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THE FULL PERIODICITY KERNEL OF THE TREFOIL

by C. LESEDUARTE and J. LLIBRE (*)

1. Introduction and main results.

Let E be a topological space. We shall study some properties of the set of periods for a class of continuous maps from E into itself. We need some notation.

The set of natural numbers, real numbers and complex numbers will be denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ respectively. For a map $f : E \to E$ we use the symbol f^n to denote $f \circ f \circ \cdots \circ f$ ($n \in \mathbb{N}$ times), f^0 or «id» denotes the identity map of E. Then, for a point $x \in E$ we define the *orbit* of x, denoted by $\mathrm{Orb}_f(x)$, as the set $\{f^n(x) : n = 0, 1, 2, \ldots\}$. We say x is a *fixed point* of f if $f(x) = x$. We say x is a *periodic point of f of period $k \in \mathbb{N}$* (or simply a *k-point*) if $f^k(x) = x$ and $f^i(x) \neq x$ for $1 \leq i < k$. In this case we say the orbit of x is a *periodic orbit of period k* (or simply a *k-orbit*). Note that if x is a k-point, then $\mathrm{Orb}_f(x)$ has exactly k elements, each of which is a k-point. We denote by $\mathrm{Per}(f)$ the set of periods of all periodic points of f.

A *connected finite regular graph* (or just a *graph* for short) is a pair consisting of a connected Hausdorff space E and a finite subspace V, whose elements are called *vertices*, such that the following conditions hold:

(1) $E \setminus V$ is the disjoint union of a finite number of open subsets $e_1, \ldots, e_k$, called *edges*. Each e_i is homeomorphic to an open interval of the real line.

(2) The boundary, $\mathrm{Cl}(e_i) \setminus e_i$, of the edge e_i consists of two distinct vertices, and the pair $(\mathrm{Cl}(e_i), e_i)$ is homeomorphic to the pair ([0,1],(0,1)).

If v and e are the number of vertices and edges respectively of E, then *the Euler characteristic of* E, is $\chi(E) = v - e$. A vertex which belongs to

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the boundary of at least three different edges is called a *branching point* of E. A vertex which belongs to a unique edge is called an *endpoint.*

An E *map* is a continuous self-map of E having fixed all branching points of E.

We say an E map f has *full periodicity* if $\mathrm{Per}(f) = \mathbb{N}$. The set $K \subset \mathbb{N}$ is the *full periodicity kernel* of E if it satisfies the following two conditions:

(1) If f is an E map and $K \subset \mathrm{Per}(f)$, then $\mathrm{Per}(f) = \mathbb{N}$.

(2) If $S \subset \mathbb{N}$ is a set such that for every E map f, $S \subset \mathrm{Per}(f)$ implies $\mathrm{Per}(f) = \mathbb{N}$, then $K \subset S$.

Note that, for a given E, if there is a full periodicity kernel, then it is unique.

From now on the topological space E will denote one of the following spaces:

$$\begin{aligned}
\mathbf{I}_i &= \{z \in \mathbb{C} : z^i \in [0,1]\},\ i = 2,3,\dots,6.\\
\mathbf{O} &= \{z \in \mathbb{C} : |z+i| = 1\},\\
\mathbf{O}_1 &= \mathbf{O} \cup \{z \in \mathbf{I}_2 : \mathrm{Re}\ z \geq 0\},\\
\mathbf{O}_2 &= \mathbf{O} \cup \mathbf{I}_2,\\
\mathbf{O}_3 &= \mathbf{O} \cup \{z \in \mathbf{I}_4 : \mathrm{Im}\ z \geq 0\},\\
\mathbf{O}_4 &= \mathbf{O} \cup \mathbf{I}_4,\\
\boldsymbol{\infty} &= \mathbf{O} \cup \{z \in \mathbb{C} : |z-i| = 1\},\\
\boldsymbol{\infty}_1 &= \boldsymbol{\infty} \cup \{z \in \mathbf{I}_2 : \mathrm{Re}\ z \geq 0\},\\
\boldsymbol{\infty}_2 &= \boldsymbol{\infty} \cup \mathbf{I}_2,\\
\mathbf{T} &= \{z \in \mathbb{C} : z = \cos(3\theta)\, e^{i\theta},\ 0 \leq \theta \leq 2\pi\}.
\end{aligned}$$

The spaces $\mathbf{I}_2, \mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}, \mathbf{O}_1, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \boldsymbol{\infty}, \boldsymbol{\infty}_1, \boldsymbol{\infty}_2$ and $\mathbf{T}$ are called the *interval* or the $\mathbf{I}$, the 3-*od* or 3-*star* or the $\mathbf{Y}$, the 4-*od* or the 4-*star*, the 5-*od* or 5-*star*, the 6-*od* or 6-*star*, the *circle*, the *sigma*, the *alpha*, the *circle with three whiskers*, the *circle with four whiskers*, the *eight*, the *eight with one whiskers*, the *eight with two whiskers* and the *trefoil* respectively.

The spaces $\mathbf{I}_3$, $\mathbf{I}_4$, $\mathbf{I}_5$, $\mathbf{I}_6$, $\mathbf{O}_1$, $\mathbf{O}_2$, $\mathbf{O}_3$, $\mathbf{O}_4$, $\boldsymbol{\infty}$, $\boldsymbol{\infty}_1$, $\boldsymbol{\infty}_2$ and $\mathbf{T}$ have exactly one branching point, namely $\mathbf{0} = 0 \in \mathbb{C}$. We also denote by $\mathbf{0}$ the $0 \in \mathbf{O}$.

The full periodicity kernel of $\mathbf{I}_2$, $\mathbf{I}_3$, $\mathbf{I}_4$, $\mathbf{I}_5$, $\mathbf{I}_6$, $\mathbf{O}$, $\mathbf{O}_1$, $\mathbf{O}_2$ and $\boldsymbol{\infty}$ are known and presented in the following theorem.

THEOREM 1.1. — *The following statements hold:*

(a) *The set* $\{3\}$ *is the full periodicity kernel of* $\mathbf{I}_2$.

(b) *The set* $\{2,3,4,5,7\}$ *is the full periodicity kernel of* $\mathbf{I}_3$.

(c) *The set* $\{2,3,4,5,6,7,10,11\}$ *is the full periodicity kernel of the* $\mathbf{I}_4$.

(d) *The set* $\{2,3,4,5,6,7,8,9,10,11,13,14,16,17,18,21,23\}$ *is the full periodicity kernel of the* $\mathbf{I}_5$.

(e) *The full periodicity kernel of the* $\mathbf{I}_6$ *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29\}.$$

(f) *The set* $\{1,2,3\}$ *is the full periodicity kernel of* $\mathbf{O}$.

(g) *The set* $\{2,3,4,5,7\}$ *is the full periodicity kernel of* $\mathbf{O}_1$.

(h) *The set* $\{2,3,4,5,6,7,10,11\}$ *is the full periodicity kernel of* $\mathbf{O}_2$.

(i) *The set* $\{2,3,4,5,6,7,8,10,11\}$ *is the full periodicity kernel of* ∞.

- Theorem 1.1 (a) is due to Sharkovskii [Sh] (see also [LY]),
- Theorem 1.1 (b) was shown by Mumbrú [M] (see also [ALM1]),
- Theorem 1.1 (c) has been proved by Alsedà and Moreno [AM] and independently by Leseduarte and Llibre [LL2],
- Statements (d) and (e) of Theorem 1.1 are due to Alsedà and Moreno [AM],
- Theorem 1.1 (f) is due to Block [Bc1] (see also [LR]),
- Theorem 1.1 (g) has been proved by Llibre, Paraños and Rodríguez [LPR1] (see also [LL1]),
- Statements (h) and (i) of Theorem 1.1 are due to Leseduarte and Llibre [LL2].

Our main goal in this paper is to characterize the full periodicity kernel of $\mathbf{O}_3$, $\mathbf{O}_4$, ∞_1, ∞_2 and $\mathbf{T}$. Thus, our main results are the following:

THEOREM 1.2. — *The full periodicity kernel of* $\mathbf{O}_3$ *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,13,14,16,17,18,21,23\}.$$

THEOREM 1.3. — *The full periodicity kernel of* $\mathbf{O}_4$ *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29\}.$$

THEOREM 1.4. — *The full periodicity kernel of* ∞_1 *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,21,23\}.$$

THEOREM 1.5. — *The full periodicity kernel of* ∞_2 *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,21,22,23,28,29\}.$$

THEOREM 1.6. — *The full periodicity kernel of* $\mathbf{T}$ *is the set*

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,21,22,23,28,29\}.$$

Theorems 1.2, 1.3, 1.4, 1.5 and 1.6 are proved in Sections 6, 7, 10, 11 and 12 respectively. Sections from 2 to 5 present preliminary definitions and results that are necessary for proving these five main theorems. In Section 13 we compare our results on the full periodicity kernel with related results of Blokh. Finally, in Section 14 we comment that full periodicity implies positive topological entropy for continuous self-maps on a graph.

The tools for studying the set of periods and the full periodicity kernel change strongly when we consider maps with some discontinuity points, see for instance [ALMT].

2. Intervals and basic intervals.

From now on we shall talk about the whiskers and the circles of E. A *circle* of E is the closure of a connected component of $E \setminus \{\mathbf{0}\}$ which is homeomorphic to $\mathbf{O}$. A *whiskers* of E is the closure of a connected component of $E \setminus \{\mathbf{0}\}$ which is homeomorphic to $\mathbf{I}_2$.

A *closed* (respectively *open*, *half-open* or *half-closed*) interval J of E is a subset of E homeomorphic to the closed interval $[0,1]$ (respectively $(0,1),[0,1)$). Notice that an interval cannot be a single point.

Let J be a closed interval of E, and let $h : [0,1] \longrightarrow J$ be a homeomorphism. Then $h(0) = a$ and $h(1) = b$ are called the *endpoints* of J. If a and b belong to the same whiskers of E, then J will be denoted by $[a,b]$ or $[b,a]$. We take an orientation, that we call *counterclokwise*, in each circle of E. If a and b belong to the same circle of E, then we write $[a,b]$ to denote the closed interval from a counterclockwise to b.

Note that it is possible that two different intervals of a circle of E have the same endpoints. But two different points of a whiskers of E always determine a unique closed interval.

Now we define a special class of subintervals of E. Let $Q = \{q_1, q_2, \ldots, q_n\}$ be a finite subset of E containing $\mathbf{0}$. For each pair q_i, q_j such that $q_i \neq q_j$ we say that the interval $[q_i, q_j]$ (respectively $[q_j, q_i]$) is *basic* if and only if $(q_i, q_j) \cap Q = \emptyset$ (respectively $(q_j, q_i) \cap Q = \emptyset$). The set of all these basic intervals is called the *set of basic intervals associated to Q*.

3. Loops and f-graphs.

Let $f : E \to E$ be an E map. If K and J are closed intervals of E, then we say that K *f-covers* J or $K \to J$ (or $J \leftarrow K$), if there is a closed subinterval M of K such that $f(M) = J$. If K does not f-cover J we write $K \not\to J$.

A *path of length m* is any sequence $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m$, where $J_0, J_1, \ldots, J_m$ are closed subintervals of E (in general, basic intervals). Furthermore, if $J_0 = J_m$, then this path is called a *loop of length m*. Such a loop will be called *non-repetitive* if there is no integer i, $0 < i < m$, such that i divides m and $J_{j+i} = J_j$ for all j, $0 \le j \le m - i$. We say that we *add* or we *concatenate* the loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_0$ to the loop $K_0 \to K_1 \to \cdots \to K_{n-1} \to K_0$ if they have a common vertex $J_0 = K_0$ and we form the new loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to K_0 \to K_1 \to \cdots \to J_0$. A loop which cannot be formed by adding two loops will be called *elementary*.

Let Q be a finite subset of E containing $\mathbf{0}$. An *f-graph of Q* is a graph with the basic intervals associated to Q as vertices, and such that if K and J are basic intervals and K f-covers J, then there is an arrow from K to J. Note that the f-graph of Q is unique up to labeling of the basic intervals. Hence from now on we shall talk about *the f-graph* of Q (or just *the f-graph* for short). The next three lemmas are well-known in one dimensional dynamics, see for instance [ALM2]. We only prove the third one because we will use its proof later.

LEMMA 3.1. — *Let f be an E-map and let K, J, L be closed subintervals of E. If $L \subset J$ and K f-covers J, then K f-covers L.*

LEMMA 3.2. — *Let f be an E map and let J be a closed subinterval of E such that J f-covers J. Then f has a fixed point in J.*

LEMMA 3.3. — *Let f be an E map and let $J_0, J_1, \ldots, J_{n-1}$ be closed subintervals of E such that $J_i \to J_{i+1}$ for $i = 0, 1, \ldots, n-2$ and $J_{n-1} \to J_0$. Then there exists a fixed point x of f^n in J_0 such that $f^i(x) \in J_i$ for $i = 1, 2, \ldots, n-1$.*

Proof. — We shall use backward induction. Let $K_{n-1} \subset J_{n-1}$ be a closed interval such that $f(K_{n-1}) = J_0$, and suppose we have constructed $K_i \subset J_i$ for some $i > 0$, $i \leq n-1$ such that $f(K_i) = K_{i+1}$ if $i < n-1$ and $f(K_i) = J_0$ if $i = n-1$. Then, by Lemma 3.1, J_{i-1} f-covers K_i and therefore there exists an interval $K_{i-1} \subset J_{i-1}$ such that $f(K_{i-1}) = K_i$. Let g be as follows:

$$g = f_{|K_{n-1}} \circ \cdots \circ f_{|K_1} \circ f_{|K_0}.$$

Then $K_0 \subset J_0$ and $g(K_0) = J_0$. Consequently $f^n(K_0) = J_0$. By continuity of f^n and Lemma 3.2 f^n has a fixed point $x \in K_0 \subset J_0$, such that $f^i(x) \in K_i \subset J_i$ for $i = 1, 2, \ldots, n-1$. □

Let J be an interval of E. Then $\mathrm{Int}(J)$ and $\mathrm{Cl}(J)$ denote the interior and the closure of J respectively.

PROPOSITION 3.4. — *Let f be an E map having a k-orbit P. Consider the set of basic intervals associated to $P' = P \cup \{\mathbf{0}\}$. Let $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ be a non-repetitive loop of length m of the f-graph of P' such that at least one J_i does not contain $\mathbf{0}$. If $m \neq 2k$, then $m \in \mathrm{Per}(f)$.*

Proof. — By Lemma 3.1 J_0 f^m-covers J_0. Then by Lemma 3.2 there exists $x \in J_0$ such that $f^m(x) = x$. If x has period m we are done. So suppose that x has period s, $0 < s < m$. Thus s divides m.

It is not possible that $x = \mathbf{0}$ because $\mathbf{0}$ is a fixed point and some $f^i(x) \in J_i$ with $J_i \cap \{\mathbf{0}\} = \emptyset$.

If $x \in \mathrm{Int}(J_0)$, then $\mathrm{Orb}_f(x) \cap P = \emptyset$. So each $f^i(x)$ is exactly in one basic interval, and consequently the loop is repetitive (because $s < m$ and s divides m). Hence, x must be a point of P. So $\mathrm{Orb}_f(x) \subset P$. Without loss of generality we can assume that $s = k$.

Let $K_0 \subset J_0$ be the interval constructed in the proof of Lemma 3.3, then $f^i(x) \in f^i(K_0) \subset J_i$ for $i = 0, 1, \ldots, m$. Since $x = f^s(x) \in f^s(K_0) \subset J_s$ it follows that J_0 and J_s have a common endpoint x.

Assume that $J_0 = J_s$. Both sets K_0 and $f^s(K_0)$ are contained in J_0 and contain x, an endpoint of J_0. Therefore $L = K_0 \cap f^s(K_0)$ is an interval (in fact it is either K_0 or $f^s(K_0)$). Clearly $f^i(L) \subset f^i(K_0) \subset J_i$, $f^i(L) \subset f^{s+i}(K_0) \subset J_{s+i}$, and $f^i(L)$ is an interval for $0 \leq i \leq s$. Thus $J_i = J_{s+i}$ for $i = 0, 1, \ldots, s-1$.

