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Frank Sottile

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PIERI'S FORMULA FOR FLAG MANIFOLDS AND SCHUBERT POLYNOMIALS

by Frank SOTTILE

1. Introduction.

Schubert polynomials had their origins in the study of the cohomology of flag manifolds by Bernstein-Gelfand-Gelfand [3] and Demazure [7]. They were later defined by Lascoux and Schützenberger [17], who developed a purely combinatorial theory.

For each permutation w in the symmetric group S_n there is a Schubert polynomial $\mathfrak{S}_w$ in the variables $x_1, \ldots, x_{n-1}$. When evaluated at certain Chern classes, a Schubert polynomial gives the cohomology class of a Schubert subvariety of the manifold of complete flags in $\mathbb{C}^n$. In this way, the collection $\{\mathfrak{S}_w \mid w \in S_n\}$ of Schubert polynomials determines a basis for the integral cohomology of the flag manifold. Thus there exist integer structure constants c^w_{uv} defined by the identity

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c^w_{uv} \mathfrak{S}_w.$$

No combinatorial formula is known, or even conjectured, for these constants. There are, however, a few special cases in which they are known.

One important case is Monk's formula [21], which characterizes the algebra of Schubert polynomials. While this is usually attributed

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to Monk, Chevalley simultaneously established the analogous formula for generalized flag manifolds in a manuscript that was only recently published [6]. Let s_k be the transposition interchanging k and $k+1$. Then $\mathfrak{S}_{s_k} = x_1 + \cdots + x_k = s_1(x_1, \ldots, x_k)$, the first elementary symmetric polynomial. For any permutation $w \in S_n$, Monk's formula states

$$\mathfrak{S}_w \cdot \mathfrak{S}_{s_k} = \mathfrak{S}_w \cdot s_1(x_1, \ldots, x_k) = \sum \mathfrak{S}_{wt_{a\,b}},$$

where t_{ab} is the transposition interchanging a and b, and the sum is over all $a \leq k < b$ where $w(a) < w(b)$ and if $a < c < b$, then $w(c)$ is not between $w(a)$ and $w(b)$.

The classical Pieri's formula computes the product of a Schur polynomial by either a complete or an elementary symmetric polynomial. Our main result is a formula for Schubert polynomials and the cohomology of flag manifolds which generalizes both Monk's formula and the classical Pieri's formula.

Let $s_m(x_1, \ldots, x_k)$ and $s_{1^m}(x_1, \ldots, x_k)$ be respectively the complete and elementary symmetric polynomials of degree m in the variables $x_1, \ldots, x_k$. When evaluated at certain Chern classes, they become the cohomology classes of special Schubert varieties. Let $\ell(w)$ be the length of a permutation w. We will show:

THEOREM 1. — *Let k, m, n be positive integers, and let $w \in S_n$.*

1. $\mathfrak{S}_w \cdot s_m(x_1, \ldots, x_k) = \sum_v \mathfrak{S}_v$, *the sum over all $v = wt_{a_1\,b_1} \cdots t_{a_m\,b_m}$, where $a_i \leq k < b_i$ and $\ell(wt_{a_1\,b_1} \cdots t_{a_i\,b_i}) = \ell(w) + i$ for $1 \leq i \leq m$ with the integers $b_1, \ldots, b_m$ distinct.*

2. $\mathfrak{S}_w \cdot s_{1^m}(x_1, \ldots, x_k) = \sum_v \mathfrak{S}_v$, *the sum over all v as in 1, except that now the integers $a_1, \ldots, a_m$ are distinct.*

Theorem 1 computes some of the structure constants in the cohomology ring of the flag manifold. If n is taken large enough, equivalently, if the index of summation is over $v \in S_{n+m}$, then these cohomological formulas become identites among the Schubert polynomials.

These formulas were stated in a different form by Lascoux and Schützenberger in [17], where an algebraic proof was outlined. They were later independently conjectured in yet another form by Bergeron and Billey [2]. Our formulation facilitates our proofs. Using geometry, we expose a surprising connection to the classical Pieri's formula (Lemma 11), from which we deduce Theorem 1. In Theorem 5 this connection is used to determine additional structure constants. Theorem 8 utilizes the formulas of Theorem 1 to give a formula for the multiplication of a Schubert polynomial by a hook Schur polynomial, indicating a relation between multiplication of Schubert polynomials and paths in the Bruhat order on S_n. This is exploited in Corollary 9 to deduce an enumerative result about the Bruhat order on S_n.

This exposition is organized as follows: Section 2 contains preliminaries about Schubert polynomials while Section 3 is devoted to the flag manifold. In Section 4 we deduce our main results from a geometric lemma proven in Section 5. We remark that while our results are stated in terms of the integral cohomology of the complex manifold of complete flags, our results and proofs are valid for the Chow rings of flag varieties defined over any field.

We would like to thank Nantel Bergeron and Sara Billey for suggesting these problems and Jean-Yves Thibon for showing us the work of Lascoux and Schützenberger.

2. Schubert polynomials.

In [3], [7] cohomology classes of Schubert subvarieties of the flag manifold were obtained from the class of a point using repeated correspondences in $\mathbb{P}^1$-bundles, which may be described algebraically as "divided differences." Subsequently, Lascoux and Schützenberger [17] found explicit polynomial representatives for these classes. We outline Lascoux and Schützenberger's construction of Schubert polynomials. For a more complete account, see [20].

For an integer $n > 0$, let S_n be the group of permutations of $[n] = \{1, 2, \ldots, n\}$. Let $t_{a\,b}$ be the transposition interchanging $a < b$. Adjacent transpositions $s_i = t_{i\,i+1}$ generate S_n. The length, $\ell(w)$, of a permutation w is characterized by $\ell(wt_{a\,b}) = \ell(w)+1$ if and only if $w(a) < w(b)$ and whenever $a < c < b$, either $w(c) < w(a)$ or $w(b) < w(c)$.

For each integer $n > 1$, let $R_n = \mathbb{Z}[x_1, \ldots, x_n]$. The group S_n acts on R_n by permuting the variables. For $f \in R_n$, the polynomial $f - s_i f$ is antisymmetric in x_i and x_{i+1}, and so is divisible by $x_i - x_{i+1}$. Thus we may define the linear divided difference operator

$$\partial_i = (x_i - x_{i+1})^{-1}(1 - s_i).$$

If $w = s_{a_1} s_{a_2} \cdots s_{a_p}$ is a factorization of w into adjacent transpositions with minimal length ($p = \ell(w)$), then the composition of divided differences $\partial_{a_1} \circ \cdots \circ \partial_{a_p}$ depends only upon w, defining an operator ∂_w for each $w \in S_n$. Let w_0 be the longest permutation in S_n, that is $w_0(j) = n+1-j$. For $w \in S_n$, define the *Schubert polynomial* $\mathfrak{S}_w$ by

$$\mathfrak{S}_w = \partial_{w^{-1}w_0} \left(x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \right).$$

The degree of ∂_i is -1, so $\mathfrak{S}_w$ is homogeneous of degree $\binom{n}{2} - \ell(w^{-1}w_0) = \ell(w)$.

Let $\mathcal{S} \subset R_n$ be the ideal generated by the non-constant symmetric polynomials. The set $\{\mathfrak{S}_w \mid w \in S_n\}$ of Schubert polynomials is a basis for $\mathbb{Z}\{x_1^{i_1} \cdots x_{n-1}^{i_{n-1}} \mid i_j \leq n-j\}$, a transversal to $\mathcal{S}$ in R_n. Thus Schubert polynomials are explicit polynomial representatives of an integral basis for the ring $H_n = R_n/\mathcal{S}$.

Recently, other descriptions have been discovered for Schubert polynomials [1], [4], [10], [11]. One may define Schubert polynomials $\mathfrak{S}_w$ for all $w \in S_\infty = \bigcup_{n=1}^{\infty} S_n$. Then $\{\mathfrak{S}_w \mid w \in S_\infty\}$ is an integral basis for the polynomial ring $\mathbb{Z}[x_1, x_2, \ldots]$. While our methods involve cohomology calculations and so are *a priori* valid only in the rings H_n, they imply identities among Schubert polynomials in the ring $\mathbb{Z}[x_1, x_2, \ldots]$.

