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ON THE POLYNOMIAL-LIKE BEHAVIOUR OF CERTAIN ALGEBRAIC FUNCTIONS

by Charles FEFFERMAN(*) & Raghavan NARASIMHAN

0. Introduction.

The purpose of this paper is to prove an extension theorem which, in particular, implies that certain families of algebraic functions have the growth and smoothness properties of polynomials. Our work was motivated by that of Parmeggiani [P] on pseudodifferential operators. We begin with a few words about [P] and how algebraic functions enter there.

It is known from the work of Stein and his collaborators [FS], [RS], [NSW] that a subelliptic differential operator is governed by a family of non-Euclidean balls. The purpose of [P] is to associate non-Euclidean balls in the cotangent bundle to a pseudodifferential operator $P(x, D)$. In dimensions one and two, the results in [P] provide a geometrical understanding of these non-Euclidean balls. It would be of interest to extend Parmeggiani's work to higher dimensions.

The method used in [P] requires writing the symbol $P(x, \xi)$ in a normal form; it is likely that extending the results in [P] to higher dimensions will also require doing this. It is this normal form which brings in algebraic functions. More precisely, fix a point $(x^0, \xi^0) \in \mathbb{R}^n \times \mathbb{R}^n$ where $P(x^0, \xi^0) = 0$. We assume that $P(x, \xi) \geq 0$. Expanding $P(x, \xi)$ in a Taylor

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series to a high order at (x^0, ξ^0) and rescaling, we obtain a polynomial $p(x,\xi)$ bounded *a priori* on the unit cube $Q = \{(x_1,\dots,x_n,\xi_1,\dots,\xi_n) \mid |x_j| \le 1,\ |\xi_j| \le 1,\ 1 \le j \le n\}$. Here $p(0,0) = 0$ and $p \ge 0$ on Q modulo a small error, which we ignore. Suppose that the Hessian $\left(\dfrac{\partial^2 p}{\partial\xi_j\,\partial\xi_k}\right)$ has rank r at the origin. Rotating coordinates, we may assume that $\left(\dfrac{\partial^2 p}{\partial\xi_j\,\partial\xi_k}\right)_{1\le j,k\le r}$ is positive definite at 0.

In the neighborhood of the origin, we can then write $p(x,\xi)$ in the normal form

$$(1) \qquad p(x,\xi_1,\dots,\xi_n) = \sum_{k=1}^{r} e_k(x,\xi)\,[\xi_k - \theta_k(x,\xi_{r+1},\dots,\xi_n)]^2 + q(x,\xi_{r+1},\dots,\xi_n),$$

where e_k, θ_k, q are smooth, and $e_k > 0$.

Here, $q(x,\xi_{r+1},\dots,\xi_n)$ is an algebraic function. In fact, let V be the (real) algebraic variety $V = \left\{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n \mid \dfrac{\partial p}{\partial\xi_k} = 0,\ 1\le k\le r\right\}$, and let $\pi : V \to \mathbb{R}^{2n-r}$ be the projection $(x_1,\dots,x_n,\xi_1,\dots,\xi_n) \mapsto (x_1,\dots,x_n,\xi_{r+1},\dots,\xi_n)$. Near the origin, V is smooth, π is a diffeomorphism, and

$$(2) \qquad q = (p|V)\circ\pi^{-1}.$$

To get useful information from (1), we need to know that q has the growth and smoothness properties of a polynomial. If we are allowed to restrict attention to a tiny neighborhood of the origin, then one can simply Taylor-expand q to obtain a polynomial. However, the method followed in [P] requires an understanding of q on a neighborhood of fixed size, and one is forced to study functions of the form (2) on such a neighborhood.

We can formulate the basic problem in a simple, general setting. Suppose that we are given a variety $V \subset \mathbb{R}^n$, defined as the set of common zeros of polynomials $P_1,\dots,P_r$, $1 \le r \le n-1$. We make the following:

ASSUMPTIONS.

(I) *The polynomials P_j have degree at most D, and their coefficients have absolute value at most C.*

(II) *We have $P_1(0)=\dots=P_r(0) = 0$ and* $\left|\det\left(\dfrac{\partial P_j}{\partial x_k}(0)\right)_{1\le j,k\le r}\right| \ge c > 0.$

Let $\pi : V \to \mathbb{R}^{n-r}$ be the projection $(x_1, \dots, x_n) \mapsto (x_{r+1}, \dots, x_n)$. In view of (II), π has a smooth local inverse $\pi^{-1} : B(0, \delta_0) \to V$ defined on a small ball. Our goal is then to understand functions of the form $F = p \circ \pi^{-1}$ where p is a polynomial of degree at most D on $\mathbb{R}^n$, and to do this uniformly with respect to the polynomials P_j satisfying our assumptions.

The following result (a somewhat weaker form of the main theorem 1.1 of this paper) provides very good control over this family of functions.

THEOREM 1. — *Let $P_1, \dots, P_r$ be polynomials satisfying assumptions* (I) *and* (II), *and let $V = \{x \in \mathbb{R}^n \mid P_1(x) = \cdots = P_r(x) = 0\}$. Then, there exist constants $\delta_*, C_*, D_* > 0$ depending only on the constants n, c, C, D occurring in* (I), (II) *such that if p is a polynomial of degree at most D on $\mathbb{R}^n$, we can find polynomials f and g with the following properties:*

(A) *f, g have degree at most D_*;*

(B) *On $B(0, 2\,\delta_*)$, we have $\frac{1}{2} < g < 2$ and $|f| \le C_* \sup_{V \cap B(0,\delta_*)} |p|$;*

(C) *$p = f/g$ on $V \cap B(0, 2\,\delta_*)$.*

We note explicitly that *the denominator g in this theorem cannot be taken* $\equiv 1$; see [FN]. We can, however, take g to depend only on the P_j, not on p.

As an easy consequence of this extension theorem, we prove the following result which shows that the family of algebraic functions mentioned above behaves like a family of polynomials of bounded degree.

THEOREM 2. — *Let V be as in Theorem 1, and let $\pi : V \to \mathbb{R}^{n-r}$ be the projection. There exist constants δ_*, $C_* > 0$ depending only on n, c, C, D (the constants in* (I), (II)*) such that π has a smooth inverse $\pi^{-1} : B(0, \delta_*) \to V$ and such that if p is a polynomial of degree $\le D$ and $F = p \circ \pi^{-1}$ on $B(0, \delta_*)$, then the following inequalities hold:*

(A) *Polynomial Growth. For $0 < 2\,\delta < \delta_*$,*

$$\sup_{B(0,2\,\delta)} |F| \le C_* \sup_{B(0,\delta)} |F|.$$

(B) *Bernstein's Inequality. If ∇F denotes the gradient of F, then*

$$\sup_{B(0,\delta)} |\nabla F| \le \frac{C_*}{\delta} \sup_{B(0,\delta)} |F| \quad \text{for} \quad 0 < \delta < \delta_*.$$

(C) *Equivalence of Norms.*

$$\sup_{B(0,\delta)} |F| \leq \frac{C_*}{\delta^{n-r}} \int_{B(0,\delta)} |F| \text{ for } 0 < \delta < \delta_*.$$

For a fixed V and δ, estimates of this type are obvious consequences of the fact that any two norms on a finite dimensional vector space are equivalent. This simple remark gives no control over the constant C_*, whereas the extension theorem enables us to control the constants easily. This control is, in turn, crucial in understanding the function q in (1) and (2).

Let us mention also that the extension theorem clearly enables us to estimate the function F of Theorem 2 in a *complex* ball of radius δ_* around 0 in terms of its values on $B(0,\delta_*) \subset \mathbb{R}^{n-r}$.

The plan of the proof of Theorem 1 given here is as follows. Let W be the space of all $\overrightarrow{P} = (P_1, \dots, P_r)$ satisfying assumptions (I) and (II). [We work with a very slightly different space W in the body of the paper.] Fix $\overrightarrow{P_0} \in W$. We say that *Theorem* 1 *holds locally at* $\overrightarrow{P_0}$ if the conclusions of the theorem hold for all $\overrightarrow{P}$ in a sufficiently small neighborhood of $\overrightarrow{P_0}$, with a constant C_* depending on $\overrightarrow{P_0}$ (but with δ_* and D_* independent of $\overrightarrow{P_0}$). The space W is compact. It is therefore sufficient to prove that Theorem 1 holds locally at any $\overrightarrow{P_0} \in W$. We shall take degeneracies into account by showing, by downward induction on s, $0 \leq s \leq \dim W$, that Theorem 1 holds locally at $\overrightarrow{P_0}$ for any $\overrightarrow{P_0}$ outside a semi-algebraic subset $V_s \subset W$ whose dimension is $< s$. Since $V_0 = \emptyset$ if $s = 0$ (since $\dim V_0 < 0$), this shows that Theorem 1 holds locally everywhere on W, thus completing the proof.

Clearly, the main work lies in the induction step. This is modeled on a simpler induction in our earlier paper [FN] in which the theorem is proved for hypersurfaces (i.e. for $r = 1$). Unlike the argument in [FN] in which we described the sets V_s in terms of factorizing the equation defining V, we are unable to describe the sets V_s explicitly, and resort instead to extensive use of the general structure of semi-algebraic sets.

It is clear from the statement of Theorem 1 that one needs to analyze polynomials vanishing on *one* connected component of the smooth part of the set of common zeros of polynomials $P_1, \dots, P_r$ satisfying (I) and (II). This analysis, essentially obvious in [FN], has to be done differently here, and one of our main steps in carrying out the induction outlined above is to establish the following result.

THEOREM 3. — *Suppose that* $(P_1, \ldots, P_r)$ *satisfy* (I) *and* (II). *Let* $V_0(P_1, \ldots, P_r)$ *be the connected component, containing 0, of the smooth part of the zero set of* $(P_1, \ldots, P_r)$.

Then, there exists an auxiliary polynomial q *on* $\mathbb{R}^n$ *with the following properties:*

(A) $q(0) = 1$, *and the degree and coefficients of* q *are bounded a priori in terms of the constants in* (I) *and* (II).

(B) *If* f *is any polynomial vanishing on* $V_0(P_1, \ldots, P_r)$, *then we can write* $q\,f$ *in the form*

$$q\,f = G_1 P_1 + \cdots + G_r P_r,$$

with polynomials G_ν *whose* degrees *are bounded a priori in terms of* $\deg f$ *and the constants in* (I) *and* (II).

In the case of hypersurfaces, i.e. when $r = 1$, this is a simple lemma (see [FN]). In the general case, we prove Theorem 3 by using Hörmander's L^2-estimates for the $\overline{\partial}$-operator on $\mathbb{C}^n$.

For a given $(P_1, \ldots, P_r)$, the polynomials g in Theorem 2 will be constructed as a product of polynomials q from Theorem 3 associated to finitely many $(P_1^{(j)}, \ldots, P_r^{(j)})$ that lie near $(P_1, \ldots, P_r)$.

Note that Theorem 3 asserts no *a priori* bounds on the coefficients of the polynomials $(G_1, \ldots, G_r)$ in (B) above. While we need bounds on their degree, we need no bounds on the coefficients. It would, however, be interesting to decide what the optimal estimates on these coefficients might be; in the case of a weaker variant of Theorem 3, one can obtain good *a priori* estimates (see Theorem 5.2 below).

It would of course be of interest to estimate the constant C_* in Theorems 1 and 2 as a function of n, c, C, D. This seems difficult to do. However, for the application to pseudodifferential operators (in particular for the results of Parmeggiani), the crucial point is to have a constant C_* independent of the polynomials P_j (as long as assumptions (I) and (II) are satisfied). For this reason, the work in the earlier sections, leading to a proof of Theorem 3 stated above (Theorem 5.5 in the text) has to be uniform in the P_j.

We are grateful to the referee who has pointed out that the methods

we have used in sections 2–5 have been applied before to related questions. We have added some of the references he gave us in the rélevant places.

1. Notation and statement of the Main Theorem.

Let $n \geq 2$ be an integer. If $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we set, as usual, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$; if $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ is an n-tuple of non-negative integers, we set $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

If $f \in \mathbb{C}[z_1, \ldots, z_n]$, $f = \sum_\alpha f_\alpha z^\alpha$ is a polynomial with complex coefficients in n variables, we define $\|f\|$ by

$$\|f\|^2 = \sum_\alpha |f_\alpha|^2.$$

If $f = (f_1, \ldots, f_r)$ is an r-tuple of polynomials, we set $\|f\|^2 = \sum \|f_j\|^2$.

We shall have occasion to use this same notation for polynomials in more than n variables; thus if $m \geq 1, f \in \mathbb{C}[z_1, \ldots, z_m]$, $f = \sum f_\alpha z^\alpha$, then $\|f\|^2 = \sum |f_\alpha|^2$.

If A is an $n \times n$ matrix over $\mathbb{C}$, we denote by $\|A\|$ the operator norm of the linear map $A : \mathbb{C}^n \to \mathbb{C}^n$.

