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FINITE RANK APPROXIMATION AND SEMIDISCRETENESS FOR LINEAR OPERATORS

by Christian LE MERDY

1. Introduction.

Approximation properties and factorizations through matrix spaces have often played a major role in the study of C*-algebras. One of the most remarkable illustrations of that fact is the result by Choi-Effros ([6]) and Kirchberg ([22]) which says that a C*-algebra B is nuclear (i.e. $A\otimes_{\min} B = A\otimes_{\max} B$ for every C*-algebra A) if and only if there exists a net of diagrams $B \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} B$ such that α_i, β_i are completely positive contractions, and $\beta_i\alpha_i$ converges to the identity mapping I_B in the point-norm topology. Very recently, Pisier ([31]) gave a new proof of that result relying upon operator space theory. At the same time, he could extend the latter to the general framework of C*-algebra-valued completely bounded maps. This extension involves decomposable operators, that is linear combinations of completely positive maps, and the associated decomposable norm $\| \ \|_{\rm dec}$ introduced by Haagerup in [18]. Let B and Y be a C*-algebra and an operator space respectively, and let $u\colon Y \to B$ be a completely bounded map. Given a positive constant C, let us say that u is C-nuclear provided that for any C*-algebra A, the tensor map $I_A \otimes u$ extends to a bounded map from $A\otimes_{\min} Y$ into $A\otimes_{\max} B$, with norm less than C. Pisier showed in [31] that this holds if and only if there exists a net of diagrams $Y \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} B$ such that $\|\alpha_i\|_{cb}\|\beta_i\|_{\rm dec} \leqslant C$ and $\beta_i\alpha_i$ converges to u in the point-norm topology. Furthermore, if Y is a unital C*-algebra, if B is unital, and if u is

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unital completely positive (u.c.p. in short) and 1-nuclear, the above net can be constructed in such a way that the α_i's and β_i's are completely positive. The Choi-Effros-Kirchberg characterization of nuclearity can therefore be recovered by applying Pisier's result to the identity mapping of unital C*-algebras.

In the category of von Neumann algebras, the priviledged notion replacing nuclearity is that of semidiscreteness, introduced by Effros and Lance in [11]. Let M be a von Neumann algebra, and let us say that M is semidiscrete if $A \otimes_{\min} M = A \otimes_{\mathrm{nor}} M$ for every C*-algebra A. It is proved in [6], [11] that this holds if and only if there exists a net $M \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} M$ such that α_i, β_i are normal u.c.p. mappings, and $\beta_i\alpha_i$ converges to the identity mapping I_M in the point-w^* topology. The main purpose of this paper is to investigate a natural notion of semidiscrete linear operators, and to study possible relationships between nuclearity and semidiscreteness for operators. Let Z be an operator space, and let $u\colon Z \to M$ be a linear map. We will say that u is C-semidiscrete if for any C*-algebra A, the tensor map $I_A \otimes u$ extends to a bounded map from $A \otimes_{\min} Z$ into $A \otimes_{\mathrm{nor}} M$, with norm less than C. We will show the following analogue of Pisier's Theorem. The linear map $u\colon Z \to M$ is C-semidiscrete if and only if there exists a net $Z \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} M$ such that $\|\alpha_i\|_{cb}\|\beta_i\|_{\mathrm{dec}} \leqslant C$ and $\beta_i\alpha_i$ converges to u in the point-w^* topology. We will also show that in that characterization, the α_i's can be chosen w^*-continuous if Z is a dual operator space, and that the α_i's and the β_i's can be chosen u.c.p. if Z is an operator system and u is u.c.p. and 1-semidiscrete.

A fundamental result on operator algebras is that semidiscreteness is equivalent to injectivity for von Neumann algebras ([8], [33]). This led to the following equivalence, established by Choi and Effros in [5]: a C*-algebra B is nuclear if and only if its bidual B^{**} is semidiscrete. It is then natural to compare the nuclearity of a linear map with the semidiscreteness of its biadjoint. Let $u\colon Y \to B$ be a completely bounded map from an operator space into a C*-algebra. We will show that if $u^{**}\colon Y^{**} \to B^{**}$ is C-semidiscrete, then u is C-nuclear. Moreover the converse holds if Y is a locally reflexive operator space.

In fact it is possible to extend the definition of C-nuclearity (resp. C-semidiscreteness) to completely bounded maps valued in a possibly non self-adjoint operator algebra (resp. dual operator algebra), admitting a contractive approximate identity (resp. a unit). The results outlined above will be established in Section 4 in this broader context. The proofs will rely upon new properties of Pisier's delta norm that we shall establish in Section 3. The so-called delta norm is a norm which can be defined on the tensor product of an operator space E and of an operator algebra B admitting a contractive approximate identity. The resulting completion is denoted by $E \overset{\delta}{\otimes} B$. It was introduced and used in [31] to establish the characterization of C-nuclear operators mentioned at the beginning of this introduction. In our Section 3, we shall introduce a normal version $\overset{\sigma\delta}{\otimes}$ of this delta norm, adapted to dual operator algebra, and shall investigate its main features and its relationships with $\overset{\delta}{\otimes}$. For instance, it will have the property that $(E \overset{\delta}{\otimes} B)^{**} = E^{**} \overset{\sigma\delta}{\otimes} B^{**}$ for any E and B as above.

The delta norm, originally introduced to study nuclearity, has also been used recently by M. Junge and the author in [21] to prove that given a finite rank operator $u: A \to B$ between two C*-algebras, we have $\|u\|_{\text{dec}} = \inf\{\|\alpha\|_{cb}\|\beta\|_{\text{dec}}\}$, where the infimum runs over all factorizations $u = \beta\alpha$ with $A \xrightarrow{\alpha} M_n \xrightarrow{\beta} B$. In Section 5, we will combine the latter result will results from Section 3 to establish local reflexivity properties of the decomposable norm. More precisely, let R be a von Neumann algebra and let B be a C*-algebra. Identifying $R_* \otimes B$ with the space of w^*-to-norm continuous finite rank operators from R into B we may define $R_* \otimes_{\text{dec}} B$ as its completion under the decomposable norm. With this notation, we will show that $R^* \otimes_{\text{dec}} B^{**}$ embeds isometrically into $(R_* \otimes_{\text{dec}} B)^{**}$.

In the next Section 2, we give preliminaries on operator spaces, operator algebras, and tensor products, as well as a review of Pisier's delta norm and C-nuclear operators.

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2. Operator spaces, tensor norms, and nuclear operators.

We will assume that readers are familiar with the basics of operator space theory and completely bounded maps, for which we refer to [27], [2], [3], [12], [14], [16], [32] and [31]. We shall use the following standard notation and terminology. Given a completely bounded map $u: X_1 \to X_2$ between operator spaces, we will denote its completely bounded norm by $\|u\|_{cb}$. We will say that u is completely contractive (or is a complete contraction) if $\|u\|_{cb} \leqslant 1$ and that it is completely isometric (or is a complete isometry) if $I_{M_n} \otimes u$ is an isometry from $M_n(X_1)$ into $M_n(X_2)$ for all $n \geqslant 1$. The notation $\otimes_{\min}$ and $\otimes_h$ will stand for the minimal and the Haagerup tensor product respectively. We will use the notation $\| \ \|_{\min}$ and $\| \ \|_h$ for the corresponding norms.

The following convenient notation will be useful in our upcoming definitions of tensor norms. Given a pair E, F of operator spaces, a Hilbert space $\mathcal{H}$, and two completely bounded maps $u: E \to B(\mathcal{H})$ and $v: F \to B(\mathcal{H})$, we shall denote by $u\cdot v: E \otimes F \to B(\mathcal{H})$ the linear mapping defined by letting

$$u\cdot v\Big(\sum_k e_k \otimes f_k\Big) = \sum_k u(e_k)v(f_k) \tag{2.1}$$

for any finite families $(e_k)_k \subset E$, $(f_k)_k \subset F$.

By an operator algebra, we will simply mean a closed subalgebra $A \subset B(H)$ of the C*-algebra of all bounded operators on a Hilbert space H. We will be mainly concerned with operator algebras A which are unital, or merely admit a contractive approximate identity (c.a.i. in short). This means that there exists a net $(e_t)_t$ in A such that $\|e_t\| \leqslant 1$ and, for any $a \in A$, $\lim_t \|e_t a - a\| = \lim_t \|a e_t - a\| = 0$. Note that the class of operator algebras with a c.a.i. includes C*-algebras. We now recall the definition of the maximal tensor product of possibly non self-adjoint operator algebras, introduced in [28]. Let A and B be operator algebras and let $y \in A \otimes B$. Using notation (2.1), we let

$$\|y\|_{\max} = \sup\big\{\|\rho\cdot \pi(y)\|\big\}, \tag{2.2}$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$, and all pairs of completely contractive homomorphisms $\rho: A \to B(\mathcal{H})$, $\pi: B \to B(\mathcal{H})$ which have commuting ranges (i.e. $\rho(a)\pi(b) = \pi(b)\rho(a)$ for any $a \in A$, $b \in B$). The completion of $A \otimes B$ for the norm $\| \ \|_{\max}$ is denoted by $A \otimes_{\max} B$ and called the maximal tensor product of A and B. Note that if A and B are C*-algebras, the latter definition coincides with the classical definition of the maximal C*-norm defined on $A \otimes B$. Indeed, a bounded homomorphism between C*-algebras is completely contractive if and only if it is a $*$-representation.

We now turn to dual structures. By definition, an operator space X is called a dual operator space if there exists another operator space

Y such that $X = Y^*$ completely isometrically. Given a Hilbert space H, we shall denote by $S_1(H)$ the space of all trace class operators on H. It has a natural operator space structure for which $S_1(H)^* = B(H)$ completely isometrically. Indeed, it can be defined for example by regarding $S_1(H)$ as the dual operator space of the C*-algebra of compact operators on H. In particular, $B(H)$ is a dual operator space. More generally, let $X \subset B(H)$ be w^*-closed. Then X is the dual operator space of the quotient operator space $S_1(H)/X_\perp$. In that situation, we shall simply set $X_* = S_1(H)/X_\perp$. Note that conversely, any dual operator space can be represented completely isometrically as a w^*-closed subspace of $B(H)$ for some Hilbert space H (see e.g. [2, Proposition 2.1] or [12, Proposition 5.1]). Accordingly, an operator algebra M is called a dual operator algebra if it can be represented algebraically and completely isometrically as a w^*-closed subalgebra of $B(H)$ for some Hilbert space H. Note for further reference that in particular, any dual operator algebra is a dual operator space (see [25] and Remark 3.5 below for a converse). The classical definition of the normal tensor product ([11]) can be extended to our non self-adjoint context, as follows. Let A be an operator algebra and let M be a dual operator algebra. Then for any $y \in A \otimes M$, we let

$$\|y\|_{\mathrm{nor}} = \sup\{\|\rho\cdot \pi(y)\|\}, \tag{2.3}$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$, and all pairs of completely contractive homomorphisms $\rho\colon A \to B(\mathcal{H})$, $\pi\colon M \to B(\mathcal{H})$ with commuting ranges, such that π is w^*-continuous. The completion of $A\otimes M$ for the norm $\| \ \|_{\mathrm{nor}}$ will be denoted by $A \otimes_{\mathrm{nor}} M$.

Still concerning dual structures, we will need the following well-known result going back to [2].