A *partition* λ is a decreasing sequence $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ of positive integers, called the *parts* of λ. Given a partition λ with at most k parts, there is a Schur polynomial $s_\lambda = s_\lambda(x_1, \ldots, x_k)$, which is symmetric in the variables $x_1, \ldots, x_k$. For a more complete treatment of Schur polynomials, see [19].

The collection of Schur polynomials forms a basis for the ring of symmetric polynomials, $\mathbb{Z}[x_1, \ldots, x_k]^{S_k}$. The Littlewood-Richardson rule is a formula for the structure constants $c^\lambda_{\mu\nu}$ for this basis, called *Littlewood-Richardson coefficients*, which are defined by the identity

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu} \, s_\lambda.$$

If λ and μ are partitions satisfying $\lambda_i \geq \mu_i$ for all i, we write $\lambda \supset \mu$. This defines a partial order on the collection of partitions, called Young's lattice. Since $c^\lambda_{\mu\nu} = 0$ unless $\lambda \supset \mu$ and $\lambda \supset \nu$ (*cf.* [19]), we see that $\mathcal{I}_{n,k} = \{s_\lambda \mid \lambda_1 \geq n-k\}$ is an ideal. Let $A_{n,k}$ be the quotient ring $\mathbb{Z}[x_1, \ldots, x_k]^{S_k}/\mathcal{I}_{n,k}$.

To a partition λ we may associate its *Young diagram*, also denoted λ, which is a left-justified array of boxes in the plane with λ_i boxes in the ith row. If $\lambda \supset \mu$, then the Young diagram of μ is a subset of that of λ, and the skew diagram λ/μ is the set theoretic difference $\lambda - \mu$. If each column of λ/μ is either empty or a single box, then λ/μ is a *skew row* of *length* m, where m is the number of boxes in λ/μ. The transpose μ^t of a partition μ is the partition whose Young diagram is the transpose of that of μ. We call the transpose of a skew row a *skew column*. The map defined by $s_\lambda \mapsto s_{\lambda^t}$ is a ring isomorphism $A_{n,k} \to A_{n,n-k}$.

If w has only one *descent* (k such that $w(k) > w(k+1)$), then w is said to be *Grassmannian* of descent k and $\mathfrak{S}_w$ is the Schur polynomial $s_\lambda(x_1, \ldots, x_k)$. Here λ is the *shape* of w, the partition with k parts where $\lambda_{k+1-j} = w(j)-j$. For integers k, m define $r[k,m]$ and $c[k,m]$ to be the Grassmannian permutations of descent k with shapes $(m, 0, \ldots, 0) = \underline{m}$ and $(1^m, 0, \ldots, 0) = 1^m$, respectively. These are the $m+1$-cycles

$$r[k,m] = (k{+}m\ k{+}m{-}1\ \ldots\ k{+}2\ k{+}1\ k)$$

$$c[k,m] = (k{-}m{+}1\ k{-}m{+}2\ \ldots\ k{-}1\ k\ k{+}1).$$

3. The flag manifold.

Let V be an n-dimensional complex vector space. A *flag* $F_\bullet$ in V is a sequence

$$\{0\} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = V,$$

of linear subspaces with $\dim_{\mathbb{C}} F_i = i$. The set of all flags is a $\frac{1}{2}n(n-1)$ dimensional complex manifold, called the flag manifold and denoted $\mathbb{F}(V)$. Over $\mathbb{F}(V)$, there is a tautological flag $\mathcal{F}_\bullet$ of bundles whose fibre at a point $F_\bullet$ is the flag $F_\bullet$. Let $-x_i$ be the Chern class of the line bundle $\mathcal{F}_i/\mathcal{F}_{i-1}$. Then the integral cohomology ring of $\mathbb{F}(V)$ is $H_n = \mathbb{Z}[x_1, \ldots, x_n]/\mathcal{S}$, where $\mathcal{S}$ is the ideal generated by those non-constant polynomials which are symmetric in $x_1, \ldots, x_n$. This description is due to Borel [5].

Given a subset $S \subset V$, let $\langle S \rangle$ be its linear span and for linear subspaces $W \subset U$ let $U - W$ be their set theoretic difference. An ordered basis $f_1, f_2, \ldots, f_n$ for V determines a flag $E_\bullet$; set $E_i = \langle f_1, \ldots, f_i \rangle$ for $1 \le i \le n$. In this case, write $E_\bullet = \langle f_1, \ldots, f_n \rangle$. A fixed flag $F_\bullet$ gives a decomposition due to Ehresmann [9] of $\mathbb{F}(V)$ into affine cells indexed by permutations w of S_n. The cell determined by w is

$$X_w^\circ F_\bullet = \{E_\bullet = \langle f_1, \ldots, f_n \rangle \mid f_i \in F_{n+1-w(i)} - F_{n-w(i)},\ 1 \le i \le n\}.$$

The complex codimension of $X_w^\circ F_\bullet$ is $\ell(w)$ and its closure is the Schubert subvariety $X_w F_\bullet$. Thus the cohomology ring of $\mathbb{F}(V)$ has an integral basis given by the cohomology classes[(1)] $[X_w F_\bullet]$, called *Schubert classes*, of the Schubert subvarieties.

Independently, Bernstein-Gelfand-Gelfand [3] and Demazure [7] related this description to Borel's, showing $[X_w F_\bullet] = \partial_{w^{-1}w_0}[X_{w_0} F_\bullet]$. Later, Lascoux and Schützenberger [17] defined Schubert polynomials, and since $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ equals $[X_{w_0}F_\bullet]$, the class of a point, showed that $[X_w F_\bullet] = \mathfrak{S}_w(x_1,\ldots,x_n)$. We adopt the convention of writing $\mathfrak{S}_w$ for the Schubert class $[X_w F_\bullet]$. Since the composition

$$\mathbb{Z}[x_1,\ldots,x_n] \hookrightarrow \mathbb{Z}[x_1,\ldots,x_{n+m}] \twoheadrightarrow H_{n+m}$$

is an isomorphism in low degrees, one may deduce identities of Schubert polynomials from product formulas for Schubert classes.

This Schubert basis for cohomology diagonalizes the intersection pairing; if $\ell(w)+\ell(v) = \dim \mathbb{F}(V) = \frac{1}{2}n(n-1)$, then

$$\mathfrak{S}_w \cdot \mathfrak{S}_v = \begin{cases} \mathfrak{S}_{w_0} & \text{if } v = w_0 w \\ 0 & \text{otherwise.} \end{cases}$$

For each $k \leq n = \dim V$, the set of all k-dimensional subspaces of V is a $k(n-k)$ dimensional complex manifold, called the Grassmannian of k-planes in V, written $G_k V$. A fixed flag $F_\bullet$ gives a decomposition of $G_k V$ into cells indexed by partitions λ with k parts, none exceeding $n-k$. The closure of such a cell is the Schubert variety

$$\Omega_\lambda F_\bullet = \left\{H \in G_k V \mid \dim H \bigcap F_{n-k+j-\lambda_j} \geq j \text{ for } 1 \leq j \leq k\right\},$$

whose codimension is $\lambda_1 + \cdots + \lambda_k = |\lambda|$.

The evaluation of a symmetric polynomial in k variables at the Chern roots $x_1,\ldots,x_n$ of the dual of the tautological k-plane bundle on $G_k V$ identifies $H^* G_k V$ with the ring $A_{n,k}$ of §2. The classes $[\Omega_\lambda F_\bullet]$ form a basis for the cohomology ring of $G_k V$ and $[\Omega_\lambda F_\bullet]$ is $s_\lambda(x_1,\ldots,x_k)$. We will write s_λ for the Schubert class $[\Omega_\lambda F_\bullet]$.

If $Y \subset V$ has codimension d, then $G_k Y \subset G_k V$ is a Schubert subvariety whose indexing partition is d^k, the partition with k parts each equal to d. It follows that $\Omega_{(n-k)^k} F_\bullet = \{F_k\}$, so $s_{(n-k)^k}$ is the class of a point.