Let $d \geq 1$ be any integer. We define

$$H^d = \{f \in \mathbb{R}[x_1, \ldots, x_n] \mid \text{ degree } (f) \leq d\},$$
$$\mathcal{H}^d = \{f \in \mathbb{C}[z_1, \ldots, z_n] \mid \text{ degree } (f) \leq d\}.$$

For $\rho > 0$, let

$$Q_\rho = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid |x_j| \leq \rho\},\ Q = Q_1.$$

If $a \in \mathbb{C}^n$ and $\rho > 0$, we set

$$Q(a, \rho) = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid \\ |\operatorname{Re}(z_j - a_j)| \leq \rho,\ |\operatorname{Im}(z_j - a_j)| \leq \rho,\ j = 1, \ldots, n\}.$$

If $a \in \mathbb{C}^n, R > 0$, we set

$$B(a, R) = \{z \in \mathbb{C}^n \mid |z - a| < R\};$$

it is the open ball of radius R centered at a; $\overline{B}(a, R)$ will denote its closure. We use the same notation also in $\mathbb{R}^n$. However, since we shall use balls in

$\mathbb{R}^k$ or $\mathbb{C}^k$ with varying k, we write $B_k(a,R) = \Big\{(x_1,\ldots,x_k) \in \mathbb{R}^k(\mathbb{C}^k) \mid \sum_{j=1}^{k} |x_j - a_j|^2 < R^2\Big\}$ when this is relevant.

Fix integers $n \geq 2$, $1 \leq r \leq n-1$, $D \geq 1$ and a constant $C_1 > 0$. *These will remain fixed throughout the paper.*

If $p_1,\ldots,p_r \in \mathbb{R}[x_1,\ldots,x_n](resp.\mathbb{C}[z_1,\ldots,z_n])$, we set

$$J_P(x) = \det\Big(\frac{\partial p_j}{\partial x_k}(x)\Big)_{1\leq j,\, k\leq r}, x \in \mathbb{R}^n$$
$$(\text{resp. } J_P(z) = \det\Big(\frac{\partial p_j}{\partial z_k}(z)\Big)_{1\leq j,k\leq r},\ z \in \mathbb{C}^n).$$

Our basic space is:

(1) $W = \{P = (p_1,\ldots,p_r) \in H^D \times \cdots \times H^D \mid P(0) = 0, \|P\| \leq C_1, J_P(0) = 1\}$.

If $P = (p_1,\ldots,p_r)$, $p_j \in \mathbb{R}[x_1,\ldots,x_n]$, we set

(2) $Z(P) = \{x \in \mathbb{R}^n \mid p_1(x) = \ldots \cdot p_r(x) = 0\}$.

If $P(0) = 0$ and $J_P(0) \neq 0$, let $V^0(P)$ be the connected component of $Z(P) \cap \{x \in \mathbb{R}^n \mid J_P(x) \neq 0\}$ containing 0. We let $V(P)$ be the closure in $\mathbb{R}^n$ of $V^0(P)$. Note that $V(P)$ is not necessarily an algebraic set.

We use similar notation over $\mathbb{C}$. If $P = (p_1,\ldots,p_r), p_j \in \mathbb{C}[z_1,\ldots,z_n]$, we set

(3) $\mathcal{Z}(P) = \{z \in \mathbb{C}^n \mid P(z) = 0\}$.

If $P(0) = 0, J_P(0) \neq 0, \mathcal{V}^0(P)$ will stand for the connected component of $\mathcal{Z}(P) \cap \{z \in \mathbb{C}^n \mid J_P(z) \neq 0\}$ containing 0, and $\mathcal{V}(P)$ will denote its closure in $\mathbb{C}^n$. In this case, $\mathcal{V}(P)$ is the irreducible component of $\mathcal{Z}(P)$ containing 0 (unique since $\mathcal{Z}(P)$ is smooth at 0 because $J_P(0) \neq 0$). In particular, $\mathcal{V}(P)$ is an algebraic variety.

(4) There is a constant ρ_1, $0 < \rho_1 \leq 1$ depending only on C_1, n, D such that for any $P \in W$, we have $Q_{\rho_1} \cap Z(P) \subset V^0(P)$ (see the remark following Lemma 2.1). For $0 < \rho \leq \rho_1$, we set

$$V_\rho(P) = Q_\rho \cap Z(P) = Q_\rho \cap V(P) = \{x \in Q_\rho \mid P_1(x) = \ldots \cdot P_r(x) = 0\}.$$

(5) For $P \in W$, we denote by $\mathcal{G}(P)$ the space of germs at 0 of functions defined on some neighborhood of 0 on $Z(P)$ (or $V(P)$).

We can now state our main theorem.

THEOREM 1.1 (The Extension Theorem).

Part 1. *There exist constants $D' \geq 1, C' > 0$, $m > 0$, depending only on C_1, n, D such that the following holds.*

Given $P \in W$, $\rho > 0$ $(0 < \rho \leq \rho_1)$, there exists $q \in H^{D'}$ with $q(0) = 1$, $\|q\| \leq C'$ such that, if $f \in H^D$, we can find $F \in H^{D'}$ for which

(i) $F = qf$ *on* $V_{\rho_1}(P)$,

and

(ii) $\|F\| \leq C'\rho^{-m} \sup\limits_{V_\rho(P)} |f|$.

Part 2. *There exists $\rho_0 > 0$ depending only on C_1, n, D such that if $P \in W$, $f \in H^D$ and $0 < \rho \leq \rho_0$, then, there exist $F, q \in H^{D'}$ with the following properties:*

(a) $\dfrac{1}{2} < q < 2$ *on* $Q_{2\rho}$

(b) $f = F/q$ *on* $V_\rho(P)$

(c) $\sup\limits_{Q_{2\rho}}|F| \leq C' \sup\limits_{V_\rho(P)} |f|$;

here, as in Part 1, *C', D' depend only on C_1, n, D.*

2. Preliminaries.

Let $n \geq 2$, $1 \leq r \leq n-1$, $D \geq 1$, $C_1 > 0$ be given. Consider the space $\mathcal{W}_0$ of r-tuples $P = (p_1, \ldots, p_r)$ with $p_j \in \mathbb{C}[z_1, \ldots, z_n]$, $\deg p_j \leq D$ (i.e. $p_j \in \mathcal{H}^D$), with

$$P(0) = 0,\ \|P\| \leq C_1,\ J_P(0) = 1\Big(J_P(z) = \det\Big(\frac{\partial p_j}{\partial z_k}(z)\Big)_{1\leq j,k\leq r}\Big).$$

(1) We introduce a new variable z_0 and write $\tilde{z} = (z_0, z) \in \mathbb{C}^{n+1}$ with $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$. If $P \in \mathcal{W}_0$, define $p_0(\tilde{z}) = p_0(z_0, \ldots, z_n) = (z_0 + 1)J_P(z) - 1$. We have $p_0(0) = 0$.

Consider the system

$$\tilde{P} = (p_0, p_1, \ldots, p_r).$$

The set $\mathcal{Z}(\tilde{P})$ of common zeroes of $p_0, \dots, p_r$ in $\mathbb{C}^{n+1}$ is a smooth affine variety isomorphic to $\mathcal{Z}(P) \cap \{z \in \mathbb{C}^n \mid J_P(z) \neq 0\} \subset \mathbb{C}^n$ (in the category of algebraic varieties). Moreover, if $\tilde{J}(\tilde{z}) = \det\Big(\dfrac{\partial p_i}{\partial z_j}\Big)_{0 \le i,\, j \le r}$, we have $\tilde{J}(\tilde{z}) = (J_P(z))^2$, and $\tilde{J}$ satisfies the equation

$$(z_0+1)^2 \tilde{J}(\tilde{z}) - p_0(z_0, \dots, z_n)(1 + (z_0+1)J_P(z)) = 1.$$

Hence

$$\begin{aligned} 1 &\le (|z_0+1|^4 + |1 + (z_0+1)J_P(z)|^2)(|\tilde{J}(\tilde{z})|^2 + |p_0(\tilde{z})|^2) \\ &\le C(1+|\tilde{z}|^2)^N (|\tilde{J}(\tilde{z})|^2 + \sum_{j=0}^{r} |p_j(\tilde{z})|^2), \end{aligned}$$

where C, N depend only on C_1, n, D.

This leads us to consider the following space:

(1) Let $n \ge 2$, $1 \le r \le n-1$, $D \ge 1$, $C_1 > 0$, $c_0 > 0$ and $N \ge 1$ be given.

Define $\mathcal{W}$ to be the following space:

$\mathcal{W}$ is the space of $P = (p_1, \dots, p_r), p_j \in \mathcal{H}^D$, such that

(a) $P(0) = 0$, $\|P\| \le C_1$, $\deg p_j \le D$ (i.e. $p_j \in \mathcal{H}^D$).

(b) $J_P(0) = 1$.

(c) $|J_P(z)|^2 + |p_1(z)|^2 + \cdots + |p_r(z)|^2 \ge c_0(1+|z|^2)^{-N}$, $z \in \mathbb{C}^n$.

(Recall that $J_P(z) = \det\Big(\dfrac{\partial p_j}{\partial z_k}(z)\Big)_{1 \le j,k \le r}$.)

We shall need certain estimates for these polynomials which can be obtained by examining the inverse function theorem. We do this for holomorphic mappings, but remark, for later use, that they remain valid for smooth maps (see remark at the end of Lemma 2.1). In particular, the statement (4) in §1 is an easy consequence of the smooth version of Lemma 2.1.

The results in this section are closely related to those in [BT] and [BY]. See, in particular, [BT] for a version of Lemma 2.6.

LEMMA 2.1. — *Let $R_1, R_2 > 0$, and let r, s be integers ≥ 1. Let $f : B_r(0, R_1) \times B_s(0, R_2) \to \mathbb{C}^r$ be a holomorphic map ($B_k(a, \rho)$ is the ball of radius ρ in $\mathbb{C}^k$ centered at $a \in \mathbb{C}^k$).*

Suppose that $f(0,0) = 0$, and let $D(z,\zeta)$ be the matrix $\Big(\dfrac{\partial f_j}{\partial z_k}(z,\zeta)\Big)_{1 \le j,k \le r}$, $z \in B_r(0, R_1)$, $\zeta \in B_s(0, R_2)$.

Assume that we are given constants $M_1, M_2, M_3 \geq 1$ and $0 < \mu \leq 1$ such that the following inequalities hold for $|z| < R_1$, $|\zeta| < R_2$:

(a) $\left|\dfrac{\partial f_j}{\partial z_k}\right| \leq M_1$, $1 \leq j,\ k \leq r$, $|\det D(z,\zeta)| \geq \mu$.

(b) $\left|\dfrac{\partial^2 f_i}{\partial z_j\, \partial z_k}\right| \leq M_2$, $1 \leq i, j, k \leq r$.

(c) $\left|\dfrac{\partial f_j}{\partial \zeta_\ell}\right| \leq M_3$, $1 \leq j \leq r$, $1 \leq \ell \leq s$.

Then, there exists a constant γ_r, depending only on r, such that the following statements hold.

Let $0 < \theta_1, \theta_2,\ \theta < \min(R_1, R_2)$ be numbers such that

$$\theta_1 \leq \gamma_r \frac{\mu}{M_1^{r-1} M_2},\ \theta \leq \frac{1}{2}\gamma_r \frac{\mu}{M_1^{r-1}}\theta_1,\ \theta_2 < \frac{1}{2sM_3}\theta. \tag{2}$$

Then we have:

(i) For fixed $\zeta \in B_s(0, R_2)$, the map $z \mapsto f(z,\zeta)$ is injective on the ball $B_r(0,\theta_1) \subset \mathbb{C}^r$.

(ii) For fixed $\zeta \in B_s(0,R_2)$, the image under f of the ball $B_r(0,\theta_1)$ contains the ball of radius θ centered at $f(0,\zeta)$ in $\mathbb{C}^r$: $f(B_r(0,\theta_1),\zeta) \supset B_r(f(0,\zeta),\theta)$.

(iii) If $|\zeta| < \theta_2$, there is a unique point $z = z(\zeta) \in B_r(0,\theta_1)$ for which $f(z,\zeta) = 0$ and the map $\zeta \mapsto z(\zeta)$ is holomorphic. In particular, if $X = \{(z,\zeta) \in B_r(0,R_1) \times B_s(0,R_2) \mid f(z,\zeta) = 0\}$ then $X \cap B_r(0,\theta_1) \times B_s(0,\theta_2) = \{(z(\zeta),\zeta) \mid |\zeta| < \theta_2\}$ and this intersection is connected.

Proof. — During the course of this proof, we shall denote by K a constant, not necessarily the same at each occurrence, which depends only on r.

Let $F(z,\zeta) = D(0,\zeta)^{-1}\cdot(f(z,\zeta)-f(0,\zeta))$, and set $g(z,\zeta) = F(z,\zeta)-z$ ($z \in B_r(0,R_1), \zeta \in B_s(0,R_2)$).

We have $g(0,\zeta) = 0$, $\dfrac{\partial g_i}{\partial z_j}(0,\zeta) = 0$, $1 \leq i,j \leq r$, $|\zeta| < R_2$. The entries of the matrix $D(0,\zeta)^{-1}$ are given by $\pm\dfrac{d_{ij}(\zeta)}{\det D(0,\zeta)}$, where $d_{ij}(\zeta)$ is the determinant of the $(r-1)\times(r-1)$ matrix obtained from $D(0,\zeta)$ by deleting the i-th row and the j-th column. Hence, the norm of the linear map $D(0,\zeta)^{-1} : \mathbb{C}^r \to \mathbb{C}^r$ satisfies, because of assumption (a),

$$\|D(0,\zeta)^{-1}\| \leq K \cdot \frac{M_1^{r-1}}{\mu}. \tag{3}$$

Using assumption (b), the definition of g and (3), we obtain

$$\left|\frac{\partial^2 g_i}{\partial z_j\, \partial z_k}\right| \leq K \cdot \frac{M_1^{r-1}}{\mu} \cdot M_2,\ 1 \leq i,j,k \leq r,\ |z| < R_1, |\zeta| < R_2. \tag{4}$$

Since $\dfrac{\partial g_i}{\partial z_j}(0,\zeta) = 0$, this gives

$$\left|\frac{\partial g_i}{\partial z_j}(z,\zeta)\right| \leq r \cdot |z| \cdot \frac{K}{\mu} M_1^{r-1} M_2, \tag{5}$$

so that, if $|z| \leq \theta_1$ and θ_1 satisfies (2), we have

$$\left|\frac{\partial g_i}{\partial z_j}\right| \leq \frac{1}{2r},\ |z| \leq \theta_1.$$

Consequently

$$|g(z,\zeta) - g(w,\zeta)| \leq \frac{1}{2}|z-w| \quad \text{for} \quad |z|, |w| < \theta_1. \tag{6}$$

This implies that, for $|z|, |w| \leq \theta_1$, we have

$$|F(z,\zeta) - F(w,\zeta)| \geq |z-w| - |g(z,\zeta) - g(w,\zeta)| \geq \frac{1}{2}|z-w|$$

so that, for fixed ζ, $z \mapsto F(z,\zeta)$ is injective on $B_r(0,\theta_1)$, and hence so is $z \mapsto f(z,\zeta)$. This proves (i).