Repeating this process we get $J_i = J_{s+i}$ for $i = 0, 1, \ldots, m - s$. Hence the loop is repetitive because s divides m, a contradiction with the assumptions. So $J_0 \neq J_s$.

If $J_q = J_{q+s}$ for some $0 < q < m - s$, then the above arguments prove that $J_{q+i} = J_{q+s+i}$ for $i = 0, 1, \ldots, s - 1$. Repeating this process we obtain that $J_i = J_{s+i}$ for $i = 0, 1, \ldots, m - s$ and so the loop is repetitive, a contradiction with the assumptions. Therefore we can assume that $J_q \neq J_{q+s}$ for $0 \leq q < m - s$.

Since x is a periodic point of period s, if follows that $J_0 = J_{2s}$ and $J_s = J_{3s}$. By the above arguments we get $J_m = J_0 = J_{2s} = J_{4s} = \cdots$ and $J_s = J_{3s} = J_{5s} = \cdots$. In particular m must be even. Furthermore $J_i = J_{2s+i}$ for $0 \leq i \leq 2s - 1$. Hence $2s = 2k$ divides m. Since $m \neq 2k$ the loop is repetitive, in contradiction with the hypotheses. □

Under the assumptions of Proposition 3.4 and if $m = 2k$, we can prove that $m \in \mathrm{Per}(f)$ if E is different from ∞ and $\mathbf{T}$. Unfortunately we do not know under the same assumptions if $m \in \mathrm{Per}(f)$ when $m = 2k$ and E is either ∞ or $\mathbf{T}$. But this is not important for the rest of the paper.

4. Q-linear maps.

Let $G = \mathbf{I}_i$, for $i = 2, 3, \ldots, 6$. It is easy to see that any tree G has a metric μ such that if $x, y \in G$ and $z \in [x, y]$, then $\mu(x, y) = \mu(x, z) + \mu(z, y)$, this metric is called the *taxicab metric.*

Let f be an E map and let $Q = \{q_1, q_2, \ldots, q_m\}$ be an invariant subset of E under f such that $\mathbf{0} \in Q$. We assume that there are points of Q in each connected component of $E \setminus \{\mathbf{0}\}$. Let E_Q be the minimal connected subgraph of E containing all the basic intervals associated to Q. Clearly E_Q is homeomorphic to E. We say that f is *Q-linear* if the following conditions hold:

(1) $E_Q = E$; in particular the endpoints of E are points of Q;

(2) for any basic interval J associated to Q, $f(J)$ is an interval formed by the union of basic intervals of Q;

(3) $f_{|J} : J \longrightarrow f(J)$ is a linear homeomorphism with respect to the taxicab metric, *i.e.* $f_{|J}$ is a homeomorphism satisfying that for any $x, y, z \in J$ such that $\mu(x, y) = \mu(x, z) + \mu(z, y)$ we have that

$$\mu\big(f(x), f(y)\big) = \mu\big(f(x), f(z)\big) + \mu\big(f(z), f(y)\big).$$

We say an E map g is a *Q-linearization* of f if the following conditions hold:

(1) $g_{|Q} = f_{|Q}$;

(2) g is Q-linear;

(3) the g-graph of Q is a subgraph of the f-graph of Q.

Suppose that f is an E map having a k-orbit P such that P has points in each connected component of $E \setminus \{\mathbf{0}\}$. Set $P' = P \cup \{\mathbf{0}\}$. Clearly, if $E \in \{\mathbf{O}, \infty, \mathbf{T}\}$ then $E_{P'} = E$. Assume now that $E \notin \{\mathbf{O}, \infty, \mathbf{T}\}$. For each whiskers W of E we consider the endpoint $q \in W$ of E and the point $p \in W$, $p \in P$ such that $(p, q) \cap P = \emptyset$. Let E' be the new topological space obtained by shrinking the interval $[p, q]$ to the point p. Note that E' is homeomorphic to E. We define the E map $h : E' \longrightarrow E'$ by $h(x) = f(x)$ if $f(x) \in E'$ and $h(x) = p$ otherwise. Of course P is a k-orbit for h, $\mathrm{Per}(h) \subset \mathrm{Per}(f)$ and the endpoint of W belongs to P. Therefore we can assume that $E_{P'} = E$. In particular, we can talk about the *P'-linearization of f* in the above way.

In the rest of this section we assume that f is an E map having a k-orbit P and consider the set of basic intervals associated to $P' = P \cup \{\mathbf{0}\}$.

LEMMA 4.1. — *Let K and J be basic intervals and let g be a P'-linearization of f. If $x \in \mathrm{Int}(J)$, $g(x) \neq \mathbf{0}$ and $g(x) \in K$, then J g-covers K.*

Proof. — Let a, b be the endpoints of J. Since J is a basic interval associated to P', its endpoints have image in P' and so $\{f(a), f(b)\} \cap \mathrm{Int}(K) = \emptyset$. By P'-linearity, since $g(x) \in K$, $g(x) \neq \mathbf{0}$ and $x \in \mathrm{Int}(J)$, there exists an interval $L \subset J$ such that $g(L) = K$. So J g-covers K. □

Let J be a basic interval. If $\mathbf{0} \in J$, then J will be called a *branching interval*; otherwise J will be called a *non-branchig interval.*

The following proposition is the converse result of Proposition 3.4 for P'-linear maps.

PROPOSITION 4.2. — *Let g be a P'-linearization of f. If g has an m-point for $m \notin \{1, 2, 3, 4, 5, 6, k\}$, then there exists a non-repetitive loop of length m through the g-graph such that at least one basic interval of the loop does not contain $\mathbf{0}$.*

Proof. — Let x be a periodic point of period m for g. Then $\mathrm{Orb}_g(x) \cap P' = \emptyset$, so for each i, $0 \leq i < m$, there exists a unique basic interval J_i containing $g^i(x)$. Since g is P'-linear, by Lemma 4.1, $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ is a loop of the g-graph. First we shall show that this loop is non-repetitive.

Since g is P'-linear, we can define by backward induction on i, a collection of subintervals K_i of J_i such that $g : K_i \longrightarrow K_{i+1}$ is one-to-one and onto, where $K_m = J_m = J_0$. Suppose now the loop is repetitive, then there exists s, $0 < s < m$, such that s divides m and $J_i = J_{i+s}$ for $0 \leq i \leq m - s$. We take s the smallest number in such a way. We claim that $K_i \subset K_{i+s}$ for $0 \leq i \leq m - s$. To prove the claim consider $K_{m-s} \subset J_{m-s} = J_m = K_m$ and by backward induction, suppose $K_{i+1} \subset K_{i+s+1}$ and $K_i \not\subset K_{i+s}$. So, there is $a \in K_i$ such that $a \notin K_{i+s}$, and $g(a) \in K_{i+1} \subset K_{i+s+1}$. Since $K_{i+s} \to K_{i+s+1}$, there exists $b \in K_{i+s}$ (and so $b \neq a$) such that $g(b) = g(a)$. This is a contradiction with the fact that g is P'-linear and $g_{|J_i}$ is one-to-one. Hence the claim is proved.

Thus $g^s(K_0) = K_s \supset K_0$ and by Lemma 3.2, g^s has a fixed point $y \in K_0$. Since m is divisible by s, $g^m(y) = y$. Note that $x \neq y$ because x has period m, and y has period $s < m$. Hence the map $g^m : K_0 \longrightarrow K_m$ is linear and has at least two fixed points. Therefore $g^m{}_{|K_0}$ must be the identity map and so $K_0 = K_m = J_m = J_0$. Then we get $K_0 = K_s = K_{2s} = \cdots = K_m$ because $K_0 \subset K_s \subset K_{2s} \subset \cdots \subset K_m = K_0$. Now consider the linear map $g^s : K_0 \longrightarrow K_s = K_0$ which has a fixed point. Since $g^s{}_{|K_0}$ is one-to-one and onto, we have two possibilities.

- *Case 1*: $g^s{}_{|K_0} = \text{id}$.

Then $g^s(x) = x$ but x has period $m > s$, a contradiction.

- *Case 2*: $g^s{}_{|K_0} \neq \text{id}$ and $g^{2s}{}_{|K_0} = \text{id}$.

Let $x_0 \in K_0 = J_0$ be a k-point for f such that $\text{Orb}_f(x_0) \subset P$. Then $g^{2s}(x_0) = x_0$. Moreover x_0 is an endpoint of K_0 and so $k = 2s$. On the other hand, since $g^{2s}(x) = x$ and x has period $m > s$ we have $2s = m$. So $k = m$, a contradiction with the hypotheses. In short we have proved that the loop $J_0 \to J_1 \to \cdots \to J_{m-1} \to J_m = J_0$ is non-repetitive.

Suppose that all the basic intervals of the non-repetitive loop of length m contain $\mathbf{0}$. Therefore $\text{Orb}_g(x)$ is contained in the branching intervals. Since $m > 6$, there is a basic interval J_i containing at least two points of $\text{Orb}_g(x)$. Let $u, v \in \text{Orb}_g(x) \cap J_i$ such that $(\mathbf{0}, v) \cap \text{Orb}_g(x) = \emptyset$, and $(u, v) \cap \text{Orb}_g(x) = \emptyset$. Since the loop is non-repetitive, there is $J_j \neq J_i$ such that $J_j \cap \text{Orb}_g(x) \neq \emptyset$. Let $z \in J_j \cap \text{Orb}_f(x)$ such that $(z, \mathbf{0}) \cap P' = \emptyset$. Therefore there is r, $0 < r < m$ such that $g^r(u) = z$ and $g^{m-r}(z) = u$.

On the other hand $g^r|_{[u,\mathbf{0}]}$ is lineal and so $g^r|_{[u,\mathbf{0}]} = [z, \mathbf{0}]$. Furthermore $v \in (u, \mathbf{0})$ and so $g^r(v) \in (z, \mathbf{0})$ in contradiction with the fact that $(z, \mathbf{0}) \cap \mathrm{Orb}_g(x) = \emptyset$. □

COROLLARY 4.3. — *Let g be a P'-linearization of f. If $m \in \mathrm{Per}(g)$ and $m \notin \{2, 3, 4, 5, 6, k, 2k\}$, then $m \in \mathrm{Per}(f)$.*

Proof. — Both E maps f and g have points of periods 1 and k. If $m \notin \{1, 2, 3, 4, 5, 6, k\}$, then by Proposition 4.2 there exists a non-repetitive loop in the g-graph of length m such that at least one of its basic intervals does not contain $\mathbf{0}$. Therefore, since the g-graph of P' is a subgraph of the f-graph of P' and $m \neq 2k$, by Proposition 3.4, f has a periodic point of period m. □

Remark 4.4. — Suppose that f is P'-linear. Then each branching interval f-covers exactly one branching interval, and perhaps some non-branching intervals. Moreover each non-branching interval f-covers either zero or two branching intervals.

5. Preliminary results in $\mathbf{I_2, I_3, I_4, I_5, I_6, O, O_1, O_2, O_3}$ and $\mathbf{O_4}$.

We need to introduce some orderings in the set of natural numbers, adding or removing some few elements.

The *Sharkovskii ordering* $>_s$ on the set $\mathbb{N}_s = \mathbb{N} \cup \{2^\infty\}$ is given by

$$\begin{aligned}
&3 >_s 5 >_s 7 >_s \cdots >_s \\
&2 \cdot 3 >_s 2 \cdot 5 >_s 2 \cdot 7 >_s \cdots >_s \\
&2^2 \cdot 3 >_s 2^2 \cdot 5 >_s 2^2 \cdot 7 >_s \cdots >_s \\
&2^n \cdot 3 >_s 2^n \cdot 5 >_s 2^n \cdot 7 >_s \cdots >_s \\
&2^\infty >_s \cdots >_s 2^n >_s \cdots >_s 2^4 >_s 2^3 >_s 2^2 >_s 2 >_s 1.
\end{aligned}$$

We shall use the symbol $\geq_s$ in the natural way. The symbol 2^∞ ensures the existence of supremum of every subset with respect to the ordering $>_s$. For $n \in \mathbb{N}_s$ we denote

$$S(n) = \{k \in \mathbb{N} : n \geq_s k\}.$$

So

$$S(2^\infty) = \{2^i : i = 0, 1, 2, \ldots\}.$$

Now we state the Sharkovskii Theorem [Sh] (see also [St], [BGMY] and [ALM2]).

THEOREM 5.1 (Interval Theorem).

(a) *If f is an interval map, then* $\mathrm{Per}(f) = S(n)$ *for some* $n \in \mathbb{N}_s$.

(b) *If n is an element of $\mathbb{N}_s$ then there exists an interval map f such that* $\mathrm{Per}(f) = S(n)$.

If we want to get a similar result for the space $\mathbf{Y}$, we need two new orderings. The *green ordering* $>_g$ on $\mathbb{N} \setminus \{2\}$ is given by

$$\begin{aligned}
&5 >_g 8 >_g 4 >_g 11 >_g 14 >_g 7 >_g 17 >_g 20 >_g 10 >_g \cdots >_g \\
&3 \cdot 3 >_g 3 \cdot 5 >_g 3 \cdot 7 >_g \cdots >_g \\
&3 \cdot 2 \cdot 3 >_g 3 \cdot 2 \cdot 5 >_g 3 \cdot 2 \cdot 7 >_g \cdots >_g \\
&3 \cdot 2^2 \cdot 3 >_g 3 \cdot 2^2 \cdot 5 >_g 3 \cdot 2^2 \cdot 7 >_g \cdots >_g \\
&3 \cdot 2^3 >_g 3 \cdot 2^2 >_g 3 \cdot 2 >_g 3 \cdot 1 >_g 1.
\end{aligned}$$

The *red ordering* $>_r$ on $\mathbb{N} \setminus \{2, 4\}$ is given by

$$\begin{aligned}
&7 >_r 10 >_r 5 >_r 13 >_r 16 >_r 8 >_r 19 >_r 22 >_r 11 >_r \cdots >_r \\
&3 \cdot 3 >_r 3 \cdot 5 >_r 3 \cdot 7 >_r \cdots >_r \\
&3 \cdot 2 \cdot 3 >_r 3 \cdot 2 \cdot 5 >_r 3 \cdot 2 \cdot 7 >_r \cdots >_r \\
&3 \cdot 2^2 \cdot 3 >_r 3 \cdot 2^2 \cdot 5 >_r 3 \cdot 2^2 \cdot 7 >_r \cdots >_r \\
&3 \cdot 2^3 >_r 3 \cdot 2^2 >_r 3 \cdot 2 >_r 3 \cdot 1 >_r 1.
\end{aligned}$$

For $n \in \mathbb{N} \setminus \{2\}$ denote

$$G(n) = \{k \in \mathbb{N} : n \geq_g k\},$$

for $n \in \mathbb{N} \setminus \{2, 4\}$ denote

$$R(n) = \{k \in \mathbb{N} : n \geq_r k\}$$

and additionally

$$G(3 \cdot 2^\infty) = R(3 \cdot 2^\infty) = \{1\} \cup \{3n : n \in S(2^\infty)\}.$$

We also denote

$$\mathbb{N}_g = (\mathbb{N} \setminus \{2\}) \cup \{3 \cdot 2^\infty\} \quad \text{and} \quad \mathbb{N}_r = (\mathbb{N} \setminus \{2, 4\}) \cup \{3 \cdot 2^\infty\}.$$

The following theorem is due to Alsedà, Llibre and Misiurewicz [ALM1] for $\mathbf{I}_3$ maps.

THEOREM 5.2 ($\mathbf{I}_3$ Theorem).

(a) *If f is an $\mathbf{I}_3$ map, then* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$ *for some* $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$ *and* $n_r \in \mathbb{N}_r$.

(b) *If* $n_s \in \mathbb{N}_s, n_g \in \mathbb{N}_g$ *and* $n_r \in \mathbb{N}_r$, *then there exists an $\mathbf{I}_3$ map f such that* $\mathrm{Per}(f) = S(n_s) \cup G(n_g) \cup R(n_r)$.