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(1) Strictly speaking, we mean the classes Poincaré dual to the fundamental cycles in homology.

The Schubert basis diagonalizes the intersection pairing; for a partition λ, let λ^c be the partition $(n-k-\lambda_k,\ldots,n-k-\lambda_1)$. If $|\mu|+|\lambda| = k(n-k)$, then

$$s_\lambda \cdot s_\mu = \begin{cases} s_{(n-k)^k} & \text{if } \lambda^c = \mu \\ 0 & \text{otherwise.} \end{cases}$$

The Schur pólynomial s_m is the complete symmetric polynomial of degree m in $x_1,\ldots,x_k$. The Schur polynomial s_{1^m} is the mth elementary symmetric polynomial in $x_1,\ldots,x_k$. Pieri's formula is a formula for multiplying Schur polynomials by either s_m or s_{1^m}. For s_m, suppose $|\mu| + |\lambda^c| + m = k(n-k)$, then

$$s_\mu \cdot s_{\lambda^c} \cdot s_m = \begin{cases} s_{(n-k)^k} & \text{if } \lambda/\mu \text{ is a skew row of length } m \\ 0 & \text{otherwise.} \end{cases}$$

For $k \leq n$, the association $E_\bullet \mapsto E_k$ defines a map $\pi : \mathbb{F}(V) \to G_kV$. The functorial map π^* on cohomology is induced by the inclusion into H_n of polynomials symmetric in $x_1,\ldots,x_k$. That is, $A_{n,k} \hookrightarrow H_n$. If λ is a partition with k parts and w the Grassmannian permutation of descent k and shape λ, then $\pi^* s_\lambda = \mathfrak{S}_w$.

Under the Poincaré duality isomorphism between homology and cohomology groups, the functorial map π_* on homology induces a group homomorphism π_* on cohomology. While π_* is not a ring homomorphism, it does satisfy the projection formula (see Example 8.17 of [12]):

$$\pi_*(\alpha \cdot \pi^*\beta) = (\pi_*\alpha) \cdot \beta,$$

where α is a cohomology class on $\mathbb{F}(V)$ and β is a cohomology class on G_kV.

4. Pieri's formula for flag manifolds.

An open problem is to find the analog of the Littlewood-Richardson rule for Schubert polynomials. That is, determine the structure constants c^u_{wv} for the Schubert basis of the cohomology of flag manifolds, which are defined by the identity

$$\mathfrak{S}_w \cdot \mathfrak{S}_v = \sum_u c^u_{w\,v} \mathfrak{S}_u.$$

These constants are positive integers as they count the points in a suitable triple intersection of Schubert subvarieties. They are are known only in some special cases.

For example, if both w and v are Grassmannian permutations of descent k so that $\mathfrak{S}_w$ and $\mathfrak{S}_v$ are pullbacks of classes from G_kV, then the classical Littlewood-Richardson rule gives a formula for the c^w_{uv}'s.

Another case is Monk's formula, which states:

$$\mathfrak{S}_w \cdot \mathfrak{S}_{s_k} = \sum \mathfrak{S}_{wt_{a\,b}},$$

the sum over all $a \leq k < b$ with $\ell(wt_{a\,b}) = \ell(w) + 1$. We use geometry to generalize this formula, giving an analog of the classical Pieri's formula.

Let $w, v \in S_n$. Write $w \xrightarrow{r[k,m]} v$ if there exist integers $a_1, b_1, \ldots, a_m, b_m$ with

1. $v = wt_{a_1\,b_1} \cdots t_{a_m\,b_m}$,
2. $a_i \leq k < b_i$ and $\ell(wt_{a_1\,b_1} \cdots t_{a_i\,b_i}) = \ell(w) + i$ for $1 \leq i \leq m$, and
3. the integers $b_1, b_2, \ldots, b_m$ are distinct.

Similarly, $w \xrightarrow{c[k,m]} v$ if we have integers $a_1, \ldots, b_m$ as in (1) and (2) where now

(3)′ the integers $a_1, a_2, \ldots, a_m$ are distinct.

Our primary result is the following.

THEOREM 1. — *Let $w \in S_n$. Then*

1. *For all k and m with $k+m \leq n$, we have* $\mathfrak{S}_w \cdot \mathfrak{S}_{r[k,m]} = \sum\limits_{w \xrightarrow{r[k,m]} v} \mathfrak{S}_v$.
2. *For all $m \leq k \leq n$, we have* $\mathfrak{S}_w \cdot \mathfrak{S}_{c[k,m]} = \sum\limits_{w \xrightarrow{c[k,m]} v} \mathfrak{S}_v$.

Theorem 1 may be alternatively stated in terms of the structure constants $c^u_{w\,v}$.

THEOREM 1′. — *Let $w, v \in S_n$. Then*

1. *For all integers k, m with $k + m \leq n$,* $c^v_{w\,r[k,m]} = \begin{cases} 1 & \textit{if } w \xrightarrow{r[k,m]} v \\ 0 & \textit{otherwise.} \end{cases}$
2. *For all integers k, m with $m \leq k \leq n$,* $c^v_{w\,c[k,m]} = \begin{cases} 1 & \textit{if } w \xrightarrow{c[k,m]} v \\ 0 & \textit{otherwise.} \end{cases}$

We first show the equivalence of parts 1 and 2 and then establish part 1. An order $<_k$ on S_n is introduced, and we show that $c^v_{w\,r[k,m]}$ is 0 unless $w <_k v$. A geometric lemma enables us to compute $c^v_{w\, r[k,m]}$ when $w <_k v$.

LEMMA 2. — *Let w_0 be the longest permutation in S_n, and $k+m \leq n$. Then*

1. $w_0 r[k,m] w_0 = c[n-k,m]$.
2. *Let $w, v \in S_n$. Then $w \xrightarrow{r[k,m]} v$ if and only if $w_0 w w_0 \xrightarrow{c[n-k,m]} w_0 v w_0$.*
3. *The map induced by $\mathfrak{S}_w \mapsto \mathfrak{S}_{w_0 w w_0}$ is an automorphism of H_n.*
4. *Statements* 1 *and* 2 *of Theorem 1′ are equivalent.*

This automorphism $\mathfrak{S}_w \mapsto \mathfrak{S}_{w_0 w w_0}$ is the analog of the map $s_\lambda(x_1, \ldots, x_k) \mapsto s_{\lambda^t}(x_1, \ldots, x_{n-k})$ for Grassmannians.

Proof. — Statements (1) and (2) are easily verified, as $w_0(j) = n+1-j$.

Statement (3) is also immediate, as $\mathfrak{S}_w \mapsto \mathfrak{S}_{w_0 w w_0}$ leaves Monk's formula invariant and Monk's formula characterizes the algebra of Schubert polynomials.

For (4), suppose $k+m \leq n$ and $w, v \in S_n$ and let $\overline{w}$ denote $w_0 w w_0$. The isomorphism $\mathfrak{S}_v \mapsto \mathfrak{S}_{\overline{v}}$ of (3) shows $c^v_{w\, r[k,m]} = c^{\overline{v}}_{\overline{w}\, \overline{r[k,m]}}$. Part (1) shows $c^{\overline{v}}_{\overline{w}\, \overline{r[k,m]}} = c^{\overline{v}}_{\overline{w}\, c[n-k,m]}$. Then (2) shows the equality of the two statements of Theorem 1′. □

Let $<_k$ be the transitive closure of the relation given by $w <_k w t_{a\, b}$ where $a \leq k < b$ and $\ell(w\, t_{a\, b}) = \ell(w)+1$. We call $<_k$ the *k-Bruhat order*, in [18] it is the k-colored Ehresmanoëdre.

LEMMA 3. — *If $c^v_{w\, r[k,m]} \neq 0$, then $w <_k v$ and $\ell(v) = \ell(w) + m$.*

Proof. — By Monk's formula, $w <_k v$ if and only if $\mathfrak{S}_v$ appears with a non-zero (necessarily positive) coefficient when $\mathfrak{S}_w (\mathfrak{S}_{t_{k\, k+1}})^{\ell(v)-\ell(w)}$ is written as a sum of Schubert classes.