To prove (ii), we construct the inverse of F by the standard iteration scheme. Set $\varphi_0(w,\zeta) = 0 \in \mathbb{C}^r$, $|w| < \frac{1}{2}\theta_1$, and define $\varphi_\nu(w,\zeta)$ for $\nu \geq 1$, $|w| < \frac{1}{2}\theta_1$, inductively by

$$\varphi_\nu(w,\zeta) = w - g(\varphi_{\nu-1}(w,\zeta),\zeta).$$

Now, (6) with $w = 0$ implies that $|g(z,\zeta)| \leq \frac{1}{2}|z|$ for $|z| < \theta_1$, so that, if $|\varphi_{\nu-1}(w,\zeta)| < \theta_1$, we have $|\varphi_\nu(w,\zeta)| \leq |w| + \frac{1}{2}\theta_1 < \theta_1$ for $|w| < \frac{1}{2}\theta_1$. Thus φ_ν is well-defined and maps $B_r(0, \frac{1}{2}\theta_1)$ into $B_r(0,\theta_1)$. Further

$$\begin{aligned} |\varphi_{\nu+1}(w,\zeta) - \varphi_\nu(w,\zeta)| &= |g(\varphi_\nu(w,\zeta),\zeta) - g(\varphi_{\nu-1}(w,\zeta),\zeta)| \\ &\leq \frac{1}{2}|\varphi_\nu(w,\zeta) - \varphi_{\nu-1}(w,\zeta)| \text{ by (6)}, \end{aligned}$$

so that, since $|\varphi_1(w,\zeta) - \varphi_0(w,\zeta)| = |w| < \frac{1}{2}\theta_1$, we have $|\varphi_{\nu+1}(w,\zeta) - \varphi_\nu(w,\zeta)| \leq 2^{-\nu-1}\theta_1$ for $\nu \geq 0$. It follows that $\lim_{\nu\to\infty} \varphi_\nu(w,\zeta) = \varphi(w,\zeta)$ exists uniformly on $B_r(0,\frac{1}{2}\theta_1) \times B_s(0,R_2)$ so that φ is holomorphic and satisfies

$$\varphi(w,\zeta) = w - g(\varphi(w,\zeta),\zeta),\ |\varphi(w,\zeta)| \leq \theta_1 \text{ for } |w| < \frac{1}{2}\theta_1.$$

However, this implies, because of (6), that $|\varphi(w,\zeta)| < \theta_1$ for $|w| < \frac{1}{2}\theta_1$. Moreover, the equation $\varphi(w,\zeta) = w - g(\varphi(w,\zeta),\zeta)$ can be written

$$(7)\qquad F(\varphi(w,\zeta),\zeta) = w,\ f(\varphi(w,\zeta),\zeta) = f(0,\zeta) + D(0,\zeta)w,\ |w| < \frac{1}{2}\theta_1.$$

In particular, for fixed ζ with $|\zeta| < R_2$, $F(B_r(0,\theta_1),\zeta) \supset B_r\left(0,\frac{1}{2}\theta_1\right)$ and $f(B_r(0,\theta_1),\zeta) - f(0,\zeta)$ contains the image of $B_r\left(0,\frac{1}{2}\theta_1\right)$ under the linear map $D(0,\zeta)$. Since $\|D(0,\zeta)^{-1}\| \leq \frac{K}{\mu}M_1^{r-1}$ (by (3)), we have

$$D(0,\zeta)B_r\left(0,\frac{1}{2}\theta_1\right) \supset B_r\left(0,\frac{\mu}{2K}\frac{\theta_1}{M_1^{r-1}}\right) \supset B_r(0,\theta)$$

by (2). This proves (ii).

To prove (iii), we remark that $|f(0,\zeta)| = |f(0,\zeta) - f(0,0)| \leq s\,M_3\theta_2$ for $|\zeta| \leq \theta_2$ (by assumption (c)). If $sM_3\theta_2 \leq \frac{1}{2}\theta$, then 0 lies in $B_r(f(0,\zeta),\theta)$, hence in the image of $B_r(0,\theta_1)$ under $z \mapsto f(z,\zeta)$. Since by (2) and (3),

$$\begin{aligned}|D(0,\zeta)^{-1}f(0,\zeta)| &\leq \frac{K}{\mu}M_1^{r-1}sM_3\theta_2\\ &< \frac{1}{2}\frac{K}{\mu}M_1^{r-1}\theta \leq \frac{1}{2}\theta_1,\end{aligned}$$

the point $-D(0,\zeta)^{-1}f(0,\zeta)$ lies in $B_r\left(0,\frac{1}{2}\theta_1\right)$, and the point $z(\zeta) \in B_r\left(0,\frac{1}{2}\theta_1\right)$ with $f(z,\zeta) = 0$, unique by (i), is given by

$$z(\zeta) = \varphi(-D(0,\zeta)^{-1}f(0,\zeta),\zeta).$$

This proves (iii).

Remark. — Lemma 2.1 remains valid if we replace $\mathbb{C}^r, \mathbb{C}^s$ by $\mathbb{R}^r, \mathbb{R}^s$ and holomorphic maps by smooth (or real-analytic) maps. The proof above implies that the function φ is continuous. However, the standard inverse function theorem implies that it is smooth (or real-analytic). In particular, the connectedness statement in Part (iii) implies that there is $\rho_1 > 0$ (depending only on n, C_1, D) such that for $P \in W$, $Q_\rho \cap Z(P) \subset V^0(P)$ for $0 < \rho \leq \rho_1$.

Recall that $\mathcal{W}$ is the space of all $P = (p_1, \ldots, p_r)$ with $p_j \in \mathcal{H}^D$, $P(0) = 0$, $\|P\| \leq C_1$, $J_P(0) = 1$ and

$$|J_P(z)|^2 + |p_1(z)|^2 + \cdots + |p_r(z)|^2 \geq c_0(1 + |z|^2)^{-N},\ z \in \mathbb{C}^n.$$

In particular, $J_P \neq 0$ on $\mathcal{Z}(P)$, so that $\mathcal{Z}(P)$ is smooth and $\mathcal{V}(P) = \mathcal{V}^0(P)$ is the connected component of $\mathcal{Z}(P)$ through the origin.

LEMMA 2.2. — *There exist constants $c_1 > 0, N_1 \geq 1$ depending only on C_1, D, n, c_0, N such that if $P \in \mathcal{W}$ and X is any connected component of $\mathcal{Z}(P)$, then, for any $z \in X$, we have*

$$\mathcal{Z}(P) \cap B(z, c_1(1 + |z|^2)^{-N_1}) \subset X.$$

Proof. — In this argument, we denote by $(\gamma, m), (\gamma_j, m_j), (\gamma', m')$, etc. constants depending only on C_1, D, n, c_0, N.

If $z \in \mathcal{Z}(P)$, we have $P(z) = 0$ so that $|J_P(z)|^2 \geq c_0(1 + |z|^2)^{-N}$. Since $p_j \in \mathcal{H}^D$, $\|P\| \leq C_1$, we have $|J_P(w)|^2 \geq \frac{1}{2}c_0(1 + |z|^2)^{-N}$ if $w \in \mathbb{C}^n$, $|w - z| \leq \gamma(1 + |z|^2)^{-m}$ (for suitable $\gamma > 0$, $m \geq 1$). We apply Lemma 2.1 with $R_1 = R_2 = \gamma(1+|z|^2)^{-m}$, $s = n-r$, $\zeta_j = z_{r+j} (1 \leq j \leq s)$ and $f_j = P_j$ (replacing the origin by z). We have $\left|\frac{\partial f_j}{\partial z_k}\right|, \left|\frac{\partial^2 f_i}{\partial z_j\, \partial z_k}\right|, \left|\frac{\partial f_j}{\partial \zeta_\ell}\right| \leq \gamma'(1+|z|^2)^{m'}$ on $B(z, 1)$ $(1 \leq i, j, k \leq r,\ 1 \leq \ell \leq s)$, and we can apply Lemma 2.1 with $\theta_1 = \gamma_1(1 + |z|^2)^{-m_1}$, $\theta_2 = \gamma_2(1 + |z|^2)^{-m_2}$, $\theta = \gamma_3(1 + |z|^2)^{-m_3}$. Part (iii) of the lemma implies that

$$\left\{w \in \mathcal{Z}(P) \mid \sum_1^r |w_j - z_j|^2 < \theta_1^2,\ \sum_{r+1}^n |w_k - z_k|^2 < \theta_2^2\right\}$$

is connected, hence contained in X since $z \in X$. The lemma follows.

DEFINITION 2.3. — *Let $E \subset \mathbb{C}^n$ and let $\gamma > 0$, $m > 0$. We set*

$$T_{\gamma,m}(E) = \bigcup_{z \in E} B(z, \gamma(1 + |z|^2)^{-m}).$$

LEMMA 2.4. — *There exist constants $\gamma, m > 0$ depending only on the constants defining $\mathcal{W}$ (viz. C_1, n, D, c_0, N) such that*

If $P \in \mathcal{W}$ and $\mathcal{V}_1, \dots, \mathcal{V}_t$ are the connected components of $\mathcal{Z}(P)$, then $T_{\gamma,m}(\mathcal{V}_i) \cap T_{\gamma,m}(\mathcal{V}_j) = \emptyset$ if $i \neq j$.

This follows easily from Lemma 2.2.

LEMMA 2.5. — *Let $\gamma_0, m_0 > 0$ be given. There exist constants $\gamma, m > 0$ depending only on γ_0, m_0 and the constants defining $\mathcal{W}$ such that*

(8) If $P \in \mathcal{W}$ and $z \notin T_{\gamma_0,m_0}(\mathcal{Z}(P))$, then,

$$|P(z)|^2 = |p_1(z)|^2 + \cdots + |p_r(z)|^2 \geq \gamma(1 + |z|^2)^{-m}.$$

Proof. — For any $z \in \mathbb{C}^n$, we have $|J_P(z)|^2 + |P(z)|^2 \geq c_0(1 + |z|^2)^{-N}$. Let $w \in \mathbb{C}^n \backslash T_{\gamma_0,m_0}(\mathcal{Z}(P))$ and suppose that

$$|P(w)|^2 \leq \frac{1}{2} c_0(1 + |w|^2)^{-N}. \tag{9}$$

Then $|J_P(w)|^2 \geq \frac{1}{2} c_0(1+|w|^2)^{-N}$. Now, since $P \in \mathcal{W}$ and $w \notin T_{\gamma_0,m_0}(\mathcal{Z}(P))$, there exist constants $\gamma', m' > 0$ depending only on γ_0, m_0 and the constants in $\mathcal{W}$ such that

(a) $|J_P(z)|^2 \geq \frac{1}{4} c_0(1 + |w|^2)^{-N}$ if $z \in B(w, \gamma'(1 + |w|^2)^{-m'})$

(b) $B(w, \gamma'(1 + |w|^2)^{-m'}) \cap \mathcal{Z}(P) = \emptyset$.

Lemma 2.1 implies that there are constants $\gamma'', m'' > 0$ (depending only on γ', m' and the constants in $\mathcal{W}$) such that the image of $B(w, \gamma'(1 + |w|^2)^{-m'})$ under the map $P : \mathbb{C}^n \to \mathbb{C}^r$ contains the ball $B(P(w), \theta)$ where $\theta = \gamma''(1 + |w|^2)^{-m''}$. By (b) above, this image does not contain $0 \in \mathbb{C}^r$, so that

$$|P(w)| \geq \gamma''(1 + |w|^2)^{-m''}. \tag{10}$$

Thus if (9) holds, so does (10), which proves the lemma.

LEMMA 2.6. — *There exist constants $\gamma_0, m_0 > 0$ depending only on the constants in $\mathcal{W}$ such that the following holds.*

Let $P \in \mathcal{W}$ and $\Omega = T_{\gamma_0,m_0}(\mathcal{Z}(P))$. Then, there exists a holomorphic map $\pi : \Omega \to \mathbb{C}^n$ with the following properties:

(i) $\pi(\Omega) \subset \mathcal{Z}(P)$, $\pi(z) = z$ if $z \in \mathcal{Z}(P)$;

(ii) If $z = (z_1, \dots, z_n)$ and $\pi(z) = (\pi_1(z), \dots, \pi_n(z))$, we have $\pi_k(z) = z_k$ for $r < k \leq n$;

(iii) $|\pi(z) - z| < 1$ for any $z \in \Omega$.