Let $\mathbf{I}_n$ be the *n-od* space define as the set $\{z \in \mathbb{C} : z^n \in [0,1]\}$. In order to obtain a generalization of the Sharkovskii Theorem for $\mathbf{I}_n$ we need to define partial ordering $\leq_n$ for $n \geq 1$. The ordering $\geq_1$ is the ordering $\geq_s$. If $n > 1$ then the ordering $\leq_n$ is defined as follows. Let m, k be positive integers.

- *Case 1:* $k = 1$. Then $m \leq_n k$ if and only if $m = 1$.
- *Case 2:* k is divisible by n. Then $m \leq_n k$ if and only if either $m = 1$ or m is divisible by n and $m/n >_s k/n$.
- *Case 3:* $k > 1$, k not divisible by n. Then $m \leq_n k$ if and only if either $m = 1$, $m = k$, or $m = ik + jn$ for some integers $i \geq 0$, $j \geq 1$.

From the definition we have that $\leq_2$ is the Sharkovskii ordering. A set Z is an *initial segment* of $\leq_n$ if whenever k is an element of Z and $m \leq_n k$, then m also belongs to Z; *i.e.* Z is closed under $\leq_n$ predecessors. The following result of Baldwin [Ba] is a generalization of the Sharkovskii Theorem and the $\mathbf{I}_3$ Theorem for arbitrary continuous self-maps of $\mathbf{I}_n$.

THEOREM 5.3 (n-od Theorem).

(a) *Let f be a continuous self-map of $\mathbf{I}_n$. Then* $\mathrm{Per}(f)$ *is a nonempty union of initial segments of* $\{\leq_p : 1 \leq p \leq n\}$.

(b) *If Z is a nonempty finite union of initial segments of* $\{\leq_p : 1 \leq p \leq n\}$, *then there is a continuous self-map of $\mathbf{I}_n$ f such that* $f(\mathbf{0}) = \mathbf{0}$ *and* $\mathrm{Per}(f) = Z$.

The n-od Theorem has been extended by Baldwin and Llibre in [BL] to continuous maps on a tree having all their branching points fixed.

We define the *Block ordering* $>_0$ on $\mathbb{N}$ as the converse of the natural ordering on $\mathbb{N} \setminus \{1\}$ and we add the 1 as the smallest element; *i.e.* $2 >_0 3 >_0 4 >_0 \cdots >_0 1$. For $n \in \mathbb{N}$, we denote

$$B(n) = \big\{k \in \mathbb{N} : n \geq_0 k\big\}.$$

Sharkovskii Theorem has been generalized by Block to the circle maps having fixed points in [Bc2].

THEOREM 5.4 (Circle Theorem).

(a) *If f is a circle map having fixed points, then* $\mathrm{Per}(f) = S(n_s) \cup B(n_b)$ *for some* $n_s \in \mathbb{N}_s$ *and* $n_b \in \mathbb{N}$.

(b) *If* $n_s \in \mathbb{N}_s$ *and* $n_b \in \mathbb{N}$, *then there exists a circle map f having fixed points such that* $\mathrm{Per}(f) = S(n_s) \cup B(n_b)$.

In [LPR2], [LPR3] the Sharkovskii Theorem has been extended to connected graphs G with zero Euler characteristic having all branching points fixed. Given a graph G, let $e(G)$ and $b(G)$ the number of its endpoints and branching points respectively.

THEOREM 5.5 (Graph Theorem). — *Let G be a connected graph such that $b(G) \neq 0$ and $\chi(G) = 0$.*

(a) *Let $f : G \to G$ be a continuous map with all branching points fixed. Then* $\mathrm{Per}(f)$ *is a nonempty finite union of initial segments of* $\{\leq_p : 0 \leq p \leq e(G) + 2\}$.

(b) *If Z is a nonempty finite union of initial segments of*

$$\{\leq_p : 0 \leq p \leq e(G) + 2\},$$

then there is a continuous map $f : G \to G$ with all the branching points fixed such that $\mathrm{Per}(f) = Z$.

We note that if G is a connected graph such that $\chi(G) = 0$ and $b(G) = 0$, then G is homeomorphic to $\mathbf{O}$. The set of periods for continuous self-maps on $\mathbf{O}$ wich have fixed points is characterized in the Circle Theorem.

6. The full periodicity kernel of $\mathbf{O}_3$.

The objective of this section is to prove Theorem 1.2.

Since $\mathbf{I}_5$ is homeomorphic to $\{z \in \mathbf{O}_3 : \operatorname{Im} z \geq -1\}$, we can consider $\mathbf{I}_5 = \{z \in \mathbf{O}_3 : \operatorname{Im} z \geq -1\}$. Let f an $\mathbf{I}_5$ map. We shall extend f to an $\mathbf{O}_3$ map $\bar{f}$ as follows. We define $\bar{f}(z) = f(z)$ if $z \in \mathbf{I}_5$ and f restricted to $\mathrm{Cl}(\mathbf{O}_3 \setminus \mathbf{I}_5)$ is any homeomorphism between $\mathrm{Cl}(\mathbf{O}_3 \setminus \mathbf{I}_5)$ and the unique closed interval in $\mathbf{I}_5$ having $f(1-i)$ and $f(-1-i)$ as endpoints. Of course $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$. From Theorem 1.1 (d) it follows that $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 16, 17, 18, 21, 23\}$ is a subset of the full periodicity kernel of $\mathbf{O}_3$. Then, to prove Theorem 1.2 it is sufficient to show the following proposition.

PROPOSITION 6.1. — *Let f be an $\mathbf{O}_3$ map. Then the following statements hold*:

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,13,14,16,17,18,21,23,28\} \subset \mathrm{Per}(f).$$

(b) *If* $13 \in \mathrm{Per}(f)$, *then* $28 \in \mathrm{Per}(f)$.

Proof. — Note that $\chi(\mathbf{O}_3) = 0$, $b(\mathbf{O}_3) = 1$ and $e(\mathbf{O}_3) = 3$. From the Graph Theorem, the set of periods of f is a nonempty finite union of initial segments of $\{\leq_p : 0 \leq p \leq 5\}$. Now we shall compute the periods forced by the periods 7 and 13 in the orderings of $\mathbf{O}_3$.

From the definition of the orderings $\leq_p$, we have that

$7 \geq_0 n$ for each $n \in \mathbb{N} \setminus \{2,3,4,5,6\}$;

$7 \geq_s n$ for each $n \in \mathbb{N} \setminus \{3,5\}$;

$7 \geq_3 n$ for each $n \in \mathbb{N} \setminus \{2,4,5,8,11,14\}$;

$7 \geq_4 n$ for each $n \in \mathbb{N} \setminus \{2,3,5,6,9,10,13,14,17,21\}$;

$7 \geq_5 n$ for each $n \in \mathbb{N} \setminus \{2,3,4,6,8,9,11,13,14,16,18,21,23,28\}$.

Therefore, if $7 \in \mathrm{Per}(f)$ then

$$\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,13,14,16,17,18,21,23,28\} \subset \mathrm{Per}(f)$$

and statement (a) holds.

On the other hand, $13 >_p 28$ for $0 \leq p \leq 5$. Consequently if $13 \in \mathrm{Per}(f)$, then $28 \in \mathrm{Per}(f)$ and statement (b) holds. □

7. The full periodicity kernel of $\mathbf{O}_4$.

The objective of this section is to prove Theorem 1.3.

Since $\mathbf{I}_6$ is homeomorphic to $\{z \in \mathbf{O}_4 : \mathrm{Im}\, z \geq -1\}$, we can consider $\mathbf{I}_6 = \{z \in \mathbf{O}_4 : \mathrm{Im}\, z \geq -1\}$. Let f an $\mathbf{I}_6$ map. We shall extend f to an $\mathbf{O}_4$ map $\bar{f}$ as follows. We define $\bar{f}(z) = f(z)$ if $z \in \mathbf{I}_6$ and $\bar{f}$ restricted to $\mathrm{Cl}(\mathbf{O}_4 \setminus \mathbf{I}_6)$ is any homeomorphism between $\mathrm{Cl}(\mathbf{O}_4 \setminus \mathbf{I}_6)$ and the unique closed interval of $\mathbf{I}_6$ having $f(1-i)$ and $f(-1-i)$ as endpoints. Of course $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$. From Theorem 1.1 (e) it follows that $\{2,3,4,5,6,7,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29\}$ is a subset of the full periodicity kernel of $\mathbf{O}_4$. Then, Theorem 1.3 is a corollary of the following proposition.

PROPOSITION 7.1. — *Let f be an $\mathbf{O}_4$ map. Then the following statements hold:*

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\{2,3,4,5,6,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29,35\} \subset \mathrm{Per}(f).$$

(b) *If* $11 \in \mathrm{Per}(f)$, *then* $35 \in \mathrm{Per}(f)$.

Proof. — Since $\chi(\mathbf{O}_4) = 0$, $b(\mathbf{O}_4) = 1$ and $e(\mathbf{O}) = 4$, by the Graph Theorem it follows that $\mathrm{Per}(f)$ is a nonempty union of initial segments of $\{\leq_p\colon 0 \leq p \leq 6\}$. We have that $7 \geq_6 n$ for each $n \in \mathbb{N} \setminus \{2,3,4,5,8,9,10,11,14,15,16,17,21,22,23,28,29,35\}$. Therefore, from the proof of Proposition 6.1 we obtain that if $7 \in \mathrm{Per}(f)$, then $\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29,35\} \subset \mathrm{Per}(f)$ and statement (a) follows.

On the other hand, $11 >_p 35$ for $0 \leq p \leq 6$. Consequently if $11 \in \mathrm{Per}(f)$, then $35 \in \mathrm{Per}(f)$ and statement (b) holds. □

8. The unfolding of ∞_1, ∞_2 and $\mathbf{T}$.

If we identify the endpoints of the segment $[0,1]$ to the point $\mathbf{0}$, then we obtain a space homeomorphic to $\mathbf{O}$.

We represent the cartesian product $\mathbf{O} \times \mathbf{O}$ (the torus) as the square $[0,1] \times [0,1]$ identifying the points $(x,0)$ and $(x,1)$ for all $x \in [0,1]$, and the points $(0,y)$ and $(1,y)$ for all $y \in [0,1]$. Thus the graph of an $\mathbf{O}$ map f is the subset $\{(x,f(x)) : x \in \mathbf{O}\}$ of $\mathbf{O} \times \mathbf{O}$, and it can be represented as in Figure 8.1.

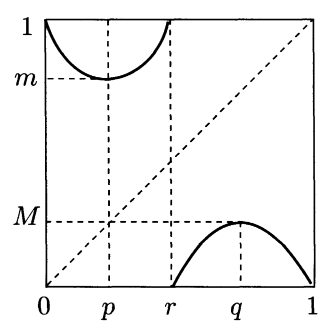

Figure 8.1. The graph of a $\mathbf{O}$ *map.*

Roughly speaking, we think the graph of an $\mathbf{O}$ map like the graph of an interval map g from $[0,1]$ into itself with the above identifications. This allows us to talk about local or absolute maximum or minimum for an $\mathbf{O}$ map in the same way as for interval maps. Thus, for instance, in the points p and q the $\mathbf{O}$ map represented in Figure 8.1 has a local minimum and maximum with values m and M respectively.

Let f be a P'-linear $\mathbf{O}$ map such that $f(\mathbf{0}) = \mathbf{0}$ and each basic interval associated to P' does not f-cover itself. Therefore the graph of f does not touch the diagonal except at $\mathbf{0}$. Let $V = [a, b]$ a closed subinterval of $\mathbf{O}$ such that $f(a) = f(b) = \mathbf{0}$, $f(c) \neq \mathbf{0}$ for all $c \in (a, b)$. Then we say that V is an *upper* (respectively *down*) *subinterval* according with they contain more local minima (respectively maxima) than local maxima (respectively minima) of f. Since f is P'-linear these upper and down subintervals are well–defined. Thus for instance the subinterval $[0, r]$ is an upper subinterval of the map f of Figure 8.1.

In the rest of this section we shall consider $E \in \{\infty_1, \infty_2, \mathbf{T}\}$ *and* f *will be a* P'*-linear map such that each basic interval associated to* P' *does not* f*-cover itself and* k *will be the period of* P*. We identify* $\mathbf{O}$ *with a circle of* E *and* $\mathbf{0} \in \mathbf{O}$ *with* $\mathbf{0} \in E$*.*

Let $V = [a, b]$ be a closed subinterval of E contained in a circle or in a whiskers of E such that $f(a) = f(b) = \mathbf{0}$, $f(c) \neq \mathbf{0}$ for all $c \in (a, b)$ and $f(V) \subsetneq \mathbf{O}$. Then in a similar way as for $\mathbf{O}$ maps, we can say as above that V is an *upper* or *down subinterval*.

Let $V \subset E$ be contained in a whiskers or in a circle of E. We say that V f*-covers* $\mathbf{O}$ (or $V \to \mathbf{O}$, or $\mathbf{O} \leftarrow V$) if one of the following statements holds:

(1) There exists $[a, b] \subset V$ with $f(a) = f(b) = \mathbf{0}$, $f(c) \neq \mathbf{0}$ for all $c \in (a, b)$ and $f([a, b]) = \mathbf{O}$.

(2) The set V is a circle of E such that $f(V) = \mathbf{O}$ and $f(x) \neq \mathbf{0}$ for all $x \neq \mathbf{0}$.

Moreover, if (1) occurs with $V = [a, b]$ or (2) occurs, then we say that V is a *crossing subset* of $\mathbf{O}$. If V does not f-cover $\mathbf{O}$, then we write $V \not\to \mathbf{O}$.

Remark 8.1. — In a similar way as in Lemma 3.1, if K and L are closed subintervals of E such that $L \subset \mathbf{O}$, $K \to \mathbf{O}$ and $\mathbf{0} \notin \mathrm{Int}(L)$, then $K \to L$.

In this section we also assume that E has no crossing subsets of $\mathbf{O}$. Then following ideas of [LPR3] and [LL1] we define the *unfolding* of ∞_1 as the graph $\infty_1^* = G_1 \cup G_2 \cup G_3$ where

$$G_1 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0 \text{ and } |z-i| = 1 \text{ or } z \in \mathbf{I}_2,\ \text{Re } z \geq 0\},$$
$$G_2 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0,\ |z+i| = 1\},$$
$$G_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = |\text{Im } z|,\ |z+i| = 1\}.$$

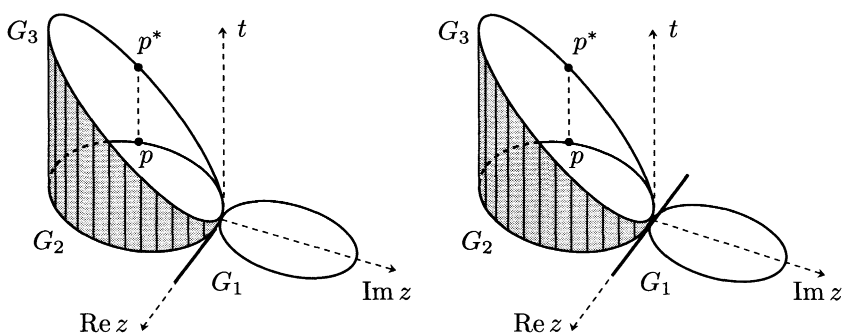

Figure 8.2. The unfoldings of ∞_1 *and* ∞_2.