Since $r[k,m] = t_{k\, k+1} \cdot t_{k\, k+2} \cdots t_{k\, k+m}$, Monk's formula shows that $\mathfrak{S}_{r[k,m]}$ is a summand of $(\mathfrak{S}_{s_k})^m$ with coefficient 1. Thus the coefficient of $\mathfrak{S}_v$ in the expansion of $\mathfrak{S}_w \cdot (\mathfrak{S}_{s_k})^m$ exceeds that of $\mathfrak{S}_v$ in $\mathfrak{S}_w \cdot \mathfrak{S}_{r[k,m]}$. Hence $c^v_{w\, r[k,m]} = 0$ unless $w <_k v$ and $\ell(v) = \ell(w) + m$. □

In Section 5 we use geometry to prove the following lemma.

LEMMA 4. — *Let $w <_k v$ be permutations in S_n. Suppose $v = wt_{a_1 b_1} \cdots t_{a_m b_m}$, where $a_i \le k < b_i$ and $\ell(wt_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(w) + i$ for $1 \le i \le m$. Let $d = n - k - \#\{b_1, \ldots, b_m\}$. Then*

1. *There is a cohomology class δ on $G_k V$ such that $\pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0 v}) = \delta \cdot s_{d^k}$.*

2. *If $w \xrightarrow{r[k,m]} v$, then there are partitions $\lambda \supset \mu$ where λ/μ is a skew row of length m whose jth row has length $\#\{i \,|\, a_i = j\}$ and $\pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0 v}) = s_\mu \cdot s_{\lambda^c} = \sum_\nu c^\lambda_{\mu\,\nu} s_{\nu^c}$.*

We first use this to compute some structure constants. For ν a partition with k parts, let $w(\nu)$ be the Grassmannian permutation of descent k and shape ν.

THEOREM 5. — *Let $w, v \in S_n$ and $k \le n$ be an integer. Suppose $w \le_k v$ and $\ell(v) = \ell(w) + m$. Let $a_1, b_1, \ldots, a_m, b_m$ be such that $v = wt_{a_1 b_1} \cdots t_{a_m b_m}$ where $a_i \le k < b_i$ and $\ell(wt_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(w) + i$ for $1 \le i \le m$. Let ν be a partition with k parts.*

1. *If $w \xrightarrow{r[k,m]} v$, the structure constant $c^v_{w\,w(\nu)}$ equals the Littlewood-Richardson coefficient $c^\lambda_{\mu\nu}$, where λ/μ is a skew row of length m whose jth row has length $\#\{i \,|\, a_i = j\}$.*

2. *If $w \xrightarrow{c[k,m]} v$, the structure constant $c^v_{w\,w(\nu)}$ equals the Littlewood-Richardson coefficient $c^\lambda_{\mu\,\nu}$, where λ/μ is a skew column of length m whose jth column has length $\#\{i \,|\, b_i = j\}$.*

Proof. — Using the involution $\mathfrak{S}_w \mapsto \mathfrak{S}_{w_0 w w_0}$, it suffices to prove part (1). Recall that $\mathfrak{S}_{w(\nu)} = \pi^*(s_\nu)$. As $\mathfrak{S}_{w_0}$ and $s_{(n-k)^k}$ are the classes of points, $\pi_* \mathfrak{S}_{w_0} = s_{(n-k)^k}$. By the projection formula and part (2) of Lemma 4,

$$\begin{aligned} c^v_{w\,w(\nu)}\, s_{(n-k)^k} = \pi_*(c^v_{w\,w(\nu)}\, \mathfrak{S}_{w_0}) &= \pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0 v} \cdot \mathfrak{S}_{w(\nu)}) \\ &= \pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0 v}) \cdot s_\nu \\ &= \left(\sum_\kappa c^\lambda_{\mu\,\kappa} s_{\kappa^c} \right) \cdot s_\nu \\ &= c^\lambda_{\mu\nu}\, s_{(n-k)^k}. \end{aligned}$$

□

Proof of Theorem 1′. — By Lemma 3, we need only show that if

$w <_k v$ and $\ell(v) - \ell(w) = m$, then

$$c^v_{w\,r[k,m]} = \begin{cases} 1 & \text{if } w \xrightarrow{r[k,m]} v \\ 0 & \text{otherwise.} \end{cases}$$

Begin by multiplying the identity $\mathfrak{S}_w \cdot \mathfrak{S}_{r[k,m]} = \sum\limits_v c^v_{w\,r[k,m]}\,\mathfrak{S}_v$ by $\mathfrak{S}_{w_0\,v}$ and use the intersection pairing to obtain:

$$\mathfrak{S}_w \cdot \mathfrak{S}_{w_0\,v} \cdot \mathfrak{S}_{r[k,m]} = c^v_{w\,r[k,m]}\,\mathfrak{S}_{w_0}.$$

Recall that $\mathfrak{S}_{r[k,m]} = \pi^* s_m(x_1, \ldots, x_k)$. Apply the map π_* and then the projection formula to obtain:

$$\begin{aligned} \pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0\,v} \cdot \pi^* s_m) &= c^v_{w\,r[k,m]}\,\pi_*(\mathfrak{S}_{w_0}) \\ \pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0\,v}) \cdot s_m &= c^v_{w\,r[k,m]}\,s_{(n-k)^k}. \end{aligned}$$

By part (1) of Lemma 4, there is a cohomology class δ on $G_k V$ with

$$\pi_*(\mathfrak{S}_w \cdot \mathfrak{S}_{w_0\,v}) \cdot s_m = \delta \cdot s_{d^k} \cdot s_m$$

But $s_{d^k} \cdot s_m = 0$ unless $d + m \leq n - k$. Since $d = n - k - \#\{b_1, \ldots, b_m\} \geq n - k - m$, we see that $c^v_{w\,r[k,m]} = 0$ unless $m = \#\{b_1, \ldots, b_m\}$, which implies $w \xrightarrow{r[k,m]} v$.

To complete the proof of Theorem 1′, suppose that $w \xrightarrow{r[k,m]} v$. By part (1) of Theorem 5, $c^v_{w\,r[k,m]} = c^\lambda_{\mu\,\underline{m}}$, where λ/μ a skew row of length m and $\underline{m} = (m, 0, \ldots, 0)$. But this equals 1 by the classical Pieri's formula for the Grassmannian. □

The formulas of Theorem 1 may be formulated as the sum over certain paths in the k-Bruhat order. We explain this formulation here. A (directed) path in the k-Bruhat order from w to v is equivalent to a choice of integers $a_1, b_1, \ldots, a_m, b_m$ with $a_i \leq k < b_i$ for $1 \leq i \leq m$ and if $w^{(0)} = w$ and $w^{(i)} = w^{(i-1)} \cdot t_{a_i\,b_i}$, then $\ell(w^{(i)}) = \ell(w) + i$ and $w^{(m)} = v$. Here, the path is

$$w = w^{(0)} <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)} = v.$$

LEMMA 6. — *Let $w, v \in S_n$ and k, m be positive integers. Then*

1. $w \xrightarrow{r[k,m]} v$ *if and only if there is a path in the k-Bruhat order of length m such that*

$$w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m).$$

2. $w \xrightarrow{c[k,m]} v$ *if and only if there is a path in the k-Bruhat order of length m such that*

$$w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m).$$

Furthermore, these paths are unique.

Proof. — If $w \xrightarrow{r[k,m]} v$, one may show that the set of values $\{w^{(i)}(a_i)\}$ and the set of transpositions $\{t_{a_i\, b_i}\}$ depend only upon w and v, and not on the particular path chosen from w to v in the k-Bruhat order.

It is also the case that rearranging the set $\{w^{(i)}(a_i)\}$ in order, as in (1), may be accomplished by interchanging transpositions $t_{a_i\, b_i}$ and $t_{a_j\, b_j}$ where $a_i \neq a_j$ (necessarily $b_i \neq b_j$). Both (1) and the uniqueness of this representation follow from these observations. Statement (2) follows for similar reasons. □

For a path γ in the k-Bruhat order, let end(γ) be the endpoint of γ. We state a reformulation of Theorem 1.

