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LINEAR HAMILTONIAN CIRCLE ACTIONS THAT GENERATE MINIMAL HILBERT BASES

by Ágúst Sverrir EGILSSON

To my grandfather, Jón Pálsson,
on his ninetieth birthday on April 4, 1999.

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1. Introduction.

Consider the orbit space $\mathbb{R}^{2k}/\mathbb{S}^1$, of a linear circle action with weights $n_1, \ldots, n_k$. It has a natural Poisson structure defined on the algebra of invariant smooth functions. The algebra $C^\infty(\mathbb{R}^{2k})^{\mathbb{S}^1}$ is "generated" by a finite number of invariant polynomials. The polynomial generators define an embedding of the singular Poisson variety $\mathbb{R}^{2k}/\mathbb{S}^1$ into a manifold M. The Poisson structure on $\mathbb{R}^{2k}/\mathbb{S}^1$ and the embedding into M induce a collection of almost Poisson(1) structures on M.

Here we address the question, asked by Cushman and Weinstein, whether among the induced almost Poisson structures there exists a Poisson structure. In [4] it is shown that this is not the case for the linear circle action $\mathbb{S}^1 \times \mathbb{R}^6 \to \mathbb{R}^6$ with weights $1, 1, \pm 2$. It is also proved that the same applies to the reduced orbit space at zero in the indefinite Hamiltonian case. Now we prove for a large class of actions and almost Poisson embeddings that among the induced almost Poisson structures there are no Poisson structures.

Keywords: Singular Poisson structures – Reduction – Hamiltonian actions.
Math. classification: 53D17 – 53D20 – 57S15.
(1) Defined in Section 6.

For any collection of relative prime weights $n_1, \ldots, n_k$ of the linear circle action we can write

$$(d_1 \cdots d_k)^{k-1} | n_1 \cdots n_k$$

(read divides) where d_i is the greatest common divisor of all the weights except the i-th one.

We look at the case when equality, modulo sign, holds. In particular for $k = 2$ we always have $d_1 d_2 = \pm n_1 n_2$ and if all the weights are ± 1 then equality holds as well. For these two cases it is known, see Examples 1 and 2, that one can extend the Poisson structure on $\mathbb{R}^{2k}/\mathbb{S}^1$ to all of M. On the other hand we prove here that if

$$(d_1 \cdots d_k)^{k-1} = \pm n_1 \cdots n_k$$

and at least three of the numbers $d_1, \ldots, d_k$ are not one then the Poisson structure on the orbit space cannot be extended to M nor can the structure on the reduced orbit space be extended.

Consider the action with weights $n_1 = 1$, $n_2 = 2$ and $n_3 = 2$. The action satisfies the first part of the condition above, i.e.,

$$(d_1 d_2 d_3)^2 = n_1 n_2 n_3,$$

since $d_1 = 2$, $d_2 = 1$ and $d_3 = 1$. But only one of the integers d_1, d_2 and d_3 is greater than one, namely d_1. It is shown in Example 3 that the Poisson structure on the orbit space $\mathbb{R}^6/\mathbb{S}^1$ of the action can be extended to a Poisson structure on M. The same is true for the derived indefinite Hamiltonian case.

In Section 3 we show that the equation $(d_1 \cdots d_k)^{k-1} = \pm n_1 \cdots n_k$ is equivalent to the existence of a Hilbert basis with k^2 elements which is the minimal size possible for a Hilbert basis, assuming k is fixed. In Section 6 we develop the methods needed to generalize results obtained for positive weights to the indefinite case. Section 7 contains the proof of our main theorem, stated in Section 5. In order to prove Theorem 1 we first consider polynomial almost Poisson structures and then extend our results to smooth structures by approximating the smooth structures with polynomial and formal almost Poisson structures. Finally Section 8 contains examples of induced Poisson structures on M.

2. Preliminaries.

The linear Hamiltonian circle action with weights $n_1,\dots,n_k$ is defined to be the $U(1)$ action on $\mathbb{R}^{2k}$ given by

$$z\star(x_1,y_1,\dots,x_k,y_k)=(z^{n_1}(x_1,y_1),\dots,z^{n_k}(x_k,y_k))$$

for a positive integer k and nonzero integers $n_1,\dots,n_k$. The set of real valued invariant polynomials for the action is denoted by $\mathbb{R}[x,y]^{U(1)}$ and it is a finitely generated $\mathbb{R}$-algebra, see e.g. [16]. A set of generators for the algebra is called a Hilbert basis for the action. Consider the space $\mathbb{R}^{2k}$ as $\mathbb{C}^k$, $U(1)$ as $\mathbb{S}^1$, and introduce, following [10], the complex coordinates $u_i=x_i+\iota y_i$ and $v_i=x_i-\iota y_i$. Then the action above is written

$$z\star(u_1,\dots,u_k)=(z^{n_1}u_1,\dots,z^{n_k}u_k).$$

The action is symplectic and the standard Poisson bracket on $\mathbb{R}^{2k}$ induces a Poisson algebra $\{,\}_{\mathbf{c}}$ on $\mathbb{C}[u_1,v_1,\dots,u_k,v_k]^{\mathbb{S}^1}$ given by the complex bivectorfield

$$\varrho_{\mathbf{c}}=-2\iota\sum_n\frac{\partial}{\partial u_n}\wedge\frac{\partial}{\partial v_n},$$

i.e., $\{f,g\}_{\mathbf{c}}=\{f,g\}_{\varrho_{\mathbf{c}}}=\varrho_{\mathbf{c}}(df\wedge dg)$, for a detailed discussion see [3]. Through the invariant smooth functions on $\mathbb{C}^k$ the orbit space $\mathbb{C}^k/\mathbb{S}^1$ inherits a singular Poisson structure and by restrictions to the reduced orbit space it also becomes a Poisson variety , see [1] for details.

The algebra $\mathbb{C}[u_1,v_1,\dots,u_k,v_k]^{\mathbb{S}^1}$ of invariant polynomials in the variables u_i and v_i is isomorphic to the semigroup ring $\mathbb{C}[S_n]$ where $S_n=S_{n_1,\dots,n_k}$ is the semigroup of all solutions $(a,b)\in\mathbb{N}^k\times\mathbb{N}^k$ to the equation

$$a_1n_1+\dots+a_kn_k=b_1n_1+\dots+b_kn_k.$$

The isomorphism

$$\mathbb{C}[S_n]\cong\mathbb{C}[u,v]^{\mathbb{S}^1}$$

is determined by

$$X^{(a,b)}\mapsto u_1^{a_1}v_1^{b_1}\cdots u_k^{a_k}v_k^{b_k}.$$

A set of generators for the semigroup S_n is mapped by the above isomorphism to a complex Hilbert basis for the action.

When all the weights $n_1,\dots,n_k$ are equal to one the semigroup S_n has a simple symmetric structure. In that case the invariant complex polynomials under the action form the algebra $\mathbb{C}[u_i v_j]_{1\leq i,j\leq k}$. In particular the circle action has a Hilbert basis with k^2 elements. In other words this Hilbert basis for the action is minimal in size among all Hilbert bases for linear Hamiltonian circle actions with k nonzero weights. In this paper we mainly consider linear Hamiltonian circle actions with k nonzero weights having a Hilbert basis with k^2 elements. Weights that satisfy this minimal condition are for example $n_1 = 1, n_2 = 2$ and $n_3 = 2$ or $n_1 = 6, n_2 = 10$ and $n_3 = 15$. Also any action with two nonzero weights n_1, n_2 will work. An example where we cannot find a Hilbert basis with k^2 elements is $n_1 = 1, n_2 = 1$ and $n_3 = 2$.

3. Observations.

Here we look at some of the properties of the weights that will be useful later. Let $n_1, \dots, n_k$ be nonzero relative prime[(2)] integers. Define $d_1, \dots, d_k$ by

$$d_i = \gcd(n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k)$$

and the integers $\sigma_1, \dots, \sigma_k \in \{-1, 1\}$ by $\sigma_i = \mathrm{sign}(n_i)$ for $i = 1, \dots, k$.

LEMMA 1. — *An element $r \in \mathbb{Z}^k$ is in the integer-hyperplane*

$$n^{\perp} = \{r \in \mathbb{Z}^k : r_1 n_1 + \cdots + r_k n_k = 0\}$$

iff there exist integers $t_1, \dots, t_k$ such that $\sigma_1 t_1 + \cdots + \sigma_k t_k = 0$ and $n_i r_i = \sigma_i d_1 \cdots d_k t_i$ for $i = 1, \dots, k$.

Proof. — Assume that $r \in n^{\perp}$. It follows from the definition of the integers $d_1, \dots, d_{i-1}, d_{i+1}, \dots, d_k$ that they all divide n_i. Since $n_i r_i \in n_1\mathbb{Z} + \cdots + n_{i-1}\mathbb{Z} + n_{i+1}\mathbb{Z} + \cdots + n_k\mathbb{Z} = d_i\mathbb{Z}$, we obtain that d_i divides $n_i r_i$. The integers $d_1, \dots, d_k$ are relative primes two-and-two since $n_1, \dots, n_k$ are relative primes. We can therefore define the integers t_i, by $\sigma_i d_1 \cdots d_k t_i = n_i r_i$, satisfying the above conditions. □

Another point we need to establish is how to find generators for the semigroup S_n. To that end, we introduce ordering on the lattice $\mathbb{N}^k \times \mathbb{N}^k$ as follows:

$$(a, b) \leq (a', b') \text{ iff } a_i \leq a'_i \text{ and } b_i \leq b'_i \text{ for } i = 1, \dots, k.$$

(2) $\gcd(n_1, \dots, n_k) = 1$.

A set of generators for the semigroup S_n is obtained by taking all the minimal elements in $S_n \setminus 0$ with respect to the above ordering. In particular a set of minimal elements, and thus generators, for $S_{1,\dots,1}$ consists of the elements $e_i + \overline{e}_j$ for $i,j = 1,\dots,k$ where $e_1,\dots,e_k,\overline{e}_1,\dots,\overline{e}_k$ denotes the usual basis for $\mathbb{Z}^k \times \mathbb{Z}^k$. Similarly, for S_n, it follows that we need at least k^2 generators.

The morphism $n^\perp \to \sigma^\perp$ between the integer-hyperplanes $n^\perp$ and $\sigma^\perp = \{t \in \mathbb{Z}^k : \sigma_1 t_1 + \cdots + \sigma_k t_k = 0\}$ which maps r to t satisfying $n_i r_i = \sigma_i d_1 \cdots d_k t_i$ for $i = 1,\dots,k$ is denoted by ι.

LEMMA 2. — *Assume that $k > 1$ and $n_1,\dots,n_k$ are nonzero relative prime integers. Then the following is equivalent:*

i) *The morphism $\iota : n^\perp \to \sigma^\perp$ is an isomorphism.*

ii) $n_i = \sigma_i d_1 \cdots d_{i-1} d_{i+1} \cdots d_k$ *for $i = 1,\dots,k$.*

iii) *The linear Hamiltonian circle action with weights $n_1,\dots,n_k$ has a Hilbert basis of minimal size.*

Proof. — First we prove that i) and ii) are equivalent. Assume that ι is an isomorphism. Let $d = d_1 \cdots d_k$. The inverse ι^{-1} maps $(t_1,\dots,t_k)$ to $(r_1,\dots,r_k)$ with $r_i = t_i\sigma_i d/n_i$ and since we can always find a $t \in \sigma^\perp$ with $t_i = 1$ for any fixed i, we conclude that n_i divides d. Now, let $q_i = n_i/d_1 \cdots d_{i-1}d_{i+1} \cdots d_k$ and observe that it is an integer. We also conclude that $\gcd(q_i,d_i) = 1$ from the following calculations: $\gcd(q_i,d_i) = \gcd(q_i d_1 \cdots d_{i-1}d_{i+1} \cdots d_k, d_i) = \gcd(n_i,d_i) = \gcd(n_1,\dots,n_k) = 1$. But $d_i/q_i = d/n_i$ is an integer by the above. Hence $q_i = \pm 1$. On the other hand if the k equations, $n_i = \sigma_i d_1 \cdots d_{i-1} d_{i+1} \cdots d_k$, hold then we can construct the inverse of ι directly.