Define the *unfolding* of ∞_2 as the graph $\infty_2^* = G_1 \cup G_2 \cup G_3$ where

$$G_1 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0 \text{ and } |z-i| = 1 \text{ or } z \in \mathbf{I}_2\},$$
$$G_2 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0,\ |z+i| = 1\},$$
$$G_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = |\text{Im } z|,\ |z+i| = 1\}.$$

Define the *unfolding* of the trefoil as the graph $\mathbf{T}^* = G_1 \cup G_2 \cup G_3$ where

$$G_1 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0,\ z = \cos(3\theta)\, e^{i\theta},\ \tfrac{1}{6}\pi \leq \theta \leq \tfrac{11}{6}\pi\},$$
$$G_2 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = 0,\ z = \cos(3\theta)\, e^{i\theta},\ -\tfrac{1}{6}\pi \leq \theta \leq \tfrac{1}{6}\pi\},$$
$$G_3 = \{(z,t) \in \mathbb{C} \times \mathbb{R} : t = \text{Re } z,\ z = \cos(3\theta)\, e^{i\theta},\ -\tfrac{1}{6}\pi \leq \theta \leq \tfrac{1}{6}\pi\}.$$

Clearly in the three cases G_2 and G_3 are homeomorphic to $\mathbf{O}$, moreover $G_1 \cup G_2$ is homeomorphic to E, so we identify $\mathbf{O}$ with G_2 and $G_1 \cup G_2$ with E (see Figures 8.2 and 8.3). Consider the projection $\pi : E^* \to E$ defined by $\pi(z,t) = (z,0)$. We denote by p^* the unique point of G_3 such that $\pi(p^*) = p$.

Since f is P'-linear, f has finitely many local extrema; and consequently finitely many upper and down subintervals. Moreover from the fact that there are no crossing subsets of $\mathbf{O}$, it follows that there exists a finite «partition» of E into upper subintervals, down subintervals and subintervals with image in $\mathrm{Cl}(E \setminus \mathbf{O})$. Now for the given E map f we define $f^* : E \to E^*$ as follows. If $p \in E$ then $f^*(p)$ is either $f(p)^*$ if $f(p) \in \mathbf{O}$ and p belongs to an upper subinterval, or $f(p)$ otherwise. Clearly f^* is well-defined. We remark that $f = \pi \circ f^* : E \to E$. Define $F = f^* \circ \pi : E^* \to E^*$. In the rest of this section we shall study the relationship between the periods of f and F.

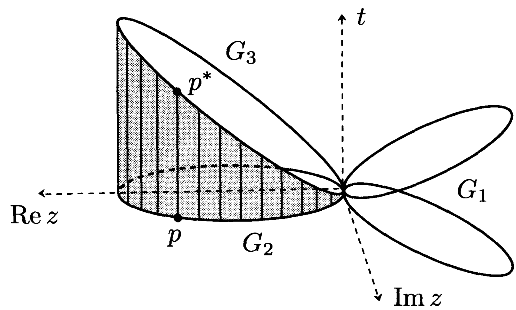

Figure 8.3. The unfolding of the trefoil.

LEMMA 8.2. — *Assume that there are no crossing subsets of* $\mathbf{O}$. *If* $q \in E^*$ *is a periodic point of* F *of period* n, *then* $p = \pi(q)$ *is a fixed point of* f^n.

Proof. — Since

$$q = F^n(q) = (f^* \circ \pi)^n(q) = f^* \circ (\pi \circ f^*)^{n-1} \circ \pi(q) = f^*\big(f^{n-1}(p)\big),$$

we get that $p = \pi(q) = f^n(p)$. □

LEMMA 8.3. — *Assume that there are no crossing subsets of* $\mathbf{O}$. *Then the following statements hold*:

(a) *If* $p = \pi(q)$ *is a periodic point of* f *of period* n, *then* $p = \pi(F^n(q))$.

(b) *Si* $p \in G_1$ *is a periodic point of* f *of period* n, *then* p *is a fixed point of* F^n.

Proof. — Statement (a) follows from the equalities

$$p = \pi(q) = f^n\big(\pi(q)\big) = (\pi \circ f^*)^n\big(\pi(q)\big) = \pi \circ (f^* \circ \pi)^n(q) = (\pi \circ F^n)(q).$$

If p is a periodic point of f of period n, we have that

$$p = f^n(p) = f^n\big(\pi(p)\big) = (\pi \circ f^*)^n\big(\pi(p)\big) = \pi \circ (f^* \circ \pi)^n(p) = (\pi \circ F^n)(p).$$

Since $p \in G_1$, we get that $F^n(p) = p$, and statement (b) is proved. □

PROPOSITION 8.4. — *Suppose that there are no crossing subsets of* **O**. *Then the following statements hold*:

(a) *If q is an n-point for F, then $p = \pi(q)$ is an n-point for f.*

(b) *If p is an n-point for f and $p \in G_1$, then p is an n-point for F.*

Proof. — We prove (a). Let q be an n-point for F. By Lemma 8.2, $p = \pi(q)$ is a fixed point of f^n. Therefore, there is a divisor s of n such that p is an s-point for f. If $s = n$, then we are done. So, assume that $s < n$. By Lemma 8.3 (a), $p = \pi(F^s(q))$. Since $s < n$, $F^s(q) = p'$ with $p' \neq q$, and of course p' belongs to the F-periodic orbit of q. Then

$$\begin{aligned} q &= F^n(q) = (f^* \circ \pi)^n(q) = (f^* \circ \pi)^{n-1} \circ f^*\big(\pi(q)\big) \\ &= (f^* \circ \pi)^{n-1} \circ f^*(p) = (f^* \circ \pi)^{n-1} \circ f^*\big(\pi\big(F^s(q)\big)\big) \\ &= (f^* \circ \pi)^n\big(F^s(q)\big) = F^n(p') = p', \end{aligned}$$

which is a contradiction. Hence $s = n$ and (a) is proved.

Now we show (b). Let p be an n-point for f and $p \in G_1$. By Lemma 8.3 (b), $p = F^n(p)$. Again, there is a divisor s of n such that p is an s-point for F. If $s = n$, then we are done. So, assume that $s < n$. Then $F^s(p) = p$. By Lemma 8.2, since $p \in G_1$ we get that $p = f^s(p)$, a contradiction. Then the lemma follows. □

9. More results in ∞, ∞_1, ∞_2 and T.

Now we add some results for P'-linear maps which we will use for the computation of the full periodicity kernel of ∞_1, ∞_2 and **T**. The following proposition follows from Section 13 of [LL2].

PROPOSITION 9.1. — *Let f be an ∞ map. The following statements hold*:

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\mathrm{Per}(f) \supset \{2,3,4,5,6,8,9,10,11,12,13,14,17,21\}.$$

(b) *If* $11 \in \mathrm{Per}(f)$, *then* $35 \in \mathrm{Per}(f)$.

(c) *If* $13 \in \mathrm{Per}(f)$, *then* $28 \in \mathrm{Per}(f)$.

If U is a finite subset of E, we shall denote by $\mathrm{Card}(U)$ the cardinal of U.

PROPOSITION 9.2. — *Let* $E \in \{\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4, \infty_1, \infty_2, \mathbf{T}\}$. *Let* f *be an* E *map having a* k*-orbit* P. *Suppose that* f *is* P'*-linear. Assume that each basic interval is* f*-covered by some basic interval different from itself and that there is a basic interval* J_0 *such that* $J_0 \to J_0$. *Then* $\{n \in \mathbb{N} : n \geq k+3\} \setminus \{2k\} \subset \mathrm{Per}(f)$.

Proof. — We denote by S the set of all basic intervals associated to P'. Notice that $\mathrm{Card}(S) = k$ if E is any n-star, $\mathrm{Card}(S) = k+1$ if $E \in \{\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$, $\mathrm{Card}(S) = k+2$ if $E \in \{\infty_1, \infty_2\}$, and $\mathrm{Card}(S) = k+3$ if $E = \mathbf{T}$. Since each basic interval is f-covered by some basic interval we get that $f(E) = E$.

Set $K_i = f^i(J_0)$ for $i \geq 0$. Note that each K_i is a connected set and $\mathrm{Card}(K_1 \cap P) \geq 2$. We consider two cases.

- *Case 1*: $E \in \{\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6, \mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4\}$.

From the fact that P is a periodic orbit and $f(E) = E$, it follows that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_r = E$, and $\mathrm{Card}(K_i \cap P) \geq i+1$ for $i < r$. Since P has period k we have that $r \leq \mathrm{Card}(K_{r-1} \cap P) \leq k$. Since each basic interval is f-covered by some basic interval different from itself, for each $J_i \in S$, $J_i \subset K_i \setminus K_{i-1}$ there exists $J_{i-1} \in S$, $J_{i-1} \subset K_{i-1} \setminus K_{i-2}$ such that $J_{i-1} \to J_i$. By hypothesis there exists $M \in S$, $M \neq J_0$ such that $M \to J_0$. Hence there is a loop of length $\ell \leq r+1 \leq k+1$ containing J_0. By construction, this loop is formed by pairwise different basic intervals and so is non-repetitive. The above loop of length ℓ together with the loop $J_0 \to J_0$ give us a non-repetitive loop of length n for each $n \geq k+1$ containing J_0.

We claim that at least one of the intervals of the above loop of length n does not contain $\mathbf{0}$. If $\mathbf{0} \notin J_0$, then we are done. So suppose that $\mathbf{0} \in J_0$. Since $J_0 \to J_0$, $f(\mathbf{0}) = \mathbf{0}$ and f is P'-linear we get that the basic intervals different from J_0 of K_1 do not contain $\mathbf{0}$ (see Remark 4.4). So the claim is proved. Hence by Proposition 3.4 the result follows.

- *Case 2*: $E \in \{\infty_1, \infty_2, \mathbf{T}\}$.

From the facts that P is a periodic orbit, K_i is a connected set and $f(E) = E$, we have that there exists an integer r such that $K_0 \subsetneq K_1 \subsetneq \cdots \subsetneq K_r = E'$, and one of the following statements holds:

(1) $E' = E$;

(2) $E \in \{\infty_1, \infty_2\}$, E' is homeomorphic to one of the spaces $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$, and $E \setminus \mathrm{Int}(E')$ is formed by two basic intervals J_1, J_2 contained in different circles of E such that $J_1 \rightleftarrows J_2$;

(3) $E = \mathbf{T}$, E' is homeomorphic to one of the spaces $\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4$, and $E \setminus \mathrm{Int}(E')$ is formed by two basic intervals J_1, J_2 contained in different circles of E such that $J_1 \rightleftarrows J_2$;

(4) $E = \mathbf{T}$, E' is homeomorphic to one of the spaces $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$, and $E \setminus \mathrm{Int}(E')$ is formed by three basic intervals J_1, J_2, J_3 contained in different circles of E such that $J_1 \to J_2 \to J_3 \to J_1$.

First we suppose that statement (1) holds. We remark that if $r \leq k$, then the result follows as in Case 1. So, since $\mathrm{Card}(S) \in \{k+2, k+3\}$, we can assume that $r \in \{k+1, k+2\}$. In the same way as in Case 1 we obtain a loop of length $\ell \leq r+1 \leq k+3$ containing J_0 and consequently $\{n \in \mathbb{N} : n \geq k+3\} \setminus \{2k\} \subset \mathrm{Per}(f)$.

Finally we assume that statement (2), (3) or (4) holds. Note that $P \subset E'$. Consider the E' map g defined as $g = f_{|E'}$. Clearly g is well-defined because f is P'-linear. Of course g is either an $\mathbf{I}_i$ map for $i = 3, \ldots, 6$, or an $\mathbf{O}_j$ map for $j = 2, 3, 4$. Moreover $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Then the result follows as in Case 1. □

The next lemmas will be used in Sections 10, 11 and 12.

LEMMA 9.3. — *Set $E \in \{\infty_1, \infty_2\}$. Let f be an E map having a k-orbit P. Suppose that f is P'-linear. Then each basic interval J contained in a whiskers of E is f-covered by some basic interval different from itself.*

Proof. — Let $p \neq \mathbf{0}$ be the endpoint of the whiskers of E containing J. Since f is P'-linear, we have that $p \in P$. Moreover, from the facts that $\mathbf{0}$ is a fixed point, f is P'-linear and $f(E)$ is connected, it follows that each basic interval of E contained in the whiskers of E is f-covered by some basic interval.

Suppose that $J \to J$, otherwise we are done. Since $[p, \mathbf{0}]$ is a whiskers of E, we can consider a total ordering $<$ on $[p, \mathbf{0}]$ such that $\mathbf{0}$ is the largest element and p the smallest one. Set $J = [p_j, p_k]$, with $p \leq p_j < p_k \leq \mathbf{0}$. Now, since f is P'-linear we can consider two cases.

- *Case 1:* $p \leq f(p_j) < p_j < p_k$ and $f(p_k) \notin [p, p_k)$.

If there are no basic intervals $K \neq J$ such that $K \to J$, then $f(P \cap [p, p_j]) \subset P \cap [p, p_j]$ with $P \cap [p, p_j] \neq \emptyset$. This is a contradiction because P is a periodic orbit not contained into $[p, \mathbf{0}]$.

- *Case 2:* $p \leq f(p_k) \leq p_j < p_k$ and $f(p_j) \notin [p, p_k)$.

Then $p_k < \mathbf{0}$, and clearly

$$f([p_k, \mathbf{0}]) \supset [f(p_k), f(\mathbf{0})] \supset [f(p_k), \mathbf{0}] \supset [p_j, \mathbf{0}] \supset J.$$

Therefore, there is a basic interval $J_1 \subset [p_k, \mathbf{0}]$ which f-covers J and $J_1 \neq J$. □

LEMMA 9.4. — *Set $E \in \{\infty_1, \infty_2\}$. Let f be an E map having a k-orbit P. Suppose that f is P'-linear. If J_0 is a closed subinterval of E with endpoints elements of P' and contained in a whiskers of E, then there is a loop of length k in the f-graph containing J_0 formed by closed subintervals of E.*

Proof. — Let $J_0 = [x, y]$ with $x, y \in P'$ and $[x, y]$ contained in a whiskers of E. For each i, $0 < i \leq k$, we define J_i recursively as the closed subinterval with endpoints $f^i(x)$ and $f^i(y)$ such that $J_{i-1} \to J_i$. Then $J_k = J_0$ because J_0 is contained in a whiskers. Then we have the loop $J_0 \to J_1 \to \cdots \to J_k = J_0$ of length k. Of course, in general the intervals J_i are not basic and the loop can be repetitive or non-repetitive. □

LEMMA 9.5. — *Set $E \in \{\infty_1, \infty_2, \mathbf{T}\}$. Let f be an E map having a k-orbit P. Suppose that f is P'-linear. Let J and K be basic intervals (eventually $J = K$) such that J f^m-covers K for some $m \geq 1$. Then there is a path of length m starting at J and ending at K.*

Proof. — If $m = 1$ it is trivial. So suppose that $m > 1$. For $1 < i \leq m$, given $J_i \in S$, $J_i \subset f^i(J)$, since f is P'-linear, we can select $J_{i-1} \in S$ such that $J_{i-1} \subset f^{i-1}(J)$ and $J_{i-1} \to J_i$. Then, by induction assumption, the path $J_0 = J \to J_1 \to \cdots \to J_{m-1} \to J_m = K$ proves the lemma. □

LEMMA 9.6. — *Set $E \in \{\infty_1, \infty_2, \mathbf{T}\}$. Let f be an E map having a k-orbit P with $k \in \{7, 11, 13\}$. Suppose that f is P'-linear and that each basic interval associated to P' is f-covered by some different basic interval. Let J and K be basic intervals. Then at least one of the following statements holds:*

(a) *If* $E = \infty_1$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,21,23\} \subset \mathrm{Per}(f).$$

If $E = \infty_2$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29\} \subset \mathrm{Per}(f).$$

If $E = \mathbf{T}$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29\} \subset \mathrm{Per}(f).$$

(b) *There is a path of length* m *starting at* J *and ending at* K, *where* $1 \le m \le k+1$ *if* $E \in \{\infty_1, \infty_2\}$ *and* $1 \le m \le k+2$ *if* $E = \mathbf{T}$.