COROLLARY 7 (Path formulation of Theorem 1). — *Let $w \in S_n$.*

1. $\mathfrak{S}_w \cdot \mathfrak{S}_{r[k,m]} = \sum_\gamma \mathfrak{S}_{\mathrm{end}(\gamma)}$, *the sum over all paths γ in the k-Bruhat order which start at w such that*

$$w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m),$$

where γ is the path $w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)}$.

Equivalently, $c^v_{w\, r[k,m]}$ counts the number of paths γ in the k-Bruhat order from w to v such that

$$w^{(1)}(a_1) < w^{(2)}(a_2) < \cdots < w^{(m)}(a_m).$$

2. $\mathfrak{S}_w \cdot \mathfrak{S}_{c[k,m]} = \sum_\gamma \mathfrak{S}_{\mathrm{end}(\gamma)}$, *the sum over all paths γ in the k-Bruhat order which start at w such that*

$$w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m),$$

where γ is the path $w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)}$.

Equivalently, $c^v_{w\, r[k,m]}$ counts the number of paths γ in the k-Bruhat order from w to v such that

$$w^{(1)}(a_1) > w^{(2)}(a_2) > \cdots > w^{(m)}(a_m).$$

This is the form of the conjectures of Bergeron and Billey [2], and it exposes a link between multiplying Schubert polynomials and paths in the Bruhat order. Such a link is not unexpected. The Littlewood-Richardson rule for multiplying Schur functions may be expressed as a sum over certain paths in Young's lattice of partitions. A connection between paths in the Bruhat order and the intersection theory of Schubert varieties is described in [14]. We believe the eventual description of the structure constants c^w_{uv} will be in terms of counting paths of certain types in the Bruhat order on S_n, and that there will be appropriate generalizations for the other classical groups. This should yield new enumerative results about the Bruhat orders on their respective Weyl groups, in the spirit of Corollary 9 below.

Using multiset notation for partitions, $(p, 1^{q-1})$ is the hook shape partition whose Young diagram is the union of a row of length p and a column of length q. Define $h[k; p, q]$ to be the Grassmannian permutation of descent k and shape $(p, 1^{q-1})$. Then $\mathfrak{S}_{h[k;\,p,q]} = \pi^* s_{(p,1^{q-1})}$. This permutation, $h[k; p, q]$, is the $p+q$-cycle

$$(k{-}q{+}1\ k{-}q{+}2\ \ldots\ k{-}1\ k\ k{+}p\ k{+}p{-}1\ \ldots\ k{+}1).$$

THEOREM 8. — *Let $q \leq k$ and $k{+}p \leq n$ be integers. Set $m = p{+}q{-}1$. For $w \in S_n$,*

$$\mathfrak{S}_w \cdot \mathfrak{S}_{h[k;\,p,q]} = \sum \mathfrak{S}_{\mathrm{end}(\gamma)},$$

the sum over all paths $\gamma : w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)}$ in the k-Bruhat order with

$$w^{(1)}(a_1) < \cdots < w^{(p)}(a_p) \text{ and } w^{(p)}(a_p) > w^{(p+1)}(a_{p+1}) > \cdots > w^{(m)}(a_m).$$

Alternatively, the sum over those paths γ with

$$w^{(1)}(a_1) > \cdots > w^{(q)}(a_q) \quad\text{and}\quad w^{(q)}(a_q) < \cdots < w^{(m)}(a_m).$$

Setting either $p = 1$ or $q = 1$, we recover Theorem 1. If we consider the coefficient $c^v_{w\,h[k;\,p,q]}$ of $\mathfrak{S}_v$ in the product $\mathfrak{S}_w \cdot \mathfrak{S}_{h[k;\,p,q]}$, we obtain:

COROLLARY 9. — *Let $w, v \in S_n$, and p, q be positive integers where $\ell(v) - \ell(w) = p + q - 1 = m$. Then the number of paths $w <_k w^{(1)} <_k w^{(2)} <_k \cdots <_k w^{(m)} = v$ in the k-Bruhat order from w to v with*

$$w^{(1)}(a_1) < \cdots < w^{(p)}(a_p) \quad\text{and}\quad w^{(p)}(a_p) > w^{(p+1)}(a_{p+1}) > \cdots > w^{(m)}(a_m)$$

equals the number of paths with

$$w^{(1)}(a_1) > \cdots > w^{(q)}(a_q) \quad\text{and}\quad w^{(q)}(a_q) < \cdots < w^{(m)}(a_m).$$

Proof of Theorem 8. — By the classical Pieri's formula,

$$s_p \cdot s_{1^{(q-1)}} = s_{(p+1,1^{q-2})} + s_{(p,1^{q-1})}.$$

Expressing these as Schubert classes on the flag manifold (applying π^*), we have:

$$\mathfrak{S}_{r[k,p]} \cdot \mathfrak{S}_{c[k,q-1]} = \mathfrak{S}_{h[k;\,p+1,q-1]} + \mathfrak{S}_{h[k;\,p,q]}.$$

Induction on either p or q (with m fixed) and Corollary 7 completes the proof. □

5. Geometry of intersections.

We deduce Lemma 4 by studying certain intersections of Schubert varieties. A key fact we use is that if $X_wF_\bullet$ and $X_vG_\bullet$ intersect generically transversally, then

$$\Big[X_wF_\bullet \bigcap X_vG_\bullet\Big] = [X_wF_\bullet] \cdot [X_vG_\bullet\Big] = \mathfrak{S}_w \cdot \mathfrak{S}_v$$

in the cohomology ring. Flags $F_\bullet$ and $G_\bullet$ are *opposite* if for $1 \leq i \leq n$, $F_i + G_{n-i} = V$. The set of pairs of opposite flags form the dense orbit of the general linear group $GL(V)$ acting on the space of all pairs of flags. Using this observation and Kleiman's Theorem concerning the transversality of a general translate [16], we conclude that for any $w, v \in S_n$ and opposite flags $F_\bullet$ and $G_\bullet$, $X_wF_\bullet$ and $X_vG_\bullet$ intersect generically transversally.

Deodhar [8] studies the intersection of two Schubert cells

$$X^\circ_w F_\bullet \bigcap X^\circ_{w_0 v} G_\bullet.$$

He shows the intersection is non-empty precisely when $w \leq v$ in the (ordinary) Bruhat order. In this case, that intersection is decomposed into locally closed subvarieties $D_{\underline{\sigma}}$, each isomorphic to $(\mathbb{C}^\times)^a \times \mathbb{C}^b$, where $\underline{\sigma}$ runs over certain subexpressions of reduced words of v, with a and b satisfying $\ell(v) - \ell(w) = a + 2b$, and with a unique index $\underline{\sigma}'$ with $b = 0$. It follows that $X_wF_\bullet \bigcap X_{w_0 v}G_\bullet$ is irreducible with a dense subset $D_{\underline{\sigma}'} \simeq (\mathbb{C}^\times)^{\ell(v)-\ell(w)}$.

These facts hold for the Schubert subvarieties of G_kV as well. Namely, if λ and μ are any partitions with $\mu \subset \lambda$ and $F_\bullet$ and $G_\bullet$ are opposite flags, then $\Omega_\mu F_\bullet \bigcap \Omega_{\lambda^c} G_\bullet$ is an irreducible, generically transverse intersection containing a dense subset isomorphic to $(\mathbb{C}^\times)^{|\lambda|-|\mu|}$.

Let $F_\bullet$ and $F_\bullet{}'$ be opposite flags in V. Let $e_1, \ldots, e_n$ be a basis for V such that e_i generates the one dimensional subspace $F_{n+1-i} \bigcap F'_i$. We deduce Lemma 4 from the following two results.