Next we prove that iii) is equivalent to i) and ii). But first define a bijection $m : \mathbb{N}^k \times \mathbb{N}^k \to \mathbb{N}^k \times \mathbb{Z}^k$ by $m(a,b) = (m_1,\dots,m_k) \times (a-b)$ where $m_i = \min\{a_i,b_i\}$. Then the composition

$$S_{n_1,\dots,n_k} \xrightarrow{m} \mathbb{N}^k \times n^\perp \xrightarrow{\mathrm{id}\times\iota} \mathbb{N}^k \times \sigma^\perp \xrightarrow{m^{-1}} S_{\sigma_1,\dots,\sigma_k}$$

defines an injective map $S_n \to S_\sigma$. The restrictions $m : S_n \to \mathbb{N}^k \times n^\perp$ and $m^{-1} : \mathbb{N}^k \times \sigma^\perp \to S_\sigma$ used in the diagram are both bijective.

Now assume that i) and ii) hold. Also assume for simplicity that $\sigma_1 = \cdots = \sigma_k = 1$. Using condition i) and the formula in ii) we calculate that $\iota^{-1}(t) = (d_1 t_1,\dots,d_k t_k)$. This allows us to calculate $\mathbb{C}[S_n]$ directly as

$$\mathbb{C}[S_n] = \mathbb{C}[u_1 v_1,\dots,u_k v_k, u_i^{d_i} v_j^{d_j}]_{i\neq j}.$$

In other words $\mathbb{C}[S_n]$ can be generated by k^2 elements.

Now assume that iii) holds, i.e., that there exists a Hilbert basis with k^2 generators for $\mathbb{C}[S_n]$. Let m be the ideal in $\mathbb{C}[S_n]$ of elements that vanish at the origin. The minimal elements in $S_n \setminus 0$ determine a $\mathbb{C}$ vector spaces basis for m/m^2. On the other hand since $\mathbb{C}[S_n]$ has a Hilbert basis with k^2 elements, which we may assume that are in m, then those generate m/m^2 also. From this we conclude that the semigroup S_n can be generated by k^2 minimal elements. Without loss of generality we will now assume that all the weights are positive. The k^2 minimal generators in $\mathbb{S}_n \setminus 0$ must be elements

$$\frac{n_j}{\gcd(n_i, n_j)} e_i + \frac{n_i}{\gcd(n_i, n_j)} \overline{e}_j.$$

In order to show that either i) or ii) holds we proceed by induction on the number of weights k. For $k = 2$ condition ii) holds for any choice of relative prime weights n_1 and n_2.

Now assume that $k > 2$ and that iii) implies i) and ii) for any choice of $k-1$ relative prime weights. Write $n_i = q_i d_1 \cdots d_{i-1} d_{i+1} \cdots d_k$ where q_i is a positive integer. By reordering the weights we can assume that $q_1 \leq q_2 \leq \cdots \leq q_k$. The equation,

$$n_2\mathbb{Z} + \cdots + n_k\mathbb{Z} = d_1\mathbb{Z},$$

implies that there must be a minimal element $(a, b) \in S_n \setminus 0$ with $a_1 = d_1$ and $b_1 = 0$. Since we just listed, above, all the minimal elements we conclude that there must be a $j > 1$ such that

$$d_1 = \frac{n_j}{\gcd(n_1, n_j)}.$$

The right hand side of this equation is just $q_j d_1 / \gcd(q_1, q_j)$ and therefore q_j must divide q_1, but we assumed that $q_1 \leq q_2 \leq q_j$ so actually $q_1 = q_2$, since $j > 1$. Now let $n' = (n'_2, \ldots, n'_k)$ with each $n'_i = n_i/d_1$. We canonically consider $S_{n'}$ a sub-semigroup of S_n and it follows immediately that the weights $n'_2, \ldots, n'_k$ satisfy condition iii) of the lemma, since $n_1, \ldots, n_k$ satisfy condition iii). To verify this, notice that there are $(k-1)^2$ minimal elements listed above that are both in S_n and $S_{n'}$ and these form a semigroup basis for $S_{n'}$.

By induction we have that condition ii) is satisfied by the weights $n'_2, \ldots, n'_k$. So let $d'_i = \gcd(n'_2, \ldots, n'_{i-1}, n'_{i+1}, \ldots, n'_k)$. For $i > 2$, since $q_1 = q_2$, we calculate that

$$d'_i = \gcd(q_1 d_i, q_2 d_i, \ldots, q_{i-1} d_i, q_{i+1} d_i, \ldots, q_k d_i).$$

The right hand side is equal to $\gcd(n_1,\ldots,n_{i-1},n_{i+1},\ldots,n_k)$ or d_i since n_j and d_j are relative primes and so are any two of the numbers $d_1,\ldots,d_k$. We have just shown that $d_i' = d_i$ for $i > 2$. On the other hand we need to calculate d_2' separately, i.e., $d_2' = \gcd(n_3',\ldots,n_k') = d_2 \gcd(q_3,\ldots,q_k)$. By the induction hypothesis the weights $n_2',\ldots,n_k'$ satisfy condition ii), i.e.,

$$n_i' = d_2'd_3'\cdots d_{i-1}'d_{i+1}'\cdots d_k',$$

for $i = 2$ we calculate that $n_2' = q_2 d_3' \ldots d_k'$ so $q_2 = 1$, for $i > 2$ we calculate $n_i' = q_i d_2 \ldots d_{i-1} d_{i+1} \ldots d_k = (q_i/\gcd(q_3,\ldots,q_k))d_2' \ldots d_{i-1}'d_{i+1}' \ldots d_k'$. Therefore $q_i/\gcd(q_3,\ldots,q_k) = 1$ for $i > 2$ which implies that $q_3 = \cdots = q_k$. In other words we have shown that $q_1 = q_2 = 1$ and that $q_3 = \cdots = q_k =: q$. Now we demonstrate that $q = 1$. If $q > 1$ then,

$$d_1 e_1 + (q-1)d_2 e_2 + d_3 \bar{e}_3,$$

is a minimal element in $S_n \backslash 0$ but not among the list of k^2 minimal elements above. Therefore we must have that $q = 1$. By induction we have shown that $q_1 = \cdots = q_k = 1$ which is just condition ii) when all the weights are positive. The other cases follow immediately. This completes the proof of Lemma 2. $\square$

4. The reduced orbit space.

Marsden and Weinstein in [7] define, for a Hamiltonian action of a Lie group G on a symplectic manifold (M,ω) with an equivariant momentum map J, the reduced orbit space M_μ, for a value μ in the dual of the Lie algebra of G. The reduced orbit space M_μ is defined as the quotient space $J^{-1}(\mu)/G_\mu$ where G_μ is the isotropy group of μ with respect to the coadjoint action of G. For weakly regular values μ of J, if G_μ acts freely and properly on the manifold $J^{-1}(\mu)$, M_μ is a manifold and there is a unique symplectic structure on M_μ which lifts to $i_\mu^*\omega$ where i_μ is the inclusion map $i_\mu : J^{-1}(\mu) \to M$.

The orbit space M/G is assigned a smooth structure $C^\infty(M/G)$ by the G-invariant functions on M and M/G inherits a Poisson bracket from M making M/G a Poisson variety. When $\mu \in J(M)$ is not a weakly regular value the reduced orbit space M_μ has a smooth structure defined by restricting functions in $C^\infty(M/G)$ to $J^{-1}(\mu)$. In [1] it is shown how the space M_μ inherits, by restriction, the structure of a Poisson variety from M/G. For a compact Lie group G, Sjamaar and Lerman show in [15] that the reduced orbit space M_0 is a union of symplectic manifolds and moreover a stratified symplectic space. Their results are then further extended in [2] by Bates and Lerman.

For the linear Hamiltonian circle action with weights $n_1, \ldots, n_k$ a momentum map J is given by

$$J = \frac{1}{2} \sum_1^k n_i(x_i^2 + y_i^2).$$

5. Embeddings and relations.

The orbit space $\mathbb{R}^{2k}/\mathbb{S}^1$, of the linear Hamiltonian circle action, has a smooth structure, as discussed above, $C^\infty(\mathbb{R}^{2k}/\mathbb{S}^1)$, defined as the set of invariant smooth functions on the phase space $\mathbb{R}^{2k}$,

$$C^\infty(\mathbb{R}^{2k}/\mathbb{S}^1) = C^\infty(\mathbb{R}^{2k})^{\mathbb{S}^1}.$$

A Hilbert embedding, F, corresponding to a Hilbert basis $f_1, \ldots, f_{\mathfrak{N}}$ for the action is defined by

$$F : \mathbb{R}^{2k}/\mathbb{S}^1 \to \mathbb{R}^{\mathfrak{N}};\ F = (f_1, \ldots, f_{\mathfrak{N}}).$$

It follows from a theorem by Schwarz, [13], that

$$F^* : C^\infty(\mathbb{R}^{\mathfrak{N}}) \to C^\infty(\mathbb{R}^{2k}/\mathbb{S}^1),$$

where F^* is the pullback of F, i.e., $F^*(g) = g \circ F$, is surjective and we have, see [9], that $F : \mathbb{R}^{2k}/\mathbb{S}^1 \to \mathbb{R}^{\mathfrak{N}}$ is a proper embedding.

The kernel of the pullback map, $\mathcal{R} = F^{*-1}(0)$, is called the ideal of relations and accordingly elements in $\mathcal{R}$ are called relations.

In Section 3 we used minimal elements among the nonzero elements in S_n to determine a complex Hilbert basis for the action. The relationship between complex and real Hilbert bases and in general the transition back and forth between the real and the complex description is discussed in details in [3].

Assume that the weights $n_1, \ldots, n_k$ are positive and satisfy the conditions in Lemma 2. A complex Hilbert basis is then given by the k^2 elements $u_1v_1, \ldots, u_kv_k$ and $u_i^{d_i}v_j^{d_j}$ for $i \neq j$. A real Hilbert basis in this case is given by the k^2 elements $x_1^2+y_1^2, \ldots, x_k^2+y_k^2$ and $\mathrm{Re}(x_i+\iota y_i)^{d_i}(x_j - \iota y_j)^{d_j}$, $\mathrm{Im}(x_i + \iota y_i)^{d_i}(x_j - \iota y_j)^{d_j}$ for $i < j$.

The corresponding Hilbert embedding is determined by

$$F : \mathbb{C}^k/\mathbb{S}^1 \to \mathbb{R}^{k^2}; (u,v) \mapsto (u_1v_1, \dots, u_kv_k) \times (u_i^{d_i} v_j^{d_j})_{i<j}.$$

Its pullback restricts to complex polynomials and defines an algebra morphism, τ_n, onto the invariant complex polynomials, i.e.,

$$\tau_n : \mathbb{C}[\mathbb{N}^{k^2}] \to \mathbb{C}[S_n].$$

In this context the usual basis for $\mathbb{Z}^{k^2}$ is denoted by $\{e_{ij} : 1 \le i,j \le k\}$ and the above epimorphism τ_n is derived from the map $\mathbb{N}^{k^2} \to S_n$ given by

$$e_{ii} \mapsto e_i + \bar{e}_i \text{ and } e_{ij} \mapsto d_ie_i + d_j\bar{e}_j \text{ if } i \neq j$$

which is also denoted by τ_n.