Proof. — Since each basic interval is f-covered by some basic interval, we get that $f(E) = E$. Set $K_i = f^i(J)$ for $i \ge 0$. Moreover since P is a periodic orbit, there is an integer $r \ge 1$ such that

$$\bigcup_{i=0}^{r} K_i = \bigcup_{i=0}^{r+1} K_i = E' \neq \bigcup_{i=0}^{r-1} K_i\,,$$

and one of the following statements holds:

(1) $E' = E$;

(2) $E \in \{\infty_1, \infty_2\}$, E' is homeomorphic to one of the spaces $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$, and $E \setminus E'$ is formed by two basic intervals J_1, J_2 contained in different circles of E such that $J_1 \rightleftarrows J_2$;

(3) $E = \mathbf{T}$, E' is homeomorphic to one of the spaces $\mathbf{O}_2, \mathbf{O}_3, \mathbf{O}_4$, and $E \setminus E'$ is formed by two basic intervals J_1, J_2 contained in different circles of E such that $J_1 \rightleftarrows J_2$;

(4) $E = \mathbf{T}$, E' is homeomorphic to one of the spaces $\mathbf{I}_3, \mathbf{I}_4, \mathbf{I}_5, \mathbf{I}_6$, and $E \setminus E'$ is formed by three basic intervals J_1, J_2, J_3 contained in different circles of E such that $J_1 \to J_2 \to J_3 \to J_1$.

We denote by S the set of all basic intervals associated to P'. Since $\mathrm{Card}(S) = k+2$ if $E \in \{\infty_1, \infty_2\}$ and $\mathrm{Card}(S) = k+3$ if $E = T$, we get that $r \le k+1$ if $E \in \{\infty_1, \infty_2\}$ and $r \le k+2$ if $E = T$. Clearly $P \subset E'$.

First we suppose that statement (2), (3) or (4) holds. Then we consider the E' map $g = f_{|E'}$. Since $\bigcup_{i=0}^{r} K_i = \bigcup_{i=0}^{r+1} K_i = E'$, g is well-defined. Of course P is a k-orbit for g. Then from the n-odd Theorem and the Graph Theorem statement (a) of Lemma 9.6 holds and we are done.

Finally suppose that $E' = E$. Therefore J f^m-covers K, for some $1 \le m \le k+1$ if $E \in \{\infty_1, \infty_2\}$ and $1 \le m \le k+2$ if $E = T$. Thus by Lemma 9.5 there is a path of length m starting at J and ending at K and statement (b) of Lemma 9.6 follows. □

From now on we shall denote by C_1 and C_2 two different circles of E.

LEMMA 9.7. — *Set $E \in \{\infty_1, \infty_2, \mathbf{T}\}$. Let f be an E map having a k-orbit P with $k \in \{7, 11, 13\}$. Suppose that f is P'-linear and that each basic interval associated to P' is f-covered by some different basic interval. If $C_1 \rightleftarrows C_2$ then at least one of the following statements holds:*

(a) *If $E = \infty_1$, then*

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,21,23\} \subset \mathrm{Per}(f).$$

If $E = \infty_2$, then

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29\} \subset \mathrm{Per}(f).$$

If $E = \mathbf{T}$, then

$$\mathbb{N} \setminus \{2,3,4,5,6,7,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29\} \subset \mathrm{Per}(f).$$

(b) $\mathrm{Per}(f) \supset \{n \in \mathbb{N} : n \ge 2k+1\}$ *if $E \in \{\infty_1, \infty_2\}$.*
$\mathrm{Per}(f) \supset \{n \in \mathbb{N} : n \ge 2k+1 \text{ odd}\} \cup \{n \in \mathbb{N} : n \ge 2k+4 \text{ even}\}$ *if $E = \mathbf{T}$.*

Proof. — By hypotheses we have $C_1 \rightleftarrows C_2$. We claim that there are two basic intervals $L \subset C_1$ and $M \subset C_2$ such that $L \rightleftarrows M$. Now we prove the claim. If there is a 2-orbit $\{x, y\}$ with $x \in C_1$ and $y \in C_2$, then we consider the basic intervals $L \subset C_1$ and $M \subset C_2$ containing x and y respectively. From the linearity of f we get that $L \rightleftarrows M$. Now we suppose that there are no 2-orbits $\{x, y\}$ with $x \in C_1$ and $y \in C_2$. Since k is not even and $C_1 \rightleftarrows C_2$, without loss of generality we can assume that there is a closed subinterval $K \subset C_1$ such that K is a crossing subset of C_2. Let $K_2 \subset C_2$ be a minimal closed subinterval f-covering K. Let $K_1 \subset C_1$ be a minimal closed subinterval f-covering K_2. In particular $K_1 \rightleftarrows K_2$. Since there are no 2-orbits $\{x, y\}$ with $x \in C_1$ and $x \in C_2$, from Lemma 3.3 it follows that $\mathbf{0} \in K_1 \cap K_2$ and the branching intervals $L \subset K_1$, $M \subset K_2$ verify $L \rightleftarrows M$. So the claim is proved.

First we suppose that L or M f^k-covers itself. Without loss of generality we can assume that L f^k-covers L. Then, by Lemma 9.5 there exists a loop of length k containing L. Therefore the above loop of length k together with the loop $L \rightleftarrows M$ give us a loop γ of length n for each $n > k$ odd and each $n \geq 2k + 2$ even. Note that γ is non-repetitive because k is not multiple of 2. We claim that we can construct γ containing some non-branching interval. Now we prove the claim. If $\mathbf{0} \notin L$ or $\mathbf{0} \notin M$, then we are done. So suppose that $\mathbf{0} \in L \cap M$. Then the only branching intervals f-covered by L and M are L and M (see Remark 4.4). Hence γ contains some non-branching interval and the claim is proved. By Proposition 3.4 the result follows.

Now we can assume that L and M do not f^k-cover itself. Thus, since P has period k, we get that L f^k-covers J for each $J \in S$ with $J \subset C_1$ and M f^k-covers J for each $J \in S$ with $J \subset C_2$.

Without loss of generality we have three possibilities for the basic intervals L and M: either $\mathbf{0} \in L \cap M$; or $\mathbf{0} \in L$ and $\mathbf{0} \notin M$; or $\mathbf{0} \notin L \cup M$. If $\mathbf{0} \in L \cap M$, then without loss of generality we can assume that there is a basic interval $M_1 \subset C_2 \setminus \mathrm{Int}(M)$ such that $L \to M_1$. Moreover, since f is P'-linear, $\mathbf{0} \notin M_1$. If $\mathbf{0} \in L$ and $\mathbf{0} \notin M$, then since $f(\mathbf{0}) = \mathbf{0}$ and $L \to M$, we have that there is a basic interval $M_1 \subset C_2 \setminus \mathrm{Int}(M)$ such that $L \to M_1$. Finally, since $k \geq 7$, if $\mathbf{0} \notin L \cup M$, then again we can assume that there is a basic interval $M_1 \subset C_2 \setminus \mathrm{Int}(M)$ such that $L \to M_1$.

In short, we get that there is $M_1 \in S$ such that $M_1 \subset C_2 \setminus \mathrm{Int}(M)$, $L \to M_1$ and $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$. Therefore M f^k-covers M_1. From Lemma 9.5 there is a path $M \to \cdots \to M_1$ of length k. If Lemma 9.6 (a) holds, then Lemma 9.7 (a) follows, and we are done. Otherwise, from Lemma 9.6 (b) we can assume that there is a path $M_1 \to \cdots \to M$ of length $m \leq k + 1$ if $E \in \{\infty_1, \infty_2\}$ and $m \leq k + 2$ if $E = \mathbf{T}$. We can suppose that m is the shortest length of all paths from M_1 to M. This path of length m together with the path $M \to L \to M_1$ and the path $M \to \cdots \to M_1$ of length k give us two loops of lengths $m + 2$ and $k + m$. Note that both loops contain M and M_1.

First suppose that m is odd. Then the loop $M \to L \to M_1 \to \cdots \to M$ of length $m + 2$ and the loop $M \rightleftarrows L$ allow us to construct a non-repetitive loop of length n for each $n \geq k + 2$ odd if $E \in \{\infty_1, \infty_2\}$ and $n \geq k + 4$ odd if $E = \mathbf{T}$. This loop contains M and M_1. On the other hand, the loops $M \to \cdots \to M_1 \to \cdots \to M$ of length $k + m$ and the loop $M \rightleftarrows L$ allow us to construct a non-repetitive loop of length n for each $n \geq 2k + 2$ even if $E \in \{\infty_1, \infty_2\}$ and $n \geq 2k+4$ even if $E = \mathbf{T}$. This loop contains M and M_1. Since $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$, by Proposition 3.4 the result follows.

Finally suppose that m is even. The loop $M \to L \to M_1 \to \cdots \to M$ of length $m+2$ and the loop $M \rightleftarrows L$ give us a non-repetitive loop of length n for each $n \geq k+3$ even containing M and M_1. Moreover, the loop $M \to \cdots \to M_1 \to \cdots \to M$ of length $k+m \leq 2k+1$ and the loop $M \rightleftarrows L$ give us a non-repetitive loop of length n for each $n \geq 2k+1$ odd containing M and M_1. Since $\mathbf{0} \notin M$ or $\mathbf{0} \notin M_1$, from Proposition 3.4, statement (b) of Lemma 9.7 holds and we are done. □

The following lemma will be used in the rest of this section and in Section 12.

LEMMA 9.8. — *Set $E \in \{\infty_1, \infty_2, \mathbf{T}\}$. Let f be an E map having a k-orbit P. Suppose that f is P'-linear. Let $\beta : J = K_0 \to K_1 \to \cdots \to K_r \to J$ and $\gamma : J = M_0 \to M_1 \to \cdots M_s \to J$ be two different loops in the f-graph having a common basic interval J. Then the loop obtained by concatenating β and γ contains some non-branching interval.*

Proof. — If some basic interval of β or γ does not contain $\mathbf{0}$, then we are done. So suppose that all basic intervals of β and γ are branching intervals. From Remark 4.4, J f-covers exactly one branching interval, and perhaps some non-branching interval. Hence $K_1 = M_1$. Since all basic intervals are branching intervals, repeating this argument we get that β and γ are the same loop, in contradiction with the hypotheses. □

The following lemma follows from ideas of [Ba] and [LPR2], [LPR3].

LEMMA 9.9. — *Let $E \in \{\infty_1, \infty_2\}$. Let f be an E map having a k-orbit P with $k \in \{7, 11, 13\}$. Suppose that f is P'-linear. Assume that there are $p_1, \ldots, p_t$ points of P for some $t \in \{3, 4\}$ such that*

$$[\mathbf{0}, p_1] \to [\mathbf{0}, p_2] \to \cdots \to [\mathbf{0}, p_t] \to [\mathbf{0}, p_1],$$

$$(\mathbf{0}, p_i) \cap P = \emptyset \text{ for } i = 1, \ldots, t$$

(i.e. $[\mathbf{0}, p_i]$ is a branching interval), and if $i \neq j$, then $[\mathbf{0}, p_i]$ and $[\mathbf{0}, p_j]$ are contained in different components of $E \setminus \{\mathbf{0}\}$. Then

$$\{n = ki + tj,\ i \geq 0,\ j \geq 1\} \subset \operatorname{Per}(f).$$

Proof. — Without loss of generality we can assume that $[\mathbf{0}, p_1]$ is contained in a whiskers W of E. By Lemma 9.4 there is a loop of length k

containing $[\mathbf{0},p_1]$. This loop together with the loop $[\mathbf{0},p_1] \to [\mathbf{0},p_2] \to \cdots \to [\mathbf{0},p_1]$ of length t give us a loop of length n for each $n = ki + tj$, $i \geq 1$, $j \geq 1$. Since k is not divisible by t and from Lemma 9.8, we get that the loop of length n is non-repetitive and at least one of its intervals does not contain $\mathbf{0}$. Consequently $\{n = ki + tj, i \geq 1, j \geq 1\} \subset \mathrm{Per}(f)$. Now we need to prove that all multiple of t also belongs to $\mathrm{Per}(f)$.

From the hypotheses, $(\mathbf{0},p_{j+1}] \subset f(\mathbf{0},p_j]$ for $0 < j < t$ and $(\mathbf{0},p_1] \subset f(\mathbf{0},p_t]$. Thus $f^j(\mathbf{0},p_1] \subset f^{t+j}(\mathbf{0},p_1]$ for all j. On the other hand, since k is not divisible by t, $f^i(\mathbf{0},p_1]$ contains elements of different components of $E \setminus \{\mathbf{0}\}$ and so there must be an integer i such that $\mathbf{0} \in f^i(\mathbf{0},p_1]$. We fix the least such i. Consider two cases.

- *Case 1: $i > t$.*

Let r be the largest positive integer such that $i > rt$. From the facts that $[\mathbf{0},p_1] \subset W$, W is an interval and f is P'-linear, we have that $f^{(r-1)t}(\mathbf{0},p_1] = (\mathbf{0},u] \subset W$ and $f^{rt}(\mathbf{0},p_1] = f^t(\mathbf{0},u] = (\mathbf{0},v] \subset W$ for some $u,v \in P$ with $u \in (\mathbf{0},v)$. Then there exists $a \in (\mathbf{0},u]$ such that $f^t(a) = v$. Since $\mathbf{0} \in f^i(\mathbf{0},p_1]$ and r is the largest positive integer such that $i > rt$, by the minimality of i it follows that there exists $b \in (\mathbf{0},v]$ such that $f^t(b) = \mathbf{0}$. Note that $b \notin (\mathbf{0},u]$ because $\mathbf{0} \notin f^{rt}(\mathbf{0},p_1] = f^t(\mathbf{0},u]$. Since W is an interval, we get $f^t[\mathbf{0},a] \supset [\mathbf{0},a] \cup [a,b]$, $f^t[a,b] \supset [\mathbf{0},a] \cup [a,b]$ and $f^t(a) \neq a$. Then by well-known results for interval maps (see Proposition 1.2.9 of [ALM2]) we obtain that f^t has points of all periods in $[\mathbf{0},b]$, and consequently f has periodic points of each multiple of t.

- *Case 2: $i \leq t$.*

Since $f^t(\mathbf{0},p_1] \supset [\mathbf{0},p_1]$, there is $a \in (\mathbf{0},p_1)$ such that $f^t(a) = p_1$ and $f^t(y) \neq p_1$ for all $y \in (\mathbf{0},a)$. Moreover, from the linearity of f it follows that $f^t|_{[\mathbf{0},a]}$ is linear. Since $\mathbf{0} \in f^i(\mathbf{0},p_1]$ and $i \leq t$, there exists $b \in (\mathbf{0},p_1)$ such that $f^t(b) = \mathbf{0}$. Note that $b \notin (\mathbf{0},a]$ because f^t is linear in $[\mathbf{0},a]$, $f^t(\mathbf{0}) = \mathbf{0}$ and $f^t(a) = p_1$. Hence $f^t[\mathbf{0},a] \supset [\mathbf{0},a] \cup [a,b]$ and $f^t[a,b] \supset [\mathbf{0},a] \cup [a,b]$. Thus the proof follows as in Case 1. □

LEMMA 9.10. — *Let $E \in \{\infty_1, \infty_2\}$. Let f be an E map having a k-orbit P with $k \in \{7,11,13\}$ such that f is P'-linear. Suppose that there are t closed subintervals $K_1,\dots,K_t$ with $t \in \{2,3,4\}$ such that K_j is contained in the closure of a connected component R_j of $E \setminus \{\mathbf{0}\}$ for each $j = 1,\dots,t$, and if $K_i \neq K_j$ then $R_i \neq R_j$. Assume that R_1 is a whiskers of E. If $K_1 \to \cdots \to K_t \to K_1$, then*

$$\mathrm{Per}(f) \supset \{n = ki + tj : i \geq 1,\ j \geq 1\}.$$

Proof. — If there is a t-orbit $\{x, \ldots, f^{t-1}(x)\}$ such that $x \in K_1, \ldots, f^{t-1}(x) \in K_t$, then let $J_1 \subset K_1, \ldots, J_t \subset K_t$ be the basic intervals containing $x, \ldots, f^{t-1}(x)$ respectively. From the linearity of f we have that $J_1 \to \cdots \to J_t \to J_1$. If there are no t-orbits in $K_1 \cup \cdots \cup K_t$, then by Lemma 3.3 there exist t branching intervals $J_1 \subset K_1, \ldots, J_t \subset K_t$ such that $J_1 \to \cdots \to J_t \to J_1$.

