism $\sigma_{[n]}$ such that for all A-R sequences ω the sequence $\sigma_{[n]}(\omega)$ contains two subwords u_n and v_n of equal length with

$$\begin{pmatrix} |u_n|_i \\ |u_n|_j \\ |u_n|_k \end{pmatrix} = \begin{pmatrix} a_n \\ b_n + n \\ c_n \end{pmatrix}$$

and

$$\begin{pmatrix} |v_n|_i \\ |v_n|_j \\ |v_n|_k \end{pmatrix} = \begin{pmatrix} a_n + 1 \\ b_n \\ c_n + n - 1 \end{pmatrix}$$

for some choice of distinct $i, j, k \in \{1, 2, 3\}$. The result is trivially true for $n = 1, 2$; for $n = 2$ we can take $\sigma_{[2]} = \sigma_1\sigma_2$, $u_2 = 212$, and $v_2 = 131$. Although the case $n = 3$ will follow from induction, we can take $\sigma_{[3]} = \sigma_1^2\sigma_2^2\sigma_1\sigma_3$. It is readily verified that for all A-R sequences ω the sequence $\sigma_{[3]}(\omega)$ contains the two factors 211211121121121112112 and 113112112111211211311 of length twenty one. Note the first factor contains three more occurrences of 2 than the other.

We suppose the result is true at stage n and we show it is true at stage $n+1$. Let $u_n, v_n, a_n, b_n, c_n, i, j, k$, and $\sigma_{[n]}$ be as above. Define $\sigma_{[n+1]} = \sigma_k^n \circ \sigma_i^n \circ \sigma_{[n]}$. To construct u_{n+1} we apply $\sigma_k^n \circ \sigma_i^n$ to v_n with a few minor modifications. Let ω be any A-R sequence and u a non-empty factor of ω. For each $a \in \{1, 2, 3\}$ the word $\sigma_a(u)a$ is a factor of $\sigma_a(\omega)$ which begins in the letter a. We define $\sigma_{(a,+)}(u) = \sigma_a(u)a$ and $\sigma_{(a,-)}(u)$ to be $\sigma_a(u)$ deprived of its initial letter a. Then we set $u_{n+1} = \sigma_{(k,-)}^n \circ \sigma_{(i,+)}^n(v_n)$

and $v_{n+1} = \sigma^n_{(k,+)} \circ \sigma^n_{(i,-)}(u_n)$. It is readily checked that

$$\begin{pmatrix} |u_{n+1}|_i \\ |u_{n+1}|_j \\ |u_{n+1}|_k \end{pmatrix} = \begin{pmatrix} a_n + n(b_n + c_n + n - 1) + n + 1 \\ b_n \\ c_n + n - 1 + n\big(a_n + n(b_n + c_n + n) + b_n\big) \end{pmatrix}$$

while

$$\begin{pmatrix} |v_{n+1}|_i \\ |v_{n+1}|_j \\ |v_{n+1}|_k \end{pmatrix} = \begin{pmatrix} a_n + n(b_n + c_n + n - 1) \\ b_n + n \\ c_n + n\big(a_n + n(b_n + n + c_n) + b_n + 1\big) \end{pmatrix}.$$

The result now follows by taking

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} c_n + n - 1 + n\big(a_n + n(b_n + c_n + n) + b_n\big) \\ a_n + n(b_n + c_n + n - 1) \\ b_n \end{pmatrix}.$$

Note that clearly σ does have a fixed point, since, by construction, $\sigma(1)$, $\sigma(2)$ and $\sigma(3)$ all begin with the same letter. □

PROPOSITION 2.3. — *For each positive integer N and for each A-R morphism σ there exists a primitive A-R morphism σ' such that for all A-R sequences ω the A-R sequence $\sigma \circ \sigma'(\omega)$ is not N-balanced.*

Proof. — Let σ be a primitive A-R morphism on the alphabet $\{1,2,3\}$ and $N > 0$. Let

$$M_\sigma = \begin{pmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \\ p_1 & p_2 & p_3 \end{pmatrix}$$

denote the incidence matrix of σ. Thus the i'th column of M_σ is the weight vector of $\sigma(i)$. By replacing σ by $\sigma \circ \sigma_i$ for some $i \in \{1,2,3\}$ and permuting the letters $\{1,2,3\}$ if necessary we can assume that

$$|\sigma(3)| > |\sigma(1)| > |\sigma(2)|.$$

Set

$$f = \frac{(n_1 + m_1 + p_1) - (n_2 + m_2 + p_2)}{n_3 + m_3 + p_3}$$

so that $0 < f < 1$.