There exists a unique functional(3) $\mathcal{J} : \mathbb{R}^{k^2} \to \mathbb{R}$ satisfying

$$J = \mathcal{J} \circ F$$

and F maps the reduced orbit space $J^{-1}(0)/\mathbb{S}^1$ into the hyperplane

$$V_J = \mathcal{J}^{-1}(0) \subset \mathbb{R}^{k^2}.$$

The smooth structure on the reduced orbit space $J^{-1}(0)/\mathbb{S}^1$ is obtained, see [1], by restricting the invariant functions to $J^{-1}(0)$, and the algebra $C^\infty(J^{-1}(0)/\mathbb{S}^1)$ satisfies,

$$C^\infty(J^{-1}(0)/\mathbb{S}^1) = (\eta \circ F)^* C^\infty(V_J)|_{J^{-1}(0)}$$

where η is the linear projection

$$\eta : \mathbb{R}^{k^2} \to V_J$$

onto V_J along the vector

$$\mathcal{J}^* = \frac{1}{2}\sum_1^k n_ie_{ii}.$$

Eventually we will prove the following theorem.

THEOREM 1. — *Let $\mathbb{S}^1 \times \mathbb{C}^k \to \mathbb{C}^k$ be a linear Hamiltonian circle action with relative prime weights $n_1, \dots, n_k$. Assume that the action generates a Hilbert basis of minimal size. Denote the greatest common divisor of all the weights except n_i by d_i. Assume further that at least three of the numbers $d_1, \dots, d_k$ are not one. Then the Poisson structure on the singular orbit space $\mathbb{C}^k/\mathbb{S}^1$, embedded into $\mathbb{R}^{k^2}$ by the Hilbert embedding, cannot be extended to $\mathbb{R}^{k^2}$. Furthermore for the action with indefinite Hamiltonian the Poisson structure on the reduced orbit space cannot either be extended to the hyperplane V_J.*

(3) Remember that $J = \sum_1^k \frac{1}{2} n_iu_iv_i$.

6. Tools of the trade.

For a detailed discussion about the material contained in this section see [3], in particular Chapter 7 therein.

Let $n_1, \dots, n_k$ be relative prime weights. Before we consider the action with weights $n_1, \dots, n_k$ we consider the linear Hamiltonian circle action with positive weights $|n_1|, \dots, |n_k|$. Assume that $|F| : \mathbb{R}^{2k} \to \mathbb{R}^{\mathfrak{N}}$ is a Hilbert embedding corresponding to a Hilbert basis derived from a set of minimal generators for the semigroup $S_{|n|}$ for the positive weights. Denote as before the sign of n_i by σ_i. Let $|\tau_{\mathbf{c}}|$ be the (complex) pullback of $|F|$. The minimal generators for $S_{|n|}$ are elements of the form $e_1 + \overline{e}_1, \dots, e_k + \overline{e}_k, f_1, \overline{f}_1, \dots, f_N, \overline{f}_N$ where $f_1, \dots, f_N$ are invariant monomials in $S_{|n|}$. So, in particular $\mathfrak{N} = k + 2N$. The conjugate operator used here (in $\overline{f}$) just interchanges e_i and $\overline{e}_i$. This allows us to describe the pullback $|\tau_{\mathbf{c}}|$ as the morphism determined by

$$L_i \mapsto u_i v_i, \ Z_j \mapsto f_j \ \text{ and } \ W_j \mapsto \overline{f}_j,$$

for a collection of coordinates $L_1, \dots, L_k, Z_1, W_1, \dots, Z_N, W_N$ on the ambient space. Denote the real $\mathfrak{N}$-dimensional vector space with the above coordinate functionals L, Z and W by $V_{\mathbf{c}}$. The algebra of multivector fields in the variables L, Z and W with formal (power series) coefficients is denoted by $\mathcal{X}^*[[V_c]]$. Define

$$\omega^\sigma : \{L, Z, W\} \to \{1, -1, \iota, -\iota\}$$

by the formula

$$\omega^\sigma_{(\gamma)} = \prod_{\{j:\sigma_j=-1\}} \iota^{\deg_{u_j} |\tau_{\mathbf{c}}|(\gamma)} \cdot \iota^{\deg_{v_j} |\tau_{\mathbf{c}}|(\gamma)},$$

and use ω^σ to define an isomorphism

$$\psi^\sigma_{\mathbf{c}} : \mathbb{C} \otimes \mathcal{X}^*[[V_{\mathbf{c}}]] \to \mathbb{C} \otimes \mathcal{X}^*[[V_{\mathbf{c}}]],$$

by the formulas

- $L_s \mapsto \omega^\sigma_{(L_s)} L_s, \ Z_t \mapsto \omega^\sigma_{(Z_t)} Z_t$ and $W_t \mapsto \omega^\sigma_{(W_t)} W_t$
- $\frac{\partial}{\partial L_s} \mapsto \frac{1}{\omega^\sigma_{(L_s)}} \frac{\partial}{\partial L_s}, \ \frac{\partial}{\partial Z_t} \mapsto \frac{1}{\omega^\sigma_{(Z_t)}} \frac{\partial}{\partial Z_t}$ and $\frac{\partial}{\partial W_t} \mapsto \frac{1}{\omega^\sigma_{(W_t)}} \frac{\partial}{\partial W_t}$.

The $k + 2N$ independent functionals L, Z and W on $V_{\mathbf{c}}$ have "real" counterparts. Assume that $L_1, \dots, L_k, R_1, I_1, \dots, R_N, I_N$ are also $k + 2N$ independent functionals on $\mathbb{R}^{k+2N}$ and denote the $k+2N$ dimensional real vector space with those new coordinate functionals by V. We define the coordinate functionals L, R and I such that the pullback, $|\tau|$, of the real Hilbert embedding $|F|$ is given by

$$|\tau|(L_i) = x_i^2 + y_i^2,\ |\tau|(R_j) = \mathrm{Re}(f_j) \text{ and } |\tau|(I_j) = \mathrm{Im}(f_j).$$

Using the above define a $\mathbb{C}$-algebra isomorphism

$$\kappa : \mathbb{C} \otimes \mathcal{X}^*[[V]] \to \mathbb{C} \otimes \mathcal{X}^*[[V_{\mathbf{c}}]]$$

by the formulas

- $L_s \mapsto L_s,\ R_t \mapsto \frac{1}{2}(Z_t + W_t)$ and $I_t \mapsto \frac{1}{2\iota}(Z_t - W_t)$
- $\frac{\partial}{\partial L_s} \mapsto \frac{\partial}{\partial L_s},\ \frac{\partial}{\partial R_t} \mapsto \frac{\partial}{\partial Z_t} + \frac{\partial}{\partial W_t}$ and $\frac{\partial}{\partial I_t} \mapsto \iota\big(\frac{\partial}{\partial Z_t} - \frac{\partial}{\partial W_t}\big)$.

The inverse of κ is determined by

- $L_s \mapsto L_s,\ Z_t \mapsto R_t + \iota I_t$ and $W_t \mapsto R_t - \iota I_t$
- $\frac{\partial}{\partial L_s} \mapsto \frac{\partial}{\partial L_s},\ \frac{\partial}{\partial Z_t} \mapsto \frac{1}{2}\big(\frac{\partial}{\partial R_t} - \iota\frac{\partial}{\partial I_t}\big)$ and $\frac{\partial}{\partial W_t} \mapsto \frac{1}{2}\big(\frac{\partial}{\partial R_t} + \iota\frac{\partial}{\partial I_t}\big)$.

Finally we define the real counterpart of $\psi_{\mathbf{c}}^\sigma$ by

$$\psi^\sigma = \kappa^{-1} \circ \psi_{\mathbf{c}}^\sigma \circ \kappa.$$

The Schouten-Nijenhuis, see [11], [12], bracket for multivector fields on a smooth manifold M

$$[,] : \mathcal{X}^*(M) \times \mathcal{X}^*(M) \to \mathcal{X}^*(M)$$

is the unique bilinear extension of the Lie derivative satisfying

- $[f, g] = 0$ if $f, g \in \mathcal{X}^0(M)$,
- $[X, f] = X(f)$ and $[X, Y] = L_X(Y)$ if $f \in \mathcal{X}^0(M)$ and $X, Y \in \mathcal{X}^1(M)$,
- $[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{xy+y} Y \wedge [X, Z]$ and
- $[X, Y] = (-1)^{xy}[Y, X]$ if $X \in \mathcal{X}^x(M),\ Y \in \mathcal{X}^y(M)$ and $Z \in \mathcal{X}^z(M)$.

The Schouten-Nijenhuis bracket can canonically be considered a formal operator on $\mathcal{X}^*[[V]]$ and $\mathcal{X}^*[[V_{\mathbf{c}}]]$ as well. The reason we consider the Schouten-Nijenhuis bracket here is because of its relationship with Poisson brackets.

A skew-symmetric bivectorfield $\Pi \in \mathcal{X}^2(M)$ defines a skew-symmetric bilinear bracket

$$\lfloor , \rfloor_\Pi : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \text{ by } \lfloor f, g \rfloor_\Pi = \Pi(df \wedge dg)$$

satisfying Leibniz identity[(4)], and vice versa. A skew-symmetric bilinear bracket on $C^\infty(M)$ satisfying Leibniz identity is referred to as an almost

(4) $\lfloor fg, h \rfloor = f\lfloor g, h \rfloor + \lfloor f, h \rfloor g$.

Poisson bracket or structure. If additionally the bracket satisfies Jacobi identity[(5)] then it is a Poisson bracket. We often refer to the bivectorfield as an almost Poisson structure instead of the bracket. The reader is referred to [14] for a treatment of (almost) Poisson structures.

Using the Schouten-Nijenhuis bracket $[,]$ there is the following characterization of Poisson brackets, see [5], [11]. For $\Pi \in \mathcal{X}^2(M)$ the condition

$$[\Pi, \Pi] = 0$$

is equivalent to $\{,\} = \lfloor , \rfloor_\Pi$ being a Poisson bracket. Accordingly we call a bivectorfield Π Poisson if $[\Pi, \Pi] = 0$. The standard Poisson structure on $\mathbb{R}^{2k}$ is determined by the bivectorfield

$$\varrho = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \cdots + \frac{\partial}{\partial x_k} \wedge \frac{\partial}{\partial y_k}.$$

Let Π be a bivectorfield on V. A smooth map $G : \mathbb{R}^{2k} \to V$ is an almost Poisson morphism with respect to the almost Poisson bracket $\lfloor , \rfloor_\Pi$ on V and the Poisson bracket $\{,\} = \lfloor , \rfloor_\varrho$ on $\mathbb{R}^{2k}$, i.e., $\{f \circ G, h \circ G\} = \lfloor f, h \rfloor_\Pi \circ G$ for smooth functions f, h on V, if and only if ϱ and Π are G-related. In other words the formula,

$$G_* \circ \varrho = \Pi \circ G,$$

is equivalent to G being compatible with the brackets $\lfloor , \rfloor_\varrho$ and $\lfloor , \rfloor_\Pi$.

A Hilbert embedding F for the action with weights $n_1, \ldots, n_k$ is given by

$$F = |F| \circ \Psi^\sigma \quad \text{where} \quad \Psi^\sigma : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$$

is defined by the formula

$$\begin{pmatrix} x_1, y_1 \\ x_2, y_2 \\ \cdots \\ \cdots \\ x_k, y_k \end{pmatrix} \longmapsto \begin{pmatrix} x_1, \sigma_1 y_1 \\ x_2, \sigma_2 y_2 \\ \cdots \\ \cdots \\ x_k, \sigma_k y_k \end{pmatrix}$$

Notice, that the Hilbert embedding F also corresponds to a Hilbert basis derived from a minimal set of generators for the semigroup S_n and that given such a Hilbert embedding F we can define the above $|F|$ by $|F| = F \circ \Psi^{\sigma^{-1}}$.