Proof. — Since $|J_P(w)|^2 \geq c_0(1+|w|^2)^{-N}$ if $w \in \mathcal{Z}(P)$, we can choose $\gamma_1, m_1 > 0$, depending only on the constants in $\mathcal{W}$, such that

$$|J_P(z)|^2 \geq \frac{1}{2}c_0(1+|z|^2)^{-N} \quad \text{for} \quad z \in T_{\gamma_1,m_1}(\mathcal{Z}(P)).$$

By Lemma 2.1, there exist $(\gamma_2, m_2), (\gamma_3, m_3)$ (depending only on the constants in $\mathcal{W}$) such that if $w \in \mathcal{Z}(P)$ and we set

$$\Delta(w) = \left\{(z_1, \dots, z_n) \in \mathbb{C}^n \mid \sum_1^r |z_j - w_j|^2 < \theta_1^2, \sum_{r+1}^n |z_k - w_k|^2 < \theta_2^2\right\}$$

with $\theta_1 = \gamma_2(1+|w|^2)^{-m_2}$, $\theta_2 = \gamma_3(1+|w|^2)^{-m_3}$, then

$$\Delta(w) \cap \mathcal{Z}(P) = \Big\{(\varphi_1^{(w)}(z_{r+1}, \dots, z_n), \dots, \varphi_r^{(w)}(z_{r+1}, \dots, z_n), z_{r+1}, \dots, z_n) \mid \sum_{r+1}^n |z_k - w_k|^2 < \theta_2^2\Big\}$$

where $\varphi^{(w)} = (\varphi_1^{(w)}, \dots, \varphi_r^{(w)})$ is a holomorphic map of $B_{n-r}(w'', \theta_2) \subset \mathbb{C}^{n-r}$ into $B_r(w', \theta_1) \subset \mathbb{C}^r$; here we have written $w' = (w_1, \dots, w_r), w'' = (w_{r+1}, \dots, w_n)$. Moreover, if $\tilde{w} \in \mathcal{Z}(P)$ and $\Delta(w) \cap \Delta(\tilde{w}) \neq \emptyset$, Part (i) of Lemma 2.1 implies that $(\varphi^{(w)}, z_{r+1}, \dots, z_n) = (\varphi^{(\tilde{w})}, z_{r+1}, \dots, z_n)$ on $\Delta(w) \cap \Delta(\tilde{w})$.

Thus, we may define a holomorphic map

$$\pi : \bigcup_{w \in \mathcal{Z}(P)} \Delta(w) \to \mathcal{Z}(P)$$

by setting $\pi(z_1, \dots, z_n) = (\varphi_1^{(w)}(z_{r+1}, \dots, z_n), \dots, \varphi_r^{(w)}(z_{r+1}, \dots, z_n), z_{r+1}, \dots, z_n)$ for $z \in \Delta(w)$. If we choose γ_0, m_0 such that $\gamma_0 < 1$ and $T_{\gamma_0,m_0}(\mathcal{Z}(P)) \subset \bigcup_{w \in \mathcal{Z}(P)} \Delta(w)$, then $\pi \mid T_{\gamma_0,m_0}(\mathcal{Z}(P))$ has the properties stated.

LEMMA 2.7. — *Let $0 < \gamma \leq 1$ and m be an integer ≥ 1. Let $U = \{z \in \mathbb{C}^n \mid |\operatorname{Re} z_j| < 1, |\operatorname{Im} z_j| < 1, j = 1, \dots, n\}$.*

There exists a constant C_* depending only on n and a sequence $\{U_\nu\}_{\nu \geq 0}$ of open sets of the form

$$U_\nu = a_\nu + \rho_\nu U, \quad a_\nu \in \mathbb{C}^n, \quad \rho_\nu > 0,$$

with the following properties:

(a) $\bigcup_\nu U_\nu = \mathbb{C}^n$.

(b) $C_*^{-m}\gamma(1+|a_\nu|^2)^{-m} \le \rho_\nu \le C_*^m\gamma(1+|a_\nu|^2)^{-m}$, $\nu \ge 0$.

(c) *If, for $t \ge 1$, we set $U_\nu(t) = a_\nu + t\rho_\nu U$ when $U_\nu = a_\nu + \rho_\nu U$, then, for any $t \ge 1$ and $\nu \ge 0$, the number of μ such that $U_\nu(t) \cap U_\mu(t) \ne \emptyset$ is $\le C_*^{tm}$.*

Proof. — Let $N_0 = [2/\gamma]$ be the integral part of $2/\gamma$. We have $\frac{1}{\gamma} \le \frac{2}{\gamma} - 1 < N_0 \le \frac{2}{\gamma}$. For $k \ge 0$, let $N_k = 2^{2km}N_0$. Let $Q = \{z \in \mathbb{C}^n \mid |\operatorname{Re} z_j| \le 1, |\operatorname{Im} z_j| \le 1, j = 1, \ldots, n\}$ be the closed unit cube in $\mathbb{C}^n$ and, for $k \ge 0$, let $Q^{(k)}$ be the closed cube of side 2^{k+1} in $\mathbb{C}^n : Q^{(k)} = 2^k Q = \{z \in \mathbb{C}^n \mid |\operatorname{Re} z_j| \le 2^k, |\operatorname{Im} z_j| \le 2^k\}$; we set $Q^{(-1)} = \emptyset$.

Divide the interval $\{x \in \mathbb{R} \mid -2^k \le x \le 2^k\}$ into equal intervals $I_\ell^{(k)}$, $\ell = 1, \ldots, L$, of length N_k^{-1} (so that $L = 2^{k+1}N_k$). Consider the cubes of side N_k^{-1} in $\mathbb{C}^n$ obtained by taking products of the $I_\ell^{(k)}$: $\operatorname{Re} z_j \in I_{\ell_j}^{(k)}$, $\operatorname{Im} z_j \in I_{\ell'_j}^{(k)}$, $1 \le \ell_j, \ell'_j \le L$.

Let $Q_\alpha^{(k)}$, $\alpha = 1, \ldots, \alpha_k$ be a list of those cubes of side N_k^{-1} described above which are not contained in $Q^{(k-1)}$. Then $\bigcup_k Q_\alpha^{(k)} = Q^{(k)} \setminus \operatorname{int}(Q^{(k-1)})$.

Now, if $z \in Q = Q^{(0)}$, $1 \le 1 + |z|^2 \le 2n + 1$, while $1 + 2^{2(k-1)} \le 1 + |z|^2 \le (2n+1)2^{2k}$ for $z \in Q^{(k)} \setminus \operatorname{int}(Q^{(k-1)})$, $k \ge 1$. In particular, if $Q_\alpha^{(k)} = a + \rho Q$, then

$$2^{2k-2} \le 1 + |a|^2 \le (2n+1)2^{2k},\ 2\rho = N_k^{-1},\ \frac{1}{\gamma}2^{2km} \le N_k \le \frac{1}{\gamma}2^{2km+1}.$$

It follows that if $Q_\alpha^{(k)} = a + \rho Q$, then

$$2^{-2m-2}\gamma(1+|a|^2)^{-m} \le \rho \le (2n+1)^m\gamma(1+|a|^2)^{-m}. \tag{11}$$

Let $0 < \epsilon < 1$, and set $U_\alpha^{(k)} = U_\alpha^{(k)}(1+\epsilon) = a + (1+\epsilon)\rho U$ if $Q_\alpha^{(k)} = a + \rho Q$; then $Q_\alpha^{(k)} \subset U_\alpha^{(k)}$; if $\{U_\nu\}_{\nu\ge 0}$ is an enumeration of $\{U_\alpha^{(k)}, k \ge 0, \alpha = 1, \ldots, \alpha_k\}$, then, since $\bigcup_{k\ge 0}\ \bigcup_{1\le\alpha\le\alpha_k} Q_\alpha^{(k)} = \mathbb{C}^n$, it follows that $\{U_\nu\}_{\nu\ge 0}$ covers $\mathbb{C}^n$ which is (a). Moreover, (11) implies that $\{U_\nu\}_{\nu\ge 0}$ satisfies (b) with $C_* = 4(n+2)$ (since $0 < \epsilon < 1$).

It remains to prove that $\{U_\nu\}_{\nu\ge 0}$ satisfies (c).

We first remark that if $t \ge 1$, and $Q_\alpha^{(k)}(t) \cap Q_\beta^{(\ell)}(t) \ne \emptyset$ (where $Q_\alpha^{(k)}(t) = a + t\rho Q$ if $Q_\alpha^{(k)} = a + \rho Q$), then we must have $|k - \ell| \le t + 2$; in fact, if $\ell \ge k + t + 2$, then $Q_\alpha^{(k)}(t)$ is contained in the cube $|\operatorname{Re} z_j|$, $|\operatorname{Im} z_j| \le 2^k + t - 1$ (since any cube $Q_\alpha^{(k)}$ has side ≤ 1) while $Q_\beta^{(\ell)}(t)$ lies outside $|\operatorname{Re} z_j|$, $|\operatorname{Im} z_j| < 2^{\ell-1} - t + 1$. Hence, if $2^{\ell-1} - t + 1 > 2^k + t - 1$, in particular if $\ell \geq k + t + 2$, we must have $Q_\alpha^{(k)}(t) \cap Q_\beta^{(\ell)}(t) = \emptyset$.

Fix k, α. From the remark above, it follows that if $Q_\alpha^{(k)}(t) \cap Q_\beta^{(\ell)}(t) \neq \emptyset$, then $Q_\beta^{(\ell)} \subset Q_\alpha^{(k)}(A_1^{tm})$ where A_1 is an absolute constant. Moreover, the side of $Q_\beta^{(\ell)}$ is $\geq A_2^{-tm}\rho$, where ρ is the side of $Q_\alpha^{(k)}$, and A_2 is also an absolute constant. The number of cubes of side $\geq A_2^{-tm}\rho$ contained in a cube of side $A_1^{tm}\rho$ and no two of which have interior points in common is $\leq C_*^{tm}$. Replacing t by $(1+\epsilon)t \leq 2t$ we obtain (c).

This proves the lemma.

LEMMA 2.8. — *Let $0 < \gamma_0 \leq 1$ and let m_0 be an integer ≥ 1. There exists a sequence $C_k = C_k(n, \gamma_0, m_0)$ of constants, $k = 0, 1, \ldots$ (C_k depends only on k, n, γ_0, m_0) such that the following holds.*

Let $E \subset \mathbb{C}^n$ be any subset. For a function φ, write $\operatorname{supp}(\varphi)$ for its support.

We can find a C^∞ function $\varphi : \mathbb{C}^n \to \mathbb{R}$ with the following properties:

(a) $0 \leq \varphi \leq 1$, $\varphi \mid E \equiv 1, \operatorname{supp}(\varphi) \subset T_{\gamma_0, m_0}(E)$.

(b) *If* $D^{(k)} = \dfrac{\partial^k}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n} \partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}}$, $\alpha_1 + \cdots + \alpha_n + \beta_1 + \cdots + \beta_n = k$, *denotes any differentiation of order $k \geq 0$, we have*

$$|D^{(k)}\varphi(z)| \leq C_k(n, \gamma_0, m_0)(1 + |z|^2)^{km_0}, \quad \forall z \in \mathbb{C}^n.$$

Proof. — Let C_* be the constant in parts (b), (c) of Lemma 2.7. Set $m = m_0$, and choose a constant $\gamma > 0$ (depending only on C_*, n, γ_0, m) with the following property:

(12) Let $Q(a, \rho) = a + \rho Q$ (Q = closed unit cube in $\mathbb{C}^n$). If $b \in Q(a, \rho)$ and $\rho \leq 2C_*^m \gamma (1 + |a|^2)^{-m}$, then $Q(a, \rho) \subset B(b, \rho_1)$ with $\rho_1 = \gamma_0 (1 + |b|^2)^{-m}$.

Let U be the open unit cube in $\mathbb{C}^n$ and $\{U_\nu\}_{\nu \geq 0}$, $U_\nu = a_\nu + \rho_\nu U$, be an open covering of $\mathbb{C}^n$ with the properties given in Lemma 2.7. We set $V_\nu = a_\nu + 2\rho_\nu U$, $\nu \geq 0$.

Let $\psi \in C^\infty(\mathbb{C}^n)$, $0 \leq \psi \leq 1$ be such that $\psi = 1$ on Q and $\operatorname{supp}(\psi) \subset 2U$.

Define $\psi_\nu(z) = \psi\left(\dfrac{z - a_\nu}{\rho_\nu}\right)$. Then $\psi_\nu = 1$ on $Q(a_\nu, \rho_\nu)$ and

$\mathrm{supp}(\psi_\nu) \subset V_\nu$. Moreover, if $D^{(k)}$ is a differentiation of order k, we have

$$|D^{(k)}\psi_\nu(z)| \leq \rho_\nu^{-k} \sup_{\mathbb{C}^n} |D^{(k)}\psi|,\ z \in \mathbb{C}^n.$$

Since $\mathrm{supp}(\psi_\nu) \subset V_\nu$ and $C_*^{-m}\gamma(1+|a_\nu|^2)^{-m} \leq \rho_\nu \leq C_*^m\gamma(1+|a_\nu|^2)^{-m}$ (property (b) in Lemma 2.7), this implies that

$$|D^{(k)}\psi_\nu(z)| \leq \overline{C}_k(\gamma, m, n)(1+|z|^2)^{km},\ z \in \mathbb{C}^n.$$

Let S be the set of $\nu \geq 0$ such that $V_\nu \cap E \neq \emptyset$, and define $\Psi(z) = \sum\limits_{\nu=0}^{\infty} \psi_\nu(z)$, $\varphi(z) = \dfrac{1}{\Psi(z)} \sum\limits_{\nu \in S} \psi_\nu(z)$.

Ψ is C^∞ since $\{V_\nu\}$ is locally finite, and $\Psi \geq 1$ on $\mathbb{C}^n$ since $\psi_\nu = 1$ on U_ν and $\cup U_\nu = \mathbb{C}^n$.