LEMMA 10. — *Let $w, v \in S_n$ with $w <_k v$ and $\ell(v) - \ell(w) = m$. Suppose that $v = wt_{a_1 b_1} \cdots t_{a_m b_m}$ with $a_i \leq k < b_i$ and $\ell(wt_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(w) + i$ for $1 \leq i \leq m$. Let $\pi : \mathbb{F}(V) \to G_k V$ be the canonical projection. Define $Y = \langle e_{w(j)} \,|\, j \leq k \text{ or } w(j) \neq v(j) \rangle$. Then Y has codimension $d = n - k - \#\{b_1, \ldots, b_m\}$ and*

$$\pi\left(X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}'\right) \subset G_k Y.$$

Also, if $E_\bullet \in X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}'$, then there exist a basis $f_1, \ldots, f_n$ for V with $E_\bullet = \langle f_1, \ldots, f_n \rangle$, where, if $j > k$ with $w(j) = v(j)$, then $f_j = e_{w(j)}$.

LEMMA 11. — *Let $w, v \in S_n$ with $w \xrightarrow{r[k,m]} v$ and let $a_1, \ldots, b_m$ be as in the statement of Lemma 10. Then there exist opposite flags $G_\bullet$ and $G_\bullet{}'$ and partitions $\lambda \supset \mu$, with λ/μ a skew row of length m whose jth row has length $\#\{i \,|\, a_i = j\}$ such that*

$$\pi(X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}') = \Omega_\mu G_\bullet \bigcap \Omega_{\lambda^c} G_\bullet{}',$$

and the map $\pi|_{X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}'} : X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}' \to \Omega_\mu G_\bullet \bigcap \Omega_{\lambda^c} G_\bullet{}'$ has degree 1.

Lemma 11 vividly exhibits the connection to the classical Pieri's formula that was mentioned in the Introduction. A typical geometric proof of Pieri's formula for Grassmannians (see [13], [15]) involves showing a triple intersection of Schubert varieties

$$\Omega_\mu G_\bullet \bigcap \Omega_{\lambda^c} G_\bullet{}' \bigcap \Omega_m G_\bullet{}'' \tag{1}$$

is transverse and consists of a single point, when $G_\bullet, G_\bullet{}'$, and $G_\bullet{}''$ are in suitably general position.

One could construct a proof of Theorem 1 along those lines, studying a triple intersection of Schubert subvarieties

$$X_w G_\bullet \bigcap X_{w_0 v} G_\bullet{}' \bigcap X_{r[k,m]} G_\bullet{}'', \tag{2}$$

where $G_\bullet, G_\bullet{}'$, and $G_\bullet{}''$ are in suitably general position. Doing so, one observes that the geometry of the intersection of (2) is governed entirely by the geometry of an intersection similar to that in (1). In part, that is because $X_{r[k,m]} G_\bullet{}'' = \pi^{-1} \Omega_m G_\bullet{}''$. This is the spirit of our method, which may be seen most vividly in Lemmas 14 and 15.

Proof of Lemma 4. — Since $F_\bullet$ and $F_\bullet{}'$ are opposite flags,

$$X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}'$$

is a generically transverse intersection, so in the cohomology ring

$$\left[X_wF_\bullet \bigcap X_{w_0v}F_\bullet'\right] = [X_wF_\bullet]\cdot[X_{w_0v}F_\bullet'] = \mathfrak{S}_w\cdot\mathfrak{S}_{w_0v}.$$

Let Y be the subspace of Lemma 10. Since $\pi(X_wF_\bullet\bigcap X_{w_0v}F_\bullet') \subset G_kY$, the class $\pi_*(\mathfrak{S}_w\cdot\mathfrak{S}_{w_0v})$ is a cohomology class on G_kY. However, all such classes are of the form $\delta\cdot[G_kY]$, for some cohomology class δ on G_kV. Since d is the codimension of Y, we have $[G_kY] = s_{d^k}$, establishing part (1) of Lemma 4.

For part (2), suppose further that $w \xrightarrow{r[k,m]} v$. If ρ is the restriction of π to $X_wF_\bullet\bigcap X_{w_0v}F_\bullet'$, then

$$\pi_*(\mathfrak{S}_w\mathfrak{S}_{w_0v}) = \pi_*\left(\left[X_wF_\bullet\bigcap X_{w_0v}F_\bullet'\right]\right) = \deg\rho\left[\pi\left(X_wF_\bullet\bigcap X_{w_0v}F_\bullet'\right)\right].$$

By Lemma 11, $\deg\rho = 1$ and $\pi(X_wF_\bullet\bigcap x_{w_0v}F_\bullet') = \Omega_\mu G_\bullet\bigcap\Omega_{\lambda^c}G_\bullet'$. Since $G_\bullet$ and $G_\bullet'$ are opposite flags, we have

$$\begin{aligned}\pi_*(\mathfrak{S}_w\cdot\mathfrak{S}_{w_0v}) = 1\cdot\left[\Omega_\mu G_\bullet\bigcap\Omega_{\lambda^c}G_\bullet'\right] &= [\Omega_\mu G_\bullet]\cdot[\Omega_{\lambda^c}G_\bullet'] = s_\mu\cdot s_{\lambda^c}\\ &= \sum_\nu c^\lambda_{\mu\,\nu}s_{\nu^c}.\end{aligned}$$

The last equality follows by the Littlewood-Richardson rule and the identity $c^\lambda_{\mu\,\nu} = c^{\nu^c}_{\mu\,\lambda^c}$. □

We deduce Lemma 10 from two additional lemmas. We first make a definition. Let $W \subsetneq V$ be a codimension 1 subspace and let $e\in V-W$ so that $V = \langle W,e\rangle$. For $1\le p\le n$, define an expanding map $\psi_p : \mathbb{F}(W)\to\mathbb{F}(V)$ as follows:

$$(\psi_pE_\bullet)_i = \begin{cases} E_i & \text{if } i<p\\ \langle E_{i-1},e\rangle & \text{if } i\ge p.\end{cases}$$

Note that if $E_\bullet = \langle f_1,\ldots,f_{n-1}\rangle$, then $\psi_pE_\bullet = \langle f_1,\ldots,f_{p-1},e,f_p,\ldots,f_{n-1}\rangle$.

For $w\in S_n$ and $1\le p\le n$, define $w|_p\in S_{n-1}$ by

$$w|_p(j) = \begin{cases} w(j) & \text{if } j<p \text{ and } w(j)<w(p)\\ w(j{+}1) & \text{if } j\ge p \text{ and } w(j)<w(p)\\ w(j)-1 & \text{if } j<p \text{ and } w(j)>w(p)\\ w(j{+}1)-1 & \text{if } j\ge p \text{ and } w(j)>w(p).\end{cases}$$

If we represent permutations as matrices, $w|_p$ is obtained by crossing out the pth row and $w(p)$th column of the matrix for w.

LEMMA 12. — *Let $W\subsetneq V$ and $e\in V-W$ with $V=\langle W,e\rangle$. Let $G_\bullet$ be a complete flag in W. For $1\le p\le n$ and $w\in S_n$,*

$$\psi_p\left(X_{w|_p}G_\bullet\right)\subset X_w\left(\psi_{w_0w(p)}(G_\bullet)\right).$$

Proof. — Let $E_\bullet \in X_{w|_p} G_\bullet$. Then W has a basis $f_1, \ldots, f_{n-1}$ with $E_\bullet = \langle f_1, \ldots, f_{n-1}\rangle$ and for each $1 \leq i \leq n-1$, $f_i \in G_{n-w|_p(i)}$. Then we necessarily have $\psi_p(E_\bullet) = \langle \phi_1, \ldots, \phi_n\rangle = \langle f_1, \ldots, f_{p-1}, e, f_p, \ldots, f_{n-1}\rangle$. Noting

$$\left(\psi_{w_0 w(p)}(G_\bullet)\right)_{n+1-j} = \begin{cases} G_{n+1-j} & \text{if } \ j > w(p) \\ \langle e, G_{n-j}\rangle & \text{if } \ j \leq w(p), \end{cases}$$

we see that $\phi_i \in \left(\psi_{w_0 w(p)}(G_\bullet)\right)_{n+1-w(i)}$. Thus

$$\psi_p\left(X_{w|_p} G_\bullet\right) \subset X_w\left(\psi_{w_0 w(p)}(G_\bullet)\right). \qquad \square$$