(5) $\lfloor \lfloor f, g \rfloor, h \rfloor + \lfloor \lfloor g, h \rfloor, f \rfloor + \lfloor \lfloor h, f \rfloor, g \rfloor = 0.$

LEMMA 3. — *Let $\mathbb{S}^1 \times \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ be a linear Hamiltonian circle action with relative prime weights.*

i) *A bivectorfield $\Pi \in \mathcal{X}^2[[V]]$ is Poisson if and only if $\psi^\sigma(\Pi)$ is a Poisson bivectorfield.*

ii) *Let $\Pi \in \mathcal{X}^2[[V]]$ be a bivectorfield and $F : \mathbb{R}^{2k} \to V$ be a Hilbert embedding, $F = |F| \circ \Psi^\sigma$, derived from a set of minimal generators for the semigroup S_n. Then ϱ and Π are $|F|$-related if and only if ϱ and $\psi^\sigma(\Pi)$ are F-related.*

iii) *A polynomial R is an $|F|$ relation if and only if $\psi^\sigma(R)$ is an F relation.*

Proof (sketch). — The isomorphism ψ^σ restricts to a map $\mathbb{C} \otimes \mathcal{X}^*[V] \to \mathbb{C} \otimes \mathcal{X}^*[V]$ between the multivector fields with polynomial coefficients. According to Lemma 4 in [3] this restriction is a Schouten-Nijenhuis morphism. It follows immediately that the same is true for the multivector fields with formal coefficients. For a bivectorfield Π in $\mathcal{X}^2[[V]]$ we conclude that $[\Pi, \Pi] = 0$ iff $[\psi^\sigma(\Pi), \psi^\sigma(\Pi)] = 0$ which confirms the first part of the lemma. The second part of the lemma also follows from a similar statement, Lemma 3 in [3], about multivector fields with polynomial coefficients and from the existence of bivector fields with polynomial coefficients which are F (or $|F|$) related to ϱ. Finally the third part is just Lemma 5 in [3]. □

7. Poisson algebras.

As explained earlier, the algebra $(\mathbb{C}[S_n], \{,\}_{\mathbf{c}})$ is a Poisson algebra with the Poisson structure, $\{,\}_{\mathbf{c}}$, derived from the usual Poisson structure on $\mathbb{R}^{2k}$. When $n = (\sigma_1, \ldots, \sigma_k)$ with each $\sigma_i = \pm 1$ or when $k \leq 2$ then there exists a natural Poisson structure on the algebra $\mathbb{C}[\mathbb{N}^{k^2}]$ such that the above(6) morphism $\tau_n : \mathbb{C}[\mathbb{N}^{k^2}] \to \mathbb{C}[S_n]$ is a Poisson morphism. For the first case $n = (\sigma_1, \ldots, \sigma_k)$ the Poisson structure on $\mathbb{C}[\mathbb{N}^{k^2}]$ can be chosen to be linear. If we consider the field of rational invariant functions $\mathbb{C}(S_n)$ instead of $\mathbb{C}[S_n]$ and consider a morphism $\tau : \mathbb{C}(\mathbb{N}^{\mathfrak{N}}) \to \mathbb{C}(S_n)$ defined similarly, $\tau(X^{e_i}) = f_i$, from a minimal Hilbert basis $f_1, \ldots, f_{\mathfrak{N}}$ then one can always choose a Poisson structure on $\mathbb{C}(\mathbb{N}^{\mathfrak{N}})$ such that τ is a Poisson morphism for any weights $n_1, \ldots, n_k$.

(6) See Section 5.

Given a Hilbert basis $f_1, \ldots, f_{\mathfrak{N}}$ we can define an almost Poisson bracket $\lfloor , \rfloor$ on $\mathbb{C}[\mathbb{N}^{\mathfrak{N}}]$ compatible with the Poisson structure on $\mathbb{C}[S_n]$ under the morphism τ as follows: Start by defining[(7)] $\lfloor X^{e_i}, X^{e_j} \rfloor \in \mathbb{C}[\mathbb{N}^{\mathfrak{N}}]$ such that $\tau(\lfloor X^{e_i}, X^{e_j} \rfloor) = \{f_i, f_j\}_{\mathbf{c}}$ for $1 \leq i < j \leq \mathfrak{N}$. Then extend $\lfloor , \rfloor$ to all the polynomials in $\mathbb{C}[\mathbb{N}^{\mathfrak{N}}]$ by requiring $\lfloor , \rfloor$ to be bilinear, skew-symmetric and satisfy Leibniz identity.

Assume that the weights $n_1, \ldots, n_k$ satisfy the conditions of Lemma 2. Let $\tau_n : \mathbb{C}[\mathbb{N}^{k^2}] \to \mathbb{C}[S_n]$ be the pullback of a Hilbert embedding derived from a set of minimal generators for the semigroup S_n as before.

For each element $t \in S_n$ define a $\mathbb{C}$-linear function

$$f_t : \mathbb{C}[\mathbb{N}^{k^2}] \to \mathbb{C}[\mathbb{N}^{k^2}]; r = \sum_s c_s X^s \mapsto \sum_{\tau_n(s)=t} c_s X^s.$$

In other words $f_t(r)$ contains only the monomial terms of r that are above t.

It is straightforward to describe the Poisson structure on $\mathbb{C}[S_n]$. For elements $a, b \in S_n$, the Poisson bracket $\{X^a, X^b\}_{\mathbf{c}}$ is given by the simple formula,

$$\{X^a, X^b\}_{\mathbf{c}} = -2\iota \sum_1^k (a_i \bar{b}_i - \bar{a}_i b_i) X^{a+b-(e_i+\bar{e}_i)},$$

where we have written $a = a_1 e_1 + \bar{a}_1 \bar{e}_1 + \cdots + a_k e_k + \bar{a}_k \bar{e}_k \in \mathbb{N}^{2k}$ and are using a similar expression for b.

LEMMA 4. — *Assume, like above, that the linear Hamiltonian circle action with weights $n_1, \ldots, n_k$ for $k > 2$ generates a Hilbert basis of minimal size and that at least three of the constants $d_1, \ldots, d_k$ are not equal to one. Let $\lfloor , \rfloor$ be any choice of an almost Poisson structure on $\mathbb{C}[\mathbb{N}^{k^2}]$ compatible with the Poisson structure on $\mathbb{C}[S_n]$ under τ_n. Then $\lfloor , \rfloor$ does not satisfy Jacobi identity.*

Proof. — By Lemma 3 we can assume that all the weights are positive, without loss of generality. By re-ordering the weights we can also assume that d_1, d_2 and d_3 are all greater than one. Consider the polynomial

$$Z = \lfloor \lfloor X^{e_{12}}, X^{e_{21}} \rfloor, X^{e_{31}} \rfloor + \lfloor \lfloor X^{e_{21}}, X^{e_{31}} \rfloor, X^{e_{12}} \rfloor + \lfloor \lfloor X^{e_{31}}, X^{e_{12}} \rfloor, X^{e_{21}} \rfloor,$$

we will show that for,

$$t = (d_1 - 2)(e_1 + \bar{e}_1) + d_2(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1,$$

(7) τ is surjective.

the image of Z under f_t is independent of the choice of the almost Poisson bracket and is given by

$$f_t Z = -4d_1^3(d_1-1)X^{(d_1-2)e_{11}}(X^{d_2e_{22}}X^{e_{31}} - X^{e_{21}}X^{e_{32}}).$$

Jacobi identity is thus not satisfied by $\lfloor , \rfloor$.

Start by establishing the notation. Let $A = X^{e_{12}}$, $B = X^{e_{21}}$, $C = X^{e_{31}}$, $D = X^{e_{32}}$ and $L_1 = X^{e_{11}}$, $L_2 = X^{e_{22}}$. The bracket $\lfloor , \rfloor$ satisfies the following equations:

$$\begin{aligned}
&\lfloor A, B\rfloor = -2\iota L_1^{d_1-1}L_2^{d_2-1}(d_1^2L_2 - d_2^2L_1) + R_{A,B}, && \\
&\lfloor B, C\rfloor = R_{B,C}, && \lfloor C, A\rfloor = 2\iota d_1^2 L_1^{d_1-1}D + R_{C,A},\\
&\lfloor D, B\rfloor = 2\iota d_2^2 L_2^{d_2-1}C + R_{D,B}, && \lfloor L_1, C\rfloor = -2\iota d_1 C + R_{L_1,C},\\
&\lfloor L_2, C\rfloor = R_{L_2,C}, && \lfloor L_1, B\rfloor = -2\iota d_1 B + R_{L_1,B},
\end{aligned}$$

where we have written $R_{A,B}$, $R_{B,C}$, $R_{C,A}$, $R_{D,B}$, $R_{L_1,C}$, $R_{L_2,C}$ and $R_{L_1,B}$ for a choice of relations, i.e., elements in $\mathcal{R}_n = \ker(\tau_n)$. It is possible to write the bracket $\lfloor , \rfloor$ in this way for these values since for two different brackets $\lfloor , \rfloor_1$ and $\lfloor , \rfloor_2$ we always have that $\lfloor f, g\rfloor_1$ and $\lfloor f, g\rfloor_2$ differ by a relation, i.e., $\tau_n(\lfloor f, g\rfloor_1 - \lfloor f, g\rfloor_2) = \{\tau_n(f), \tau_n(g)\}_{\mathbf{c}} - \{\tau_n(f), \tau_n(g)\}_{\mathbf{c}} = 0$. Before calculating $f_t Z$ we need to understand better properties of elements in the ideal of relations $\mathcal{R}_n = \ker(\tau_n)$. Now we list some of these properties and their consequences for the problem at hand.

A relation contains no linear part. This follows from the observation that the k^2 minimal generators for $\mathbb{C}[S_n]$ are linearly independent irreducible elements in $\mathbb{C}[S_n]$. Using this we can simplify $f_t\lfloor\lfloor A, B\rfloor, C\rfloor$, $f_t\lfloor\lfloor B, C\rfloor, A\rfloor$ and $f_t\lfloor\lfloor C, A\rfloor, B\rfloor$ as follows:

i) $f_t\lfloor\lfloor A, B\rfloor, C\rfloor = -4d_1^3(d_1-1)L_1^{d_1-2}L_2^{d_2}C + f_t\lfloor R_{A,B}, C\rfloor$,

ii) $f_t\lfloor\lfloor B, C\rfloor, A\rfloor = f_t\lfloor R_{B,C}, A\rfloor$ and

iii) $f_t\lfloor\lfloor C, A\rfloor, B\rfloor = 4d_1^3(d_1-1)L_1^{d_1-2}BD + f_t\lfloor R_{C,A}, B\rfloor$.

If W is a non-constant monomial term in a relation R then W is divisible by XY where X and Y are two different monomials from the collection $\mathcal{D} = \{X^{d_1e_{11}}, \dots, X^{d_ke_{kk}}, X^{e_{ij}} : i \neq j\}$. We will prove this below, but first use it to show that

$$f_t\lfloor R, Q\rfloor = 0$$

for any relation R and $Q \in \{A, B, C\}$.

Assume, for a moment, that $f_t\lfloor R, Q\rfloor \neq 0$ then there exists a non-constant monomial term W in R such that $f_t\lfloor W, Q\rfloor \neq 0$. Write $W = FGX$ where F and G are two different monomials from $\mathcal{D}$.

By direct calculations, using Leibniz identity, we obtain that

$$\lfloor FGX, Q\rfloor = FG\lfloor X, Q\rfloor + FX\lfloor G, Q\rfloor + GX\lfloor F, Q\rfloor.$$

Observe that any monomial which contains the product of three elements in $\mathcal{D}$ is in the kernel of f_t. Therefore adding a relation to $\lfloor X, Q\rfloor$, $\lfloor G, Q\rfloor$ or $\lfloor F, Q\rfloor$ will not change the outcome for $f_t\lfloor FGX, Q\rfloor$.