LEMMA 2.1. — *Let H be a Hilbert space, let $Z \subset B(H)$ be w^*-closed, let $n \geqslant 1$ be an integer, and let $v\colon Z \to M_n$ be some w^*-continuous and completely bounded operator. Then for any $\varepsilon > 0$, there exists a w^*-continuous operator $\widetilde{v}\colon B(H) \to M_n$ extending v, with $\|\widetilde{v}\|_{cb} \leqslant (1+\varepsilon)\|v\|_{cb}$.*

As announced in our introduction, we will use decomposable operators and Haagerup's decomposable norm ([18]). Let A and B be two C*-algebras. We recall that by definition a linear mapping $u\colon A \to B$ is called a decomposable operator if it lies in the linear span of completely positive maps. Equivalently, u is decomposable if it can be written as $u = (u_1 - u_2) + i(u_3 - u_4)$, with $u_1, u_2, u_3, u_4\colon A \to B$ completely positive. We shall denote by $DEC(A,B)$ the space of all such operators. For any

$u \in DEC(A,B)$, we let $\|u\|_{\rm dec} = \inf\bigl\{\max\{\|S_1\|\|S_2\|\}\bigr\}$, where the infimum runs over all completely positive maps $S_1\colon A \to B$ and $S_2\colon A \to B$ such that the operator $v\colon A \to M_2(B)$ defined by

$$v(x) = \begin{pmatrix} S_1(x) & u(x^*)^* \\ u(x) & S_2(x) \end{pmatrix} \tag{2.4}$$

is completely positive. It is shown in [18] that $\|\ \|_{\rm dec}$ is well defined and is a complete norm on $DEC(A,B)$. Moreover the inequality $\|u\|_{cb} \leqslant \|u\|_{\rm dec}$ holds for any $u \in DEC(A,B)$, and $\|u\| = \|u\|_{cb} = \|u\|_{\rm dec}$ if u is completely positive.

We now turn to a brief review on Pisier's delta norm and nuclearity for linear maps. We mainly report on [31, Section 6] but we warn the reader that some of the results given below are only implicit in [31], or given in a slightly different form. We hope that the references and complements we include will make the situation clear.

Definition 2.2 ([31]). — *Let E be an operator space and let B be an operator algebra with a c.a.i.. For any $z \in E \otimes B$, we set*

$$\delta(z) = \inf\Bigl\{ \|[e_{pq}]\|_{M_n(E)} \Bigl\|\sum_{p=1}^{n} a_p a_p^*\Bigr\|^{1/2} \Bigl\|\sum_{q=1}^{n} b_q^* b_q\Bigr\|^{1/2} \Bigr\},$$

where the infimum runs over all decompositions of z of the form $z = \sum_{1\leqslant p,q\leqslant n} e_{pq} \otimes a_p b_q$, with arbitrary $n \geqslant 1$, $e_{pq} \in E$, and a_p, $b_q \in B$. The completion of $E \otimes B$ for the norm δ is denoted by $E \overset{\delta}{\otimes} B$.

This definition was originally given in [31, Theorem 6.3.1] for unital operator algebras but it makes sense for operator algebras with a c.a.i. as well. Indeed, let B be such an operator algebra. Then by Cohen's factorization theorem (see [19]), any $c \in B$ can be written as a product $c = ab$ for some a, $b \in B$. Likewise, the following theorem is proved in [31] for unital operator algebras only. More precisely part (2) corresponds to [31, Theorem 6.3.1] whereas part (1) is implicit in the proof of the latter. However this proof readily extends to the framework of operator algebras with a c.a.i..

Theorem 2.3 ([31]). — *Let E and B be as in Definition 2.2.*

(1) *For any $\xi \in \bigl(E \overset{\delta}{\otimes} B\bigr)^*$, there exist a Hilbert space $\mathcal{H}$, two vectors h_1, h_2 in $\mathcal{H}$, and two completely contractive maps $\theta\colon E \to B(\mathcal{H})$ and*

$\pi: B \to B(\mathcal{H})$ with commuting ranges such that π is a homomorphism, $\|h_1\|\,\|h_2\| \leqslant \|\xi\|$, and

$$\forall e \in E,\, b \in B, \quad \xi(e \otimes b) = \langle \theta(e)\pi(b)h_1, h_2\rangle.$$

Conversely, any $\xi: E \otimes B \to \mathbb{C}$ of this form extends to a bounded functional on $\big(E \overset{\delta}{\otimes} B\big)^*$, with norm less than $\|h_1\|\,\|h_2\|$.

(2) *For any $z \in E \otimes B$,*

$$\delta(z) = \sup\big\{\|\theta\cdot \pi(z)\|\big\},$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$, all completely contractive maps $\theta: E \to B(\mathcal{H})$, and all completely contractive homomorphisms $\pi: B \to B(\mathcal{H})$ with commuting ranges.

Throughout the paper, given any $n \geqslant 1$, we shall denote by $(E_{pq})_{1\leqslant p,q\leqslant n}$ the canonical basis (= matrix units) of M_n. The delta norm is related to factorization through matrix spaces by the following simple observation. Let X be an operator space and let B be an operator algebra with a c.a.i.. We shall use the canonical identification between $X^* \otimes B$ and the space of all finite rank operators from X into B. So let $u: X \to B$ be any finite rank operator, and let $z \in X^* \otimes B$ be associated to u. Assume that we have a decomposition $z = \sum e_{pq} \otimes a_p b_q$, for some $[e_{pq}] \in M_n(X^*)$ and $(a_p)_{1\leqslant p\leqslant n}$, $(b_q)_{1\leqslant q\leqslant n}$ in B. Then let $\alpha: X \to M_n$ and $\beta: M_n \to B$ be the linear mappings defined by $\alpha(x) = [\langle e_{pq}, x\rangle]$ for any $x \in X$, and

$$\beta(E_{pq}) = a_p b_q \qquad \text{for any } 1 \leqslant p, q \leqslant n. \tag{2.5}$$

Then the factorization $u = \beta\alpha$ holds and, by the definition of operator space duality, we have $\|\alpha\|_{cb} = \|[e_{pq}]\|_{M_n(X^*)}$. One can therefore deduce that

$$\delta(z) = \inf\Big\{\|\alpha\|_{cb}\,\Big\|\sum_p a_p a_p^*\Big\|^{1/2}\,\Big\|\sum_q b_q^* b_q\Big\|^{1/2}\Big\}, \tag{2.6}$$

where the infimum runs over all factorizations

$$u = \beta\alpha, \qquad X \xrightarrow{\alpha} M_n \xrightarrow{\beta} B, \tag{2.7}$$

with β defined by (2.5). It is shown in [31, Corollary 6.3.5] that if B is a C*-algebra, one actually obtains

$$\delta(z) = \inf\big\{\|\alpha\|_{cb}\|\beta\|_{\mathrm{dec}} \,:\, \alpha,\, \beta \text{ satisfy } (2.7)\big\}. \tag{2.8}$$

We can now introduce one of the key notions of this paper, namely that of C-nuclear operator, defined as follows.

DEFINITION 2.4. — *Let Y be an operator space, let B be an operator algebra with a c.a.i., and let $u: Y \to B$ be a bounded operator. Given any constant $C > 0$, we say that u is C-nuclear if whenever A is an operator algebra, then $I_A \otimes u$ extends to a bounded operator from $A \otimes_{\min} Y$ into $A \otimes_{\max} B$, with*

$$\big\| I_A \otimes u: A \otimes_{\min} Y \longrightarrow A \otimes_{\max} B \big\| \leqslant C. \tag{2.9}$$

In the C*-algebra literature, the name of "nuclear operator" is often used to denote a u.c.p. map between C*-algebras which lies in the point-norm closure of the set of all finite rank u.c.p. maps. Our terminology is consistent with the usual one since Theorem 2.6 below shows that a u.c.p. map is nuclear in the classical sense if and only if it is 1-nuclear in the sense of Definition 2.4.

Remark 2.5. — Assume that B is a C*-algebra and let $u: Y \to B$ be a bounded operator. Then u is C-nuclear provided that (2.9) holds for any C*-algebra A.

Indeed let us assume that property, and let A be an arbitrary operator algebra. Let $z = \sum a_k \otimes y_k \in A \otimes Y$, for some finite families $(a_k)_k \subset A$ and $(y_k) \subset Y$. We wish to estimate $\|(I_A \otimes u)(z)\|_{\max} = \|\sum_k a_k \otimes u(y_k)\|_{\max}$. According to (2.2), we let $\rho: A \to B(\mathcal{H})$ and $\pi: B \to B(\mathcal{H})$ be commuting completely contractive homomorphisms, for some Hilbert space $\mathcal{H}$. Since B is self-adjoint, π is a $*$-representation hence its range $\pi(B)$ is self-adjoint. Therefore its commutant $[\pi(B)]' \subset B(\mathcal{H})$ is self-adjoint as well. Now observe that the $\rho(a_k)$'s belong to $[\pi(B)]'$. Hence applying our assumption that (2.9) holds for C*-algebras to $[\pi(B)]'$, we obtain

$$\Big\|\sum_k \rho(a_k) \otimes u(y_k)\Big\|_{[\pi(B)]' \otimes_{\max} B} \leqslant C \Big\|\sum_k \rho(a_k) \otimes y_k\Big\|_{[\pi(B)]' \otimes_{\min} Y}.$$

Since ρ is completely contractive we deduce

$$\begin{aligned} \big\|\rho \cdot \pi (I_A \otimes u)(z)\big\| &= \Big\|\sum_k \rho(a_k)\pi\big(u(y_k)\big)\Big\| \\ &\leqslant C \Big\|\sum_k \rho(a_k) \otimes y_k\Big\|_{\min} \leqslant C\|z\|_{\min}. \end{aligned}$$

Taking the supremum over all pairs (ρ, π) as above yields $\|(I_A \otimes u)(z)\|_{\max} \leqslant C\|z\|_{\min}$, which exactly means that u is C-nuclear in the sense of Definition 2.4.

The next statement summarizes Pisier's work on C-nuclear operators contained in [31, Section 6]. In the latter paper, the author is concerned with bounded operators between C*-algebras. Modulo the easy Remark 2.5 above, he proves the equivalence of the assertions (i), (ii), (iii)', and (iv) of Theorem 2.6 below, in the case when Y and B are both C*-algebras ([31, Corollary 6.3.6]). Likewise, he proves the assertion (4) below in the case when Y and B are unital C*-algebras ([31, Corollary 6.3.8]). However it is easy to check that the proofs of [31, Corollary 6.3.6] and [31, Corollary 6.3.8] can be adapted to our more general setting to show (1) and (4) respectively. Note moreover that (2) immediately follows from (1) by applying (2.8) above. Since no significant argument is needed to obtain these generalizations, we omit their proofs.