LEMMA 13. — *Let $W \subsetneq V$ and $e \in V - W$ with $V = \langle W, e\rangle$ and let $G_\bullet$ and $G_\bullet'$ be opposite flags in W. Suppose that $w <_k v$ are permutations in S_n and $p > k$ an integer such that $w(p) = v(p)$. Let $w_0^{(j)}$ be the longest permutation in S_j. Then*

1. *$\ell(v|_p) - \ell(w|_p) = \ell(v) - \ell(w)$ and $w|_p <_k v|_p$.*
2. *$\psi_p\left(X_{w|_p} G_\bullet \bigcap X_{w_0^{(n-1)}(v|_p)} G_\bullet'\right)$*
   $= X_w\left(\psi_{w_0^{(n)} w(p)}(G_\bullet)\right) \bigcap X_{w_0^{(n)} v}\left(\psi_{v(p)}(G_\bullet')\right)$.
3. *If $E_\bullet \in X_w\left(\psi_{w_0^{(n)} w(p)}(G_\bullet)\right) \bigcap X_{w_0^{(n)} v}\left(\psi_{v(p)}(G_\bullet')\right)$, then $E_p = \langle E_{p-1}, e\rangle$.*
4. *If $F_\bullet$ and $F_\bullet'$ are opposite flags in V and $E_\bullet \in X_w F_\bullet \bigcap X_{w_0^{(n)} v} F_\bullet'$, then $E_k \subset F_{n-w(p)} + F'_{w(p)-1}$.*

Proof. — First recall that $\ell(vt_{a\,b}) = \ell(v)+1$ if and only if $v(a) < v(b)$ and if $a < j < b$, then $v(j)$ is not between $v(a)$ and $v(b)$. Thus if $\ell(vt_{a\,b}) = \ell(v)+1$ and $p \notin \{a, b\}$, we have $\ell(vt_{a\,b}|_p) = \ell(v|_p)+1$. Statement (1) follows by induction on $\ell(v) - \ell(w)$.

For (2), since $(w_0^{(n)} v)|_p = w_0^{(n-1)}(v|_p)$ and $w_0^{(n)} w_0^{(n)} v = v$, Lemma 12 shows

$$\psi_p\left(X_{w|_p} G_\bullet \bigcap X_{w_0^{(n-1)}(v|_p)} G_\bullet'\right) \\ \subset X_w\left(\psi_{w_0^{(n)} w(p)}(G_\bullet)\right) \bigcap X_{w_0^{(n)} v}\left(\psi_{v(p)}(G_\bullet')\right).$$

The flags $\psi_{w_0^{(n)} w(p)}(G_\bullet)$ and $\psi_{v(p)}(G_\bullet')$ are opposite flags in V, since $G_\bullet$ and $G_\bullet'$ are opposite flags in W. Then part (1) shows both sides have the same dimension. Since ψ_p is injective, they are equal.

To show (3), let $E_\bullet \in X_w\left(\psi_{w_0^{(n)}w(p)}(G_\bullet)\right)\bigcap X_{w_0^{(n)}v}\left(\psi_{v(p)}(G_\bullet')\right)$. By (2), there is a flag $E_\bullet' \in X_{w|_p}G_\bullet \bigcap X_{w_0^{(n-1)}(v|_p)}G_\bullet'$ with $\psi_p(E_\bullet') = E_\bullet$, so $E_p = \langle E'_{p-1}, e\rangle = \langle E_{p-1}, e\rangle$.

For (4), let $W = F_{n-w(p)} + F'_{v(p)-1}$ and e any nonzero vector in the one dimensional space $F_{n+1-w(p)}\bigcap F'_{v(p)}$. The distinct subspaces in $F_\bullet \bigcap W$ define a flag $G_\bullet$, and those in $F_\bullet' \bigcap W$ define a flag $G_\bullet'$. In fact, $\psi_{w_0^{(n)}w(p)}(G_\bullet) = F_\bullet$ and $\psi_{w(p)}(G_\bullet') = F_\bullet'$, and $G_\bullet$ and $G_\bullet'$ are opposite flags in W. By (2),

$$\psi_p\left(X_{w|_p}G_\bullet \bigcap X_{w_0^{(n-1)}(v|_p)}G_\bullet'\right) = X_wF_\bullet \bigcap X_{w_0^{(n)}v}F_\bullet'.$$

Thus flags in $X_wF_\bullet \bigcap X_{w_0^{(n)}v}F_\bullet'$ are in the image of ψ_p. As $k < p$, $(\psi_pE_\bullet)_k = E_k \subset W$, establishing part (4). □

Proof of Lemma 10. — Let $F_\bullet$ and $F_\bullet'$ be opposite flags in V, let $w <_k v$ and let $E_\bullet \in X_wF_\bullet \bigcap X_{w_0v}F_\bullet'$. Define a basis $e_1, \ldots, e_n$ for V by $F_{n+1-j}\bigcap F'_j = \langle e_j\rangle$ for $1 \le j \le n$. Suppose $v = wt_{a_1 b_1}\cdots t_{a_m b_m}$ with $a_i \le k < b_i$. Let $\{p_1, \ldots, p_d\}$ be the complement of $\{b_1, \ldots, b_m\}$ in $\{k+1, \ldots, n\}$. For $1 \le i \le d$, let $Y_i = \langle e_1, \ldots, e_{w(p_i)-1}, e_{w(p_i)+1}, \ldots, e_n\rangle$. Since $w(p_i) = v(p_i)$ and $k < p_i$, we see that $Y_i = F_{n-w(p_i)} + F'_{w(p_i)-1}$, so part (4) of Lemma 13 shows $E_k \subset Y_i$. Thus

$$E_k \subset \bigcap_{i=1}^{d} Y_i = \langle e_{w(j)} \mid j < k \text{ or } j = b_i\rangle = Y.$$

Since $w(p_i) = v(p_i)$ for $1 \le i \le d$, we have $E_{p_i} = \langle E_{p_i-1}, e_{w(p_i)}\rangle$, by part (3) of Lemma 13. So if $E_\bullet = \langle f_1, \ldots, f_n\rangle$, we may assume that $f_{p_i} = e_{w(p_i)} \in F_{n+1-w(p_i)}\bigcap F'_{v(p_i)}$ for $1 \le i \le d$, completing the proof. □

To prove Lemma 11, we begin by describing an intersection in a Grassmannian. Recall that $\Omega_\lambda F_\bullet = \{H \in G_kV \mid \dim H\bigcap F_{k-j+\lambda_j} \ge j \text{ for } 1 \le j \le k\}$.

LEMMA 14. — *Suppose that $L_1, \ldots, L_k, M \subset V$ with*

$$V = M \bigoplus L_1 \bigoplus \cdots \bigoplus L_k.$$

Let $r_j = \dim L_j - 1$ and $m = r_1 + \cdots + r_k$. Then there are opposite flags $F_\bullet$ and $F_\bullet'$ and partitions $\lambda \supset \mu$ with $\lambda_j - \mu_j = r_j$ and λ/μ a skew row of length m such that in G_kV,

$$\Omega_\mu F_\bullet \bigcap \Omega_{\lambda^c}F_\bullet' = \{H \in G_kV \mid \dim H\bigcap L_j = 1 \text{ for } 1 \le j \le k\}.$$

Proof. — Let $\mu_k = 0$ and $\mu_j = r_{j+1} + \cdots + r_k$ for $1 \le j < k$ and $\lambda_j = r_j + \mu_j$ for $1 \le j \le k$. Choose a basis $e_1, \ldots, e_n$ for V such that

$$\begin{aligned} L_j &= \langle e_{k+1-j+\mu_j}, e_{k+2-j+\mu_j}, \ldots, e_{k+1+r_j-j+\mu_j} = e_{k+1-j+\lambda_j} \rangle \\ M &= \langle e_{m+k+1}, \ldots, e_n \rangle. \end{aligned}$$

Let $F_\bullet = \langle e_n \ldots, e_1 \rangle$ and $F_\bullet{}' = \langle e_1, \ldots, e_n \rangle$. Then

$$F_{n-k+j-\mu_j} = M \bigoplus L_1 \bigoplus \cdots \bigoplus L_j$$

$$F'_{n-k+(k+1-j)-\lambda^c_{k+1-j}} = F'_{k+1-j+\lambda_j} = L_j \bigoplus \cdots \bigoplus L_k.$$