Part I: Here we verify that

$$f_t FG\lfloor X, Q\rfloor = 0.$$

Assume this is not the case, i.e., $f_t FG\lfloor X, Q\rfloor \neq 0$. Since $\lfloor , \rfloor$ extends the almost Poisson structure on $\mathbb{R}^{2k}$ we have that

$$\tau_n\lfloor X, Q\rfloor = \{x, q\}_{\mathbf{c}} \text{ where } x = \tau_n(X) \text{ and } q = \tau_n(Q).$$

By considering that $\tau_n(FG)$ must divide t one concludes that the only choice for $\tau_n(FG)$ is $d_2(e_2 + \overline{e}_2) + d_3 e_3 + d_1\overline{e}_1$. On the other hand the polynomial $\{x, q\}_c$ is an invariant say, $\{x, q\} = c_1 m_1 + \cdots + c_p m_p$ for different invariant monomials $m_1, \ldots, m_p$ and each c_i nonzero. If we lift each of these monomials to monomials $M_1, \ldots, M_p$ respectively, i.e., $\tau_n(M_i) = m_i$, then we obtain an expression for $\lfloor X, Q\rfloor$ as

$$\lfloor X, Q\rfloor = c_1 M_1 + \cdots + c_p M_p + R$$

where R is a relation. We have already remarked above that ignoring R will not effect $f_t FG\lfloor X, Q\rfloor$. But from the definition of the Poisson structure on $\mathbb{C}[S_n]$ it is immediate that each of the terms $m_1, \ldots, m_k$ contains at least one of the monomials $d_1 e_1$, $d_2\overline{e}_2$, $d_2 e_2$, $d_1\overline{e}_1$, $d_3 e_3$ or $d_1\overline{e}_1$ since $q = \tau_n(Q) \in \{d_1 e_1 + d_2\overline{e}_2, d_2 e_2 + d_1\overline{e}_1, d_3 e_3 + d_1\overline{e}_1\}$. Combining this and the expression for $\tau_n(FG)$ above, one concludes that the equation

$$\tau_n(FGM_i) = t$$

never holds, hence $f_t FG\lfloor X, Q\rfloor = 0$.

Part II: Now we verify that

$$f_t FX\lfloor G, Q\rfloor = 0.$$

Let $q = \tau_n(Q) = d_i e_i + d_j\overline{e}_j$ and $g = \tau_n(G) = d_{i'} e_{i'} + d_{j'}\overline{e}_{j'}$. Direct calculations reveal that

$$\lfloor G, Q\rfloor = 2\iota\delta_{ij'} d_i^2 X^{e_{ii}(d_i - 1)} H_1 - 2\iota\delta_{ji'} d_j^2 X^{e_{jj}(d_j - 1)} H_2 + R,$$

for a relation R and monomials H_1 and H_2 in $\mathcal{D}$. Since

$$q \in \{d_1 e_1 + d_2\overline{e}_2,\ d_2 e_2 + d_1\overline{e}_1,\ d_3 e_3 + d_1\overline{e}_1\}$$

the choices for i and j are $i = 1,2,3$ and $j = 1,2$. Both of the pullbacks $\tau_n(X^{e_{11}(d_1-1)}) = (d_1-1)(e_1+\overline{e}_1)$ and $\tau_n(X^{e_{33}(d_3-1)}) = (d_3-1)(e_3+\overline{e}_3)$ do not divide t since d_1 and d_3 are greater than 1. We can therefore ignore any product of $X^{e_{11}(d_1-1)}$ and $X^{e_{22}(d_2-1)}$ in $\lfloor G,Q \rfloor$ when calculating $f_t FX\lfloor G,Q\rfloor$. The other possible monomials, ignoring relations, in the expression for $\lfloor G,Q\rfloor$ above, are of the form $\pm 2\iota d_2^2 X^{e_{22}(d_2-1)}H$ where H is in $\mathcal{D}$. In order for the expression $f_t(\pm FX2\iota d_2^2 X^{e_{22}(d_2-1)}H)$ to be nonzero we must have

$$t = \tau_n(FXH) + (d_2-1)(e_2+\overline{e}_2)$$

but if $d_2 > 1$ then this equation has no solution for F and H in $\mathcal{D}$. We have now shown that $f_t FX\lfloor G,Q\rfloor$ (and $f_t GX\lfloor F,Q\rfloor$) is zero. Notice that here is the only place in the proof that we actually used that $d_2 > 1$ and that $d_3 > 1$.

The calculations above confirm the formula given for $f_t Z$ at the beginning of the proof. Still, we have yet to prove that if W is a non-constant monomial term in a relation R then W is divisible by XY where X and Y are two different monomials from the collection $\mathcal{D} = \{X^{d_1e_{11}},\ldots,X^{d_ke_{kk}},X^{e_{ij}} : i\neq j\}$. The only (monic) monomials that are not divisible by such a product are of the form $X^{e_{11}m_1}\cdots X^{e_{kk}m_k}X^{me_{ij}}$ where $m_1,\ldots,m_k$ and m are integers with $0\le m_1<d_1,\ldots,0\le m_k<d_k$. Assume, for a moment, that W is of this form and let

$$z = \tau_n(W) = m_1(e_1+\overline{e}_1)+\cdots+m_k(e_k+\overline{e}_k)+m(d_ie_i+d_j\overline{e}_j).$$

It is immediate that W is the only monomial in $\mathbb{C}[\mathbb{N}^{k^2}]$ that pullbacks to z under τ_n. In particular such a term, W, cannot be a part of relation R since it would not cancel when R is pulled back to $\mathbb{C}[S_n]$ by τ_n. This completes the proof of Lemma 4. □

We need to extract more from the above proof. The comments below will be useful in the proof of Corollary 2.

Comment I: Above we used that the relations do not contain any linear part to obtain simple expressions for

$$f_t\lfloor\lfloor A,B\rfloor,C\rfloor,\ f_t\lfloor\lfloor B,C\rfloor,A\rfloor \text{ and } f_t\lfloor\lfloor C,A\rfloor,B\rfloor.$$

A closer look reveals that this only depends on specific linear parts, i.e., scalar products of B and C. To summarize we can alter the almost Poisson bracket $\lfloor,\rfloor$ by adding polynomials of the form $\mathcal{J}P$ to the relations, above, and still obtain similar expressions

i) $f_t\lfloor\lfloor A,B\rfloor,C\rfloor = -4d_1^3(d_1-1)f_t(L_1^{d_1-2}L_2^{d_2}C)+f_t\lfloor R_{A,B}+\mathcal{J}P_{A,B},C\rfloor,$

ii) $f_t\lfloor\lfloor B,C\rfloor,A\rfloor = f_t\lfloor R_{B,C}+\mathcal{J}P_{B,C},A\rfloor$ and

iii) $f_t\lfloor\lfloor C,A\rfloor,B\rfloor = 4d_1^3(d_1-1)f_t(L_1^{d_1-2}BD)+f_t\lfloor R_{C,A}+\mathcal{J}P_{C,A},B\rfloor$.

The above formulas hold for any t such that

$$t_{\min} \le t < t_{\max}$$

where $t_{\min} = d_2(e_2+\overline{e}_2)+d_3e_3+d_1\overline{e}_1$ and

$$t_{\max} = (d_1-2)(e_1+\overline{e}_1)+(d_2+1)(e_2+\overline{e}_2)+d_3e_3+d_1\overline{e}_1.$$

Furthermore if $(d_1-2)(e_1+\overline{e}_1)$ does not divide t, i.e., $(d_1-2)(e_1+\overline{e}_1)\not\leq t$ then the above formulas i) – iii) also hold. Here $\mathcal{J}$ is the linear polynomial

$$\mathcal{J} = \frac{1}{2}\sum_1^k n_iX^{e_{ii}}.$$

Comment II: The arguments leading to and contained in Part I and Part II of the proof, used to verify that

$$f_t\lfloor Q,R\rfloor = 0,$$

actually hold for all $t \le t_{\max *}$ where

$$t_{\max *} = (d_1-2)(e_1+\overline{e}_1)+2(d_2-1)(e_2+\overline{e}_2)+d_3e_3+d_1\overline{e}_1.$$

The following is a stronger version of Lemma 4, it contains what we actually proved.

COROLLARY 1. — *Under the assumptions of Lemma 4, define the Jacobiator* $\mathfrak{J}$ *by*

$$\mathfrak{J}(A,B,C) = \lfloor\lfloor A,B\rfloor,C\rfloor+\lfloor\lfloor B,C\rfloor,A\rfloor+\lfloor\lfloor C,A\rfloor,B\rfloor$$

for $A,B,C\in\mathbb{C}[\mathbb{R}^{k^2}]$ *and let* $t\in S_n$ *be the element*

$$t = (d_1-2)(e_1+\overline{e}_1)+d_2(e_2+\overline{e}_2)+d_3e_3+d_1\overline{e}_1,$$

then

$$f_t\circ\mathfrak{J}\neq 0.$$

LEMMA 5. — *Let* F *be a Hilbert embedding*

$$F:\mathbb{C}^k/\mathbb{S}^1\to\mathbb{R}^{\mathfrak{N}}$$

corresponding to a Hilbert basis of $\mathfrak{N}$ *minimal generators in* S_n. *Assume that there exists a Poisson structure* $\{,\}$ *on* $\mathbb{R}^{\mathfrak{N}}$ *such that* F *is a Poisson embedding. Let* $\mathfrak{T}$ *be a finite collection of elements in* S_n. *Then there* *exists an almost Poisson bracket* $\lfloor , \rfloor$ *on* $\mathbb{C}[\mathbb{N}^{\mathfrak{N}}]$, *compatible with the Poisson structure on* $\mathbb{C}[S_n]$, *under the pullback of* F, *and such that*

$$f_t(\lfloor\lfloor A,B\rfloor,C\rfloor+\lfloor\lfloor B,C\rfloor,A\rfloor+\lfloor\lfloor C,A\rfloor,B\rfloor)=0$$

for all $A,B,C\in\mathbb{C}[\mathbb{N}^{\mathfrak{N}}]$ *and* $t\in\mathfrak{T}$.

Proof (sketch). — Let

$$\pi=\sum_{1\le i<j\le\mathfrak{N}}P_{ij}\frac{\partial}{\partial x_i}\wedge\frac{\partial}{\partial x_j}$$

be the Poisson bivectorfield corresponding to the Poisson structure on $\mathbb{R}^{\mathfrak{N}}$, i.e., $\{x_i,x_j\}=P_{ij}$. By choosing the almost Poisson bracket on $\mathbb{R}^{\mathfrak{N}}$ such that the polynomials $\lfloor x_i,x_j\rfloor$ and the Taylor series of P_{ij} agree up to a large degree we can guarantee that the identity

$$\lfloor\lfloor A,B\rfloor,C\rfloor+\lfloor\lfloor B,C\rfloor,A\rfloor+\lfloor\lfloor C,A\rfloor,B\rfloor$$

vanishes also up to any desired degree and thereby that the above equations hold, see [3], e.g. Lemma 12, for details. □

We will need the following to analyze the Poisson structure on the reduced orbit space in the the indefinite Hamiltonian case.

LEMMA 6. — *Let* $\mathbb{S}^1\times\mathbb{C}^k\to\mathbb{C}^k$ *be a linear Hamiltonian circle action with nonzero relative prime weights* $n_1,\dots,n_k$. *Assume that the action has an indefinite Hamiltonian, i.e., not all the weights have the same sign. Let* J *be the momentum map*

$$J=\frac{1}{2}\sum_1^k n_i(x_i^2+y_i^2),$$

and $F:\mathbb{C}^k/\mathbb{S}^1\to\mathbb{R}^{\mathfrak{N}}$ *a Hilbert embedding corresponding to a Hilbert basis of* $\mathfrak{N}$ *minimal generators in* S_n. *Let* $f\in C^\infty(\mathbb{R}^{2k})^{\mathbb{S}^1}$ *be an invariant smooth function such that*

$$f|_{J^{-1}(0)}=0.$$

Then the Taylor series of f *at zero, denoted by* $T(f)$, *can be written*

$$T(f)=J\cdot Q$$

where Q *is an invariant formal power series* $Q\in\mathbb{R}[[x,y]]^{\mathbb{S}^1}$.