If $z \in E$ and $\nu \notin S$, then $z \notin V_\nu$, so that $z \notin \mathrm{supp}(\psi_\nu)$. Hence, for $z \in E$, $\sum\limits_{\nu \in S} \psi_\nu(z) = \Psi(z)$, so that $\psi(z) = 1$.

Further, if $z \in \mathrm{supp}(\varphi)$, then $z \in \mathrm{supp}(\psi_\nu)$ for some $\nu \in S$ (since $\{V_\mu\}$ is locally finite) so that $z \in V_\nu$ for some ν with $V_\nu \cap E \neq \emptyset$. The choice (12) above of γ shows that we then have $V_\nu \subset B(b, \rho_1)$, $b \in E$, $\rho_1 = \gamma_0(1+|b|^2)^{-m}$, so that $V_\nu \subset T_{\gamma_0,m_0}(E)$. Thus $\mathrm{supp}(\varphi) \subset T_{\gamma_0,m_0}(E)$.

We have $|D^{(k)}\psi_\nu(z)| \leq \overline{C}_k(\gamma, m, n)(1+|z|^2)^{km}$; by property (c) in Lemma 2.7, we therefore have

$$|D^{(k)}\Psi(z)| \leq C_*^{2m}\overline{C}_k(\gamma, m, n)(1+|z|^2)^{km},$$

and the same argument, applied to the numerator, gives

$$|D^{(k)} \sum_{\nu \in S} \psi_\nu(z)| \leq C_*^{2m}\overline{C}_k(\gamma, m, n)(1+|z|^2)^{km}.$$

These inequalities, and the fact that $\Psi \geq 1$ on $\mathbb{C}^n$ imply that

$$|D^{(k)}\varphi(z)| \leq C_k'(\gamma, m, n)(1+|z|^2)^{km},$$

thus proving the lemma.

3. Separating components of a smooth algebraic variety.

Let $n \geq 2$, $1 \leq r \leq n-1$, $D \geq 1$, $C_1 > 0$, $c_0 > 0$ and $N \geq 1$ be given. They define the space $\mathcal{W}$ introduced in §2:

$\mathcal{W}$ is the space of r-tuples $P = (p_1, \ldots, p_r)$, $p_j \in \dot{\mathcal{H}}^D$, $P(0) = 0$, $\|P\| \leq C_1$, $J_P(0) = 1$ such that

$$|J_P(z)|^2 + |P(z)|^2 \geq c_0(1+|z|^2)^{-N}$$

(where $J_P(z) = \det\left(\frac{\partial p_j}{\partial z_k}\right)_{1\leq j,k\leq r}$).

The aim of this section is the proof of the following theorem.

The formalism of the Koszul complex is also developed in [KT] and applied to a closely related problem.

THEOREM 3.1. — *There exist constants $C', D' > 0$ depending only on n, D, C_1, c_0, N such that the following holds.*

Let $P \in \mathcal{W}$ and let X be any connected component of $\mathcal{Z}(P) = \{z \in \mathbb{C}^n \mid P(z) = 0\}$.

We can find a polynomial $F(z_1, \dots, z_n)$ with the following properties:

(i) $\deg F \leq D'$, $\|F\| \leq C'$.

(ii) $F \mid X = 1$, $F \mid \mathcal{Z}(P) \backslash X = 0$.

We begin with some algebraic preliminaries.

Let $r \geq 1$ be an integer and let $E_p = \overset{p}{\wedge}\, \mathbb{C}^r$ be the p-th exterior power of $\mathbb{C}^r$ ($E_0 = \mathbb{C}, E_1 = \mathbb{C}^r$ and $E_p = 0$ if $p > r$).

Let $e_1, \dots, e_r$ be the standard basis of $\mathbb{C}^r [e_i = (0, \dots, 1, \dots, 0)$ with 1 in the i-th place].

If I is an increasing p-tuple $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < \cdots < i_p \leq r$, set $e_I = e_{i_1} \wedge \cdots \wedge e_{i_p} \in E_p$. (If I is not in increasing order, we use the convention that e_I is alternating in the indices $i_1, \dots, i_p$.) The $\{e_I\}, I$ increasing, form a basis of E_p.

(1) Let $x_1, \dots, x_r \in \mathbb{C}$. We define a map $\kappa = \kappa_p = \kappa_p(x)$,

$$\kappa : E_p \longrightarrow E_{p-1}$$

as follows: κ is the $\mathbb{C}$-linear map such that

$$(2) \qquad \kappa(e_I) = \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} e_{I\backslash\{i_k\}}, e_{I\backslash\{i_k\}} = e_{i_1} \wedge \cdots \wedge \hat{e}_{i_k} \wedge \cdots \wedge e_{i_p},$$

where the hat over e_{i_k} indicates that it is to be deleted. On $E_1, \kappa : E_1 \to \mathbb{C}$ is the map $\sum u_i e_i \mapsto \sum x_i u_i$. One checks easily that the map $\kappa \circ \kappa : E_{p+1} \to E_{p-1} (p \geq 1)$ is 0:

$$\kappa^2 = 0,$$

so that one has an algebraic complex, usually called the *Koszul complex* of $(x_1, \dots, x_r)$:

$$(3) \qquad 0 \to E_r \xrightarrow{\kappa} E_{r-1} \to \cdots \to E_1 \xrightarrow{\kappa} E_0 \to 0.$$

Let $y_1, \dots, y_r$ be any r complex numbers. We define a map $h = h_p = h_p(y)$:

$$h : E_p \to E_{p+1} (p \geq 0)$$

by

$$h(v) = \Big(\sum_{j=1}^{r} y_j e_j\Big) \wedge v,\ v \in E_p. \tag{4}$$

LEMMA 3.2. — *Let $\xi = \sum_{j=1}^{r} x_j \cdot y_j$. We have:*

$$\kappa h(v) = \xi \cdot v \textit{ if } v \in E_0; \tag{5}$$

$$(\kappa h + h\kappa)(v) = \xi \cdot v \textit{ if } v \in E_p,\ p > 0. \tag{6}$$

Proof. — This is a standard fact about the Koszul complex. The verification runs as follows.

If $v \in E_0 = \mathbb{C}, h(v) = vy_1e_1 + \dots + vy_r\, e_r$, and $\kappa(h(v)) = \sum x_j y_j \cdot v$.

Let $p > 0$. It is enough to check (6) when $v = e_I$ for some $I = (i_1 < \dots < i_p)$. We have

$$\begin{aligned} h\kappa(e_I) &= \Big(\sum_{j=1}^{r} y_j e_j\Big) \wedge \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} e_{I\setminus\{i_k\}} \\ &= \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} y_{i_k} e_{i_k} \wedge e_{I\setminus\{i_k\}} + \sum_{j\notin I} \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} y_j e_j \wedge e_{I\setminus\{i_k\}} \\ &= \Big(\sum_{k=1}^{p} x_{i_k} y_{i_k}\Big) e_I + \sum_{j\notin I} \sum_{k=1}^{p} (-1)^{k-1} x_{i_k} y_j e_j \wedge e_{I\setminus\{i_k\}} \end{aligned}$$

while

$$\begin{aligned} \kappa h(e_I) &= \kappa\Big(\sum_{j=1}^{r} y_j e_j \wedge e_I\Big) = \sum_{j\notin I} y_j \kappa(e_j \wedge e_I) \\ &= \sum_{j\notin I} x_j y_j e_I + \sum_{j\notin I} y_j \sum_{k=1}^{p} (-1)^k x_{i_k} e_j \wedge e_{I\setminus\{i_k\}}; \end{aligned}$$

adding these two equations, we get (6) for $v = e_I$, thus proving the lemma.

Let $0 \leq q \leq n$, $0 \leq p \leq r$. We denote by $\mathcal{A}^q(E_p)$ the space of C^∞ forms of type $(0, q)$ on $\mathbb{C}^r$ with values in E_p. If $J = (j_1 < \dots < j_q)$ is an increasing q-tuple of integers j_k, $1 \leq j_k \leq n$, and we set $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, an element $\omega \in \mathcal{A}^q(E_p)$ can be written uniquely in the form

$$\omega = \sum_J v_J \otimes d\bar{z}^J, \ v_J = \sum_I v_J^I e_I, \ I = (i_1 < \cdots < i_p),$$

where the v_J^I are C^∞ functions on $\mathbb{C}^n$. We set

$$|\omega(z)|^2 = \sum_J |v_J(z)|^2 = \sum_J \sum_I |v_J^I(z)|^2.$$

The $\overline{\partial}$-operator extends to a map (denoted again by $\overline{\partial}$)

$$\overline{\partial} : \mathcal{A}^q(E_p) \to \mathcal{A}^{q+1}(E_p), \ p, q \geq 0.$$

If $\omega = \sum_{J,I} v_J^I e_I \otimes d\bar{z}^J$, we have

$$\overline{\partial}\omega = \sum_{I,J} \sum_{\nu=1}^n \frac{\partial v_J^I}{\partial \bar{z}_\nu} e_I \otimes d\bar{z}_\nu \wedge d\bar{z}^J.$$

If $f_1, \ldots, f_r$ are C^∞ functions on $\mathbb{C}^n$, we define a map $\kappa = \kappa_p^q(f) : \mathcal{A}^q(E_p) \to \mathcal{A}^q(E_{p-1})$ by

$$\kappa\Big(\sum_{I,J} v_J^I e_I \otimes d\bar{z}^J\Big)(z) = \sum_{I,J} v_J^I(z) \sum_{k=1}^p (-1)^{k-1} f_{i_k}(z) e_{I\setminus\{i_k\}} \otimes d\bar{z}^J. \tag{7}$$

Note that if $f_1, \ldots, f_r$ are holomorphic on $\mathbb{C}^n$, then

$$\kappa\overline{\partial}(\omega) = \overline{\partial}\kappa(\omega), \ \omega \in \mathcal{A}^q(E_p).$$

In fact, in terms of bases, $\kappa(\omega)$ is given by multiplication by a matrix of *holomorphic* functions, and this operation commutes with $\overline{\partial}$.

If $\Omega_0 \subset \mathbb{C}^n$ is open, and $g_1, \ldots, g_r \in C^\infty(\Omega_0)$, and we denote by $\mathcal{A}_0^q(E_p)$ the subspace of $\mathcal{A}^q(E_p)$ consisting of forms ω with $\operatorname{supp}(\omega) \subset \Omega_0$, then, we can define a map $h = h_p^q(g) : \mathcal{A}_0^q(E_p) \to \mathcal{A}_0^q(E_{p+1})$ by:

$$h\Big(\sum_{I,J} v_J^I e_I \otimes d\bar{z}^J\Big) = \sum_J \sum_I \sum_{j=1}^r g_j v_J^I (e_j \wedge e_I) \otimes d\bar{z}^J \tag{8}$$

(i.e. we just operate on the coefficients of the $d\bar{z}^J$). Lemma 3.2 implies that

$$\kappa h = \text{ multiplication by } \sum_1^r f_j g_j \text{ on } \mathcal{A}_0^q(E_0),\ q \geq 0, \tag{9}$$

and

$$\kappa h + h\kappa = \text{ multiplication by } \sum_1^r f_j g_j \text{ on } \mathcal{A}_0^q(E_p),\ p > 0, q \geq 0. \tag{10}$$

We shall use the following L^2-existence theorem for the $\overline{\partial}$-operator due to Hörmander [H].

THEOREM 3.3. — *Let $q \geq 1, m \geq 1$, and let ω be a C^∞ form of type $(0,q)$ on $\mathbb{C}^n$ with $\overline{\partial}\omega = 0$ and such that*

$$\int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda < \infty \quad (d\lambda = \text{ Lebesgue measure}).$$

Then, there exists a C^∞ form α of type $(0,q-1)$ on $\mathbb{C}^n$ with

$$\overline{\partial}\alpha = \omega, \int_{\mathbb{C}^n} |\alpha|^2(1+|z|^2)^{-m-2} d\lambda \leq \int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda.$$

Note. — The existence of a form α with L^2_{loc} coefficients with these properties is proved in Hörmander's book [H], §4.4 (the second edition). If H is the Hilbert space of forms β of type $(0,q-1)$ with $\int |\beta|^2(1+|z|^2)^{-m-2}d\lambda < \infty$, we can choose α such that it is orthogonal, in H, to all forms $\beta \in H$ with $\overline{\partial}\beta = 0$ [since if α' is the projection of α onto the orthogonal complement of $\{\beta \in H, \overline{\partial}\beta = 0\}$, then $\overline{\partial}\alpha = \overline{\partial}\alpha'$ and $\int |\alpha'|^2(1+|z|^2)^{-m-2}d\lambda \leq \int |\alpha|^2(1+|z|^2)^{-m-2}d\lambda$].

Now, if $q = 1$, any solution of $\overline{\partial}\alpha = \omega$ is smooth. If $q > 1$, α is, in particular, orthogonal to any form $\overline{\partial}\varphi$, where φ is of type $(0, q-2)$ and is C^∞ with compact support: $\int(\alpha, \overline{\partial}\varphi)(1+|z|^2)^{-m-2}d\lambda = 0$. This implies that $\overline{\partial}^*((1+|z|^2)^{-m-2}\alpha) = 0$, where $\overline{\partial}^*$ is the formal adjoint of $\overline{\partial}$ in the Euclidean metric, viz., if $f = \sum\limits_{|J|=q-1} f_J d\overline{z}^J$, then

$$\overline{\partial}^*(f) = \sum_{|K|=q-2} \Big(\sum_{j=1}^n \frac{\partial f_{jK}}{\partial z_j}\Big) d\overline{z}^K.$$

The equations

$$\overline{\partial}\alpha = \omega, \ \omega \text{ smooth}, \overline{\partial}^*((1+|z|^2)^{-m-2}\alpha) = 0$$

are sufficient to guarantee the smoothness of α (see the proof of Theorem 4.2.5 in Hörmander's book [H]). In fact, they can be written $\overline{\partial}\alpha = \omega$, $\overline{\partial}^*\alpha = L_0(\alpha)$, L_0 an operator of order 0 with smooth coefficients. In particular

$$(\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial})\alpha = (\overline{\partial}L_0)(\alpha) + \overline{\partial}^*\omega;$$

since $\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is elliptic of order 2 and $\overline{\partial}L_0$ is of order 1 with smooth coefficients, once can simply apply the standard regularity theorem for elliptic operators.