Consider the three points

$$S = \{(n_1, n_2 + n_3 f); (m_1, m_2 + m_3 f); (p_1, p_2 + p_3 f)\}$$

in $\mathbb{R}^2$. If any two points of S lie on the line $y = x$, then so does the third; but this would imply a linear dependence between the columns of M_σ

contradicting the fact that the determinant of M_σ is equal to 1. It follows immediately from this and the definition of f that two points of S lie on opposite sides of the line $y = x$. Without loss of generality we suppose that

$$** \begin{cases} n_1 > n_2 + n_3 f \\ m_1 < m_2 + m_3 f. \end{cases}$$

Let K be an integer greater than the maximum of $\{\frac{1}{1-f}; \frac{n_1+n_2+n_3+N}{n_1-n_2-n_3 f}; \frac{m_2+N}{m_2+m_3 f-m_1}\}$ and such that $fK \in \mathbb{Z}$. From the proof of the previous proposition there exist nonnegative integers a, b, c and a primitive A-R morphism σ' such that for any A-R sequence ω the sequence $\sigma'(\omega)$ contains two factors u and v of equal length with

$$\begin{pmatrix} |u|_1 \\ |u|_2 \\ |u|_3 \end{pmatrix} = \begin{pmatrix} a+K \\ b \\ c \end{pmatrix}$$

and

$$\begin{pmatrix} |v|_1 \\ |v|_2 \\ |v|_3 \end{pmatrix} = \begin{pmatrix} a \\ b+(K-1) \\ c+1 \end{pmatrix}.$$

Moreover we can write $\sigma' = \sigma_3^{K-1} \circ \sigma''$ for some A-R morphism σ'' (see the proof of Proposition 2.2).

Set $K' = fK \in \mathbb{Z}$. Since $K' < K - 1$, for any factor v' of $\sigma'(\omega)$ of length K' we have

$$\begin{pmatrix} |v'|_1 \\ |v'|_2 \\ |v'|_3 \end{pmatrix} = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ K'-1+\epsilon_3 \end{pmatrix}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \in \{0, 1\}$ and $\epsilon_1 + \epsilon_2 + \epsilon_3 = 1$. Fix a factor v' of length K' such that vv' is a factor of $\sigma'(\omega)$. We claim: i) $|\sigma(u)|_1 - |\sigma(vv')|_1 > N$ and ii) $|\sigma(vv')|_2 - |\sigma(u)|_2 > N$.(1) To see i) we have

$$|\sigma(u)|_1 = n_1 a + n_2 b + n_3 c + n_1 K$$

$$|\sigma(vv')|_1 = n_1 a + n_2 b + n_3 c + n_2 K - n_2 + n_3 + n_1\epsilon_1 + n_2\epsilon_2 + n_3\epsilon_3 + n_3 K' - n_3$$

whence

$$\begin{aligned} |\sigma(u)|_1 - |\sigma(vv')|_1 &= (n_1 - n_2 - n_3 f)K + n_2 - n_1\epsilon_1 - n_2\epsilon_2 - n_3\epsilon_3 \\ &> (n_1 - n_2 - n_3 f)\, K - n_1 - n_2 - n_3 \\ &> (n_1 - n_2 - n_3 f)\left(\frac{n_1 + n_2 + n_3 + N}{n_1 - n_2 - n_3 f}\right) - n_1 - n_2 - n_3 \\ &= N. \end{aligned}$$

(1) The imbalances with respect to the symbols 1, 2 are a consequence of the choice made in **. Had we chosen for ** inequalities involving m and p instead of n and m, the resulting imbalances would involve the symbols 2, 3.

Similarly to see ii) we have

$$|\sigma(u)|_2 = m_1a + m_2b + m_3c + m_1K$$
$$|\sigma(vv')|_2 = m_1a + m_2b + m_3c + m_2K - m_2 + m_3 + m_1\epsilon_1 + m_2\epsilon_2 + m_3\epsilon_3 + m_3K' - m_3$$

whence

$$\begin{aligned}|\sigma(vv')|_2 - |\sigma(u)|_2 &= (m_2 + m_3f - m_1)K - m_2 + m_1\epsilon_i + m_2\epsilon_2 + m_3\epsilon_3\\ &> (m_2 + m_3f - m_1)K - m_2\\ &> (m_2 + m_3f - m_1)\left(\frac{m_2 + N}{(m_2 + m_3f - m_1)K}\right) - m_2\\ &= N.\end{aligned}$$

The proposition now follows from Lemma 2.1. □

THEOREM 2.4. — *There exists an A-R sequence ω_* which is not N-balanced for any N.*

Proof. — By Proposition 2.3 there exist primitive A-R morphisms $\sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)} \ldots$ such that for each positive integer N and for each A-R sequence ω, the A-R sequence $\sigma^{(1)} \circ \sigma^{(2)} \circ \cdots \circ \sigma^{(N)}(\omega)$ is not N-balanced. Thus the A-R sequence $\omega_* = \sigma^{(1)} \circ \sigma^{(2)} \circ \sigma^{(3)} \circ \cdots \omega$ is not N-balanced for any N. □