Proof. — Let $f\in C^\infty(\mathbb{R}^{2k})^{\mathbb{S}^1}$ be an invariant smooth function that vanishes on $J^{-1}(0)$. In order to prove the lemma we assume, without loss of generality, that there are precisely m negative weights, among them being n_k, and that $m \leq k/2$, if not, we can always reorder the weights or replace J with $-J$ and Q with $-Q$.

Consider J as a polynomial in x_k and use the Malgrange-Mather division theorem, see [6], [8], to write

$$f(x,y) = J(x,y)q(x,y) + x_k r_1(\hat{x},y) + r_2(\hat{x},y)$$

with q, r_1 and r_2 smooth functions, and where $\hat{x}$ simply denotes that x_k is missing, i.e., $\hat{x} = (x_1, \dots, x_{k-1})$. For values $(\hat{x},y)$ with $J((\hat{x},0),y) > 0$ there are two nonzero solutions t and $-t$ to the equation $J((\hat{x},t),y) = 0$, from this and since we are assuming that $f|_{J^{-1}(0)} = 0$ it follows immediately that

$$r_1(\hat{x},y) = r_2(\hat{x},y) = 0 \;\text{ if }\; J(x,y) \geq 0.$$

Now introduce a linear change of coordinates $L(s,t) = (x,y)$ (use that $m \leq k/2$) such that $J \circ L(s,t) \geq 0$ in the positive 2^{2k}-tant $s_i \geq 0,\ t_i \geq 0$ for $i = 1, \dots, k$. The Taylor series at 0 of $f \circ L(s,t)$, and therefore also of $f(x,y)$, is determined by taking limits in the positive 2^{2k}-tant from which it follows that $T(f) = J \cdot T(q)$. Applying an averaging operator M for the circle action,

$$M(h) = \frac{1}{2\pi}\int_0^{2\pi} h(e^{\iota\theta}(x,y))d\theta,$$

to each term of the Taylor series $T(q)$ results in an invariant formal series $Q = M(T(q))$ satisfying

$$T(f) = J \cdot Q,$$

as desired. □

By taking a closer look at the proof of Lemma 4 and also by using the above lemma we obtain the following.

Corollary 2. — *Let $\mathbb{S}^1 \times \mathbb{C}^k \to \mathbb{C}^k$ be a linear Hamiltonian circle action with nonzero relative prime weights $n_1, \dots, n_k$ and such that the action generates a Hilbert basis of minimal size. Assume that the action has an indefinite Hamiltonian, $k > 2$ and that among $d_1, \dots, d_k$ there are at least 3 elements that are not one. Let*

$$J = \frac{1}{2}\sum_1^k n_i(x_i^2 + y_i^2),$$

and assume $F : \mathbb{C}^k/\mathbb{S}^1 \to \mathbb{R}^{k^2}$ is a Hilbert embedding corresponding to the Hilbert basis of k^2 minimal generators in S_n, let $\mathcal{J}$ be the functional *induced by $\mathcal{J} \circ F = J$. Then the Poisson structure on the reduced orbit space $J^{-1}(0)/\mathbb{S}^1$ cannot be extended to the hyperplane $V_J = \mathcal{J}^{-1}(0)$.*

Proof. — Assume that $\{,\}_{V_J}$ is a Poisson bracket on V_J extending the Poisson structure on the reduced orbit space

$$\mu_0 = J^{-1}(0)/\mathbb{S}^1.$$

We will show that this leads to a contradiction.

Extend the Poisson bracket to all of $V = \mathbb{R}^{k^2}$ by requiring $\mathcal{J}$ to be a Casimir[(8)] function, i.e., let $v_1, \dots, v_{k^2-1}$ be a linear coordinate system for V_J and extend the bracket to all of V by requiring $\{\mathcal{J}, v_i\}_V = 0$ and $\{v_i, v_j\}_V = \{v_i, v_j\}_{V_J}$. Let $Z = J^{-1}(0)$. The bracket $\{,\}_{V_J}$ extends the structure on μ_0, by definition, so we have that $\{v_i, v_j\}_{V_J} \circ F$ and $\{v_i \circ F, v_j \circ F\}$ agree on Z. Since $J = \mathcal{J} \circ F$ is Casimir in the Poisson algebra $C^\infty(R^{2k})^{\mathbb{S}^1}$ the identity extends to all the coordinate functions $v_1, \dots, v_{k^2-1}, \mathcal{J}$ on V and eventually to all of $C^\infty(V)$, i.e.,

$$\{S, T\}_V \circ F|_Z = \{S \circ F, T \circ F\}|_Z,$$

for all smooth functions S, T on V.

Fix a choice of an almost Poisson bracket with polynomial coefficients, $\lfloor , \rfloor$, on V extending the Poisson structure on $\mathbb{R}^{2k}$. We can choose $\lfloor , \rfloor$ such that $\mathcal{J}$ is Casimir. The brackets $\lfloor , \rfloor_V$ and $\lfloor , \rfloor$ are related through the equation

$$\lfloor S, T \rfloor \circ F|_Z = \{S, T\}_V \circ F|_Z,$$

for smooth functions S, T on V. Given smooth functions S and T on V, it now follows from Lemma 6 that the Taylor series of $\{S, T\}_V - \lfloor S, T \rfloor$ is of the form

$$R_{ST} + \mathcal{J} Q_{ST}$$

where R_{ST} is a formal relation on V and Q_{ST} is a power series on V.

Now consider the Poisson bivectorfield π determined by $\{,\}_V$, i.e., $\pi(df \wedge dg) = \{f, g\}_V$ for smooth functions f and g on V. Let Π be the formal bivectorfield obtained by replacing the coefficients of π by its Taylor series, at zero, in the coordinates L, Z, W, i.e.,

$$\Pi = T(\pi) \in \mathcal{X}^2[[V_c]].$$

According to the above the coefficients of Π are of the form $\lfloor S, T \rfloor + R_{ST} + \mathcal{J} Q_{ST}$ for formal relations R_{ST} and power series Q_{ST}.

(8) f is Casimir if $\{f, g\} = 0$ for all smooth functions g.

Now consider the formal bivectorfield $|\Pi|$ where

$$\Pi = \psi^\sigma(|\Pi|)$$

and ψ^σ is defined as in Lemma 3 with $\sigma_i = \operatorname{sign}(n_i)$. By Lemma 3 the formal bivectorfield $|\Pi|$ is Poisson. Repeat the above process for the almost Poisson bracket, $\lfloor , \rfloor$ on V. Let ζ by the bivectorfield determined by $\lfloor , \rfloor$, i.e., $\lfloor , \rfloor = \lfloor , \rfloor_\zeta$. Define $|\zeta|$ by $\zeta = \psi^\sigma(|\zeta|)$. Then by Lemma 3, $|\zeta|$ determines an almost Poisson structure compatible with the Poisson structure on $\mathbb{C}[S_{|n|}]$.

For simplicity assume that d_1, d_2 and d_3 are greater than one. Going back to the notation in the proof of Lemma 4, let $A = X^{e_{12}}$, $B = X^{e_{21}}$, $C = X^{e_{31}}$, $D = X^{e_{32}}$ and $L_1 = X^{e_{11}}$, $L_2 = X^{e_{22}}$ where the complex coordinate functionals $X^{e_{ij}}$ are chosen with respect to the action with positive weights $|n_1|, \ldots, |n_k|$, i.e., such that

$$X^{e_{ij}} \circ |F| = u_i^{d_i} v_j^{d_j} \quad \text{for } i \neq j \text{ and } X^{e_{ii}} \circ |F| = u_i v_i.$$

By the definition of ψ^σ we have that for coordinate functions X and Y

$$\{X, Y\}_\Pi = \frac{1}{\omega_X^\sigma \omega_Y^\sigma} \psi_{\mathbf{c}}^\sigma(\{X, Y\}_{|\Pi|}).$$

From which it follows that

$$\{X, Y\}_{|\Pi|} = \omega_X^\sigma \omega_Y^\sigma \psi_{\mathbf{c}}^{\sigma-1}(\{X, Y\}_\Pi).$$

Now we calculate

$$\{A, B\}_{|\Pi|} = \lfloor A, B \rfloor_{|\zeta|} + \omega_A^\sigma \omega_B^\sigma \psi_{\mathbf{c}}^{\sigma-1}(R_{AB}) + \omega_A^\sigma \omega_B^\sigma \psi_{\mathbf{c}}^{\sigma-1}(\mathcal{J}) \psi_{\mathbf{c}}^{\sigma-1}(Q_{AB})$$

and after observing by direct calculations that

$$|\mathcal{J}| \stackrel{\text{def}}{=} \psi_{\mathbf{c}}^{\sigma-1}(\mathcal{J}) = \frac{1}{2} \sum_1^k |n_i| X^{e_{ii}}$$

and since that by Lemma 3

$$\psi_{\mathbf{c}}^{\sigma-1}(R_{AB}) \circ |F| = 0$$

we can write

$$\{A, B\}_{|\Pi|} = \lfloor A, B \rfloor_{|\zeta|} + R'_{AB} + |\mathcal{J}| Q'_{AB}$$

where Q'_{AB} is a power series and R'_{AB} is a formal relation with respect to the action with positive weights $|n_1|, \ldots, |n_k|$.

Next we define a bivectorfield Ξ on V with polynomial coefficients. It is an approximation to the Poisson bivector $|\Pi|$ as follows. We require that Ξ satisfies the Jacobi identity up to a high degree, so, for example,

$$f_t Z = 0$$

where

$$Z = \lfloor\lfloor A, B\rfloor_\Xi, C\rfloor_\Xi + \lfloor\lfloor B, C\rfloor_\Xi, A\rfloor_\Xi + \lfloor\lfloor C, A\rfloor_\Xi, B\rfloor_\Xi$$

and

$$t \le t_{\max} = (d_1 - 2)(e_1 + \bar{e}_1) + (d_2 + 1)(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1.$$

We also require that $|\mathcal{J}|$ is Casimir up to a high degree with respect to Ξ, so, for example,

$$f_t \lfloor A, |\mathcal{J}|\rfloor_\Xi = 0.$$

This is achieved by defining Ξ from $|\Pi|$ using equations of the form

$$\lfloor A, B\rfloor_\Xi = \lfloor A, B\rfloor_{|\zeta|} + R''_{AB} + |\mathcal{J}| Q''_{AB}$$

where the polynomials R''_{AB} and Q''_{AB} are obtained by approximating R'_{AB} up to a high degree with a polynomial relation R''_{AB} and Q'_{AB} with a polynomial approximation Q'_{AB}. Similar equations can be assume for $\lfloor B, C\rfloor_\Xi$, $\lfloor C, A\rfloor_\Xi$, $\lfloor D, B\rfloor_\Xi$, $\lfloor L_1, C\rfloor_\Xi$, $\lfloor L_2, C\rfloor_\Xi$ and $\lfloor L_1, B\rfloor_\Xi$ as well. By Comment I after Lemma 3 we now obtain that

i) $f_t \lfloor\lfloor A, B\rfloor_\Xi, C\rfloor_\Xi = -4d_1^3(d_1 - 1) f_t(L_1^{d_1-2} L_2^{d_2} C) + f_t \lfloor R''_{AB} + |\mathcal{J}| Q''_{AB}, C\rfloor_\Xi$,

ii) $f_t \lfloor\lfloor B, C\rfloor_\Xi, A\rfloor_\Xi = f_t \lfloor R''_{BC} + |\mathcal{J}| Q''_{BC}, A\rfloor_\Xi$ and

iii) $f_t \lfloor\lfloor C, A\rfloor_\Xi, B\rfloor_\Xi = 3d_1^3(d_1 - 1) f_t(L_1^{d_1-2} BD) + f_t \lfloor R''_{CA} + |\mathcal{J}| Q''_{CA}, B\rfloor_\Xi$.

for

$$t = (d_1 - 2)(e_1 + \bar{e}_1) + d_2(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1$$

or if t is not divisible by $(d_1 - 2)(e_1 + \bar{e}_1)$.