Since J_1 is contained in a whiskers of E, fom Lemma 9.4 there is a loop of length k containing J_1. This loop together with the loop $J_1 \to \cdots \to J_t \to J_1$ give us a loop γ of length n for each $n = ki + tj$ with $i \geq 1$, $j \geq 1$. The loop γ is non-repetitive because k is not divisible by t and by Lemma 9.8 at least one of its intervals does not contain $\mathbf{0}$. Hence by Proposition 3.4 the result holds. □

10. The full periodicity kernel of ∞_1.

The goal of this section is to prove Theorem 1.4.

Since $\mathbf{I}_5$ is homeomorphic to $\{z \in \infty_1 : -1 \leq \text{Im } z \leq 1\}$, we can consider $\mathbf{I}_5 = \{z \in \infty_1 : -1 \leq \text{Im } z \leq 1\}$. Set $A = \{z \in \infty_1 : \text{Im } z \geq 1\}$ and $B = \{z \in \infty_1 : \text{Im } z \leq -1\}$. Let f be a $\mathbf{I}_5$ map. We shall extend f to an ∞_1 map $\bar{f}$ as follows. We define $\bar{f}(z) = f(z)$ if $z \in \mathbf{I}_5$, $\bar{f}_{|A}$ is any homeomorphism between A and the unique closed interval in $\mathbf{I}_5$ having $f(1+i)$ and $f(-1+i)$ as endpoints; and finally $\bar{f}_{|B}$ is any homeomorphism between B and the unique closed interval in $\mathbf{I}_5$ having $f(1-i)$ and $f(-1-i)$ as endpoints. Of course $\text{Per}(f) = \text{Per}(\bar{f})$. From Theorem 1.1 (d), the set $\{2,3,4,5,6, 7,8,9,10,11,13,14,16,17,18,21,23\}$ is a subset of the full periodicity kernel of ∞_1. Then to prove Theorem 1.4 it is sufficient to show the following two propositions.

Proposition 10.1. — *Let f be an ∞_1 map such that*

$$\{2,3,4,5,6,7,8,9,10,11,12,13,14,16,17,18,21,23\} \subset \text{Per}(f).$$

Then $\text{Per}(f) = \mathbb{N}$.

Proposition 10.2. — *There is an ∞_1 map g such that*

$$\text{Per}(g) = \mathbb{N} \setminus \{12\}.$$

Proposition 10.1 will be a corollary of the following proposition.

PROPOSITION 10.3. — *Let f be an ∞_1 map. Then the following statements hold*:

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,12,13,14,16,17,18,21,23,28,35\}.$$

(b) *If* $11 \in \mathrm{Per}(f)$, *then* $35 \in \mathrm{Per}(f)$.

(c) *If* $13 \in \mathrm{Per}(f)$, *then* $28 \in \mathrm{Per}(f)$.

In the rest of this section we fix the ∞_1 map f having a k-orbit P with $k \in \{7, 11, 13\}$ and the set of the basic intervals S associated to P'.

This fixed ∞_1 map will be called the *standard ∞_1 map*.

LEMMA 10.4. — *Let f be the standard ∞_1 map. If the periodic orbit P does not have points in each connected component of $\infty_1 \setminus \{\mathbf{0}\}$ then Proposition* 10.3 *holds.*

Proof. — Let E' be the union of the closures of the connected components of $\infty_1 \setminus \{\mathbf{0}\}$ having points of P. Of course $E' \subset \infty_1$. Then we define the map $g : E' \longrightarrow E'$ as follows. For $z \in E'$, $g(z) = f(z)$ if $f(z) \in E'$; and $g(z) = \mathbf{0}$ otherwise. Notice that g is either an $\mathbf{I}_2$, $\mathbf{O}$, $\mathbf{O}_1$ or ∞ map. Moreover $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence from the Interval Theorem, the Circle Theorem, the Graph Theorem and Proposition 9.1 the result follows. □

Remark 10.5. — From Lemma 10.4 we can assume that the periodic orbit P has points into each connected component of $\infty_1 \setminus \{\mathbf{0}\}$. Furthermore, by Corollary 4.3 in what follows we can suppose that the standard map f will be P'-linear.

LEMMA 10.6. — *Let f be the standard ∞_1 map. Suppose that there is a basic interval J such that no basic interval of $S \setminus \{J\}$ f-covers J. Then Proposition* 10.3 *follows.*

Proof. — By Lemma 9.3 each basic interval of the whiskers of ∞_1 is f-covered by some different basic interval. Therefore J is contained in a circle of ∞_1. Consider the map $g = f_{|\infty_1 \setminus \mathrm{Int}(J)}$. Clearly g is well-defined because f is P'-linear and no basic interval of $S \setminus \{J\}$ f-covers J. Moreover g is either a $\mathbf{O}_2$ or $\mathbf{O}_3$ map such that $\mathrm{Per}(g) = \mathrm{Per}(f)$. Hence from the Graph Theorem the lemma follows. □

Remark 10.7. — From Lemma 10.6 we can assume that each basic interval is f-covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval which f-covers itself, then Proposition 10.3 holds. So, from now on, we can suppose that each basic interval does not f-cover itself.

LEMMA 10.8. — *Let f be the standard ∞_1 map. Identify* $\mathbf{O}$ *with a circle of ∞_1. If there are no crossing subsets of* $\mathbf{O}$, *then Proposition* 10.3 *holds.*

Proof. — With the notation of Section 8, by Proposition 8.4 (b) we have that $k \in \mathrm{Per}(F)$. Since P has elements on each component of $\infty_1 \setminus \{\mathbf{0}\}$ and there are no crossing subsets of $\mathbf{O}$, we get that $F(\infty_1^*)$ is homeomorphic to $\mathbf{O}_2$ or $\mathbf{O}_3$. So from the Graph Theorem we obtain that if $7 \in \mathrm{Per}(F)$, then $\mathrm{Per}(F) \supset \mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,13,14,16,17,18,21,23,28\}$; if $11 \in \mathrm{Per}(F)$, then $35 \in \mathrm{Per}(F)$; and if $13 \in \mathrm{Per}(F)$, then $28 \in \mathrm{Per}(F)$. Now from Proposition 8.4(a) the result follows. □

Remark 10.9. — If there are no subsets of ∞_1 f-covering C_1 or C_2, from Lemma 10.8, Proposition 10.3 holds. So from now on we can assume that there are crossing subsets of C_1 and C_2.

Proof of Proposition 10.3. — Denote by W and p_1 the whiskers and the endpoint of ∞_1 respectively. From Remark 10.7 each basic interval does not f-cover itself. Therefore $f^{k-1}(p_1) \notin W$. Without loss of generality we can asume that $f^{k-1}(p_1) \in C_1$. Moreover, from the fact that $f(\mathbf{0}) = \mathbf{0}$, it follows that there are two closed subintervals M_1, M_2 contained in C_1 such that $\mathbf{0} \notin \mathrm{Int}(M_1)$, $\mathbf{0} \notin \mathrm{Int}(M_2)$, $\mathrm{Int}(M_1) \cap \mathrm{Int}(M_2) = \emptyset$, and $M_1 \to W \leftarrow M_2$.

If $W \to C_1$, then since $\mathbf{0} \notin \mathrm{Int}(M_1)$, from Remark 8.1 $W \rightleftarrows M_1$, and from Lemma 9.10 the result holds. So, in what follows we can suppose that $W \not\to C_1$.

By Remarks 10.7 and 10.9, and since $W \not\to C_1$ we have that $C_2 \to C_1$. If $C_1 \to C_2$ from Lemma 9.7 the result follows. So from now on we can suppose that $C_1 \not\to C_2$.

By Remarks 10.7 and 10.9 we get that $W \to C_2$, then there exist closed subintervals $K_1, K_2 \subset C_2$ such that $W \to K_1 \to M_1 \to W$ and $W \to K_2 \to M_2 \to W$. If $k \in \{11, 13\}$ by Lemma 9.10 the result holds. If $k = 7$, from Lemma 9.10 we have that $\mathrm{Per}(f) \supset \{n = 7i + 3j : i \geq 1, j \geq 1\}$. So we need to prove that $15 \in \mathrm{Per}(f)$. Concatenating the loops $W \to K_1 \to M_1 \to W$ and $W \to K_2 \to M_2 \to W$ we obtain a non-repetitive loop of length $3i$ for $i \geq 1$. If $\mathbf{0} \notin W \cap K_1 \cap K_2 \cap M_1 \cap M_2$ then the result holds. Otherwise, there are three branching intervals $J_1 \subset W, J_2 \subset C_2$ and $J_3 \subset C_1$ such that $J_1 \to J_2 \to J_3 \to J_1$. From Lemma 9.9 the result holds. □

Proof of Proposition 10.2. — Let Z be the union of the following initial segments: $\{n \leq_5 11\}$, $\{n \leq_s 8\}$, $\{n \leq_3 6\}$ and $\{n \leq_5 14\}$. This is $Z = \mathbb{N} \setminus \{7, 9, 12, 13, 17, 18, 22, 23, 28\}$. From the n-od Theorem there exists an $\mathbf{I}_5$ map f_1 such that $\mathrm{Per}(f_1) = Z$. We shall extend f_1 to an ∞_1 map g as follows. As in the beginning of this section, consider $\infty_1 = \mathbf{I}_5 \cup A \cup B$. Let $g(z) = f_1(z)$ if $z \in \mathbf{I}_5$. Then we need to define $g_{|A\cup B}$.

We choose seven points a_i, $i = 1, 2, \ldots, 7$, in $A \cup B$ as follows. We consider A as the union of the following intervals which have pairwise disjoint interiors: $J_1 = [1+i, a_1], J_2 = [a_1, a_3], J_3 = [a_3, a_5], J_4 = [a_5, a_7]$ and $J_5 = [a_7, -1+i]$. Set B as the union of the following intervals which have pairwise disjoint interiors: $J_6 = [1-i, a_2], J_7 = [a_2, a_4], J_8 = [a_4, a_6]$ and $J_9 = [a_6, -1-i]$. Set $P = \{a_i : i = 1, 2, \ldots, 7\}$. Define $g(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, 6$ and $g(a_7) = a_1$. Clearly P is a 7-orbit. Let g be restricted to each J_i the unique linear map with respect to the taxicab metric such that the only elementary loops in the $g_{|A\cup B}$-graph are $J_1 \rightleftarrows J_6$, $J_1 \to J_6 \to J_2 \to J_7 \to J_3 \to J_8 \to J_4 \to J_1$ and $J_1 \to J_6 \to J_2 \to J_7 \to J_3 \to J_8 \to J_4 \to J_9 \to J_5 \to J_1$. Since $\mathbf{I}_5$ is an invariant set under g, from Propositions 3.4 and 4.2 we get that $\mathrm{Per}(g_{|A\cup B}) = \{n \in \mathbb{N} : n \geq 7 \text{ odd}\} \cup \{n \in \mathbb{N} : n \geq 16 \text{ even}\}$. Of course $\mathrm{Per}(g) = \mathrm{Per}(f_1) \cup \mathrm{Per}(g_{|A\cup B}) = \mathbb{N} \setminus \{12\}$. □

11. The full periodicity kernel of ∞_2.

The goal of this section is to prove Theorem 1.5.

Since $\mathbf{I}_6$ is homeomorphic to $\{z \in \infty_2 : -1 \leq \operatorname{Im} z \leq 1\}$, we can consider $\mathbf{I}_6 = \{z \in \infty_2 : -1 \leq \operatorname{Im} z \leq 1\}$. Set $A = \{z \in \infty_2 : \operatorname{Im} z \geq 1\}$ and $B = \{z \in \infty_2 : \operatorname{Im} z \leq -1\}$. Let f be an $\mathbf{I}_6$ map. We shall extend f to an ∞_2 map $\bar{f}$ as follows. We define $\bar{f}(z) = f(z)$ if $z \in \mathbf{I}_6$; $\bar{f}_{|A}$ is any homeomorphism between A and the unique closed interval in $\mathbf{I}_6$ having $f(1+i)$ and $f(-1+i)$ as endpoints; and finally $\bar{f}_{|B}$ is any homeomorphism between B and the unique closed interval of $\mathbf{I}_6$ having $f(1-i)$ and $f(-1-i)$ as endpoints. Of course $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$. From Theorem 1.1 (e), $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29\}$ is a subset of the full periodicity kernel of ∞_2. On the other hand, $\infty_1 \subset \infty_2$. Let

f be an ∞_1 map. We shall extend f to an ∞_2 map as follows. Set $\bar{f}(z) = f(z)$ if $z \in \infty_1$ and $f(z) = \mathbf{0}$ otherwise. Of course $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$. By Theorem 1.4, 12 belongs to the full periodicity of ∞_2. Then, to prove Theorem 1.5 it is sufficient to show the following proposition.

PROPOSITION 11.1. — *Let f be an ∞_2 map. Then the following statements hold*:

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29,35\} \subset \mathrm{Per}(f).$$

(b) *If* $11 \in \mathrm{Per}(f)$, *then* $35 \in \mathrm{Per}(f)$.

In the rest of this section we fix the ∞_2 map f having a k-orbit P with $k \in \{7, 11\}$ and the set of the basic intervals S associated to P'.

This fixed ∞_2 map will be called the *standard ∞_2 map.*

LEMMA 11.2. — *Let f be the standard ∞_2 map. If the periodic orbit P does not have points in each connected component of $\infty_2 \setminus \{\mathbf{0}\}$, then Proposition* 11.1 *holds.*

Proof. — Let E' be the union of the closures of the connected components of $\infty_2 \setminus \{\mathbf{0}\}$ having points of P. Of course $E' \subset \infty_2$. Then we define the map $g : E' \to E'$ as follows. For $z \in E'$, $g(z) = f(z)$ if $f(z) \in E'$; and $g(z) = \mathbf{0}$ otherwise. Notice that g is either a $\mathbf{I}_2, \mathbf{O}, \mathbf{O}_1, \mathbf{O}_2, \infty$ or ∞_1 map. Moreover $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence from the Interval Theorem, the Graph Theorem, Proposition 9.1 and Proposition 10.3 the result follows. □

Remark 11.3. — From Lemma 11.2 we can assume that the periodic orbit P has points in each connected component of $\infty_2 \setminus \{\mathbf{0}\}$. Furthermore, by Corollary 4.3 in what follows we can suppose that the standard map f will be P'-linear.

LEMMA 11.4. — *Let f be the standard ∞_2 map. Suppose that there is a basic interval J such that there are no basic intervals of $S \setminus \{J\}$ f-covering J. Then Proposition* 11.1 *holds.*

Proof. — By Lemma 9.3 we get that J is contained in a circle of ∞_2. Consider the map $g = f_{|\infty_2 \setminus \mathrm{Int}(J)}$. Clearly g is well–defined because f is P'-linear. Moreover g is either an $\mathbf{O}_3$ or $\mathbf{O}_4$ map such that $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence from the Graph Theorem the lemma follows. □

Remark 11.5. — From Lemma 11.4 we can assume that each basic interval is f-covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval f-covering itself, then Proposition 11.1 holds. So, from now on we can assume that each basic interval does not f-cover itself.

LEMMA 11.6. — *Let f be the standard ∞_2 map. Identify $\mathbf{O}$ with a circle of ∞_2. If there are no crossing subsets of $\mathbf{O}$ then Proposition* 11.1 *holds.*

Proof. — With the notation of Section 8, by Proposition 8.4 (b) we have that $k \in \operatorname{Per}(F)$. Since P has elements in each component of $\infty_2 \setminus \{\mathbf{0}\}$ and there are no crossing subsets of $\mathbf{O}$, we get that $F(\infty_2^*)$ is homeomorphic to $\mathbf{O}_3$ or $\mathbf{O}_4$. So from the Graph Theorem we obtain that if $7 \in \operatorname{Per}(F)$, then $\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,13,14,15,16,17,18,21,22,23,28,29,35\} \subset \operatorname{Per}(F)$, and if $11 \in \operatorname{Per}(F)$, then $35 \in \operatorname{Per}(F)$. Now by Proposition 8.4 (a) the result follows. $\square$

Remark 11.7. — If there are no subsets of ∞_2 f-covering C_1 or C_2, from Lemma 11.6, Proposition 11.1 holds. So from now on we can assume that there are crossing subsets of C_1 and C_2.