If $H \in \Omega_\mu F_\bullet \bigcap \Omega_{\lambda^c} F_\bullet{}'$, then $\dim H \bigcap F_{n-k+j-\mu_j} \ge j$ for $1 \le j \le k$ and

$$\dim H \bigcap F'_{n-k+(k+1-j)-\lambda^c_{k+1-j}} \ge k+1-j,$$

for $1 \le j \le k$. Thus for $1 \le j \le k$,

$$\dim H \bigcap F_{n-k+j-\mu_j} \bigcap F'_{n-k+(k+1-j)-\lambda^c_{k+1-j}} \ge 1.$$

But $F_{n-k+j-\mu_j} \bigcap F'_{n-k+(k+1-j)-\lambda^c_{k+1-j}} = L_j$, so $\dim H \bigcap L_j \ge 1$ for $1 \le j \le k$. Since $L_j \bigcap L_i = \{0\}$ if $j \ne i$, we see that $\dim H \bigcap L_j = 1$. Thus

$$\Omega_\mu F_\bullet \bigcap \Omega_{\lambda^c} F_\bullet{}' \subset \left\{ H \in G_k V \mid \dim H \bigcap L_j = 1 \ \text{ for } \ 1 \le j \le k \right\}.$$

We show these varieties have the same dimension, establishing their equality: since $|\lambda| = |\mu| + m$, and $F_\bullet$ and $F_\bullet{}'$ are opposite flags, $\Omega_\mu F_\bullet \bigcap \Omega_{\lambda^c} F_\bullet{}'$ has dimension m. But the map $H \mapsto (H \bigcap L_1, \ldots, H \bigcap L_k)$ defines an isomorphism between $\{H \in G_k V \mid \dim H \bigcap L_j = 1 \ \text{ for } \ 1 \le j \le k\}$ and $\mathbb{P}L_1 \times \cdots \times \mathbb{P}L_k$, which has dimension $\sum_j (\dim L_j - 1) = m$. Here, $\mathbb{P}L_j$ is the projective space of one dimensional subspaces of L_j. □

We relate this to intersections of Schubert varieties in the flag manifold.

LEMMA 15. — *Suppose that $w \xrightarrow{r[k,m]} v$ and $v = w t_{a_1 b_1} \cdots t_{a_m b_m}$ with $a_i \le k < b_i$ and $\ell(w t_{a_1 b_1} \cdots t_{a_i b_i}) = \ell(w) + i$ for $1 \le i \le m$. Let $F_\bullet$ and $F_\bullet{}'$ be opposite flags in V and let $\langle e_i \rangle = F_{n+1-i} \bigcap F'_i$. Define*

$$\begin{aligned} L_j &= \langle e_j, e_{w(b_i)} \mid a_i = j \rangle \\ M &= \langle e_{w(p)} \mid k < p \ \text{ and } \ w(p) = v(p) \rangle. \end{aligned}$$

Then

1. $\dim L_j = 1 + \#\{i \mid a_i = j\}$ *and* $V = M \bigoplus L_1 \bigoplus \cdots \bigoplus L_k$.

2. *If* $E_\bullet \in X_wF_\bullet \bigcap X_{w_0v}F_\bullet{}'$, *then* $\dim E_k \bigcap L_j = 1$ *for* $1 \leq j \leq k$.

3. *Let* π *be the map induced by* $E_\bullet \mapsto E_k$. *Then*

$$\pi : X_wF_\bullet \bigcap X_{w_0v}F_\bullet{}' \to \left\{H \in G_kV \mid \dim H \bigcap L_j = 1 \quad \text{for} \quad 1 \leq j \leq k\right\}$$

is surjective and of degree 1.

Proof. — Part (1) is immediate.

For (2) and (3), note that both $\{H \in G_kV \mid \dim H \bigcap L_j = 1 \text{ for } 1 \leq j \leq k\}$ and $X_wF_\bullet \bigcap X_{w_0v}F_\bullet{}'$ are irreducible and have dimension m. We exhibit an m dimensional subset of each over which π is an isomorphism.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{C}^\times)^m$ be an m-tuple of nonzero complex numbers. We define a basis $f_1, \ldots, f_n$ of V depending upon α as follows.

$$f_j = \begin{cases} e_{w(j)} + \sum\limits_{i\,:\,a_i=j} \alpha_i e_{w(b_i)} & \text{if } j \leq k \\ e_{w(j)} & \text{if } \ j > k \quad \text{and} \quad j \notin \{b_1, \ldots, b_m\} \\ \sum\limits_{\substack{i:a_i=a_q \\ w(b_i)\geq w(j)}} \alpha_i e_{w(b_i)} & \text{if } \ j = b_q > k. \end{cases}$$

Let $i_1 < \cdots < i_s$ be those integers i_l with $a_{i_l} = j$. Since $t_{a_i\,b_i}$ lengthens the permutation $wt_{a_1\,b_1} \cdots t_{a_{i-1}\,b_{i-1}}$, we see that

$$\begin{array}{ccccccc} w(j) & < & w(b_{i_1}) & < & \cdots & < & w(b_{i_s}) \\ \| & & \| & & & & \| \\ v(b_{i_1}) & < & v(b_{i_2}) & < & \cdots & < & v(j). \end{array}$$

Thus the first term in f_j is proportional to $e_{w(j)}$. Hence $f_j \in F_{n+1-w(j)} - F_{n-w(j)}$, and so $f_1, \ldots, f_n$ is a basis of V and the flag $E_\bullet(\alpha) = \langle f_1, \ldots, f_n\rangle$ is in $X_wF_\bullet$.

Note that $f_1', \ldots, f_n'$ is also a basis for $E_\bullet(\alpha)$, where f_j' is given by

$$f_j' = \begin{cases} f_j & \text{if } \ j \leq k \\ f_j & \text{if } \ j > k \quad \text{and} \quad j \notin \{b_1, \ldots, b_m\} \\ f_{a_q} - f_j & \text{if } \ j = b_q > k. \end{cases}$$

Here, the last term in each f_j' is proportional to $e_{v(j)}$, so $f_j' \in F'_{v(j)} = F'_{n+1-w_0v(j)}$, showing that $E_\bullet(\alpha) \in X_{w_0v}F_\bullet{}'$.

Since $f_j \in L_j$ for $1 \leq j \leq k$, we have $\dim E_\bullet(\alpha) \bigcap L_j = 1$ for $1 \leq j \leq k$. As $\{E_\bullet(\alpha) \mid \alpha \in (\mathbb{C}^\times)^m\}$ is a subset of $X_wF_\bullet \bigcap X_{w_0v}F_\bullet{}'$ of dimension m, it is dense. Thus if $E_\bullet \in X_wF_\bullet \bigcap X_{w_0v}F_\bullet{}'$, then $\dim E_k \bigcap L_j = 1$ for $1 \leq j \leq k$.

The set $\{(E_\bullet(\alpha))_k \mid \alpha \in (\mathbb{C}^\times)^m\}$ is a dense subset of

$$\{H \in G_k V \mid \dim H \bigcap L_j = 1 \quad \text{for} \quad 1 \leq j \leq k\} \simeq \mathbb{P}L_1 \times \cdots \times \mathbb{P}L_k.$$

Since π is an isomorphism of this set with $\{E_\bullet(\alpha) \mid \alpha \in (\mathbb{C}^\times)^m\}$, the map

$$\pi : X_w F_\bullet \bigcap X_{w_0 v} F_\bullet{}' \to \left\{H \in G_k V \mid \dim H \bigcap L_j = 1 \quad \text{for} \quad 1 \leq j \leq k\right\}$$

is surjective of degree 1, proving the lemma. □

We note finally that Lemma 11 is an immediate consequence of Lemmas 14 and 15(3).

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Frank SOTTILE,
Department of Mathematics
University of Toronto
100 St. George Street
Toronto, Ontario M5S 1A1 (Canada).
sottile@math.toronto.edu