Note that we are only using Hörmander's theorem for $\mathbb{C}^n$, not for arbitrary pseudoconvex domains. His method is very simple in this case.

COROLLARY 3.4. — *Let $q \geq 1$ and $\omega \in \mathcal{A}^q(E_p)$, $p \geq 0$. Assume that $\overline{\partial}\omega = 0$ and that $\int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda < \infty$. Then, there exists $\alpha \in \mathcal{A}^{q-1}(E_p)$ with*

$$\overline{\partial}\alpha = \omega, \int_{\mathbb{C}^n} |\alpha|^2(1+|z|^2)^{-m-2} d\lambda \leq \int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda.$$

If $\omega \in \mathcal{A}^q(E_p)$ and $\int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda < \infty$, we write

$$\|\omega\|_m^2 = \int_{\mathbb{C}^n} |\omega|^2(1+|z|^2)^{-m} d\lambda.$$

Proof of Theorem 3.1. — In the course of this proof, all constants $\gamma, \gamma', \gamma_j, m, m', m_j, C^{(j)}, N_j, \ldots$ will, unless otherwise stated, depend only on the constants n, D, C_1, r, c_0, N defining $\mathcal{W}$; dependence on other parameters will be explicitly indicated.

Let $P \in \mathcal{W}$ and let $X = X_1, X_2, \ldots, X_t$ be the connected components of $\mathcal{Z}(P) = \{z \in \mathbb{C}^n \mid P(z) = 0\}$. [The number t is $\leq D^r$ as can be proved using Bezout's Theorem; see the remark following Lemma 5.7. We shall not need this fact.]

By Lemma 2.4, we can find $\gamma > 0$, $m \geq 1$ so that

$$T_{\gamma,m}(X_i) \cap T_{\gamma,m}(X_j) = \emptyset \text{ if } i \neq j.$$

Now, if $S \subset \mathbb{C}^n$ is connected, so also is $T_{\gamma,m}(S)$ (for any $\gamma > 0, m > 0$). Thus

$$T_{\gamma,m}(\mathcal{Z}(P)) = \bigcup_i T_{\gamma,m}(X_i)$$

is the decomposition of $T_{\gamma,m}(\mathcal{Z}(P))$ into connected components.

Choose $\gamma' > 0$, $\gamma' < \gamma$ such that if $E = T_{\gamma',m}(\mathcal{Z}(P))$, then $T_{\gamma',m}(E) \subset T_{\gamma,m}(\mathcal{Z}(P))$. Set $\Omega = T_{\gamma',m}(X) = T_{\gamma',m}(X_1)$. Then $T_{\gamma',m}(\Omega) \subset T_{\gamma',m}(E) \subset \bigcup_i T_{\gamma,m}(X_i)$ and $T_{\gamma',m}(\Omega)$ is connected; since the $T_{\gamma,m}(X_i)$ are pointwise disjoint and connected, and $T_{\gamma',m}(\Omega) \cap T_{\gamma,m}(X_1) \neq \emptyset$, it follows that

$$T_{\gamma',m}(\Omega) \subset T_{\gamma,m}(X).$$

Hence, using Lemma 2.8, we obtain the following:

(11) There exists $\varphi \in C^\infty(\mathbb{C}^n)$, $0 \leq \varphi \leq 1$, such that

$$\varphi = 1 \text{ on } \Omega, \ \mathrm{supp}(\varphi) \subset T_{\gamma,m}(X),$$

and such that, for $k \geq 0$, there are constants C_k depending only on k, γ, γ', m, n with

$$|D^{(k)}\varphi(z)| \leq C_k(1+|z|^2)^{km},\ z \in \mathbb{C}^n,\ k \geq 0$$

for any differentiation $D^{(k)}$ of order k.

Clearly $\varphi \mid X_j = 0$ if $j \geq 2$.

Lemma 2.5 gives us:

(12) There exist constants $\gamma_1 > 0, N_1 \geq 1$ such that

$$\begin{aligned} |P(z)|^2 &= \sum_{j=1}^{r} |p_j(z)|^2 \\ &\geq \gamma_1(1+|z|^2)^{-N_1} \text{ if } z \notin T_{\gamma',m}(\mathcal{Z}(P)) = \bigcup_{i=1}^{t} T_{\gamma',m}(X_i). \end{aligned}$$

Consider the Koszul complex defined by the functions $(p_1, \ldots, p_r)$:

$$0 \to \mathcal{A}^q(E_r) \xrightarrow{\kappa} \cdots \to \mathcal{A}^q(E_1) \xrightarrow{\kappa} \mathcal{A}^q(E_0) \to 0, \tag{13}$$

κ being defined by:

$$\kappa(e_I \otimes d\bar{z}^J) = \sum_{k=1}^{p} (-1)^{k-1} p_{i_k}(z) e_{I\setminus\{i_k\}} \otimes d\bar{z}^J,$$
$$I = (i_1 < \cdots < i_p), J = (j_1 < \cdots < j_q).$$

Let $\mathcal{A}_0^q(E_p)$ be the space of smooth $(0,q)$ forms ω on $\mathbb{C}^n$ with values in E_p such that

$$\operatorname{supp}(\omega) \cap \mathcal{Z}(P) = \emptyset.$$

We define a map $h : \mathcal{A}_0^q(E_p) \to \mathcal{A}_0^q(E_{p+1})$ as in (8) above with $\Omega_0 = \mathbb{C}^n \setminus \mathcal{Z}(P), g_j = \overline{p}_j / |P|^2$:

$$h\Big(\sum_{I,J} v_J^I e_I \otimes d\bar{z}^J\Big) = \sum_{I,J} v_J^I \Big(\sum_{j=1}^{r} \frac{\overline{p}_j}{|P|^2} e_j\Big) \wedge e_I \otimes d\bar{z}^J.$$

Since $p_1 \dfrac{\overline{p}_1}{|P|^2} + \cdots + p_r \dfrac{\overline{p}_r}{|P|^2} = 1$ on $\Omega_0 = \mathbb{C}^n \setminus \mathcal{Z}(P)$, we obtain, from (9) and (10) above, the following:

$$\begin{aligned} \kappa h &= \text{ identity on } \mathcal{A}_0^q(E_0), q \geq 0, \text{ and,} \\ \kappa h + h\kappa &= \text{ identity on } \mathcal{A}_0^q(E_p) \quad \text{for } p > 0, q \geq 0. \end{aligned} \tag{14}$$

LEMMA 3.5. — *There is a sequence $\{\overline{C}_k\}_{k\geq 0}$ of constants depending only on k and the constants in $\mathcal{W}$, and $N_2 > 0$ (depending only on the constants in $\mathcal{W}$) such that*

If $\omega \in \mathcal{A}_0^q(E_p)$ is such that $\operatorname{supp}(\omega) \cap T_{\gamma',m}(\mathcal{Z}(P)) = \emptyset$, *and if we have estimates*

$$|D^{(k)}\omega(z)| \leq A_k(1+|z|^2)^{M_k}, z \in \mathbb{C}^n, k \geq 0, A_k \leq A_{k+1}, M_k \leq M_{k+1},$$

then

$$|D^{(k)}h(\omega)(z)| \leq \overline{C}_k A_k(1+|z|^2)^{M_k+(k+1)N_2},\ k \geq 0, z \in \mathbb{C}^n;$$

here $D^{(k)}$ runs over all differentiations of order k.

Proof. — The coefficients of the form $h(\omega)$ are linear combinations of functions of the form

$$\frac{\overline{p}_j}{|P|^2}\,u,$$

u being the coefficient of $e_I \otimes d\overline{z}^J$ in ω for some I, J. Since $\|p_j\| \leq C_1$, $\deg p_j \leq D$ and $|P(z)|^2 \geq \gamma_1(1+|z|^2)^{-N_1}$ on the support of u (by (12)), we have, for $\ell \geq 0$,

$$|D^{(\ell)}\Big(\frac{\overline{p}_j}{|P|^2}\Big)| \leq C'_\ell(1+|z|^2)^{3\ell D+(\ell+1)N_1}$$

(C'_ℓ depending only on ℓ and the constants in $\mathcal{W}$), while $|D^{(k-\ell)}u| \leq A_k(1+|z|^2)^{M_k}$ $(0 \leq \ell \leq k)$. The lemma follows.

Returning to the proof of Theorem 3.1, consider the function φ described in (11) above. Since $\varphi = 1$ on $\Omega = T_{\gamma',m}(X)$ and $\operatorname{supp}(\varphi) \subset T_{\gamma,m}(X)$, we have

$$\operatorname{supp}(\overline{\partial}\varphi) \subset T_{\gamma,m}(X) \setminus \Omega.$$

We define, successively,

$$\varphi_1 = h(\overline{\partial}\varphi), \varphi_2 = h(\overline{\partial}\varphi_1), \ldots, \varphi_p = h(\overline{\partial}\varphi_{p-1}), \ldots, \varphi_r = h(\overline{\partial}\varphi_{r-1}).$$

These are well-defined since $\operatorname{supp}(\overline{\partial}\varphi) \cap \mathcal{Z}(P) = \emptyset$ and neither $\overline{\partial}$ nor h increases supports. Moreover, we have

$$\varphi_p \in \mathcal{A}_0^p(E_p) \text{ for } 1 \leq p \leq r.$$

Now $\overline{\partial}\varphi \in \mathcal{A}_0^1(E_0)$ and, by (14) $\kappa(\varphi_1) = \kappa h(\overline{\partial}\varphi) = \overline{\partial}\varphi$. Hence $\kappa(\overline{\partial}\varphi_1) = \overline{\partial}\kappa(\varphi_1) = \overline{\partial}(\overline{\partial}\varphi) = 0$, and (14) now gives

$$\overline{\partial}\varphi_1 = (\kappa h + h\kappa)(\overline{\partial}\varphi_1) = \kappa h(\overline{\partial}\varphi_1) = \kappa(\varphi_2).$$

By induction, we obtain

(15) For $1 \leq p \leq r$, $\operatorname{supp}(\varphi_p) \subset T_{\gamma,m}(X) \setminus \Omega$, and

$$\kappa(\varphi_p) = \overline{\partial}\varphi_{p-1} \text{ for } 2 \leq p \leq r,$$
$$\kappa(\varphi_1) = \overline{\partial}\varphi.$$

Moreover, by Lemma 3.5 and induction on p, we obtain [since $\operatorname{supp}(\overline{\partial}\varphi) \cap T_{\gamma',m}(\mathcal{Z}(P)) = \emptyset$ and $|D^{(k)}\overline{\partial}\varphi(z)| \le C_{k+1}(1+|z|^2)^{(k+1)m}$ by (11)]:

(16) There is a constant $N_3 > 0$ (depending only on the constants in $\mathcal{W}$) and a sequence $\{C''_k\}_{k\ge 0}$ of constants depending only on k and the constants in $\mathcal{W}$ such that

$$|D^{(k)}\varphi_p(z)| \le C''_k(1+|z|^2)^{(k+1)N_3},\ z \in \mathbb{C}^n, k \ge 0, 1 \le p \le r.$$

We now make the following remark:

(17) For $q \ge 0$, the map $\kappa : \mathcal{A}_0^q(E_r) \to \mathcal{A}_0^q(E_{r-1})$ is injective.

In fact, if $v \in E_r = \overset{r}{\wedge}\mathbb{C}^r, v \ne 0$, we can write $v = ue_1 \wedge \cdots \wedge e_r\, u \in \mathbb{C}$, $u \ne 0$ and

$$\kappa(ue_1 \wedge \cdots \wedge e_r \otimes d\overline{z}^J) = \sum_{k=1}^{r}(-1)^{k-1}p_k(z)e_1 \wedge \cdots \wedge \hat{e}_k \wedge \cdots \wedge e_r \otimes d\overline{z}^J$$

which is $\ne 0$ unless all the $p_k(z) = 0$, hence is $\ne 0$ if $z \notin \mathcal{Z}(P)$.

Now, $|P(z)|^2 \le C^{(0)}(1+|z|^2)^D$ since $p_j \in \mathcal{H}^D$ and $\|P\| \le C_1$. Hence:

$$|\kappa(\omega)(z)|^2 \le C^{(1)}(1+|z|^2)^D|\omega(z)|^2,\ z \in \mathbb{C}^n,\ \omega \in \mathcal{A}^q(E_p). \tag{18}$$

By construction, $\overline{\partial}\varphi_r \in \mathcal{A}_0^{r+1}(E_r)$ and $\kappa(\overline{\partial}\varphi_r) = \overline{\partial}(\kappa(\varphi_r)) = \overline{\partial}(\overline{\partial}\varphi_{r-1})$ (by (15)) $= 0$ [κ and $\overline{\partial}$ commute since the p_j are holomorphic]. Thus, by (17), we must have $\overline{\partial}\varphi_r = 0$. Moreover, if we take $p = r$, $k = 0$ in (16), we find that

$$\int_{\mathbb{C}^n} |\varphi_r|^2(1+|z|^2)^{-N_3-n-1}\, d\lambda \le C^{(2)}.$$

By Corollary 3.4, there exists $\psi_{r-1} \in \mathcal{A}^{r-1}(E_r)$ with

$$\overline{\partial}\psi_{r-1} = \varphi_r,\ \|\psi_{r-1}\|^2_{m_1} = \int_{\mathbb{C}^n} |\psi_{r-1}|^2(1+|z|^2)^{-m_1}d\lambda \le C^{(2)} \tag{19}$$

where $m_1 = N_3 + n + 3$.