THEOREM 2.6 ([31]). — *Let Y be an operator space and let B be an operator algebra with a c.a.i.. Let $u: Y \to B$ be a bounded operator and let $C > 0$ be a constant.*

(1) *The following three assertions are equivalent:*

(i) *u is C-nuclear.*

(ii) *For any finite dimensional operator space E, $I_E \otimes u$ extends to a bounded operator from $E \otimes_{\min} Y$ into $E \overset{\delta}{\otimes} B$, with $\|I_E \otimes u: E \otimes_{\min} Y \to E \overset{\delta}{\otimes} B\| \leqslant C$.*

(iii) *There exists a net $u_i: Y \to B$ of finite rank operators converging to u in the point-norm topology, such that letting $z_i \in Y^* \otimes B$ be associated to u_i, we have $\delta(z_i) < C$.*

(2) *If B is a C*-algebra, then (i) is equivalent to:*

(iii)′ *There exists a net $u_i: Y \to B$ of finite rank operators converging to u in the point-norm topology, such that every u_i admits a factorization $u_i = \beta_i \alpha_i$, with*

$$Y \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} B \quad \text{and} \quad \|\alpha_i\|_{cb}\|\beta_i\|_{\mathrm{dec}} \leqslant C.$$

(3) *If B and Y are C*-algebras, then (i) is equivalent to*

(iv) *There exists a net $u_i: A \to B$ of finite rank operators converging to u in the point-norm topology, with $\|u_i\|_{\mathrm{dec}} \leqslant C$.*

(4) *If B is a unital C^*-algebra, if Y is an operator system, and if u is a u.c.p. map, then the assertion (i) holds with $C = 1$ if and only if*

(v) *There exists a net $u_i \colon Y \to B$ of finite rank operators converging to u in the point-norm topology, such that every u_i admits a factorization $u_i = \beta_i \alpha_i$, with*

$$Y \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} B \quad \text{and} \quad \alpha_i,\, \beta_i \text{ are c.p. and contractive.}$$

3. The normal delta tensor product and the second dual of $E \overset{\delta}{\otimes} B$.

In this section, we shall define a normal version of the delta tensor product, in analogy with the normal Haagerup tensor product of operator spaces ([10], [15]). We give ourselves a dual operator space X and a unital dual operator algebra M. We define $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma$ as the set of all $\xi \in \big(X \overset{\delta}{\otimes} M\big)^*$ which are separately w^*-continuous. Namely ξ belongs to $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma$ provided that $x \mapsto \xi(x \otimes m_0)$ and $m \mapsto \xi(x_0 \otimes m)$ are w^*-continuous for any $m_0 \in M$ and any $x_0 \in X$. Clearly $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma \subset \big(X \overset{\delta}{\otimes} M\big)^*$ is a closed subspace. Now we define

$$X \overset{\sigma\delta}{\otimes} M = \Big(\big(X \overset{\delta}{\otimes} M\big)^*_\sigma\Big)^*, \tag{3.1}$$

and we call this space the normal delta tensor product of X and M. We can obviously regard $X \otimes M$ as a subspace of $X \overset{\sigma\delta}{\otimes} M$. The norm inherited by this possibly non dense embedding is described by the following lemma.

LEMMA 3.1. — *Let $y \in X \otimes M$ and let $K > 0$ be a constant. Then $\|y\|_{X \overset{\sigma\delta}{\otimes} M} \leqslant K$ if and only if there exists a net $(y_t)_t \subset X \otimes M$ such that $\delta(y_t) < K$ and, for any $\xi \in \big(X \overset{\delta}{\otimes} M\big)^*_\sigma$, we have $\lim_t \xi(y_t) = \xi(y)$.*

Proof. — Assume that $\|y\|_{X \overset{\sigma\delta}{\otimes} M} \leqslant K$. By Hahn-Banach there exists some $\widehat{y} \in \big(X \overset{\delta}{\otimes} M\big)^{**}$ extending y, with norm less than K. By Goldstine's Lemma, one may find a net $(y_t)_t \subset X \otimes M$ tending to $\widehat{y}$ in the w^*-topology of $\big(X \overset{\delta}{\otimes} M\big)^{**}$, with $\delta(y_t) < K$. This yields the 'only if' part, and the converse is obvious. □

Our goal is now to introduce and to show Theorem 3.3, which is the main result of this section. It gives a dual description of the norm induced on $X \otimes M$ by its embedding into $X \overset{\sigma\delta}{\otimes} M$ and should be regarded as a normal analogue of the second part of Theorem 2.3.

DEFINITION 3.2. — *Given any $y \in X \otimes M$, we set*

$$\widetilde{\delta}(y) = \sup\{\ \|\theta\cdot\pi(y)\|\ \},$$

where the supremum runs over all Hilbert spaces $\mathcal{H}$, all w^*-continuous completely contractive maps $\theta: X \to B(\mathcal{H})$, and all w^*-continuous completely contractive homomorphisms $\pi: M \to B(\mathcal{H})$ with commuting ranges. We denote by $X \overset{\widetilde{\delta}}{\otimes} M$ the completion of $X \otimes M$ under $\widetilde{\delta}$, which is clearly a norm.

THEOREM 3.3. — *For any dual operator space X, for any unital dual operator algebra M, and for any $y \in X \otimes M$, we have $\widetilde{\delta}(y) = \|y\|_{X \overset{\sigma\delta}{\otimes} M}$. Consequently,*

$$X \overset{\widetilde{\delta}}{\otimes} M \subset X \overset{\sigma\delta}{\otimes} M \quad \text{isometrically.}$$

Our proof of Theorem 3.3 requires a special form of Wittstock's factorization Theorem for completely bounded maps, that we now establish. The following extends a result due to Haagerup corresponding to the case when M is a von Neumann algebra (see [17, Proposition 2.12]).

PROPOSITION 3.4. — *Let M be a unital dual operator algebra, let H be a Hilbert space and let $u: M \to B(H)$ be w^*-continuous and completely bounded. Then there exist a Hilbert space K, two linear operators $V: H \to K$, $W: K \to H$, and a unital w^*-continuous completely contractive homomorphism $\pi: M \to B(K)$ such that $\|V\|\,\|W\| = \|u\|_{cb}$ and, for any $m \in M$, $u(m) = W\pi(m)V$.*

Proof. — Let $u: M \to B(H)$ be completely bounded. By the classical form of Wittstock's factorization Theorem ([17], [26], [34]), we may find a Hilbert space G, two linear operators $S: H \to G$, $T: G \to H$, and a unital completely contractive homomorphism $\rho: M \to B(G)$ such that $\|S\|\,\|T\| = \|u\|_{cb}$ and

$$\forall m \in M, \quad u(m) = T\rho(m)S. \tag{3.2}$$

Moreover we may clearly assume that $G = \overline{\text{Span}}\{\rho(M)S(H)\}$. Let us introduce the closed subspace

$$K = \overline{\text{Span}}\{\rho(M)^*T^*(H)\} \subset G.$$

Then let $J: K \to G$ be the canonical embedding operator, and let $P = J^*: G \to K$ be the corresponding projection. Lastly, we let $Q = JP \in B(G)$. For any $m \in M$, K is an invariant subspace of $\rho(m)^*$ hence $K^\perp$ is an invariant subspace of $\rho(m)$ hence

$$Q\rho(m)(I - Q) = 0. \tag{3.3}$$

Moreover since ρ is unital, the range of T^* is included in K hence $T^* = QT^*$ or, equivalently, $T(I-Q) = 0$. These two identities together with (3.2) yield

$$\forall m \in M, \quad u(m) = TQ\rho(m)QS. \tag{3.4}$$

Given any $m \in M$, let us define $\pi(m): K \to K$ by $\pi(m) = P\rho(m)J$. We claim that $\pi: M \to B(K)$ is a homomorphism. Indeed, let m_1, m_2 be in M, then we have

$$\begin{aligned}\pi(m_1m_2) - \pi(m_1)\pi(m_2) &= P\rho(m_1)\rho(m_2)J - P\rho(m_1)JP\rho(m_2)J \\ &= P\rho(m_1)(I - Q)\rho(m_2)J = 0 \qquad \text{by (3.3).}\end{aligned}$$

Clearly π is completely contractive and unital hence letting $V = PS$ and $W = TJ$, we deduce from (3.4) the desired relation $u(m) = W\pi(m)V$ for any $m \in M$.

It thus remains to show that π is w^*-continuous, under the assumption that u itself is w^*-continuous. We introduce

$$\mathcal{K}_1 = \text{Span}\{\pi(M)V(H)\} \quad \text{and} \quad \mathcal{K}_2 = \text{Span}\{\pi(M)^*W^*(H)\}$$

which are, by construction, two dense subspaces of K. Let a, b be in M and let x, y be in H. The multiplication mapping is separately w^*-continous on M hence according to our assumption on u, the mapping $m \mapsto u(amb)$ is w^*-continuous hence the functional $m \mapsto \langle u(amb)x, y\rangle$ is w^*-continuous on M. Now observe that for any $m \in M$,

$$\begin{aligned}\langle u(amb)x, y\rangle = \langle W\pi(amb)Vx, y\rangle &= \langle W\pi(a)\pi(m)\pi(b)Vx, y\rangle \\ &= \langle \pi(m)\pi(b)Vx, \pi(a)^*W^*y\rangle\,.\end{aligned}$$

Then taking linear combinations, we deduce that for any $k_1 \in \mathcal{K}_1$ and $k_2 \in \mathcal{K}_2$, the functional $m \mapsto \langle \pi(m)k_1, k_2\rangle$ belongs to M_*. Since $\langle \pi(m)k_1, k_2\rangle =$

$\langle \pi^*(k_1 \otimes k_2), m\rangle$, we thus obtain that $\pi^*(\mathcal{K}_1 \otimes \mathcal{K}_2) \subset M_*$. Now recall that $K \otimes K$ is dense in the space $S_1(K) = B(K)_*$ of trace class operators on K, hence $\mathcal{K}_1 \otimes \mathcal{K}_2$ is dense in $S_1(K)$ as well. By continuity, we obtain that π^* maps $S_1(K)$ into M_*, which exactly means that π is w^*-continuous. □

Remark 3.5. — From the above argument, it is easy to derive an alternate proof of the characterization of unital dual operator algebras established in [25]. The non trivial implication of that characterization says that if M is a unital operator algebra such that

(a) M is a dual operator space,

(b) the multiplication mapping on M is separately w^*-continuous,

then there exist a Hilbert space K and a unital w^*-continuous completely isometric homomorphism from M into $B(K)$.

To show that, observe that the proof of Proposition 3.4 works as well for a unital operator algebra M satisfying (a) and (b). Moreover (a) implies that there exist a Hilbert space H and a w^*-continuous completely isometric mapping $u: M \to B(H)$ (see [2, Proposition 2.1] or [12, Proposition 5.1]). Let us therefore apply Proposition 3.4 to u, and let $u(\ ) = W\pi(\ )V$ be the resulting factorization, for some unital w^*-continous completely contractive homomorphism $\pi: M \to B(K)$. For any $n \geqslant 1$ and any $[m_{pq}] \in M_n(M)$, we have

$$\|[m_{pq}]\| = \|[u(m_{pq})]\| \leqslant \|W\|\, \|[\pi(m_{pq})]\|\, \|V\| \leqslant \|[\pi(m_{pq})]\|.$$

Since $\|[\pi(m_{pq})]\| \leqslant \|[m_{pq}]\|$, we obtain that π is a complete isometry, whence the result.

The following statement is the next step in the proof of Theorem 3.3. It should be regarded as a weak*-analogue of the first part of Theorem 2.3. Its proof follows a similar scheme as that of [31, Theorem 6.3.1], with a special attention given to the w^*-continuity of the linear maps involved.

Proposition 3.6. — *Let X be a dual operator space, let M be a unital dual operator algebra, and let $\xi \in \big(X \overset{\delta}{\otimes} M\big)^*$. Then ξ belongs to $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma$ if and only if there exist a Hilbert space $\mathcal{H}$, two vectors $h_1, h_2 \in \mathcal{H}$, a w^*-continuous completely contractive map $\theta: X \to B(\mathcal{H})$ and a w^*-continuous completely contractive homomorphism $\pi: M \to B(\mathcal{H})$ such that $\|h_1\|\, \|h_2\| \leqslant \|\xi\|$, θ and π have commuting ranges, and*

$$\forall x \in X,\ m \in M, \quad \xi(x \otimes m) = \langle \theta(x)\pi(m)h_1, h_2\rangle. \tag{3.5}$$

Proof. — First note that the 'if part' clearly follows from (the easy part of) Theorem 2.3 so we only focus on the 'only if part'.