The integers $d_1, \ldots, d_k$ are relative primes two-and-two. Since none of d_1, d_2 or d_3 is equal to 1 we can assume without loss of generality that

$$1 < d_1 < d_3 < d_2.$$

Now use Comment II after Lemma 3 to establish that

$$f_t \lfloor Q, R\rfloor_\Xi = 0,$$

for $Q = A, B, C$, an $|F|$ relation R and all

$$t \le t_{\max *} = (d_1 - 2)(e_1 + \bar{e}_1) + 2(d_2 - 1)(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1.$$

In particular, for

$$t_1 = (d_1 - 2)(e_1 + \bar{e}_1) + d_2(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1$$

we can now write zero as

$$f_{t_1} Z = -4d_1^3(d_1 - 1) X^{(d_1-2)e_{11}} (X^{d_2 e_{22}} X^{e_{31}} - X^{e_{21}} X^{e_{32}}) + f_{t_1} |\mathcal{J}| \Omega$$

where $\mathfrak{Q}$ is the polynomial

$$\mathfrak{Q} = \lfloor Q''_{AB}, C \rfloor_\Xi + \lfloor Q''_{BC}, A \rfloor_\Xi + \lfloor Q''_{CA}, B \rfloor_\Xi.$$

Write

$$\mathfrak{Q} = \sum_s \mathfrak{q}_s X^s.$$

Since $f_{t_1} Z = 0$ we conclude, by comparing terms, that

$$\frac{|n_1|}{2} \mathfrak{q}_{s_1} = -4d_1^3(d_1 - 1) \neq 0,$$

where

$$s_1 = (d_1 - 3)e_{11} + e_{21} + e_{32}.$$

If $d_1 = 2$ this is not possible, since then $d_1 - 3 = -1$, and we have a contradiction. Assume therefore that $d_1 > 2$. Now let

$$t_2 = (d_1 - 3)(e_1 + \bar{e}_1) + (d_2 + 1)(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1$$

then by Comment I since $(d_1 - 2)(e_1 + \bar{e}_1)$ does not divide t_2 and by Comment II we have that

$$f_{t_2} Z = f_{t_2} |\mathcal{J}| \mathfrak{Q}.$$

The left hand side of the above equation is zero so by comparing terms we conclude that

$$\frac{|n_2|}{2} \mathfrak{q}_{s_1} X^{s_1 + e_{22}} + \frac{|n_1|}{2} \mathfrak{q}_{s_2} X^{s_2 + e_{11}} = 0$$

where

$$s_2 = (d_1 - 4)e_{11} + e_{22} + e_{21} + e_{32}.$$

Therefore since $\mathfrak{q}_{s_1} \neq 0$ we must have

$$\mathfrak{q}_{s_2} \neq 0.$$

In general, let

$$s_i = (d_1 - 2 - i)e_{11} + (i - 1)e_{22} + e_{21} + e_{32},$$

$$t_i = (d_1 - 1 - i)(e_1 + \bar{e}_1) + (d_2 + i - 1)(e_2 + \bar{e}_2) + d_3 e_3 + d_1 \bar{e}_1.$$

We can conclude that

$$f_{t_i} |\mathcal{J}| \mathfrak{Q} = 0$$

using Comment I and Comment II after the proof of Lemma 3 for

$$i = 1, \dots, d_1 - 1.$$

Calculating $f_{t_i}|\mathcal{J}|\Omega$, for $i > 1$, results in equations

$$\frac{|n_2|}{2}\mathfrak{q}_{s_{i-1}}X^{s_{i-1}+e_{22}} + \frac{|n_1|}{2}\mathfrak{q}_{s_i}X^{s_i+e_{11}} = 0,$$

and nonzero terms $\mathfrak{q}_{s_1}, \ldots, \mathfrak{q}_{s_{d_1-1}}$. On the other hand

$$X^{s_{d_1-1}}$$

is not a polynomial, it is a rational function, since

$$s_{d_1-1} = -e_{11} + (d_1 - 2)e_{22} + e_{21} + e_{32},$$

therefore

$$\mathfrak{q}_{s_{d_1-1}} = 0$$

which is a contradiction. This completes the proof of Corollary 2. □

Theorem 1 now follows from Lemma 4 (or rather Corollary 1), Lemma 5 and Corollary 2.

8. Induced Poisson structures.

Here we consider 4 relevant examples of classes of induced Poisson structures.

8.1. Linear circle actions.

Example 1. — Linear Poisson structures.

Define grading on each of the spaces of n-vector fields with polynomial coefficients

$$\mathcal{X}^n[V] = \mathcal{X}^n[V]^0 + \mathcal{X}^n[V]^1 + \mathcal{X}^n[V]^2 + \ldots$$

where $\mathcal{X}^n[V]^s$ is the space of n-vector fields with homogeneous coefficients of degree s, to be exact this means that X is in $\mathcal{X}^n[V]^s$ if $X(dg_1 \wedge \ldots \wedge dg_n)$ is homogeneous of degree s for all $g_1, \ldots, g_n \in V^*$.

As before let $F : \mathbb{R}^{2k}/\mathbb{S}^1 \to \mathbb{R}^{\mathfrak{N}}$ be the Hilbert embedding associated to a minimal homogeneous Hilbert basis $f_1, \ldots, f_{\mathfrak{N}}$ and let $\Pi \in \mathcal{X}^2[\mathbb{R}^{\mathfrak{N}}]$ be an almost Poisson bivector, i.e.,

$$\lfloor f, g \rfloor_\Pi = \Pi(df \wedge dg)$$

extending the Poisson structure on $\mathbb{R}^{2k}/\mathbb{S}^1$. In other words ϱ and Π are F-related, $F_* \circ \varrho = \Pi \circ F$.

Write $\Pi = \Pi^0 + \Pi^1 + \Pi^2 + \Pi^3 + \dots$ with each Π^s in $\mathcal{X}^2[V]^s$. The homogeneous invariants $f_1, \dots, f_{\mathfrak{N}}$ are all of degrees greater than one and consequently $(F_* \circ \varrho)_0 = 0$. Now it follows that $\Pi^0 = 0$ and therefore the "total" Jacobiator is given by

$$[\Pi, \Pi] = [\Pi^1, \Pi^1] + \epsilon^{[2]}$$

where $[\Pi^1, \Pi^1]$ is homogeneous of degree one and $\epsilon^{[2]}$ contains only terms of degrees greater than one. The Schouten-Nijenhuis bracket behaves nicely with respect to F-related multivector fields and it follows that

$$[\Pi, \Pi] \circ F = F_* \circ [\varrho, \varrho].$$

Since ϱ is Poisson, i.e., $[\varrho, \varrho] = 0$ we conclude that $[\Pi, \Pi] \circ F = 0$. As the generating set $\{f_1, \dots, f_{\mathfrak{N}}\}$ is minimal, i.e., $f_n \notin \mathbb{R}[f_1, \dots, \widehat{f_n}, \dots, f_{\mathfrak{N}}]$, and each f_n is homogeneous it follows that $[\Pi^1, \Pi^1] \circ F = 0$. Finally since $f_1, \dots, f_{\mathfrak{N}}$ are linearly independent we have that

$$[\Pi^1, \Pi^1] = 0.$$

We summarize the calculations in this example as follows, assumptions as above:

COROLLARY 3. — *Let $\Pi \in \mathcal{X}^2[\mathbb{R}^{\mathfrak{N}}]$ be an almost Poisson structure on $\mathbb{R}^{\mathfrak{N}}$ extending the Poisson structure on the orbit space $\mathbb{R}^{2k}/\mathbb{S}^1$ of a linear Hamiltonian circle action. Then the linear part of Π is a Poisson bivector.*

Example 2. — Linear circle actions $\mathbb{S}^1 \times \mathbb{C}^2 \to \mathbb{C}^2$.

Consider a linear Hamiltonian circle action with positive (for simplicity) relative prime weights n_1 and n_2. In this case $d_1 = n_2$ and $d_2 = n_1$ and the weights satisfy the conditions of Lemma 2, i.e., $n_1 n_2 = d_1 d_2$. A complex Hilbert basis for the action has four invariants $u_1 v_1$, $u_2 v_2$, $u_1^{d_1} v_2^{d_2}$ and $v_1^{d_1} u_2^{d_2}$. The corresponding Hilbert embedding is given by

$$\mathbb{R}^4/\mathbb{S}^1 \to \mathbb{R}^4; (x, y) \mapsto (u_1 v_1, u_2 v_2, u_1^{d_1} v_2^{d_2}) \quad (\in \mathbb{R}^2 \times \mathbb{C}).$$

An almost Poisson structure $\lfloor , \rfloor$ extending the Poisson structure on the orbit space is, as before, described in terms of the induced almost Lie algebra $(\mathbb{C}[X^{e_{11}}, X^{e_{22}}, X^{e_{12}}, X^{e_{21}}], \lfloor , \rfloor)$. One can choose the almost Lie algebra bracket such that

$$\lfloor X^{e_{ii}}, X^{e_{jk}} \rfloor = -2\iota c_{ijk} X^{e_{jk}}$$

where c_{ijk} is a (real) constant uniquely determined by the pullback to $\mathbb{C}[S_n]$, i.e., $\{\tau_n(X^{e_{ii}}), \tau_n(X^{e_{jk}})\}_{\mathbf{c}} = -2\iota c_{ijk}\tau_n(X^{e_{jk}})$. As always, for positive weights, $\tau_n(X^{e_{ij}}) = u_i^{d_i} v_j^{d_j}$ if $i \neq j$ and $\tau_n(X^{e_{ii}}) = u_i v_i$. The constants c_{ijk} satisfy $c_{ijk} = -c_{ikj}$ and $c_{ijj} = 0$. The only ambiguity left in the definition of the almost Lie algebra is to specify $\lfloor X^{e_{12}}, X^{e_{21}} \rfloor$. We can assume that

$$\lfloor X^{e_{12}}, X^{e_{21}} \rfloor \in \mathbb{C}[X^{e_{11}}, X^{e_{22}}].$$

These conditions are sufficient to show that the Jacobiator $\mathfrak{J}$,

$$\mathfrak{J}(A, B, C) = \lfloor \lfloor A, B \rfloor, C \rfloor + \lfloor \lfloor B, C \rfloor, A \rfloor + \lfloor \lfloor C, A \rfloor, B \rfloor$$

is zero since the only way to find three different coordinate functions A, B, C among $X^{e_{11}}$, $X^{e_{22}}$, $X^{e_{12}}$, $X^{e_{21}}$ is to include $X^{e_{11}}$ or $X^{e_{22}}$. We summarize, as follows:

COROLLARY 4. — *There always exists a Poisson structure on $\mathbb{R}^4$ extending the Poisson structure on the singular Poisson variety $\mathbb{C}^2/\mathbb{S}^1$.*

Example 3. — The action

$$\mathbb{S}^1 \times \mathbb{C}^3 \to \mathbb{C}^3; z \times (u_1, u_2, u_3) \mapsto (zu_1, z^2u_2, z^2u_3).$$

Here the linear Hamiltonian circle action has weights $n_1 = 1$, $n_2 = 2$ and $n_3 = 2$. It follows that $d_1 = 2$, $d_2 = 1$ and $d_3 = 1$ and the equation

$$(d_1 d_2 d_3)^2 = n_1 n_2 n_3$$

holds, so the action satisfies the conditions in Lemma 2. A (complex) Hilbert basis is given by the nine elements

$$u_1v_1,\ u_2v_2,\ u_3v_3,\ u_1^2v_2,\ u_1^2v_3,\ u_2v_3,\ v_1^2u_2,\ v_1^2u_3,\ v_2u_3.$$