Proof of Proposition 11.1. — Denote by W_1, W_2 the whiskers of ∞_2 and by $p_1 \in W_1, p_2 \in W_2$ its endpoints. From Remark 11.5 each basic interval does not f-cover itself. Therefore $f^{k-1}(p_1) \notin W_1$ and $f^{k-1}(p_2) \notin W_2$. We consider two cases.

- *Case 1: $f^{k-1}(p_1) \in W_2$.*

Consequently $W_2 \rightarrow W_1$. If $W_1 \rightarrow W_2$, since the whiskers are subintervals of E, from Lemma 9.10 the result follows. Hence, from now on we will assume that $W_1 \not\rightarrow W_2$. In particular $f^{k-1}(p_2) \notin W_1$. Without loss of generality we can suppose that $f^{k-1}(p_2) \in C_1$. Moreover since $f(\mathbf{0}) = \mathbf{0}$, it follows that there are two closed subintervals M_1, M_2 contained in C_1 such that $\operatorname{Int}(M_1) \cap \operatorname{Int}(M_2) = \emptyset, \mathbf{0} \notin \operatorname{Int}(M_1), \mathbf{0} \notin \operatorname{Int}(M_2)$ and $M_1 \rightarrow W_2 \leftarrow M_2$.

If $W_2 \rightarrow C_1$, then from Remark 8.1 $W_2 \rightleftarrows M_1$, and by Lemma 9.10 the result holds. So from now on we will assume that $W_2 \not\rightarrow C_1$.

If $W_1 \rightarrow C_1$, then from Remark 8.1 $W_1 \rightarrow M_1$. Thus we consider the loop $W_1 \rightarrow M_1 \rightarrow W_2 \rightarrow W_1$ and from Lemma 9.10 the result follows. So from now on we can assume that $W_1 \not\rightarrow C_1$.

Since each basic interval does not f-cover itself, $C_1 \nrightarrow C_1$. Then from Remark 11.7 we have that $C_2 \to C_1$. Hence there is a closed subinterval $K \subset C_2$ such that $K \to M_1$. If $W_2 \to C_2$, then we consider the loop $W_2 \to K \to M_1 \to W_2$ and by Lemma 9.10 the result holds. Thus we can assume that $W_2 \nrightarrow C_2$.

If $C_1 \to C_2$, from Lemma 9.7 we are done. So we can assume that $C_1 \nrightarrow C_2$.

By Remarks 11.5 and 11.7 we get that $W_1 \to C_2$. Therefore there are closed subintervals $K_1 \subset W_1$, $K_2 \subset W_2$ and $L_1, L_2 \subset C_2$ such that $K_1 \to L_1 \to M_1 \to K_2 \to K_1$ and $K_1 \to L_2 \to M_2 \to K_2 \to K_1$. If $k = 11$, then statement (b) of Proposition 11.1 follows from Lemma 9.10. If $k = 7$, by Lemma 9.10 we obtain that $\text{Per}(f) \supset \{n = 7i + 4j : i \geq 1, j \geq 1\}$. So we need to prove that $\{20, 24\} \subset \text{Per}(f)$. With the loops $K_1 \to L_1 \to M_1 \to K_2 \to K_1$ and $K_1 \to L_2 \to M_2 \to K_2 \to K_1$ we obtain a non-repetitive loop of length $4i$ for $i \geq 1$. If $\mathbf{0} \notin K_1 \cap K_2 \cap L_1 \cap L_2 \cap M_1 \cap M_2$ then the result holds. Otherwise, there are four branching intervals $J_1 \subset W_1, J_2 \subset C_2, J_3 \subset C_1$ and $J_4 \subset W_2$ such that $J_1 \to J_2 \to J_3 \to J_4 \to J_1$. Therefore statement (a) of Proposition 11.1 follows from Lemma 9.9 and we are done.

- *Case 2:* $f^{k-1}(p_1) \notin W_2$.

Without loss of generality we can assume that $f^{k-1}(p_1) \in C_1$. Moreover, since $f(\mathbf{0}) = \mathbf{0}$, there exist two closed subintervals $M_1, M_2 \subset C_1$ such that $\text{Int}(M_1) \cap \text{Int}(M_2) = \emptyset, \mathbf{0} \notin \text{Int}(M_1)$, $\mathbf{0} \notin \text{Int}(M_2)$ and $M_1 \to W_1 \leftarrow M_2$.

If $W_1 \to C_1$, then from Remark 8.1 we have $W_1 \rightleftarrows M_1$ and the result follows from Lemma 9.10. Hence from now on we can suppose that $W_1 \nrightarrow C_1$.

First assume that $C_2 \to C_1$. Then there is a closed subinterval $K \subset C_2$ such that $K \to M_1$. If $C_1 \to C_2$ the result follows from Lemma 9.7. So we can suppose that $C_1 \nrightarrow C_2$. If $W_1 \to C_2$, then there is a closed subinterval $L \subset W_1$ such that $L \to K \to M_1 \to L$. So from Lemma 9.10 the result follows. Hence we can assume that $W_1 \nrightarrow C_2$. From Remarks 11.5 and 11.7 we obtain that $W_2 \to C_2$.

If $C_1 \to W_2$ or $C_2 \to W_2$, by above arguments the proposition follows. So we can assume that $C_1 \nrightarrow W_2$ and $C_2 \nrightarrow W_1$. In particular $f^{k-1}(p_2) \notin C_1 \cup C_2$. Moreover, since $f^{k-1}(p_2) \notin W_2$ we get that $f^{k-1}(p_2) \in W_1$. Consequently the rest of the proof follows as in Case 1.

Finally assume that $C_2 \nrightarrow C_1$. From Remarks 11.5 and 11.7 we have that $W_2 \rightarrow C_1$. If $C_1 \rightarrow W_2$ or $W_1 \rightarrow W_2$ by above arguments the proposition holds. So, in particular we can assume that $f^{k-1}(p_2) \notin C_1 \cup W_1$. Therefore $f^{k-1}(p_2) \in C_2$. Moreover since $f(\mathbf{0}) = \mathbf{0}$, there are closed subintervals $L_1, L_2 \subset C_2$ such that $L_1 \rightarrow W_2 \leftarrow L_2$, $\mathrm{Int}(L_1) \cap \mathrm{Int}(L_2) = \emptyset, \mathbf{0} \notin \mathrm{Int}(L_1)$ and $\mathbf{0} \notin \mathrm{Int}(L_2)$. If $W_2 \rightarrow C_2$ or $C_1 \rightarrow C_2$ by above arguments the result follows. Hence from Remarks 11.5 and 11.7 we can suppose that $W_1 \rightarrow C_2$. Now the proposition follows as before. $\square$

12. The full periodicity kernel of the trefoil.

The goal of this section is to prove Theorem 1.6.

Let p_1 and p_2 be the two endpoints of ∞_2. Let A be a graph homeomorphic to $\mathbf{I}$ with endpoints p_1 and p_2 such that $\infty_2 \cup A$ is homeomorphic to $\mathbf{T}$. Thus in this section we shall consider $\mathbf{T} = \infty_2 \cup A$. Let f be an ∞_2 map, we shall extend f to a $\mathbf{T}$ map $\bar{f}$ as follows. We define $\bar{f}(z) = f(z)$ if $z \in \infty_2$ and $\bar{f}_{|A}$ is any homeomorphism between A and a closed subinterval in ∞_2 having $f(p_1)$ and $f(p_2)$ as endpoints. Of course $\mathrm{Per}(f) = \mathrm{Per}(\bar{f})$. From Theorem 1.5 $\{2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,21,22,23,28,29\}$ is a subset of the full periodicity kernel of $\mathbf{T}$. Then, to prove Theorem 1.6 it is sufficient to show the following proposition.

PROPOSITION 12.1. — *Let f be a $\mathbf{T}$ map. Then the following statements hold*:

(a) *If* $7 \in \mathrm{Per}(f)$, *then*

$$\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,12,13,14,\\ 15,16,17,18,21,22,23,28,29,35\} \subset \mathrm{Per}(f).$$

(b) *If* $11 \in \mathrm{Per}(f)$, *then* $35 \in \mathrm{Per}(f)$.

In the rest of this section we fix the $\mathbf{T}$ map f having a k-orbit P with $k \in \{7, 11\}$ and the set of the basic intervals S associated to P'.

This fixed $\mathbf{T}$ map will be called the *standard* $\mathbf{T}$ *map*.

LEMMA 12.2. — *Let f be the standard $\mathbf{T}$ map. If the periodic orbit P does not have points in each connected component of $\mathbf{T} \setminus \{\mathbf{0}\}$, then Proposition* 12.1 *holds.*

Proof. — Let E' be the union of the closures of the connected components of $\mathbf{T} \setminus \{\mathbf{0}\}$ having points of P. Of course $E' \subset \mathbf{T}$. We define the map $g : E' \to E'$ as follows. For $z \in E'$, $g(z) = f(z)$ if $f(z) \in E'$; and $g(z) = \mathbf{0}$ otherwise. Notice that g is either an $\mathbf{O}$ or an ∞ map. Moreover $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence from the Circle Theorem and Proposition 9.1 the result follows. □

Remark 12.3. — From Lemma 12.2 we can assume that the periodic orbit P has points into each connected component of $\mathbf{T} \setminus \{\mathbf{0}\}$. Furthermore, by Corollary 4.3 in what follows we can suppose that the standard $\mathbf{T}$ map f will be P'-linear.

LEMMA 12.4. — *Let f be the standard $\mathbf{T}$ map. If there is a basic interval J such that no basic intervals of $S \setminus \{J\}$ f-cover J, then Proposition* 12.1 *holds.*

Proof. — Consider the map $g = f_{|\mathbf{T}\setminus \mathrm{Int}(J)}$. Clearly g is well-defined because f is P'-linear. Moreover g is either an ∞_1 or an ∞_2 map such that $\mathrm{Per}(g) \subset \mathrm{Per}(f)$. Hence from Propositions 10.3 and 11.1 the lemma follows. □

Remark 12.5. — From Lemma 12.4 we can assume that each basic interval is f-covered by some different basic interval. On the other hand, Proposition 9.2 shows that if there exists some basic interval f-covering itself, then Proposition 12.1 holds. So, from now on we can assume that each basic interval does not f-cover itself.

LEMMA 12.6. — *Let f be the standard $\mathbf{T}$ map. Identify $\mathbf{O}$ with a circle of $\mathbf{T}$. If there are no crossing subsets of $\mathbf{O}$, then Proposition* 12.1 *holds.*

Proof. — With the notation of Section 8, by Proposition 8.4 (b) we have that $k \in \mathrm{Per}(F)$. Since P has elements in each component of $\mathbf{T} \setminus \{\mathbf{0}\}$ and there are no crossing subsets of $\mathbf{O}$, we get that $F(\mathbf{T}^*)$ is homeomorphic to ∞_1 or ∞_2. So from Propositions 10.3 and 11.1 we obtain that if $7 \in \mathrm{Per}(F)$, then $\mathbb{N} \setminus \{2,3,4,5,6,8,9,10,11,12,13,14,15,16,17,18,21,22,23,28,29,35\} \subset \mathrm{Per}(F)$, and if $11 \in \mathrm{Per}(F)$, then $35 \in \mathrm{Per}(F)$. Now by Proposition 8.4 (a) the result follows. □

From now on we shall denote by C_1, C_2 and C_3 the three circles of $\mathbf{T}$.

Remark 12.7. — If there are no crossing subsets of C_1, C_2 or C_3, from Lemma 12.6, Proposition 12.1 holds. So from now on we can assume that there are crossing subsets of C_1, C_2 and C_3.

LEMMA 12.8. — *Let f be the standard* $\mathbf{T}$ *map with $k = 11$. Let J_1, J_2, J_3 and J be basic intervals such that $J_i \subset C_i$ for $i = 1,2,3$, $J \subset C_2$ and $J \neq J_2$. Suppose that there are three loops $J_1 \to J_2 \to J_3 \to J_1$, $J_1 \to J \to \overset{m}{\cdots} \to J_1$ and $J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{m}{\cdots} \to J_1$ of lengths 3, $m+1$ and $12+m$ respectively for some $1 \leq m \leq 13$. Then $35 \in \mathrm{Per}(f)$.*

Proof. — In this proof we will use the fact that a loop obtained by concatenating two different loops contains at least one non-branching interval (see Lemma 9.8).

For $m = 1,2,\ldots,13$, we consider the loops $J_1 \to J_2 \to J_3 \to J_1$ and either

$J_1 \rightleftarrows J$; or
$J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{2}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{3}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{4}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{5}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{6}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{7}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{8}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{9}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{10}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J_2 \to \overset{11}{\cdots} \to \cdots \to J \to \overset{11}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{12}{\cdots} \to \cdots \to J_1$; or
$J_1 \to J \to \overset{13}{\cdots} \to \cdots \to J_1$

respectively. Since we can put 35 as $7 \cdot 3 + 7 \cdot 2$, or $7 \cdot 3 + 14$, or $3 + 8 \cdot 4$, or $3 + 16 \cdot 2$, or $6 \cdot 3 + 17$, or $7 \cdot 3 + 2 \cdot 7$, or $3 + 8 \cdot 4$, or $5 \cdot 3 + 20$, or $5 \cdot 3 + 2 \cdot 10$, or $8 \cdot 3 + 11$, or $4 \cdot 3 + 23$, or $3 \cdot 3 + 13 \cdot 2$ or $7 \cdot 3 + 14$, the result follows. □

Proof of Proposition 12.1. — If $C_i \rightleftarrows C_j$ for some $i,j \in \{1,2,3\}$, $i \neq j$, then from Lemma 9.7 the result follows. So from now on we can assume that we do not have $C_i \rightleftarrows C_j$ for $i,j \in \{1,2,3\}$, $i \neq j$.

Therefore, without loss of generality, from Remarks 12.5 and 12.7 we can assume that $C_1 \to C_2 \to C_3 \to C_1$. We claim that there are three basic intervals $J_i \subset C_i$, $i = 1,2,3$ such that $J_1 \to J_2 \to J_3 \to J_1$. Now we prove the claim. If there exists a 3-orbit $\{x,y,z\}$ with $x \in C_1$, $y \in C_2$ and $z \in C_3$, then we consider the basic intervals $J_i \subset C_i$ for $i = 1,2,3$ containing x,y and z respectively. From the linearity of f we get that $J_1 \to J_2 \to J_3 \to J_1$. Now we suppose that there are no 3-orbits $\{x,y,z\}$ with $x \in C_1, y \in C_2$

and $z \in C_3$. Since k is not multiple of 3 and $C_1 \to C_2 \to C_3 \to C_1$, without loss of generality we can assume that there is a closed subinterval $K \subset C_1$ such that K is a crossing subset of C_2. Let $K_3 \subset C_3$ be a minimal closed subinterval f-covering K. Let $K_2 \subset C_2$ be a minimal closed subinterval f-covering K_3. Finally let $K_1 \subset C_1$ be a minimal closed subinterval f-covering K_2. In particular $K_1 \to K_2 \to K_3 \to K_1$. Since there are no 3-orbits $\{x, y, z\}$ with $x \in C_1$, $y \in C_2$ and $z \in C_3$, from Lemma 3.3 it follows that $\mathbf{0} \in K_1 \cap K_2 \cap K_3$ and the branching intervals $J_i \subset K_i$ for $i = 1, 2, 3$ verify $J_1 \to J_2 \to J_3 \to J_1$. Thus the claim is proved.

Denote by γ the loop $J_1 \to J_2 \to J_3 \to J_1$.

First suppose that J_i f^k-covers itself for some $i \in \{1, 2, 3\}$. Thus from Lemma 9.5 there is a loop of length k containing J_i. This loop together with γ give us a non-repetitive loop of length $n = ki + 3j$ for $i \geq 1, j \geq 1$. Since k is no divisible by 3, the loop of length n is non-repetitive. Moreover, from Remark 4.4 at least one of its intervals does not contain $\mathbf{0}$. Hence from Proposition 3.4 the result follows.