Let ξ be in $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma$ and assume that $\|\xi\| \leqslant 1$. Let $\widehat{\xi}\colon M \otimes X \otimes M \to \mathbb{C}$ be the functional defined by letting $\widehat{\xi}(m_1 \otimes x \otimes m_2) = \xi(x \otimes m_1 m_2)$ for any $x \in X$ and $m_1, m_2 \in M$. Given $m_1^1, \dots, m_1^n, m_2^1, \dots, m_2^n$ in M and $[x_{pq}]$ in $M_n(X)$, we have

$$\begin{aligned}\sum_{1\leqslant p,q\leqslant n} \widehat{\xi}(m_1^p \otimes x_{pq} \otimes m_2^q) &= \sum_{1\leqslant p,q\leqslant n} \xi(x_{pq} \otimes m_1^p m_2^q)\\ &\leqslant \|\xi\|\, \delta\Big(\sum_{1\leqslant p,q\leqslant n} x_{pq} \otimes m_1^p m_2^q\Big)\\ &\leqslant \|\xi\|\, \|[x_{pq}]\|\, \Big\|\sum_{p=1}^n m_1^p m_1^{p*}\Big\|^{1/2} \Big\|\sum_{q=1}^n m_2^{q*} m_2^q\Big\|^{1/2}\end{aligned}$$

by Definition 2.2. This shows that $\widehat{\xi}$ extends to an element of $\big(M \otimes_h X \otimes_h M\big)^*$ with norm less than 1. By the factorization theorem of multilinear completely bounded maps ([7], [29]), there exist a Hilbert space H, two vectors $e_1, e_2 \in H$ with norms $\leqslant 1$, and three completely contractive maps $u_1\colon M \to B(H)$, $v\colon X \to B(H)$, and $u_2\colon M \to B(H)$ such that

$$\forall x \in X,\ \forall m_1, m_2 \in M, \quad \xi(x \otimes m_1 m_2) = \langle u_1(m_1) v(x) u_2(m_2) e_2, e_1\rangle. \tag{3.6}$$

Now note that since the multiplication mapping is separately w^*-continuous on M, our assumption on ξ implies that $\widehat{\xi}$ is separately w^*-continuous in each of the three variables. It therefore follows from [14, Theorem 3.1] that the factorization (3.6) can be achieved with w^*-continuous maps u_1, u_2, and v. Assuming this, we may find, by Proposition 3.4, two Hilbert spaces K_1, K_2, two unital w^*-continuous completely contractive homomorphisms $\pi_1\colon M \to B(K_1)$, $\pi_2\colon M \to B(K_2)$, and four contractions

$$V_2\colon H \longrightarrow K_2,\ W_2\colon K_2 \longrightarrow H,\ V_1\colon H \longrightarrow K_1,\ W_1\colon K_1 \longrightarrow H,$$

such that $u_2(\ ) = W_2\pi_2(\ )V_2$ and $u_1(\ ) = W_1\pi_1(\ )V_1$. Letting $e_2' = V_2 e_2$, $e_1' = W_1^* e_1$ and $v'(\ ) = V_1 v(\ ) W_2$, we can then rewrite (3.6) as follows:

$$\forall x \in X,\ \forall m_1, m_2 \in M, \quad \xi(x \otimes m_1 m_2) = \langle \pi_1(m_1) v'(x) \pi_2(m_2) e_2', e_1'\rangle\,. \tag{3.7}$$

We can now conclude as in the proof of [31, Theorem 6.3.1]. First, changing K_2 and K_1 into $\overline{\mathrm{Span}}\{\pi_2(M)e_2'\}$ and $\overline{\mathrm{Span}}\{\pi_1(M)^* e_1'\}$ if necessary, we may and do assume that $\pi_2(M)e_2'$ and $\pi_1(M)^*e_1'$ are dense in K_2 and K_1 respectively. Then we deduce from (3.7) that

$$\forall x \in X,\ \forall m \in M, \quad v'(x)\pi_2(m) = \pi_1(m)v'(x)\,. \tag{3.8}$$

Indeed, for any $x \in X$ and m, m_1, $m_2 \in M$ we have

$$\langle \pi_1(m_1 m)v'(x)\pi_2(m_2)e_2', e_1'\rangle = \langle \pi_1(m_1)v'(x)\pi_2(mm_2)e_2', e_1'\rangle$$

by (3.7), whence the equality

$$\langle \pi_1(m)v'(x)\pi_2(m_2)e_2', \pi_1(m_1)^*e_1'\rangle = \langle v'(x)\pi_2(m)\pi_2(m_2)e_2', \pi_1(m_1)^*e_1'\rangle.$$

This yields (3.8). Now let $\mathcal{H} = K_1 \oplus K_2$ and let $\pi: M \to B(\mathcal{H})$ and $\theta: X \to B(\mathcal{H})$ be defined by

$$\pi(m) = \begin{pmatrix} \pi_1(m) & 0 \\ 0 & \pi_2(m) \end{pmatrix} \quad \text{and} \quad \theta(x) = \begin{pmatrix} 0 & v'(x) \\ 0 & 0 \end{pmatrix}.$$

It follows from (3.8) that θ and π have commuting ranges. Furthermore, θ and π are w^*-continuous and completely contractive. Indeed. $v': X \to B(K_2, K_1)$ is w^*-continuous and completely contractive, since v is. Letting $h_2 = 0 \oplus e_2'$ and $h_1 = e_1' \oplus 0$ in $\mathcal{H}$, we obtain the desired factorization (3.5).

□

Proof of Theorem 3.3. — We consider $y \in X \otimes M$. Let ξ be given in $\big(X \overset{\delta}{\otimes} M\big)^*_\sigma$, with $\|\xi\| = 1$. Then let $\mathcal{H}$, h_1, h_2, θ, π be as in Proposition 3.4 such that (3.5) holds and $\|h_1\|\,\|h_2\| \leqslant 1$. We have $\xi(y) = \langle \theta\cdot\pi(y)h_1, h_2\rangle$, hence

$$|\xi(y)| \leqslant \|\theta\cdot\pi(y)\|\,\|h_1\|\,\|h_2\| \leqslant \widetilde{\delta}(y)$$

by Definition 3.2. Taking the supremum over ξ, we obtain $\|y\|_{X \overset{\sigma\delta}{\otimes} M} \leqslant \widetilde{\delta}(y)$ by (3.1).

To check the converse inequality, we let $K = \|y\|_{X \overset{\sigma\delta}{\otimes} M}$ and we give ourvelves a Hilbert space $\mathcal{H}$ and $\theta: X \to B(\mathcal{H})$ and $\pi: M \to B(\mathcal{H})$ as in Definition 3.2. It suffices to show that

$$\|\theta\cdot\pi(y)\| \leqslant K. \tag{3.9}$$

Let h_1 and h_2 be two elements of $\mathcal{H}$, with $\|h_1\| = \|h_2\| = 1$, and let $\xi \in \big(X \overset{\delta}{\otimes} M\big)^*_\sigma$ be defined by (3.5). We now consider a net $(y_t)_t \subset X \otimes M$ given by Lemma 3.1. By Theorem 2.3, $\|\theta\cdot\pi(y_t)\| \leqslant K$ for any t, hence

$|\xi(y_t)| \leqslant K$ for any t. Passing to the limit, we obtain that $|\xi(y)| \leqslant K$ and taking the supremum over all pairs (h_1, h_2), we finally have (3.9). □

Let E be an operator space and let B be an operator algebra with a c.a.i.. Our aim is to describe $\left(E \overset{\delta}{\otimes} B\right)^{**}$ in terms of the normal delta tensor product. Recall that the second dual space B^{**} is a unital dual operator algebra and that given any bounded homomorphism $\pi\colon B \to B(\mathcal{H})$, the second adjoint mapping $\pi^{**}\colon B^{**} \to B(\mathcal{H})^{**}$ is a homomorphism as well. (See [13, Section 2] and the references therein for details.)

Given any $\xi \in \left(E^{**} \overset{\delta}{\otimes} B^{**}\right)^{*}_{\sigma}$, we let $\overset{\circ}{\xi} \in \left(E \overset{\delta}{\otimes} B\right)^{*}$ be obtained by restricting ξ to $E \otimes B$. Clearly the linear mapping

$$J\colon \left(E^{**} \overset{\delta}{\otimes} B^{**}\right)^{*}_{\sigma} \longrightarrow \left(E \overset{\delta}{\otimes} B\right)^{*}; \quad \xi \mapsto \overset{\circ}{\xi},$$

is contractive and one to one. We claim that it is actually an isometric isomorphism.

Indeed, let φ be in $\left(E \overset{\delta}{\otimes} B\right)^{*}$, with $\|\varphi\| = 1$. By Theorem 2.3, we may find a Hilbert space $\mathcal{H}$, two vectors $h_1, h_2 \in \mathcal{H}$ with norms $\leqslant 1$, and two completely contractive maps $\theta\colon E \to B(\mathcal{H})$ and $\pi\colon B \to B(\mathcal{H})$ such that π is a homomorphism, θ and π have commuting ranges, and $\varphi(e \otimes b) = \langle \theta(e)\pi(b)h_1, h_2\rangle$ for any $e \in E$ and $b \in B$. Let $S_1(\mathcal{H}) = B(\mathcal{H})_*$, and let $Q\colon B(\mathcal{H})^{**} \to B(\mathcal{H})$ be the adjoint of the canonical embedding of $S_1(\mathcal{H})$ into its second dual $B(\mathcal{H})^{*}$. We introduce $\widetilde{\theta}\colon E^{**} \to B(\mathcal{H})$ and $\widetilde{\pi}\colon B^{**} \to B(\mathcal{H})$ by letting $\widetilde{\theta} = Q\theta^{**}$ and $\widetilde{\pi} = Q\pi^{**}$. By construction, each of these two operators is w^*-continuous and completely contractive. Moreover since Q is a $*$-representation and π^{**} is a homomorphism, $\widetilde{\pi}$ is a homomorphism as well. Now let us fix $x \in E^{**}$ and $m \in B^{**}$, and let $(e_t)_t \subset E$ and $(b_s)_s \subset B$ be two nets converging to x and m in the w^*-topologies of E^{**} and B^{**} respectively. Then for any $f \in S_1(\mathcal{H}) = B(\mathcal{H})_*$, we have

$$\begin{aligned}\langle \widetilde{\theta}(x)\widetilde{\pi}(m), f\rangle &= \lim_t \langle \widetilde{\theta}(e_t)\widetilde{\pi}(m), f\rangle = \lim_t \lim_s \langle \widetilde{\theta}(e_t)\widetilde{\pi}(b_s), f\rangle \\ &= \lim_t \lim_s \langle \widetilde{\pi}(b_s)\widetilde{\theta}(e_t), f\rangle = \langle \widetilde{\pi}(m)\widetilde{\theta}(x), f\rangle\,.\end{aligned}$$

Thus $\widetilde{\theta}$ and $\widetilde{\pi}$ have commuting ranges. Therefore letting $\xi(x \otimes m) = \langle \widetilde{\theta}(x)\widetilde{\pi}(m)h_1, h_2\rangle$ for $x \in E^{**}$ and $m \in B^{**}$ defines $\xi \in \left(E^{**} \overset{\delta}{\otimes} B^{**}\right)^{*}_{\sigma}$ with $\|\xi\| \leqslant 1$. Since $\overset{\circ}{\xi}$ clearly equals to φ, this completes the proof that J is an isometric isomorphism. Summing up and recalling (3.1), we have proved the following.