The pullback τ_n of the Hilbert embedding

$$\mathbb{R}^6/\mathbb{S}^1 \hookrightarrow \mathbb{R}^9$$

is determined by $\tau_n(X^{e_{ii}}) = u_i v_i$ and if $i \neq j$ we have $\tau_n(X^{e_{ij}}) = u_i^{d_i} v_j^{d_j}$ as before where $X^{e_{11}}$, $X^{e_{22}}$, $X^{e_{33}}$ and $X^{e_{12}}, X^{e_{21}}$, $X^{e_{13}}, X^{e_{31}}$, $X^{e_{23}}, X^{e_{32}}$ are the (complex) coordinate functionals on $\mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{C}^3$. An almost Poisson structure $\lfloor , \rfloor$ on $\mathbb{R}^9$ compatible with the Poisson structure on $\mathbb{R}^6/\mathbb{S}^1$ is determined uniquely by terms of the form $\lfloor X^{e_{ij}}, X^{e_{i'j'}} \rfloor$ containing non-linear parts, since the linear part of each term is uniquely determined by the compatibility condition. Let $\{ , \}$ be the almost Poisson structure on $\mathbb{R}^9$ determined by the following list of non-linear terms:

- $\{X^{e_{12}}, X^{e_{31}}\} = -8\iota X^{e_{11}} X^{e_{32}}$, $\{X^{e_{13}}, X^{e_{21}}\} = -8\iota X^{e_{11}} X^{e_{23}}$,

- $\{X^{e_{12}}, X^{e_{21}}\} = -8\iota X^{e_{11}}X^{e_{22}} + 2\iota X^{2e_{11}} - 16\iota(X^{e_{22}}X^{e_{33}} - X^{e_{23}}X^{e_{32}})$,
- $\{X^{e_{13}}, X^{e_{31}}\} = -8\iota X^{e_{11}}X^{e_{33}} + 2\iota X^{2e_{11}} - 16\iota(X^{e_{22}}X^{e_{33}} - X^{e_{23}}X^{e_{32}})$.

This almost Poisson structure $\{,\}$ actually satisfies Jacobi identity. In order to verify the identity one has to calculate (straightforward) the 84 relevant identities. One can use Corollary 3 to simplify the process and also an argument similar to the one used to verify the Jacobi identity in Example 2. The Poisson structure on $\mathbb{R}^9$, $\{,\}$ above, is obtained by lifting the Poisson structure on $\mathbb{R}^6/\mathbb{S}^1$ via the pullback τ_n. The choice of relations in $\{X^{e_{12}}, X^{e_{21}}\}$ and $\{X^{e_{13}}, X^{e_{31}}\}$, i.e., the "$-16\iota(X^{e_{22}}X^{e_{33}} - X^{e_{23}}X^{e_{32}})$" term is inspired by Part II in the proof of Lemma 5.

Let Π be the Poisson bivectorfield corresponding to the Poisson bracket $\{,\}$. Now we look at the indefinite Hamiltonian case, i.e.,

$$J = \frac{1}{2}\sigma_1(x_1^2 + y_1^2) + \sigma_2(x_2^2 + y_2^2) + \sigma_3(x_3^2 + y_3^2),$$

obtained by changing the signs of some of the weights. It follows from Lemma 3 that $\psi^\sigma(\Pi)$ is Poisson and defines a structure compatible with the structure on the orbit space of the action with weights $\sigma_1, 2\sigma_2, 2\sigma_3$. What is left to verify is that $\psi^\sigma(\Pi)$ actually is a real bivectorfield. By direct calculations for $R = X^{e_{22}}X^{e_{33}} - X^{e_{23}}X^{e_{32}}$ we find that

$$\psi^\sigma_{\mathbf{c}}\Big(R\frac{\partial}{\partial X^{e_{12}}} \wedge \frac{\partial}{\partial X^{e_{21}}}\Big) = \frac{\sigma_2\sigma_3}{\omega^\sigma_{X^{e_{12}}}\omega^\sigma_{X^{e_{21}}}} R\frac{\partial}{\partial X^{e_{12}}} \wedge \frac{\partial}{\partial X^{e_{21}}}.$$

The factor $\omega^\sigma_{X^{e_{12}}}\omega^\sigma_{X^{e_{21}}}$ is just ± 1 for symmetry reasons. Similarly we obtain that

$$\psi^\sigma\Big(R\frac{\partial}{\partial X^{e_{13}}} \wedge \frac{\partial}{\partial X^{e_{31}}}\Big) = \pm R\frac{\partial}{\partial X^{e_{13}}} \wedge \frac{\partial}{\partial X^{e_{31}}}$$

from which we can conclude that $\psi^\sigma(\Pi)$ is real. We summarize these calculations as follows:

COROLLARY 5. — *The Poisson structure on the singular orbit space $\mathbb{C}^3/\mathbb{S}^1 \subset \mathbb{R}^9$ of the linear Hamiltonian circle action with weights $\pm 1, \pm 2, \pm 2$ can be extended to all of $\mathbb{R}^9$.*

8.2. Linear discrete circle actions.

Let $n_1, \ldots, n_k$ be nonzero relative prime weights and as before let

$$d_i = \gcd(d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_k).$$

Identify the discrete circle $\mathbb{Z}_d$ with the subgroup of $\mathbb{S}^1$ generated by $e^{2\pi\iota/d}$ and $\mathbb{R}^{2k}$ with $\mathbb{C}^k$.

CLAIM 1. — *Let $\mathbb{Z}_{d_1\cdots d_k}$ act on $\mathbb{R}^{2k}$ by $z(u_1,\ldots,u_k)=(z^{n_1}u_1,\ldots,z^{n_k}u_k)$. The $\mathbb{Z}_{d_1\cdots d_k}$-invariant polynomials in variables $u_1,v_1,\ldots,u_k,v_k$ are generated by the set $\{u_1v_1,\ldots,u_kv_k,\ u_1^{d_1},v_1^{d_1},\ldots,u_k^{d_k},v_k^{d_k}\}$.*

Proof. — The polynomials in $\{u_1v_1,\ldots,u_kv_k,\ u_1^{d_1},v_1^{d_1},\ldots,u_k^{d_k},v_k^{d_k}\}$ are invariant under the action of $\mathbb{Z}_{d_1\cdots d_k}$ since the action preserves length and $d_in_i/d_1\cdots d_k$ is always an integer for $i=1,\ldots,k$. An invariant (monic) monomial is of the form $u_1^{a_1}\ldots u_k^{a_k}v_1^{b_1}\ldots v_k^{b_k}$ with $(n_1(a_1-b_1)+\cdots+n_k(a_1-b_1))/d_1\cdots d_k$ being an integer. Multiplying this equation through with n_i and by noticing that $n_in_j(a_j-b_j)/d_1\cdots d_k$ is in $\mathbb{Z}$ if $i\neq j$ one obtains that $n_i^2(a_i-b_i)/d_1\cdots d_k$ is also an integer. On the other hand d_i and n_i are relative primes so d_i must divide (a_i-b_i). From this it follows that the set $\{u_1v_1,\ldots,u_kv_k,\ u_1^{d_1},v_1^{d_1},\ldots,u_k^{d_k},v_k^{d_k}\}$ generates all the invariants in variables $u_1,v_1,\ldots,u_k,v_k$. □

The real invariant polynomials are generated by polynomials obtained from the generators in variables u and v by taking real and imaginary parts. The number of polynomials needed does not change when going from complex to real invariants. This defines an injection of $\mathbb{R}^{2k}/Z_{d_1\cdots d_k}$ into $\mathbb{R}^k\times\mathbb{R}^{2k}$ given by

$$F(u_1,\ldots,u_k)=(u_1v_1,\ldots,u_kv_k,u_1^{d_1},\ldots,u_k^{d_k})$$

where as before we have identified $\mathbb{R}^{2k}$ with $\mathbb{C}^k$ through $u_i=x_i+\iota y_i$ and $v_i=x_i-\iota y_i$. The Poisson structure on $\mathbb{R}^{2k}$ projects to a Poisson structure on $\mathbb{R}^{2k}/\mathbb{Z}_{d_1\cdots d_k}$ where the smooth structure on $\mathbb{R}^{2k}/\mathbb{Z}_{d_1\cdots d_k}$ is given by the $\mathbb{Z}_{d_1\cdots d_k}$-invariant functions on $\mathbb{R}^{2k}/\mathbb{Z}_{d_1\cdots d_k}$.

THEOREM 2. — *Let $\mathbb{Z}_{d_1\cdots d_k}$ act on $\mathbb{R}^{2k}$ by $z*(u_1,\ldots,u_k)=(z^{n_1}u_1,\ldots,z^{n_k}u_k)$. Then there is a natural induced Poisson structure on $\mathbb{R}^k\times\mathbb{R}^{2k}$ such that F is an injective Poisson map of $\mathbb{R}^{2k}/\mathbb{Z}_{d_1\cdots d_k}$ into $\mathbb{R}^k\times\mathbb{R}^{2k}$.*

Proof. — Using the real coordinates $l_1,\ldots,l_k$ on $\mathbb{R}^k$ and the complex coordinates $u_1,v_1,\ldots,u_k,v_k$ on $\mathbb{R}^{2k}$ define an almost Poisson structure $\lfloor,\rfloor$ on $C^\infty(\mathbb{R}^k\times\mathbb{R}^{2k})$ as follows. Let $\lfloor u_i,u_j\rfloor=0$, $\lfloor v_i,v_j\rfloor=0$, $\lfloor l_i,l_j\rfloor=0$ and

$$\lfloor l_i,u_i\rfloor=2\iota d_iu_i,\quad \lfloor l_i,v_i\rfloor=-2\iota d_iv_i,\quad \lfloor u_i,v_i\rfloor=-2\iota d_i^2l_i^{d_i-1}$$

furthermore for $i\neq j$ let $\lfloor l_i,u_j\rfloor=0$, $\lfloor l_i,v_j\rfloor=0$ and $\lfloor u_i,v_j\rfloor=0$. Notice, we are choosing the coordinates such that the function F is determined by $l_i \circ F = u_i v_i, u_i \circ F = u_i^{d_i}$ and $v_i \circ F = v_i^{d_i}$. It is straightforward to verify that F is an almost Poisson map between the Poisson structure on $\mathbb{R}^{2k}$ and the almost Poisson structure on $\mathbb{R}^k \times \mathbb{R}^{2k}$. In order to verify that $\lfloor , \rfloor$ satisfies the Jacobi identity, it is enough to show that for $f, g, h \in \{l_1, \ldots, l_k, u_1, v_1, \ldots, u_k, v_k\}$ the Jacobiator

$$\mathfrak{J}(f,g,h) = \lfloor\lfloor f,g\rfloor,h\rfloor + \lfloor\lfloor g,h\rfloor,f\rfloor + \lfloor\lfloor h,f\rfloor,g\rfloor$$

vanishes. It follows from the definition of $\lfloor , \rfloor$ that $\mathfrak{J}(f,g,h)$ is in the $\mathbb{C}$-linear span of the polynomials in

$$\mathfrak{P} = (u_i, v_i, u_i l_i^{d_i-2}, v_i l_i^{d_i-2}, l_i^{d_i-1})_{1\leq i\leq k}.$$

On the other hand, the pullback of F maps the set of polynomials in the collection $\mathfrak{P}$, above, injectively to normalized monomials in $u_1, v_1, \ldots, u_k, v_k$. Since F is an almost Poisson map we have that $\mathfrak{J}(f,g,h) \circ F = 0$ which is only possible, by the above, if $\mathfrak{J}(f,g,h) = 0$. In other words $\lfloor , \rfloor$ is a Poisson bracket. □

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