Let (X, T, μ) be a dynamical system; following Kesten [19] (see also [13]) a subset A of X is called a *bounded remainder set* if there exist two real numbers a and C such that, for every integer N and for μ-almost all $x \in X$, we have

(1) $$|\mathrm{Card}\{n < N \,:\, T^n(x) \in A\} - aN| < C.$$

As an immediate consequence of Theorem 2.4 we have

COROLLARY 2.5. — *Let X denote the orbit closure of the sequence $\omega_* \in \{1,2,3\}^{\mathbb{N}}$ given in Theorem 2.4. Then for some $i \in \{1,2,3\}$ the cylinder $[i] \subset X$ is not a bounded remainder set.*

Proof. — By Theorem 2.4 there exist an A-R sequence ω_* and $i \in \{1,2,3\}$ such that for each $n > 0$ there exist two factors u_n and v_n of ω with $|u_n| = |v_n|$ and $|u_n|_i - |v_n|_i > n$. We claim $[i]$ is not a bounded remainder set in the subshift generated by ω_*. In fact by taking $n > C/2$ in (1) we see that (1) cannot be simultaneously satisfied for points $x \in X$

beginning in u_n and points $x \in X$ beginning in v_n. Since each cylinder $[u_n]$ and $[v_n]$ has positive measure, it follows that $[i]$ is not a bounded remainder set. □

COROLLARY 2.6. — *There exists an A-R sequence which is not a natural coding of a rotation on the 2-torus* $\mathbb{T}^2$.

Proof. — We recall the following result of Rauzy in [28] later generalized by Ferenczi in [13]:

THEOREM 2.7 (Rauzy [28]). — *Let* (R, α) *be a rotation on the n-torus* $\mathbb{T}^n$, *and let* $A \subset \mathbb{T}^n$. *If there exists a partition of A into finitely many sets* A_i, $1 \leqslant i \leqslant s$, *such that the induced mapping S of R on A is defined by* $Sx = x + \alpha_i$ *for* $x \in A_i$, *where the* α_i *are elements of* $\mathbb{R}^2$ *which are all congruent modulo some lattice M for which A is a fundamental domain, then A is a bounded remainder set.*

Note that this theorem does not assume any other property of regularity on the set A (neither compactness, nor even measurability).

By Corollary 2.5 there exists an A-R sequence $\omega_* \in \{1,2,3\}^{\mathbb{N}}$ and a letter $i \in \{1,2,3\}$ such that the cylinder $[i]$ is not a bounded remainder set. We define the *first return words to the cylinder* $[i]$ to be all the words $\omega_j \dots \omega_k$ such that $\omega_j = \omega_{k+1} = i$ and $\omega_l \neq i$ for $j+1 \leqslant l \leqslant k$; it is proved in [30] that for each letter i there are three first return words to the cylinder $[i]$, w_1, w_2, w_3, and that the *induced sequence* $\mathcal{D}_i(\omega_*)$, obtained from ω_* by replacing each first return word w_j by the letter j, is also an A-R sequence. But then, if both ω_* and $\mathcal{D}_i(\omega_*)$ were natural codings of rotations on $\mathbb{T}^2$, then by Theorem 2.7 the cylinder $[i]$ would be a bounded remainder set. □

Remarks and Questions. — The imbalances in the A-R sequences of Propositions 2.2 and 2.3 are in some sense very particular. One can show by induction that the Tribonacci sequence is 2-balanced. More generally one can also show that if ω is a *linearly recurrent* A-R sequence (i.e., there exists a constant $K > 0$ such that for each factor u, each first return word v to u, that is $v = \omega_j \dots \omega_k$ such that $\omega_j \dots \omega_{j+|u|-1} = u$, $\omega_{k+1} \dots \omega_{k+|u|} = u$ and $\omega_l \dots \omega_{l+|u|-1} \neq u$ for $j+1 \leqslant l \leqslant k$, satisfies $K^{-1}|u| \leqslant |v| \leqslant K|u|$), then ω is N-balanced for some N. Linearly recurrent A-R sequences were completely characterized in [9] and [29].

The counterexample to the conjecture of Arnoux and Rauzy given in Corollary 2.6 naturally raises the following questions: for which A-R sequences does the conjecture hold? For instance, is the conjecture true for all linearly recurrent A-R sequences (see definition above)? Does a weaker form of the conjecture (e.g., measure-theoretic isomorphism to a rotation) hold for every A-R sequence? Of course, natural coding implies measure-theoretic isomorphism, but it is a priori stronger as one requires the translation vector to be constant on each domain of the partition. For instance, the coding of an irrational rotation of the circle with respect to the partition $\{[0,1/2[,[1/2,1[\}$ is measure-theoretically isomorphic to a rotation, but is not a natural coding.

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