Proposition 3.7. — *For any operator space E and any operator algebra B with a c.a.i.,*

(1) $\big(E \overset{\delta}{\otimes} B\big)^* = \big(E^{**} \overset{\delta}{\otimes} B^{**}\big)^*_\sigma$ *isometrically.*

(2) $\big(E \overset{\delta}{\otimes} B\big)^{**} = E^{**} \overset{\sigma\delta}{\otimes} B^{**}$ *isometrically.*

Clearly $\widetilde{\delta} \leqslant \delta$ hence combining Theorem 3.3 and Proposition 3.7 immediately leads to the following results.

Corollary 3.8. — *Let E be an operator space and let B be an operator algebra with a c.a.i.,*

(1) *We have* $E^{**} \overset{\widetilde{\delta}}{\otimes} B^{**} \subset \big(E \overset{\delta}{\otimes} B\big)^{**}$ *isometrically.*

(2) *We have* $E^{**} \overset{\delta}{\otimes} B^{**} \longrightarrow \big(E \overset{\delta}{\otimes} B\big)^{**}$ *contractively.*

(3) *If E is finite dimensional, then* $E \overset{\widetilde{\delta}}{\otimes} B^{**} = \big(E \overset{\delta}{\otimes} B\big)^{**}$ *isometrically.*

Remark 3.9. — In general, for arbitrary dual operator space X and unital dual operator algebra M,

$$X \overset{\widetilde{\delta}}{\otimes} M \neq X \overset{\delta}{\otimes} M. \tag{3.10}$$

In other words, the norm induced by the normal delta tensor product on the algebraic tensor product may be different from the delta norm. This is in sharp contrast with the corresponding situation for the Haagerup tensor product. Indeed, it is shown in [15] that for any dual operator spaces $X_1, \ldots, X_N$, the norm induced by the normal Haagerup tensor product $X_1 \overset{\sigma h}{\otimes} \cdots \overset{\sigma h}{\otimes} X_N$ on $X_1 \otimes \cdots \otimes X_N$ is the Haagerup norm itself.

Let us now explain (3.10). Let B be any nuclear C^*-algebra whose second dual B^{**} is not nuclear (for instance, B is the C^*-algebra of all compact operators on some infinite dimensional Hilbert space). Then let E be an arbitrary finite dimensional operator space. Since nuclear C^*-algebras are locally reflexive (see [1], [9]), $\big(E \otimes_{\min} B\big)^{**} = E \otimes_{\min} B^{**}$ isometrically. Moreover $E \otimes_{\min} B = E \overset{\delta}{\otimes} B$ isometrically by Theorem 2.6. These two identities together with the third part of Corollary 3.8 yield:

$$E \overset{\widetilde{\delta}}{\otimes} B^{**} = E \otimes_{\min} B^{**} \quad \text{isometrically.} \tag{3.11}$$

Let $M = B^{**}$ and assume that contrary to (3.10), one has $X \overset{\widetilde{\delta}}{\otimes} M = X \overset{\delta}{\otimes} M$ for any X. Then there is a constant $K \geqslant 1$ such that for every finite dimensional operator space E,

$$\big\|\mathrm{Id}\colon E \overset{\widetilde{\delta}}{\otimes} M \longrightarrow E \overset{\delta}{\otimes} M\big\| \leqslant K.$$

Under this assumption, (3.11) implies

$$\big\|\mathrm{Id}\colon E \otimes_{\min} M \longrightarrow E \overset{\delta}{\otimes} M\big\| \leqslant K$$

for every finite dimensional E. By Theorem 2.6 again, this implies that M is nuclear, whence a contradiction.

4. Semidiscrete operators.

In the first result of this section (Theorem 4.3), we characterize semidiscrete operators, defined below, by a suitable finite rank approximation property. We shall make use of the normal tensor product of operator algebras defined by (2.3). Then we shall specify our result for unital completely positive semidiscrete operators with domain an operator system. Lastly, we shall investigate the relationships between nuclearity and semidiscreteness for operators.

DEFINITION 4.1. — *Let Z be an operator space, let M be a unital dual operator algebra, and let $u\colon Z \to M$ be a bounded operator. Given any constant $C > 0$, we say that u is C-semidiscrete if whenever A is an operator algebra, then $I_A \otimes u$ extends to a bounded operator from $A \otimes_{\min} Z$ into $A \otimes_{\mathrm{nor}} M$, with*

$$\big\|I_A \otimes u\colon A \otimes_{\min} Z \longrightarrow A \otimes_{\mathrm{nor}} M\big\| \leqslant C. \tag{4.1}$$

Note that for any integer $n \geqslant 1$, $M_n \otimes_{\min} Z = M_n(Z)$ and $M_n \otimes_{\mathrm{nor}} M = M_n(M)$ isometrically hence any C-semidiscrete operator u is completely bounded, with $\|u\|_{cb} \leqslant C$. Furthermore, any C-nuclear operator from Z into M is automatically C-semidiscrete.

Remark 4.2. — Assume that M is a von Neumann algebra. Then arguing as in Remark 2.5, it is easy to check that a bounded operator $u\colon Z \to M$ is C-semidiscrete provided that (4.1) is fulfilled for any C^*-algebra A.

THEOREM 4.3. — *Let $u: Z \to M$ be any bounded operator from an operator space into a unital dual operator algebra, and let $C > 0$ be a constant.*

(1) *The following three assertions are equivalent:*

(i) *u is C-semidiscrete.*

(ii) *For any finite dimensional operator space X, $I_X \otimes u$ extends to a bounded operator from $X \otimes_{\min} Z$ into $X \overset{\widetilde{\delta}}{\otimes} M$, with $\|I_X \otimes u: X \otimes_{\min} Z \to X \overset{\widetilde{\delta}}{\otimes} M\| \leqslant C$.*

(iii) *There exists a net $u_i: Z \to M$ of finite rank operators converging to u in the point-w^* topology (i.e., for any $z \in Z$, $u(z) = w^* - \lim_i u_i(z)$), such that letting $y_i \in Z^* \otimes M$ be associated to u_i, we have $\delta(y_i) < C$. More explicitly, for any i, there exist $n_i \geqslant 1$, $c_{i1}, \dots, c_{in_i}, d_{i1}, \dots, d_{in_i}$ in M, and a bounded operator $\alpha_i: Z \to M_{n_i}$ such that*

$$\|\alpha_i\|_{cb} \Big\|\sum_p c_{ip}c_{ip}^*\Big\|^{1/2} \Big\|\sum_q d_{iq}^* d_{iq}\Big\|^{1/2} < C,$$

and $u_i = \beta_i \alpha_i$, where $\beta_i: M_{n_i} \to M$ is the bounded operator defined by $\beta_i(E_{pq}) = c_{ip}d_{iq}$.

(2) *Assume moreover that M is a von Neumann algebra. Then (i) is equivalent to:*

(iii)$'$ *There exists a net $u_i: Z \to M$ of finite rank operators converging to u in the point-w^* topology, such that every u_i admits a factorization $u_i = \beta_i \alpha_i$, with*

$$Z \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} M \quad \text{and} \quad \|\alpha_i\|_{cb}\|\beta_i\|_{\text{dec}} < C.$$

(3) *Assume now that M is a von Neumann algebra and that Z is a C^*-algebra. Then the assertion (i) is equivalent to:*

(iv) *There exists a net $u_i: Z \to M$ of finite rank operators converging to u in the point-w^* topology, with $\|u_i\|_{\text{dec}} < C$.*

Proof. — We only have to prove (1). Indeed, one can readily deduce (2) from (1) by means of (2.7). Moreover (3) follows from (1) and [21, Theorem 2.1]. (We observe however that the equivalence (i) $\Leftrightarrow$ (iv) can be proved without appealing to [21].)

(i) $\Longrightarrow$ (ii):

We assume that u is C-semidiscrete and we fix a (finite dimensional) operator space X. Let $y = \sum_{1\leqslant k\leqslant n} x_k \otimes z_k \in X\otimes Z$ be given, with $x_1,\dots,x_n \in X$ and $z_1,\dots,z_n \in Z$. We give ourselves a Hilbert space $\mathcal{H}$, a w^*-continuous completely contractive homomorphism $\pi\colon M\to B(\mathcal{H})$ and a completely contractive map $\theta\colon X\to B(\mathcal{H})$ commuting with π. We let $A=[\pi(M)]'\subset B(\mathcal{H})$ be the commutant algebra of the range of π. Our assumption that θ and π commute means that θ as valued in A. We have

$$(\theta\cdot\pi)\big[(I_X\otimes u)y\big] = \sum_k \theta(x_k)\pi\big(u(z_k)\big).$$

Hence according to the w^*-continuity of π and our assumption (i),

$$\begin{aligned}\big\|(\theta\cdot\pi)\big[(I_X\otimes u)y\big]\big\| &\leqslant \Big\|\sum_k \theta(x_k)\otimes u(z_k)\Big\|_{A\otimes_{\mathrm{nor}}M}\\ &\leqslant C\,\Big\|\sum_k \theta(x_k)\otimes z_k\Big\|_{A\otimes_{\min}Z}.\end{aligned}$$

Since $\|\theta\|_{cb}\leqslant 1$, we finally obtain that $\big\|(\theta\cdot\pi)\big[(I_X\otimes u)y\big]\big\| \leqslant C\,\|y\|_{\min}$. Taking the supremum over all pairs (θ,π) as above and applying Definition 3.2, we obtain the inequality $\widetilde{\delta}\big((I_X\otimes u)y\big) \leqslant C\,\|y\|_{\min}$. This shows (ii).

(ii) $\Longrightarrow$ (iii):

Let $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) be the set of all finite subsets of Z (resp. M_*), and let $\mathcal{I}=\mathcal{I}_1\times\mathcal{I}_2\times(0,1)$. We endow $\mathcal{I}$ with its canonical order, given by

$$i=(I_1,I_2,\varepsilon)\leqslant i'=(I_1',I_2',\varepsilon')\Longleftrightarrow I_1\subset I_1',\ I_2\subset I_2',\ \varepsilon\geqslant\varepsilon'.$$

We assume (ii) and fix some $i=(I_1,I_2,\varepsilon)$ in $\mathcal{I}$. We let $E\subset Z$ be the finite dimensional subspace spanned by I_1 and let $j\colon E\to Z$ be the canonical embedding. Then we denote by $X=E^*$ the dual operator space of E. Next, we let $y\in X\otimes Z$ be the tensor representing the mapping j. Note that we have $\|y\|_{\min}=\|j\|_{cb}\leqslant 1$ hence our assumption yields $\widetilde{\delta}\big((I_X\otimes u)y\big)\leqslant C$. According to Theorem 3.3 and Lemma 3.1, there exists a net $(s_t)_t\subset X\otimes M$ such that $\delta(s_t)<C$ and, for any $f\in M_*$,

$$(I_X\otimes f)\big[(I_X\otimes u)y\big] = \lim_t (I_X\otimes f)s_t.$$

It now follows from this approximation property that there exists some t (in fact, an infinity) such that

$$(4.2)\qquad \forall z\in I_1,\ \forall f\in I_2,\quad \big|\langle (I_X\otimes u)y, z\otimes f\rangle - \langle s_t, z\otimes f\rangle\big| \leqslant \varepsilon.$$

We now fix some t satisfying (4.2). Since $\delta(s_t) < C$ we may find, by Definition 2.2, an integer $n \geqslant 1$, a matrix $[x_{pq}] \in M_n(X)$, and $c_1, \ldots, c_n, d_1, \ldots, d_n \in M$ such that $s_t = \sum_{p,q} x_{pq} \otimes c_p d_q$, and

$$\big\|[x_{pq}]\big\| \Big\|\sum c_p c_p^*\Big\|^{1/2} \Big\|\sum d_q^* d_q\Big\|^{1/2} < C.$$

Let $v\colon E \to M$ be the finite rank operator associated to s_t. Then $v = a\beta$, where $a\colon E \to M_n$ is the linear mapping represented by $[x_{pq}]$, and $\beta\colon M_n \to M$ is defined by letting $\beta(E_{pq}) = c_p d_q$ for each element E_{pq} of the canonical basis of M_n. By Wittstock's extension Theorem ([17], [26], [34]), there exists a completely bounded map $\alpha\colon Z \to M_n$ extending a, with $\|\alpha\|_{cb} = \|a\|_{cb} (= \|[x_{pq}]\|)$.