Consider now $\varphi_{r-1} - \kappa(\psi_{r-1}) \in \mathcal{A}^{r-1}(E_{r-1})$. By (15), (19), we have $\overline{\partial}(\varphi_{r-1} - \kappa(\psi_{r-1})) = \overline{\partial}\varphi_{r-1} - \kappa(\overline{\partial}\psi_{r-1}) = \overline{\partial}\varphi_{r-1} - \kappa(\varphi_r) = 0$. Further, by (16), (18), (19), there exist $m_2' > 0$ and $C^{(3)} > 0$ so that

$$\|\varphi_{r-1} - \kappa(\psi_{r-1})\|^2_{m_2'} \le C^{(3)}, \tag{20}$$

and we can apply Corollary 3.4 again to find $\psi_{r-2} \in \mathcal{A}^{r-2}(E_{r-1})$ such that, with $m_2 = m_2' + 2$,

$$\overline{\partial}\psi_{r-2} = \varphi_{r-1} - \kappa(\psi_{r-1}),\ \|\psi_{r-2}\|^2_{m_2} \le C^{(3)}.$$

We can iterate this procedure to find $\psi_r = 0, \psi_{r-1}, \dots, \psi_0$, and constants $m_2 > 0$, $m_3 > 0, \dots, m_r > 0$, $C^{(2)}, C^{(3)}, \dots, C^{(r+1)} > 0$ such that

$$\begin{aligned} &\psi_{r-q} \in \mathcal{A}^{r-q}(E_{r-q+1}) \\ &\bar{\partial}\psi_{r-q} = \varphi_{r-q+1} - \kappa(\psi_{r-q+1}) \\ &\|\psi_{r-q}\|^2_{m_q} \le C^{(q+1)} \quad \text{for} \quad 1 \le q \le r. \end{aligned} \tag{21}$$

[Note that if the results in (21) have been proved for $1 \le q \le q_0 - 1$, then

$$\begin{aligned} \bar{\partial}(\varphi_{r-q_0+1} - \kappa(\psi_{r-q_0+1})) &= \bar{\partial}\varphi_{r-q_0+1} - \kappa(\bar{\partial}\psi_{r-q_0+1}) \\ &= \bar{\partial}\varphi_{r-q_0+1} - \kappa(\varphi_{r-q_0+2} - \kappa(\psi_{r-q_0+2})) \\ &= \bar{\partial}\varphi_{r-q_0+1} - \kappa(\psi_{r-q_0+2}) = 0 \end{aligned}$$

by (15), and we can apply Hörmander's theorem, Cor. 3.4, to obtain (21) for $q = q_0$.]

We have $\psi_0 \in \mathcal{A}^0(E_1)$. Consider $\varphi - \kappa(\psi_0) \in \mathcal{A}^0(E_0) = C^\infty(\mathbb{C}^n)$. Now, $\bar{\partial}(\varphi - \kappa(\psi_0)) = \bar{\partial}\varphi - \kappa(\bar{\partial}\psi_0) = \bar{\partial}\varphi - \kappa(\varphi_1 - \kappa(\psi_1)) = \bar{\partial}\varphi - \kappa(\varphi_1) = 0$ by (15). Thus:

(22) The function $F = \varphi - \kappa(\psi_0)$ is holomorphic on $\mathbb{C}^n$.

Moreover, by (18) and (21), we have

$$\int_{\mathbb{C}^n} |\kappa(\psi_0)|^2(1 + |z|^2)^{-m_r - D} d\lambda \le C^{(1)}C^{(r+1)},$$

while $\int_{\mathbb{C}^n} |\varphi|^2(1+|z|^2)^{-n-1}d\lambda \le \int_{\mathbb{C}^n}(1+|z|^2)^{-n-1}d\lambda \,(< \infty)$. Consequently, there exist constants $\overline{m}(> n+1)$ and $\overline{C} > 0$ depending only on the constants in $\mathcal{W}$ such that

$$\int_{\mathbb{C}^n} |F(z)|^2(1 + |z|^2)^{-\overline{m}} d\lambda \le \overline{C}. \tag{23}$$

Now, $\psi_0 \in \mathcal{A}^0(E_1)$, and we can write $\psi_0 = \sum_{j=1}^{r} \psi_{0j}e_j$, where $\psi_{0j} \in C^\infty(\mathbb{C}^n)$. Hence $\kappa(\psi_0) = \sum_{j=1}^{r} \psi_{0j}p_j$. In particular, $\kappa(\psi_0) \mid \mathcal{Z}(P) \equiv 0$. Thus we have

$$F = \varphi \text{ on } \mathcal{Z}(P); \text{ i.e. } F|X \equiv 1, F|X_j \equiv 0 \quad \text{for} \quad j > 1. \tag{24}$$

Thus, we have only to show that F is a polynomial satisfying the required estimates to complete the proof of Theorem 3.1.

Since F is holomorphic, the function $|F|^2$ is subharmonic. Hence, if $w \in \mathbb{C}^n$ and $R > 0$, we have

$$|F(w)|^2 \le \frac{1}{\operatorname{vol}(B(w,R))} \int_{B(w,R)} |F(z)|^2 d\lambda.$$

If $|w| \geq 1$, we take $R = \frac{1}{2}|w|$ and remark that $1 + |z|^2 \leq 3(1 + |w|^2)$ for $|z - w| < \frac{1}{2}|w|$, and $\operatorname{vol}(B(w, R)) = \operatorname{vol}(B(0, \frac{1}{2})) \cdot |w|^{2n}$. Hence

$$\begin{aligned} |F(w)|^2 &\leq c_1(\overline{m}, n) \cdot (1 + |w|^2)^{\overline{m}} |w|^{-2n} \int_{\mathbf{C}^n} \frac{|F(z)|^2}{(1 + |z|^2)^{\overline{m}}} d\lambda \\ &\leq c_2(\overline{m}, n) \cdot \overline{C}(1 + |w|^2)^{\overline{m} - n} \quad \text{(by (23))} \end{aligned}$$

where $c_1(\overline{m}, n)$ and $c_2(\overline{m}, n)$ depend only on $\overline{m}$ and n. It follows that F *is a polynomial of degree* $\leq \overline{m} - n$.

Also, for $|w| \leq n$, we have

$$\begin{aligned} |F(w)|^2 &\leq c_n \int_{|z| \leq 2n} |F(z)|^2 d\lambda \leq c_3(\overline{m}, n) \int_{\mathbf{C}^n} \frac{|F(z)|^2}{(1 + |z|^2)^{\overline{m}}} d\lambda \\ &\leq C_{\#},\ C_{\#} \text{ depending only on the constants in } \mathcal{W}. \end{aligned}$$

By Cauchy's inequalities:

$$\left| \frac{\partial^{|\alpha|} F}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}(0) \right| \leq \alpha! C_{\#},\ \alpha = (\alpha_1, \ldots, \alpha_n).$$

Since F is a polynomial of degree $\leq D' = \overline{m} - n$, this proves that $\|F\| \leq C'$ where C' depends only on the constants in $\mathcal{W}$. This and (24) complete the proof of Theorem 3.1.

A similar argument, combined with Lemma 2.6, can be used to prove the following extension theorem.

THEOREM 3.6. — *There exists a constant $D' > 0$ depending only on the constants in $\mathcal{W}$, and, for $M \geq 0$, a constant $C(M)$ depending only on M and the constants in $\mathcal{W}$ such that the following holds.*

Let $P \in \mathcal{W}$ and let f be a holomorphic function on $\mathcal{Z}(P)$. Assume that there are constants $A > 0, M \geq 0$ such that

$$|f(z)|^2 \leq A(1 + |z|^2)^M \quad \text{for } z \in \mathcal{Z}(P).$$

Then, there exists a polynomial F of degree $\leq M + D'$ such that $F \mid \mathcal{Z}(P) = f$ and $\|F\|^2 \leq C(M)A$.

Sketch of Proof. — By Lemma 2.6, we can find γ_0, m_0 and a holomorphic retraction $\pi : T_{\gamma_0,m_0}(\mathcal{Z}(P)) \to \mathcal{Z}(P)$, $|\pi(z) - z| < 1$. Choose $\gamma > 0$ ($\gamma < \gamma_0$) such that if $\Omega = T_{\gamma,m_0}(\mathcal{Z}(P))$ then $T_{\gamma,m_0}(\Omega) \subset T_{\gamma_0,m_0}(\mathcal{Z}(P))$, and $\varphi \in C^\infty(\mathbb{C}^n)$ with $\varphi = 1$ on Ω, $\operatorname{supp}(\varphi) \subset T_{\gamma_0,m_0}(\mathcal{Z}(P))$, $|D^{(k)}\varphi(z)| \leq C_k(1 + |z|^2)^{km_0}$ ($k \geq 0$). Consider the C^∞ function on

$\mathbb{C}^n : \varphi_0 = \varphi \cdot (f \circ \pi)$ on $T_{\gamma_0,m_0}(\mathcal{Z}(P))$, $\varphi_0 = 0$ outside $T_{\gamma_0,m_0}(\mathcal{Z}(P))$. The form $\omega \in \mathcal{A}^1(E_0)$ defined by $\omega = \overline{\partial}\varphi_0$ $(= (f \circ \pi) \cdot \overline{\partial}\varphi)$ has support outside Ω. We can repeat the proof of Theorem 3.1 to find a controlled function $\psi_0 \in \mathcal{A}^0(E_1)$ such that $\overline{\partial}(\varphi_0 - \kappa(\psi_0)) = 0$, so that $F = \varphi_0 - \kappa(\psi_0)$ is holomorphic and $F \mid \mathcal{Z}(P) = \varphi_0 \mid \mathcal{Z}(P) = f$. The argument in Theorem 3.1 gives the estimate

$$\int_{\mathbb{C}^n} |F|^2(1+|z|^2)^{-M-D'} d\lambda \leq C'A$$

$(C', D'$ depending only on the constants in $\mathcal{W})$, and the proof is completed as above, using the subharmonicity of $|F|^2$.

4. Polynomials vanishing on a smooth variety.

In this section, we prove the following theorem.

The referee has informed us that a stronger version of Theorem 4.1 below is contained in recent work of F. Amoroso [A].

THEOREM 4.1. — *Let $n \geq 2$, $D \geq 1$, $1 \leq r \leq n-1$, $C_1 > 0$, $c_0 > 0, N \geq 1$ be given, and let $\mathcal{W}$ be, as before, the space of r-tuples $P = (p_1, \dots, p_r)$ with $p_j \in \mathcal{H}^D, P(0) = 0$, $\|P\| \leq C_1$, $J_P(0) = 1$ and $|J_P(z)|^2 + |P(z)|^2 \geq c_0(1+|z|^2)^{-N}$, where $J_P(z) = \det\left(\dfrac{\partial p_j}{\partial z_k}\right)_{1\leq j,k \leq r}$.*

There exists a constant $D' \geq 1$ depending only on the constants defining $\mathcal{W}$, and, for $d \geq 1$ a constant $C(d)$ depending only on d and the constants in $\mathcal{W}$ such that the following holds.

Let $d \geq 1$ be an integer, and let $F \in \mathcal{H}^d$ be such that $F(z) = 0$, $\forall z \in \mathcal{Z}(P)$ $(= \{z \in \mathbb{C}^n \mid P(z) = 0\})$.

Then, there exist polynomials $F_1, \dots, F_r$ such that

(1.a) $\quad F = \sum\limits_{j=1}^{r} F_j p_j$

(1.b) $\quad \deg F_j \leq d + D'$

(1.c) $\quad \|F_j\| \leq C(d)\|F\|$.

Proof. — During the course of this proof, constants γ, γ', γ_j, m, m', m_j, C, C', $C^{(j)}$ etc. will, unless otherwise stated, depend only on the constants defining $\mathcal{W}$. If $k, d, \dots$ are other parameters, we write $C(k, \mathcal{W})$,

$C(d, \mathcal{W})$ etc. to indicate constants depending only on the parameters in question and on the constants defining $\mathcal{W}$.

If $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we write $z = (x, \zeta)$ with $x = (x_1, \dots, x_r) = (z_1, \dots, z_r)$ and $\zeta = (\zeta_1, \dots, \zeta_{n-r})) = (z_{r+1}, \dots, z_n)$.

Given $G \in \mathcal{H}^d$, we can write

$$G(x,\zeta) - G(y,\zeta) = \sum_{j=1}^{r} (x_j - y_j) A_G^{(j)}(x, y, \zeta) \tag{2}$$

where

$$A_G^{(j)}(x,y,\zeta) = \int_0^1 \frac{\partial G}{\partial x_j}(tx + (1-t)y, \zeta)dt. \tag{3}$$

Clearly

(4) $A_G^{(j)}$ is a polynomial of degree $\leq d-1$ in x, y, ζ and $\|A_G^{(j)}\| \leq c_1(d,n)\|G\|$, where $\|A_G^{(j)}\|$ denotes the norm in the space of polynomials in $n+r$ variables and $c_1(d,n)$ depends only on d and n.