Now we can assume that J_i does not f^k-cover itself for $i = 1, 2, 3$. Thus, since P has period k, we get that J_i f^k-covers K for each $K \in S \setminus \{J_i\}$, $K \subset C_i, i = 1, 2, 3$. Since $k \neq 3$, without loss of generality we can assume that $J \leftarrow J_1 \to J_2 \to J_3 \to J_1$ where $J \in S \setminus \{J_2\}$ and $J \subset C_2$. Consequently J_2 f^k-covers J and from Lemma 9.5 there exists a path of length k starting at J_2 and ending at J. On the other hand, from Lemma 9.6 we can suppose that there is a path of length m, $1 \leq m \leq k+2$ starting at J and ending at J_1. Therefore we get the loops γ, $J_1 \to J \to \overset{m}{\cdots} \to \cdots \to J_1$, and $J_1 \to J_2 \to \overset{k}{\cdots} \to \cdots \to J \to \overset{m}{\cdots} \to \cdots \to J_1$ of lengths $3, m+1$ and $k+m+1$ respectively.

If $k = 11$ the result follows from Lemma 12.8. So Proposition 12.1 (b) holds.

From now on we take $k = 7$ and we will prove statement (a) of Proposition 12.1. Denote by $\mathcal{N}$ the set $\mathbb{N} \setminus \{2, 3, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 21, 22, 23, 28, 29, 35\}$. In the rest of this section we will take into account the following facts. If L is a basic interval contained in a circle C, then L f^k-covers L or L f^k-covers M for each $M \in S \setminus \{L\}$, $M \subset C$ because L has endpoints elements of P' and f has period k. Again from Lemma 9.8, if we concatenate two different loops, the new loop contains some non-branching interval.

Concatenating the three loops containing J_1 of lengths 3, $m+1$ and

$k+m+1=8+m$ we get in a similar way to the proof of Lemma 12.8 that if $m \in \{1,3,4,6,7\}$, then $\mathcal{N} \subset \mathrm{Per}(f)$; if $m=2$, then $\mathcal{N} \setminus \{20\} \subset \mathrm{Per}(f)$; if $m=5$, then $\mathcal{N} \setminus \{20,26\} \subset \mathrm{Per}(f)$; if $m=8$, then $\mathcal{N} \setminus \{20,26,32\} \subset \mathrm{Per}(f)$; and if $m=9$, then $\mathcal{N} \setminus \{24\} \subset \mathrm{Per}(f)$.

First we suppose that J f^7-covers J_2. Hence from Lemma 9.5 there is a path $J \to \overset{7}{\cdots} \to \cdots \to J_2$, then we have the loops γ and $J_2 \to \overset{7}{\cdots} \to \cdots \to J \to \overset{7}{\cdots} \to \cdots \to J_2$ of length 3 and 14 respectively. Concatenating these loops, from Lemma 9.8 and Proposition 3.4 we get that $\{20,26,32\} \subset \mathrm{Per}(f)$ and the result follows for $m \in \{2,5,8\}$. Now, if $m=9$, we consider the loops $J \to \overset{m}{\cdots} \to J_1 \to J$ and $J \to \overset{7}{\cdots} \to \cdots \to J_2 \to \overset{7}{\cdots} \to \cdots \to J$ of lengths $m+1=10$ and 14 respectively. In the same way as above we get that $24 \in \mathrm{Per}(f)$ and the proof follows.

Finally we assume that J does not f^7-cover J_2. Thus J f^7-covers itself, and from Lemma 9.5 there is a loop $J \to \overset{7}{\cdots} \to \cdots \to J$. Therefore we get the loops γ and $J_2 \to \overset{7}{\cdots} \to \cdots \to J \to \overset{7}{\cdots} \to \cdots \to J \to \overset{m}{\cdots} \to J_1 \to J_2$ of lengths 3 and $15+m$. This last loop of length $15+m$ will be denoted by β. We note that since J_1 f-covers J_2 and J, from Remark 4.4, $\mathbf{0} \notin J \cap J_2$. In particular β contains some non-branching interval. As before, if $m=2$, then $20 \in \mathrm{Per}(f)$. If $m=5$, then concatenating the loops γ and β we get $26 \in \mathrm{Per}(f)$. Now we show that $20 \in \mathrm{Per}(f)$. Of course we can assume that $m=5$ is the minimal length of all paths from J to J_1 (otherwise we are done). From the subpath $J \to \overset{m}{\cdots} \to J_1$ of β, and by the minimality of m, it follows that β cannot be a repetition of a loop of length smaller or equal than $m=5$. So if β is repetitive, then it is twice a loop of length 10. Therefore the subpath $J \to \overset{7}{\cdots} \to \cdots \to J \to \overset{5}{\cdots} \to \cdots \to J_1$ of β must be $J \to \overset{7}{\cdots} \to \cdots \to J \to \overset{3}{\cdots} \to \cdots \to J \to \overset{2}{\cdots} \to \cdots \to J_1$. Then there is a path of length 2 from J to J_1 in contradiction with the minimality of $m=5$. Hence we obtain that if $m=5$, then $20 \in \mathrm{Per}(f)$. If $m=9$, then β has length 24. We can suppose that $m=9$ is the minimal length of all paths from J to J_1. From the path $J \to \overset{m}{\cdots} \to J_1$ and by the minimality of m we have that β is not a repetition of a loop of length smaller or equal than $m=9$. Thus if β is repetitive, then it is twice a loop of length 12. From the path $J \to \overset{7}{\cdots} \to \cdots \to J \to \overset{9}{\cdots} \to \cdots \to J_1$, we obtain $J \to \overset{7}{\cdots} \to \cdots \to J \to \overset{5}{\cdots} \to \cdots \to J \to \overset{4}{\cdots} \to \cdots \to J_1$. This is a contradiction with the minimality of m. Consequently, if $m=9$, then $24 \in \mathrm{Per}(f)$. If $m=8$, then concatenating γ and β we get $\{26,32\} \subset \mathrm{Per}(f)$.

So in the rest of this proof we shall assume that $m = 8$ and we will show that $20 \in \operatorname{Per}(f)$. We can suppose that m is the shortest length of all paths starting at J and ending at J_1; otherwise we have proved that the result follows.

Let $J \to M_1 \to M_2 \to M_3 \to M_4 \to M_5 \to M_6 \to M_7 \to J_1$ be the path φ from J to J_1 of length $m = 8$. We shall study the basic intervals M_i which form φ. Suppose that φ contains J_2. From the minimality of m and since $J_2 \to J_3 \to J_1$, we get $J_2 \neq M_i$ for $i \in \{1, \dots, 5\}$. If $J_2 = M_6$, then we obtain the loop $J \to \overset{7}{\cdots} \to \cdots \to J \to M_1 \to M_2 \to M_3 \to M_4 \to M_5 \to J_2 \to \overset{7}{\cdots} \to \cdots \to J$ of length 20. By the minimality of m this loop is non-repetitive and the result follows. If $J_2 = M_7$ then consider the loops γ and $J_1 \rightleftarrows J_2$. We note that at least one of the intervals J_1, J_2, J_3 does not contains $\mathbf{0}$, because $J_1 \leftarrow J_2 \to J_3$ and a branching interval f-covers exactly one branching interval (see Remark 4.4). So the result holds. Hence we can suppose that φ does not contain J_2. We remark that φ contains 9 different basic intervals by the minimality of m. Since $\operatorname{Card}(S) = k + 3 = 10$, φ must contain J_3. Again from the minimality of m and since $J_3 \to J_1$, we get $J_3 = M_7$. For $j \in \{1, 2, \dots, 6\}$ consider the basic interval M_j. Since $M_j \in C_i$ for some $i \in \{1, 2, 3\}$, we get the path $J_i \to \overset{7}{\cdots} \to \cdots \to M_j$. If M_j f^7-covers J_i, then we obtain the loop $J_i \to \overset{7}{\cdots} \to \cdots \to M_j \to \overset{7}{\cdots} \to \cdots \to J_i$. This loop of length 14 together with γ give us that $20 \in \operatorname{Per}(f)$. So from now on we can assume that M_j does not f^7-cover J_i and consequently we have $M_j \to \overset{7}{\cdots} \to \cdots \to M_j$ for $j = 1, 2, \dots, 6$.

Suppose that $M_6 \subset C_2$. Then we consider the loops γ, $J_2 \to \overset{7}{\cdots} \to \cdots \to M_6 \to \overset{3}{\cdots} \to \cdots \to J_2$ and $M_6 \to \overset{7}{\cdots} \to \cdots \to M_6$ and the result follows. Suppose that $M_6 \subset C_3$. Then from the loops γ and $J_3 \to \overset{7}{\cdots} \to \cdots \to M_6 \to J_3$ we get that $20 \in \operatorname{Per}(f)$. So we can assume that $M_6 \subset C_1$.

Suppose that $M_5 \subset C_1$. Then we consider the loops γ, $J_1 \to \overset{7}{\cdots} \to \cdots \to M_5 \to \overset{3}{\cdots} \to \cdots \to J_1$ and $M_5 \to \overset{7}{\cdots} \to \cdots \to M_5$ and the result follows. If $M_5 \subset C_2$, then from the loops γ, $J_2 \to \overset{7}{\cdots} \to \cdots \to M_5 \to \overset{4}{\cdots} \to \cdots \to J_2$ the result holds. So we can assume that $M_5 \subset C_3$.

If $M_4 \subset C_1$, then we consider the loops γ and $J_1 \to \overset{7}{\cdots} \to \cdots \to M_4 \to \overset{4}{\cdots} \to \cdots \to J_1$ and the result follows. If $M_4 \subset C_3$, then by the loops γ, $J_3 \to \overset{7}{\cdots} \to \cdots \to M_4 \to \overset{3}{\cdots} \to \cdots \to J_3$ and $M_4 \to \overset{7}{\cdots} \to \cdots \to M_4$ we obtain $20 \in \operatorname{Per}(f)$. Hence we can suppose that $M_4 \subset C_2$.

Suppose that $M_3 \subset C_2$. Then we consider the loops $J_2 \to \overset{7}{\cdots} \to \cdots \to M_3 \to \overset{6}{\cdots} \to \cdots \to J_2$ and $M_3 \to \overset{7}{\cdots} \to \cdots \to M_3$ and we are done. If $M_3 \subset C_3$, then from the loops γ and $J_3 \to \overset{7}{\cdots} \to \cdots \to M_3 \to \overset{4}{\cdots} \to \cdots \to J_3$ the result follows. So we can assume that $M_3 \subset C_1$.

Suppose that $M_2 \subset C_1$. Then we consider the loops $J_1 \to \overset{7}{\cdots} \to \cdots \to M_2 \to \overset{6}{\cdots} \to \cdots \to J_1$ and $M_2 \to \overset{7}{\cdots} \to \cdots \to M_2$ and the result holds. If $M_2 \subset C_2$, then from the loops γ and $J_2 \to \overset{7}{\cdots} \to \cdots \to M_2 \to \overset{7}{\cdots} \to \cdots \to J_2$ the result follows. Therefore we can assume that $M_2 \subset C_3$.

If $M_1 \subset C_1$, then from the loops γ and $J_1 \to \overset{7}{\cdots} \to \cdots \to M_1 \to \overset{7}{\cdots} \to \cdots \to J_1$ we are done. If $M_1 \subset C_3$, then we consider the loops $J_3 \to \overset{7}{\cdots} \to \cdots \to M_1 \to \overset{6}{\cdots} \to \cdots \to J_3$ and $M_1 \to \overset{7}{\cdots} \to \cdots \to M_1$ and the result holds. So we can suppose that $M_1 \subset C_2$.

Therefore the ten basic intervals of S satisfy that $\{J_1, M_3, M_6\} \subset C_1$, $\{J, J_2, M_1, M_4\} \subset C_2$ and $\{J_3, M_2, M_5\} \subset C_3$.

If J f-covers J_1, J_2 or J_3 then we have the loops γ and $J_1 \rightleftarrows J$; γ and $J_1 \to J \to J_2 \to J_3 \to J_1$; or $J \to \overset{7}{\cdots} \to \cdots \to J$ and $J \to J_3 \to J_1 \to J$ according with $J \to J_1$, $J \to J_2$ or $J \to J_3$ respectively. Thus we obtain that $20 \in \mathrm{Per}(f)$ and we are done. Clearly J does not f-cover M_j for $j \in \{2, \ldots, 6\}$ by the minimality of m. Hence we can assume that the only basic interval f-covered by J is M_1.

If M_1 f-covers some interval of $\{J, J_1, J_2, J_3\}$, then in a similar way as before we get that $20 \in \mathrm{Per}(f)$. Again from the minimality of m, M_1 does not f-cover M_j for $j > 2$. So we can assume that the only basic interval f-covered by M_1 is M_2.

If M_2 f-covers some interval of $\{J, J_1, J_2, J_3, M_1\}$, we obtain easily that $20 \in \mathrm{Per}(f)$. Clearly $M_2 \not\twoheadrightarrow M_j$ for $j > 3$. So we can assume that the only basic interval f-covered by M_2 is M_3.

If M_3 f-covers some interval of $\{J, J_1, J_2, J_3, M_1, M_2\}$, we get that $20 \in \mathrm{Per}(f)$ and we are done. By the minimality of m $M_3 \not\twoheadrightarrow M_j$ for $j > 4$. Thus we can suppose that the only basic interval f-covered by M_3 is M_4.

Since J, M_1, M_2 and M_3 f-cover a unique basic interval, namely M_1, M_2, M_3 and M_4 respectively, it follows that if J is a branching (respectively non-branching) interval, then M_1, M_2, M_3 and M_4 are branching (respectively non-branching) intervals; here we are used Remark 4.4. This is a contradiction with the fact that J, M_1 and M_4 are contained in the circle C_2 and C_2 has exactly four basic intervals. Thus statement (a) of Proposition 12.1 holds and we are done. □

13. Upper bounds of the full periodicity kernel.

Blokh proved in [Bk1], [Bk2] the existence of a natural number $L(G)$ such that if a continuous self-map on a graph G verifies that $\{1, 2, \ldots, L(G)\} \subset \text{Per}(f)$ then $\text{Per}(f) = \mathbb{N}$. This result shows that the full periodicity kernel of G is a finite set. Of course, $\{1, 2, \ldots, L(G)\}$ contains the full periodicity kernel of G.

We consider the trefoil and its proper subspaces and we shall compare the Blokh bound $L(E)$ with the best upper bound of the full periodicity kernel, which follows from Theorems 1.1, 1.2, 1.3, 1.4, 1.5 and 1.6. See Table 13.1. We note that $L(E)$ is fairly good for the n-star but it is too much large for the other spaces.

14. About the topological entropy.

The *topological entropy* of a graph map f is a non-negative real number $h(f)$ associated to f which increases with the complexity of f. For a definition and main properties see [ALM2].

Llibre and Misiurewicz [LM] obtain the next result for continuous self-maps of graphs.

THEOREM 14.1. — *Let f be a continuous self-map of a graph. Then the following statements are equivalent:*

(a) $h(f) > 0$.

(b) *There is $m \in \mathbb{N}$ such that $\{m \cdot n : n \in \mathbb{N}\} \subset \text{Per}(f)$.*

We have the following result for our spaces as a corollary of the above theorem.

COROLLARY 14.2. — *Let K_E be the full periodicity kernel of E. Let f be an E map. If $K_E \subset \text{Per}(f)$ then $h(f) > 0$.*

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E	best upper bound	$L(E)$
$\mathbf{I}_2$	3	8
$\mathbf{I}_3$	7	24
$\mathbf{I}_4$	11	32
$\mathbf{I}_5$	23	60
$\mathbf{I}_6$	29	72
$\mathbf{O}$	3	153 548 648
$\mathbf{O}_1$	7	2 643 549 795
$\mathbf{O}_2$	11	43 419 841 302
$\mathbf{O}_3$	23	571 949 175 609
$\mathbf{O}_4$	29	2 650 538 105 490
∞	11	43 419 841 302
∞_1	23	571 949 175 609
∞_2	29	2 650 538 105 490
$\mathbf{T}$	29	2 650 538 105 490

Table 13.1. *Upper bounds for the full periodicity kernel of E.*

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