For the index $i = (I_1, I_2, \varepsilon) \in \mathcal{I}$ considered here, we let $u_i = \beta\alpha$. By (2.6, 2.7), the tensor $y_i \in Z^* \otimes M$ associated to u_i satisfies $\delta(y_i) < C$. Now observe that $(I_X \otimes u)y \in X \otimes M$ represents $uj\colon E \to M$, that is the restriction of u to E. Hence (4.2) can be rewritten as follows:

$$\forall z \in I_1,\ \forall f \in I_2, \quad \big|\langle u(z), f\rangle - \langle u_i(z), f\rangle\big| \leqslant \varepsilon.$$

This shows that u_i tends to u in the point-w^* topology, and concludes the proof.

(iii) $\Longrightarrow$ (i):

Let $(u_i)_i$ be a net satisfying (iii), and let A be an arbitrary operator algebra. By Theorem 2.6, we know that for every i,

$$\big\|I_A \otimes u_i\colon A \otimes_{\min} Z \longrightarrow A \otimes_{\max} M\big\| \leqslant C. \tag{4.3}$$

Let $\pi\colon M \to B(\mathcal{H})$ and $\rho\colon A \to B(\mathcal{H})$ be two commuting completely contractive homomorphisms, and assume that π is w^*-continuous. Then let $y = \sum_k a_k \otimes z_k \in A \otimes Z$ be given, with $a_k \in X$ and $z_k \in Z$. Then

$$\Big\|\sum_k \rho(a_k)\pi\big(u_i(z_k)\big)\Big\| = \big\|(\rho\cdot\pi)\big[(I_A \otimes u_i)y\big]\big\| \leqslant C\|y\|_{\min} \tag{4.4}$$

by (4.3). For any k, we know that $u_i(z_k) \xrightarrow{w^*} u(z_k)$ hence $\pi(u_i(z_k)) \xrightarrow{w^*} \pi(u(z_k))$. Consequently,

$$\forall k, \quad \rho(a_k)\pi\big(u_i(z_k)\big) \xrightarrow{w^*} \rho(a_k)\pi\big(u(z_k)\big).$$

Passing to the limit in (4.4), we thus obtain that $\left\|\sum_k \rho(a_k)\pi\big(u(z_k)\big)\right\| \leqslant C\|y\|_{\min}$. Then taking the supremum over all pairs (ρ,π), we obtain that $\|(I_A \otimes u)y\|_{\text{nor}} \leqslant C\|y\|_{\min}$. $\square$

We now turn to unital completely positive maps. We recall that an operator system is an operator space $Z \subset B(H)$ containing the identity (that we shall simply denote by 1), and such that whenever $z \in Z$, then $z^* \in Z$. We refer e.g. to [27] for the necessary information on that notion and on completely positive maps whose domain is an operator system. Given a bounded operator $u\colon Z_1 \to Z_2$ between two operator systems, we let $u_{\#}\colon Z_1 \to Z_2$ be defined by $u_{\#}(z) = u(z^*)^*$, and we say that u is self-adjoint provided that $u = u_{\#}$. In particular, any positive map is self-adjoint. By a dual operator system, we shall mean a w^*-closed operator system $Z \subset B(H)$. Our aim is to specify Theorem 4.3 when Z is an operator system and u is 1-semidiscrete and u.c.p.. We will make use of the following decomposition result, which is somewhat implicit in the proof of [31, Lemma 6.3.7].

PROPOSITION 4.4. — *Let $Z \subset B(H)$ be an operator system and let B be a C^*-algebra. We give ourselves a finite rank self-adjoint operator $u\colon Z \to B$.*

(1) *Let $y \in Z^* \otimes B$ be representing u and let $C > \delta(y)$. Then there exist an integer $n \geqslant 1$ and three completely positive maps $\alpha\colon Z \to M_n$, $\beta\colon M_n \to B$ and $\psi\colon Z \to B$ such that $u = \beta\alpha - \psi$, $\|\beta\alpha + \psi\| \leqslant C$, $\|\alpha\| \leqslant 1$, and $\|\beta\| \leqslant C$.*

(2) *Assume moreover that Z is a dual operator system, and that $y \in Z_* \otimes B$. Let $C > \delta(y)$. Then the conclusion of (1) holds with the following additional property. There exist a Hilbert space K, a unital w^*-continuous $*$-representation $\pi\colon B(H) \to B(K)$, and a contraction $V\colon \ell_2^n \to K$ such that $\alpha(z) = V^*\pi(z)V$ for any $z \in Z$.*

Proof. — We first consider (1). By our assumption and (2.6, 2.7), there exist a completely bounded map $v\colon Z \to M_n$ and $a_1, \ldots, a_n$, $b_1, \ldots$, $b_n \in B$ such that $\|v\|_{cb} < 1$, $\|\sum_p a_p a_p^*\| < C$, $\|\sum_q b_q^* b_q\| < C$, and $u = wv$, where $w\colon M_n \to B$ is defined by $w(E_{pq}) = a_p b_q$. By Wittstock's factorization Theorem we may find a Hilbert space K, a unital $*$-representation $\pi\colon B(H) \to B(K)$, and two bounded operators V_1, $V_2\colon \ell_2^n \to K$ such that $v(z) = V_1^*\pi(z)V_2$ for any $z \in Z$, and $\|V_1\| = \|V_2\| \leqslant 1$. Then we define $v_1, v_2\colon Z \to M_n$ by the formulae $v_1(\ ) = V_1^*\pi(\ )V_1$ and $v_2(\ ) = V_2^*\pi(\ )V_2$, and we define w_1, $w_2\colon M_n \to B$ by letting $w_1(E_{pq}) = a_p a_q^*$ and

$w_2(E_{pq}) = b_p^* b_q$. Clearly these are four completely positive maps and moreover, $\|v_1\| \leqslant 1$, $\|v_2\| \leqslant 1$, $\|w_1\| \leqslant C$, and $\|w_2\| \leqslant C$. We now let

$$\varphi = (1/4)\big(w_1 v_1 + 2u + w_2 v_2\big) \quad \text{and} \quad \psi = (1/4)\big(w_1 v_1 - 2u + w_2 v_2\big).$$

By construction $u = \varphi - \psi$ whereas $\varphi + \psi = (1/2)\big(w_1 v_1 + w_2 v_2\big)$. We immediately deduce that $\|\varphi + \psi\| \leqslant C$.

We now define $\alpha\colon Z \longrightarrow M_{2n}$ and $\beta\colon M_{2n} \longrightarrow B$ as follows:

$$\forall z \in Z, \qquad \alpha(z) = (1/2) \begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \pi(z) \ \big(V_1, V_2\big), \tag{4.5}$$

and for any $1 \leqslant p, q \leqslant 2n$, $\beta(E_{pq}) = (1/2)c_p^* c_q$, where $c_k = a_k^*$ if $1 \leqslant k \leqslant n$ and $c_k = b_k$ if $n+1 \leqslant k \leqslant 2n$. A thorough examination of these definitions shows that $\beta\alpha = (1/4)(w_1 v_1 + w_\# v_\# + wv + w_2 v_2)$. However $u = wv$ is assumed self-adjoint hence $w_\# v_\# = u_\# = u$. Accordingly we have $\varphi = \beta\alpha$. Furthermore α and β are clearly completely positive, with $\|\alpha\| \leqslant 1$ and

$$\|\beta\| = \Big\|\beta\Big(\sum_{p=1}^{2n} E_{pp}\Big)\Big\| = (1/2)\Big\|\sum_{p=1}^{n} a_p a_p^* + \sum_{q=1}^{n} b_q^* b_q\Big\| \leqslant C.$$

Lastly observe that by the same reasoning as above, we may write ψ as a composition of two c.p. map, whence the property that ψ is completely positive. This proves (1).

The assertion (2) is easy to deduce from the above proof. Indeed, let us assume that Z is a dual operator system and that $y \in Z_* \otimes B$. Then the factorization $u = wv$ used above can be achieved for some w^*-continuous $v\colon Z \longrightarrow M_n$ with $\|v\|_{cb} < 1$. By Lemma 2.1, v has a w^*-continuous and completely contractive extension $\widetilde{v}\colon B(H) \longrightarrow M_n$ hence it follows from [17, Proposition 2.12] (see also Proposition 3.4 above) that the $*$-representation π can be assumed w^*-continuous. The result therefore follows from (4.5).

□

THEOREM 4.5. — *Let Z be an operator system, let M be a von Neumann algebra, and let $u\colon Z \longrightarrow M$ be a u.c.p. map. The following two assertions are equivalent:*

(i) *u is 1-semidiscrete, i.e. (by Remark 4.2):*

For any C^*-algebra A, $\qquad \big\|I_A \otimes u\colon A \otimes_{\min} Z \longrightarrow A \otimes_{\text{nor}} M\big\| = 1.$

(ii) *There exists a net $u_i: Z \to M$ of finite rank operators converging to u in the point-w^* topology, such that every u_i admits a factorization $u_i = \beta_i\alpha_i$, with*

$$Z \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} M \quad \text{and} \quad \alpha_i,\, \beta_i \ \text{are u.c.p.}$$

Proof. — Clearly by Theorem 4.3 we only have to show that (i) implies (ii). So we assume that u is 1-semidicrete. By the implication (i) $\Rightarrow$ (iii) of Theorem 4.3, there is a net $u_i: Z \to M$ of finite rank operators converging to u in the point-w^* topology such that if we let $y_i \in Z^* \otimes M$ be representing u_i, we have $\delta(y_i) < 1$. Since u is self-adjoint, we may assume that each u_i is self-adjoint as well, by changing u_i to $(1/2)(u_i + u_{i\#})$ if necessary. Let us apply part (1) of Proposition 4.4 to each u_i, with $C = 1$. Then we may write $u_i = \varphi_i - \psi_i$, where φ_i and ψ_i are finite rank completely positive maps with the following two properties. First,

$$\|\varphi_i + \psi_i\| \leqslant 1, \tag{4.6}$$

and second, there is a factorization $\varphi_i = \beta_i\alpha_i$, for some completely positive contractive maps $\alpha_i: Z \to M_{n_i}$ and $\beta_i: M_{n_i} \to M$. We claim that

$$\forall z \in Z, \qquad \psi_i(z) \xrightarrow{w^*} 0. \tag{4.7}$$

Indeed let $f \in M_*$ be a state. Since $\langle u(1), f\rangle = \langle 1, f\rangle = 1$, we know that

$$\langle \varphi_i(1), f\rangle - \langle \psi_i(1), f\rangle \longrightarrow 1.$$

However $\langle \varphi_i(1), f\rangle$ and $\langle \psi_i(1), f\rangle$ are two nonnegative real numbers whose sum is $\leqslant 1$ by (4.6). Hence we have $\langle \psi_i(1), f\rangle \longrightarrow 0$. Now let $z \geqslant 0$ be in Z. Then $0 \leqslant z \leqslant \|z\|1$ hence $0 \leqslant \psi_i(z) \leqslant \|z\|\psi_i(1)$, hence applying f, we finally obtain

$$0 \leqslant \langle \psi_i(z), f\rangle \leqslant \|z\|\langle \psi_i(1), f\rangle.$$

It therefore follows from above that $\langle \psi_i(z), f\rangle \longrightarrow 0$. Since Z (resp. M_*) is spanned by its positive elements (resp. by its states), this shows (4.7). We thus obtain that $\varphi_i = \beta_i\alpha_i$ converges to u in the point-w^* topology hence it remains to check that α_i and β_i can be modified to become unital maps.

First note that given any completely positive map $\alpha: Z \to M_n$, there exists a u.c.p. map $\widehat{\alpha}: Z \to M_n$ such that $\alpha(z) = \alpha(1)^{1/2}\widehat{\alpha}(z)\alpha(1)^{1/2}$ for any $z \in Z$. Indeed this follows either from [4, Lemma 2.2] or from Stinespring's factorization Theorem (see the proof of Lemma 4.9 below).

Moreover if $\beta: M_n \to M$ is any c.p. map and if $\widehat{\beta}: M_n \to M$ is defined by $\widehat{\beta}(m) = \beta(\alpha(1)^{1/2} m \alpha(1)^{1/2})$, then $\widehat{\beta}$ is c.p. as well and $\beta\alpha = \widehat{\beta}\widehat{\alpha}$. Furthermore if α and β are both contractive, then $\widehat{\alpha}$ and $\widehat{\beta}$ are contractive as well. This shows that in the above construction, we can now assume that each α_i is unital.

For every i, we let $\eta_i \in M_{n_i}^*$ be a state and define a new map $\beta_i': M_{n_i} \to M$ by letting $\beta_i'(m) = \beta_i(m) + \eta_i(m)(1 - \beta_i(1))$. With that definition, we clearly have $\beta_i'(1) = 1$. Note that since $\|\beta_i\| \leqslant 1$, we have $1 - \beta_i(1) \geqslant 0$ hence β_i' is a u.c.p. map. Moreover

$$1 - \beta_i(1) = 1 - \beta_i\alpha_i(1) = 1 - u_i(1) \xrightarrow{w^*} 0$$

hence $\beta_i\alpha_i - \beta_i'\alpha_i$ tends to 0 in the point-w^* topology. Changing β_i to β_i', we thus obtain the expected fact that β_i can be assumed unital. □

Remark 4.6. — Up to an obvious modification, the simple unitization process explained above can be applied as well to C-nuclear operators. Thus the assertion (v) in Theorem 2.6 can be replaced by

(v)′ There exists a net $u_i: Y \to B$ of finite rank operators converging to u in the point-norm topology, such that every u_i admits a factorization $u_i = \beta_i\alpha_i$, with

$$Y \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} B \quad \text{and} \quad \alpha_i, \beta_i \text{ are u.c.p.}$$

COROLLARY 4.7. — *Let $Z \subset B(H)$ be an operator space, let M be a unital dual operator algebra, and let $u: Z \to M$ be a bounded operator. If u is C-semidiscrete for some $C > 0$, then there exists a C-semidiscrete operator $\widetilde{u}: B(H) \to M$ extending u. Assume moreover that Z is an operator system, M is a von Neumann algebra and u is u.c.p. and 1-semidiscrete. Then u admits a u.c.p. and 1-semidiscrete extension $\widetilde{u}: B(H) \to M$.*

Proof. — Assume that u is C-semidiscrete. By Theorem 4.3, there exist nets $\alpha_i: Z \to M_{n_i}$ and $\beta_i: M_{n_i} \to M$ as in part (iii) of Theorem 4.3. For any i we may find, by Wittstock's extension Theorem, an operator $\widetilde{\alpha}_i: B(H) \to M_{n_i}$ extending α_i, with $\|\widetilde{\alpha}_i\|_{cb} = \|\alpha_i\|_{cb}$. Letting $\widetilde{u}_i = \beta_i\widetilde{\alpha}_i$, we obviously have $\|\widetilde{u}_i\| \leqslant C$. Since bounded sets of $B(B(H), M)$ are relatively compact in the point-w^* topology, we see that $(\widetilde{u}_i)_i$ admits a limit point $\widetilde{u}: B(H) \to M$ for that topology. By construction, $\widetilde{u}$ extends u and by the implication "(iii) ⇒ (i)" of Theorem 4.3, we see that $\widetilde{u}$ is C-semidiscrete.

The second part of the corollary immediately follows from the first one, since unital completely contractive maps are automatically completely positive. □

By definition ([11]), a von Neumann algebra M is semidiscrete if and only if the identity operator I_M is 1-semidiscrete in the sense of Definition 4.1. It is well known that in that case, I_M satisfies the assertion (ii) of Theorem 4.5 with the additionnal property that the α_i's are w^*-continuous (see [6]). It is thus natural to ask whether the α_i's can be assumed to be w^*-continuous in part (iii) of Theorem 4.3 (resp. in part (ii) of Theorem 4.5), when Z is a dual operator space (resp. a dual operator system). It turns out to be the case, and this is essentially a consequence of the proofs written above. We record these facts in the next statement. It should be emphasized that in either (1) or (2) below, u is not assumed w^*-continuous.

PROPOSITION 4.8

(1) *Let $u\colon Z \to M$ be C-semidiscrete from a dual operator space into a unital dual operator algebra (resp. a von Neumann algebra). Then the assertion (iii) (resp. (iii)') of Theorem 4.3 is fulfilled with the additional property that each $\alpha_i\colon Z \to M_{n_i}$ is w^*-continuous.*

(2) *Let Z be dual operator system (that is, $Z \subset B(H)$ is a w^*-closed operator system). Let M be a von Neumann algebra and let $u\colon Z \to M$ be u.c.p. and 1-semidiscrete. Then the assertion (ii) of Theorem 4.5 is fulfilled with the additional property that each $\alpha_i\colon Z \to M_{n_i}$ is w^*-continuous.*

LEMMA 4.9. — *Let H be a Hilbert space and let $\gamma\colon B(H) \to M_n$ be some completely positive, w^*-continuous, and contractive map. Then there exists a w^*-continuous u.c.p. map $\widehat{\gamma}\colon B(H) \to M_n$ such that $\gamma(z) = \gamma(1)^{1/2}\widehat{\gamma}(z)\gamma(1)^{1/2}$ for any $z \in B(H)$.*

Proof. — By the normal form of Stinespring's Theorem, there exist a Hilbert space K, a unital w^*-continuous $*$-representation $\pi\colon B(H) \to B(K)$, and a contraction $V\colon \ell_2^n \to K$ such that $\gamma(z) = V^*\pi(z)V$ for any $z \in B(H)$. Let $V = U|V|$ be the polar decomposition of V. That is, $|V| = (V^*V)^{1/2} \in M_n$, and $U\colon \ell_2^n \to K$ is a partial isometry, with $\ker U = \ker V$. Let $\eta \in B(H)_*$ be a normal state and let $\widehat{\gamma}\colon B(H) \to M_n$ be defined by

$$\widehat{\gamma}(z) = U^*\pi(z)U + \eta(z)(1 - U^*U).$$

By construction, $\widehat{\gamma}$ is u.c.p. and w^*-continuous. Moreover $\gamma(1)^{1/2} = |V|$ whereas $|V|(1-U^*U)|V| = 0$, so that $\gamma(\ ) = \gamma(1)^{1/2}\widehat{\gamma}(\ )\gamma(1)^{1/2}$ as required. □

Proof of Proposition 4.8. — To prove (1), we need to come back inside the proof of the implication "(ii) ⇒ (iii)" of Theorem 4.3, with the new assumption that Z is a dual operator space. With the notation therein, the operator space $E \subset Z$ spanned by I_1 is finite dimensional hence w^*-closed. It therefore follows from Lemma 2.1 that the operator $\alpha: Z \to M_n$ extending the operator a associated to $[x_{pq}]$ can be chosen to be w^*-continuous, with

$$\|\alpha\|_{cb}\Big\|\sum c_p c_p^*\Big\|^{1/2}\Big\|\sum d_q^* d_q\Big\|^{1/2} < C.$$

This yields (1).

To obtain (2), we re-write the first part of the proof of Theorem 4.5, using the second part of Proposition 4.4 and the property (1) we just proved. We thus obtain a net of diagrams $Z \xrightarrow{\alpha_i} M_{n_i} \xrightarrow{\beta_i} M$ such that

(a) $\beta_i\alpha_i$ converges to u in the point-w^* topology and the α_i's and β_i's are contractive and completely positive.

(b) Each α_i admits a factorization $\alpha_i(z) = V_i^*\pi_i(z)V_i$, where $V_i: \ell_2^{n_i} \to K_i$ is a contraction, and $\pi_i: B(H) \to B(K_i)$ is a unital w^*-continuous $*$-representation.

Applying Lemma 4.9 to each $\gamma_i: B(H) \to B(K_i)$ defined by $\gamma_i(\ ) = V_i^*\pi(\ )V_i$, we see that (a) can be achieved with the additional property that each α_i is w^*-continuous and u.c.p.. Arguing as in the last part of the proof of Theorem 4.5, we finally obtain that the β_i's can also be assumed unital. □

We recall that by definition ([9], [12]), an operator space Y is locally reflexive provided that

$$(4.8) \qquad \big(E \otimes_{\min} Y\big)^{**} = E \otimes_{\min} Y^{**} \quad \text{isometrically}$$

for any finite dimensional operator space E. More explicitly, given any operator spaces Y and E, with E finite dimensional, let $J_E: \big(E\otimes_{\min} Y\big)^{**} \to E \otimes_{\min} Y^{**}$ be the identity mapping. Then we always have $\|J_E\| \leqslant 1$, and Y is locally reflexive provided that $\|J_E^{-1}\| \leqslant 1$ for any E. We can obviously extend this definition by saying that Y is C-locally reflexive ($C \geqslant 1$) if $\|J_E^{-1}\| \leqslant C$ for any finite dimensional E.

In C^*-algebra Theory, local reflexivity, nuclearity, and semidiscreteness are related as follows. On one hand a C^*-algebra B is nuclear if and only if the von Neumann algebra B^{**} is semidiscrete ([5]) and on the other hand, every nuclear C^*-algebra is locally reflexive, by [1, 9]. The next corollary indicates relationships between local reflexivity, nuclearity, and semidiscreteness in the broader context of operators.

COROLLARY 4.10. — *Let Y be an operator space, let B be an operator algebra with a c.a.i., and let $u: Y \to B$ be a bounded operator.*

(1) *If $u^{**}: Y^{**} \to B^{**}$ is C-semidiscrete for some $C > 0$, then u is C-nuclear.*

(2) *If u is C-nuclear and Y is C'-locally reflexive for some $C > 0$ and $C' \geqslant 1$, then u^{**} is CC'-semidiscrete.*

(3) *If u is completely isometric, then u^{**} is 1-semidiscrete if and only if u is 1-nuclear and Y is locally reflexive.*

Proof. — We use the notation introduced before Corollary 4.10. We shall consider:

$$I_E \otimes u: E \otimes_{\min} Y \longrightarrow E \overset{\delta}{\otimes} B \quad \text{and} \quad I_E \otimes u^{**}: E \otimes_{\min} Y^{**} \longrightarrow E \overset{\widetilde{\delta}}{\otimes} B^{**}.$$

We clearly have $(I_E \otimes u)^{**} = (I_E \otimes u^{**})J_E$ by Corollary 3.8, (3). Since $\|J_E\| \leqslant 1$, we deduce that for any E,

$$\|I_E \otimes u\| = \|(I_E \otimes u)^{**}\| \leqslant \|I_E \otimes u^{**}\|.$$

Hence (1) follows from the equivalences "(i) $\Leftrightarrow$ (ii)" in Theorem 2.6 and Theorem 4.3. Similarly, we have

$$\|I_E \otimes u^{**}\| \leqslant \|J_E^{-1}\| \|\|I_E \otimes u\|$$

for any E, whence (2).

We now turn to (3). By (1), we only have to check that if u is completely isometric and u^{**} is 1-semidiscrete, then Y is locally reflexive. First observe that given an arbitrary finite dimensional E, the identity mapping is a contraction from $E \overset{\delta}{\otimes} B$ into $E \otimes_{\min} B$, hence from $E \overset{\widetilde{\delta}}{\otimes} B^{**}$ into $\big(E \otimes_{\min} B\big)^{**}$ by Corollary 3.8. Thus assuming that u^{**} is 1-semidiscrete, we then obtain that $I_E \otimes u^{**}$ is a contraction from $E \otimes_{\min} Y^{**}$ into $\big(E \otimes_{\min} B\big)^{**}$. Moreover since u is a complete isometry, u^{**} is a complete isometry as well hence $I_E \otimes u^{**}$ is a complete isometry from $E \otimes_{\min} Y^{**}$ into $E \otimes_{\min} B^{**}$. Since the identity mapping is a contraction from $\big(E \otimes_{\min} B\big)^{**}$ into $E \otimes_{\min} B^{**}$, we finally obtain that

$$(4.9) \quad I_E \otimes u^{**}: E \otimes_{\min} Y^{**} \longrightarrow \big(E \otimes_{\min} B\big)^{**} \quad \text{is a complete isometry.}$$

Using again the assumption that u is a complete isometry, we see that $I_E \otimes u$ is a complete isometry from $E \otimes_{\min} Y$ into $E \otimes_{\min} B$ hence passing to the biadjoint,

$$(4.10)\quad I_E \otimes u^{**} \colon \big(E \otimes_{\min} Y\big)^{**} \longrightarrow \big(E \otimes_{\min} B\big)^{**} \quad \text{is a complete isometry.}$$

Comparing (4.9) and (4.10), we obtain that (4.8) holds for any finite dimensional E, whence the local reflexivity of Y. This proves the result. □

Remark 4.11. — Let Y be an operator space and let $j \colon Y \to B(H)$ be a completely isometric embedding, for some Hilbert space H. Then Y is exact (in the sense of [30]) if and only if j is 1-nuclear. A remarkable theorem of Kirchberg asserts that any exact C*-algebra is locally reflexive. We mention that the more general problem whether any exact operator space is locally reflexive is open. This is equivalent to the question whether in Corollary 4.10 (3), we have u 1-nuclear $\Leftrightarrow$ u^{**} 1-semidicrete.

5. Local reflexivity properties of the decomposable norm.

We recall that given any two operator spaces E and Y, there is a natural embedding of $E^{**} \otimes Y^{**}$ into $(E \otimes_{\min} Y)^{**}$, inducing a norm at least greater than $\| \ \|_{\min}$ on $E^{**} \otimes Y^{**}$. However this norm can be different from $\| \ \|_{\min}$ and is in general quite difficult to determine. This phenomenon has been discovered before the operator space theory arose, by Archbold and Batty ([1]), and it is now known that it is related to the lack of either local reflexivity or exactness of certain operator spaces. More precisely, an operator space Y is locally reflexive if and only if $E \otimes_{\min} Y^{**} \subset (E \otimes_{\min} Y)^{**}$ isometrically for any E whereas E is exact (in the sense of [30]) if and only if $E \otimes_{\min} Y^{**} \subset (E \otimes_{\min} Y)^{**}$ isometrically for any Y. We refer to [1], [9], [16], [20] and [31, Section 6.5] for the proofs of these results and more information.

The aim of this section is to show, by combining Corollary 3.8 with the main result of [21], that the phenomenon briefly recalled above disappears if we replace the completely bounded norm, corresponding to $\| \ \|_{\min}$, by the decomposable one, in the case when Y is a C^*-algebra and E is the operator space predual of a von Neumann algebra. Let us introduce a relevant notation. We give ourselves an arbitrary C^*-algebra B. For any von Neumann algebra R, we shall denote by $R_* \otimes_{\mathrm{dec}} B$ the closure of $R_* \otimes B$

into $DEC(R,B)$, so that

$$R_* \otimes_{\text{dec}} B \subset DEC(R,B) \quad \text{isometrically.}$$

Let A be a C*-algebra and let $u: A \to B$ be any finite rank operator. Then, let $z \in A^* \otimes B$ be associated to u. It is shown in [21, Theorem 2.1] that $\|u\|_{\text{dec}} = \|z\|_{A^* \overset{\delta}{\otimes} B}$. Furthermore, if $A = R$ is a von Neumann algebra and $z \in R_* \otimes B$, then we also have $\|u\|_{\text{dec}} = \|z\|_{R_* \overset{\delta}{\otimes} B}$. Thus for any B and R as above, we have

$$R_* \otimes_{\text{dec}} B = R_* \overset{\delta}{\otimes} B, \tag{5.1}$$

whereas for any C*-algebra A, we obtain that

$$A^* \otimes_{\text{dec}} B \subset DEC(A,B) \quad \text{isometrically.} \tag{5.2}$$

THEOREM 5.1. — *For any C^*-algebra B and any von Neumann algebra R, we have*

$$R^* \otimes_{\text{dec}} B^{**} \subset \big(R_* \otimes_{\text{dec}} B\big)^{**} \quad \textit{isometrically.}$$

More explicitly, given any finite rank bounded operator $u: R \to B^{**}$, then $\|u\|_{\text{dec}} \leqslant 1$ if and only if there is a net of w^*-to- norm continuous finite rank operators $u_t: R \to B$ with $\|u_t\|_{\text{dec}} < 1$ such that for all $\xi \in \big(R_* \otimes_{\text{dec}} B\big)^*$: $\langle \xi, u\rangle = \lim_t \langle \xi, u_t\rangle$.

The proof of Theorem 5.1 will require the following elementary fact.

LEMMA 5.2. — *Let A be a C^*-algebra and let M be a von Neumann algebra. Let $u: A \to M$ be a bounded operator and let C be a constant. Assume that there exist a net $(u_t)_t \subset DEC(A,M)$ converging to u in the point-w^* topology, with $\|u_t\|_{\text{dec}} < C$. Then $u \in DEC(A,M)$, with $\|u\|_{\text{dec}} \leqslant C$.*

Proof. — By the definition of the decomposable norm (see (2.4) and above it) there exist, for each t, two completely positive maps $S_1^t: A \to M$ and $S_2^t: A \to M$ such that the mapping $v_t: A \to M_2(M)$ defined by

$$v_t(a) = \begin{pmatrix} S_1^t(a) & u_t(a^*)^* \\ u_t(a) & S_2^t(a) \end{pmatrix}$$

is completely positive, with $\|S_1^t\| \leqslant C$, $\|S_2^t\| \leqslant C$. Refining the net if necessary, we can assume that the two nets $(S_1^t)_t$ and $(S_2^t)_t$ converge in the point-w^* toplology of $B(A, M)$. That is, there exist $S_1: A \to M$, $S_2: A \to M$ such that for all $a \in A$, $S_1^t(a) \xrightarrow{w^*} S_1(a)$ and $S_2^t(a) \xrightarrow{w^*} S_2(a)$. Obviously S_1 and S_2 are completely positive with norms $\leqslant C$. By construction, the mapping $v: A \to M_2(M)$ defined by

$$v(a) = \begin{pmatrix} S_1(a) & u(a^*)^* \\ u(a) & S_2(a) \end{pmatrix}$$

is the point-w^* limit of $(v_t)_t$ hence is completely positive. This proves that $\|u\|_{\text{dec}} \leqslant C$. □

Proof of Theorem 5.1. — It follows from (5.1) applied twice that $R_* \otimes_{\text{dec}} B = R_* \overset{\delta}{\otimes} B$ and $R^* \otimes_{\text{dec}} B^{**} = R^* \overset{\delta}{\otimes} B^{**}$ isometrically. Hence applying Corollary 3.8, (2), we obtain that the norm induced by $\left(R_* \otimes_{\text{dec}} B\right)^{**}$ on $R^* \otimes B^{**}$ is dominated by the decomposable one, whence a contraction

$$J: R^* \otimes_{\text{dec}} B^{**} \longrightarrow \left(R_* \otimes_{\text{dec}} B\right)^{**}.$$

Let us denote by $j: B \to B^{**}$ the canonical embedding of B into its second dual. We let $z \in R^* \otimes B^{**}$, and we assume that $\|J(z)\| \leqslant 1$. Then by Goldstine's Lemma, there is a net $(z_t)_t \subset R_* \otimes B$ converging to $J(z)$ in the w^*-topology of $\left(R_* \otimes_{\text{dec}} B\right)^{**}$, and satisfying $\|z_t\|_{R_* \otimes_{\text{dec}} B} < 1$ for any t. We now let $u: R \to B^{**}$ be associated to z. Likewise for each t, we let $u_t: R \to B$ be associated to z_t. The w^*-convergence of z_t to $J(z)$ implies that ju_t converges to u in the point-w^* topology. Moreover we have

$$\|ju_t\|_{\text{dec}} \leqslant \|u_t\|_{\text{dec}} = \|z_t\|_{R_* \otimes_{\text{dec}} B} < 1.$$

Hence applying Lemma 5.2 with $M = B^{**}$, we find that $\|u\|_{\text{dec}} \leqslant 1$. However by (5.2) we know that $\|u\|_{\text{dec}}$ coincides with the norm of z in $R^* \otimes_{\text{dec}} B^{**}$. Thus $\|z\| \leqslant 1$ if $\|J(z)\| \leqslant 1$. This shows that J is indeed an isometry. □

We consider now the special case when B has the weak expectation property (WEP in short). We refer to [24] for the definition of that notion and to [23] for recent deep developments. If B is a C^*-algebra with the WEP, then $R_* \otimes_{\min} B = R_* \otimes_{\text{dec}} B$ for any von Neumann algebra R, by [18, Proposition 3.3]. We thus derive from Theorem 5.1 the following.

Corollary 5.3. — *Let B be a C^*-algebra with the WEP and let R be a von Neumann algebra. Then*

$$R^* \otimes_{\text{dec}} B^{**} \subset \left(R_* \otimes_{\min} B\right)^{**} \quad \textit{isometrically.}$$

It should be noticed that if B is not nuclear, then $R^* \otimes_{\text{dec}} B^{**}$ and $R^* \otimes_{\min} B^{**}$ are different in general. Indeed, this follows from [18, Theorem 2.1].

COROLLARY 5.4. — *Let A, B be two C^*-algebras, and assume that A is finite dimensional.*

(1) We have $DEC(A,B)^{**} = DEC(A,B^{**})$ isometrically.

(2) If B has the WEP, then $CB(A,B)^{**} = DEC(A,B^{**})$ isometrically.

Proof. — Apply Theorem 5.1 and Corollary 5.3. □

Note added in proof. — Extending a result of Kirchberg, Effros-Ozawa-Ruan have proved in a recent paper "On injectivity and nuclearity for operator spaces" that any exact operator space is locally reflexive. This implies (see Remark 4.11) that part (3) of our Corollary 4.10 may be replaced by "u is 1-nuclear $\Leftrightarrow$ u^{**} is 1-semidiscrete", for any completely isometric $u : Y \to B$.

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