By Lemma 2.6, we have the following:

(5) There exist $\gamma_0 > 0$, $m_0 \geq 1$ such that, if $P \in \mathcal{W}$ and $\Omega = T_{\gamma_0, m_0}(\mathcal{Z}(P))$, we can find a holomorphic map

$$\pi(x,\zeta) = (\pi_0(x,\zeta), \zeta), \ ((x,\zeta) \in \Omega)$$

of Ω into $\mathbb{C}^n$ with the properties: $\pi(\Omega) \subset \mathcal{Z}(P), \pi \mid \mathcal{Z}(P) =$ identity, and $|\pi_0(x,\zeta) - x| < 1$. (By definition, π_0 is a holomorphic map of Ω into $\mathbb{C}^r$.)

We write $\pi_0(x,\zeta) = (\pi_1(x,\zeta), \dots, \pi_r(x,\zeta))$. By Cauchy's inequalities, if $k \geq 0$ and $(x,\zeta) = z \in T_{\gamma_0/2, m_0}(\mathcal{Z}(P))$, then

$$|D^{(k)}\pi_j(x,\zeta)| \leq C_1(k, \mathcal{W})(1+|z|^2)^{km_0+1}. \tag{6}$$

We use formulae (2), (3) for p_i to write

$$p_i(x,\zeta) - p_i(y,\zeta) = \sum_{j=1}^{r} p_{ij}(x,y,\zeta)(x_j - y_j)$$

with $p_{ij}(x,y,\zeta) = \displaystyle\int_0^1 \frac{\partial p_i}{\partial x_j}(tx + (1-t)y, \zeta)dt$. If $z = (x,\zeta) \in \Omega$, we have

$$p_i(x,\zeta) = \sum_{j=1}^{r} P_{ij}(x,\zeta)(x_j - \pi_j(x,\zeta))$$

where $P_{ij}(x,\zeta) = P_{ij}(x, \pi_0(x,\zeta), \zeta)$. Using (4) with $G = p_i$ and the fact that $|\pi_0(x,\zeta) - x| < 1$, we obtain:

(7) if $z = (x,\zeta) \in \Omega$, $|P_{ij}\ (x,\zeta)| \leq C^{(1)}(1+|z|)^{D-1} \leq C^{(2)}(1+|z|^2)^{D/2}$.

Now, if $z = (x,\zeta) \in \mathcal{Z}(P)$, we have $\pi(z) = z, \pi_0(x,\zeta) = x$, so that

(8) if $(x,\zeta) \in \mathcal{Z}(P)$, then $P_{ij}(x,\zeta) = \int_0^1 \frac{\partial p_i}{\partial z_j}(tx+(1-t)x,\zeta)dt = \frac{\partial p_i}{\partial z_j}(x,\zeta)$.

Since $\left|\det\left(\frac{\partial p_i}{\partial z_j}\right)\right|^2 \geq c_0(1+|z|^2)^{-N}$ if $z \in \mathcal{Z}(P)$, we have

(9) $$|\det(P_{ij}(x,\zeta))|^2 \geq c_0(1+|z|^2)^{-N} \text{ if } z \in \mathcal{Z}(P).$$

From (7) and Cauchy's inequalities we obtain:

(10) If $z = (x,\zeta) \in T_{\gamma_0/2,m_0}(\mathcal{Z}(P))$, we have

$$|D^{(k)}P_{ij}(x,\zeta)| \leq C_2(k,\mathcal{W})(1+|z|^2)^{D/2+km_0},\ k \geq 0.$$

From (9) and (10), we deduce the following:

(11) There are constants γ_1, m_1 and $\gamma_2, m_2, \gamma_\nu > 0, m_\nu \geq 1$, such that $T_{\gamma_1,m_1}((\mathcal{Z}(P)) \subset T_{\gamma_0/2,m_0}(\mathcal{Z}(P))$ and $|\det(P_{ij}(x,\zeta))|^2 \geq \gamma_2(1+|z|^2)^{-m_2}$ for $(x,\zeta) = z \in T_{\gamma_1,m_1}(\mathcal{Z}(P))$.

From (7) and (11) we obtain:

(12) There exist constants $C^{(3)}$,$m_3 \geq 1$ such that the matrix $(P_{ij}(x,\zeta))$ is invertible if $(x,\zeta) \in T_{\gamma_1,m_1}(\mathcal{Z}(P))$ and the inverse matrix $(Q_{ij}(x,\zeta))$ satisfies

$$|Q_{ij}(x,\zeta)| \leq C^{(3)}(1+|z|^2)^{m_3},\ (x,\zeta) = z \in T_{\gamma_1,m_1}(\mathcal{Z}(P)).$$

(13) For $1 \leq i \leq r$

$$\begin{aligned}\sum_{k=1}^r Q_{ik}(x,\zeta)p_k(x,\zeta) &= \sum_{j=1}^r\sum_{k=1}^r Q_{ik}(x,\zeta)P_{kj}(x,\zeta)(x_j - \pi_j(x)) \\ &= x_i - \pi_i(x,\zeta).\end{aligned}$$

Let now F be a polynomial of degree $\leq d$ such that $F \mid \mathcal{Z}(P) = 0$. If $z = (x,\zeta) \in T_{\gamma_1,m_1}(\mathcal{Z}(P))$, we have (using (2), (3) with $G = F$)

(14) $$\begin{aligned}F(z) &= F(x,\zeta) - F(\pi_0(x,\zeta),\zeta) \\ &= \sum_{j=1}^r A_F^{(j)}(x,\pi_0(x,\zeta),\zeta)(x_j - \pi_j(x,\zeta)) \\ &= \sum_{j,k=1}^r A_F^{(j)}(x,\pi_0(x,\zeta),\zeta)Q_{jk}(x,\zeta)p_k(x,\zeta).\end{aligned}$$

Using (4) for $G = F$ and the fact that $|\pi_0(x,\zeta) - x| < 1$, we see that $|A_F^{(j)}(x, \pi_0(x,\zeta), \zeta)|^2 \leq c_3(d, \mathcal{W})\|F\|^2(1+|z|^2)^d$. This, (13) and (14) give us:

(15) There exist constants $\gamma > 0, m, \overline{m} \geq 1$ depending only on the constants in $\mathcal{W}$, and a constant $\overline{C}(d,\mathcal{W})$ (depending only on $d \geq 1$ and the constants in $\mathcal{W}$) such that the following holds.

If $P \in \mathcal{W}$, $d \geq 1$, $F \in H^d$ and $F \mid \mathcal{Z}(P) = 0$, then we can find holomorphic functions $g_1, \ldots, g_r$ on $T_{\gamma,m}(\mathcal{Z}(P))$ such that, for $z \in T_{\gamma,m}(\mathcal{Z}(P))$,

$$F(z) = g_1(z)p_1(z) + \cdots + g_r(z)p_r(z)$$

and

$$|g_1(z)|^2 + \cdots + |g_r(z)|^2 \leq \overline{C}(d,\mathcal{W})\|F\|^2(1+|z|^2)^{d+\overline{m}}.$$

We choose $(\gamma', m'), (\gamma'', m'')$ such that, if $P \in \mathcal{W}, B(w, \gamma'(1+|w|^2)^{-m'}) \subset T_{\gamma'',m''}(\mathcal{Z}(P))$ for any $w \in T_{\gamma',m'}(\mathcal{Z}(P))$ and $B(w,\gamma''(1+|w|^2)^{-m''}) \subset T_{\gamma,m}(\mathcal{Z}(P))$ for any $w \in T_{\gamma'',m''}(\mathcal{Z}(P))$. We may assume that $\gamma'' < \gamma_1, m'' > m_1$ (γ_1, m_1 as in (11)).

By Lemma 2.8, we can find $\varphi \in C^\infty(\mathbb{C}^n)$ such that $0 \leq \varphi \leq 1, \varphi = 1$ on $T_{\gamma',m'}(\mathcal{Z}(P))$, $\operatorname{supp}(\varphi) \subset T_{\gamma'',m''}(\mathcal{Z}(P))$ and $|D^{(k)}\varphi(z)| \leq C_k(\mathcal{W})(1+|z|^2)^{km'}, k \geq 0, C_k(\mathcal{W})$ being a constant depending only on k and the constants in $\mathcal{W}$.

Let $P \in \mathcal{W}$, let $F \in \mathcal{H}^d$, $F \mid \mathcal{Z}(P) = 0$. Let g_j $(1 \leq j \leq r)$ be the holomorphic functions on $T_{\gamma,m}(\mathcal{Z}(P))$ constructed in (15).

For $1 \leq j \leq r$, we define

$$\Phi_j = \varphi g_j + (1-\varphi)\frac{\overline{p}_j F}{|P|^2}.$$

Let $\gamma_3 > 0, m_3 \geq 1$ be such that

(16) $$|P(z)|^2 \geq \gamma_3(1+|z|^2)^{-m_3} \text{ if } z \in \mathbb{C}^n \setminus T_{\gamma',m'}(\mathcal{Z}(P));$$

these exist by Lemma 2.5.

LEMMA 4.2. — *The functions Φ_j defined above have the following properties:*

(i) $\Phi_j \in C^\infty(\mathbb{C}^n), \Phi_j \mid T_{\gamma',m'}(\mathcal{Z}(P))$ *is holomorphic.*

(ii) $F(z) = \sum_1^r \Phi_j(z)p_j(z)$ *for* $z \in \mathbb{C}^n$.

(iii) *If $k \geq 0$ and $D^{(k)}$ is any differentiation of order k, we have*

$$|D^{(k)}\Phi_j(z)|^2 \leq C_k(d, \mathcal{W})\|F\|^2(1+|z|^2)^{d+(k+1)N_2}.$$

Here $N_2 \geq 1$ is a constant depending only on the constants defining $\mathcal{W}$, while $C_k(d, \mathcal{W})$ depends only on k, d and the constants defining $\mathcal{W}$.

Proof of Lemma 4.2. — Since $\varphi = 1$ on $T_{\gamma',m'}(\mathcal{Z}(P))$ and g_j is holomorphic on this set, (i) is obvious.

For (ii), if $z \in \mathcal{Z}(P)$, then $\Phi_j(z) = g_j(z)$ and (ii) follows from (15). If $z \in T_{\gamma,m}(\mathcal{Z}(P))\backslash\mathcal{Z}(P)$, we have

$$\begin{aligned}\sum \Phi_j(z)p_j(z) &= \varphi(z)\sum g_j(z)p_j(z) + (1-\varphi(z))\sum \frac{\overline{p_j(z)}F(z)}{|P(z)|^2}p_j(z)\\ &= F(z).\end{aligned}$$

Finally, if $z \notin T_{\gamma,m}(\mathcal{Z}(P))$, we have

$$\sum \Phi_j(z)p_j(z) = \sum \frac{\overline{p_j(z)}F(z)}{|P(z)|^2}p_j(z) = F(z);$$

this proves (ii).

To prove (iii), we make the following remarks. We have $\mathrm{supp}(\varphi) \subset T_{\gamma'',m''}(\mathcal{Z}(P))$, and $B(w, \gamma''(1+|z|^2)^{-m''}) \subset T_{\gamma,m}(\mathcal{Z}(P))$ if $w \in T_{\gamma'',m''}(\mathcal{Z}(P))$. From the estimates for g_j on $T_{\gamma,m}(\mathcal{Z}(P))$ given in (15) and Cauchy's inequalities, we deduce that

$$|D^{(k)}g_j(z)|^2 \leq C'_k(d, \mathcal{W})\|F\|^2(1+|z|^2)^{d+\overline{m}+2km''}$$

for $z \in T_{\gamma'',m''}(\mathcal{Z}(P))$. Now, if $z \notin T_{\gamma',m'}(\mathcal{Z}(P))$ (in particular, if $z \in \mathrm{supp}(1-\varphi)$), we have $\left|D^{(k)}\dfrac{\overline{p}_j}{|P|^2}(z)\right|^2 \leq C_k(\mathcal{W})(1+|z|^2)^{(k+1)m_4}$ because of (16) (m_4 depending only on the constants in $\mathcal{W}$). These estimates, and the estimates for $D^{(k)}\varphi$ in the defining properties of φ imply (iii).

We can now use the method of proof of Theorem 3.1 to complete that of Theorem 4.1.

We consider again the space $\mathbb{C}^r$ with standard basis $e_1, \ldots, e_r$. As in §3, let $E_p = \overset{p}{\wedge}\,\mathbb{C}^r$. $\mathcal{A}^q(E_p)$ is the space of smooth $(0,q)$ forms on $\mathbb{C}^n$ with values in E_p, and, for $P \in \mathcal{W}$, $\mathcal{A}^q_0(E_p)$ is the subspace of forms with support $\subset \mathbb{C}^n\backslash\mathcal{Z}(P)$. We consider again the Koszul complex

$$(17) \qquad 0 \to \mathcal{A}^q(E_r) \xrightarrow{\kappa} \mathcal{A}^q(E_{r-1}) \to \cdots \to \mathcal{A}^q(E_1) \xrightarrow{\kappa} \mathcal{A}^q(E_0) \to 0$$

with κ defined by

$$(18)\quad \kappa(e_I \otimes d\bar{z}^J) = \sum_{k=1}^{p} (-1)^{k-1} p_{i_k}(z) e_{I\setminus\{i_k\}} \otimes d\bar{z}^J,$$
$$I = (i < \cdots < i_p), J = (j_1 < \cdots < j_q).$$

We also have the homotopy operator

$$h : \mathcal{A}_0^q(E_p) \to \mathcal{A}_0^q(E_{p+1}) : h(\omega) = \Big(\sum_{j=1}^{r} \frac{\overline{p}_j}{|P|^2} e_j\Big) \wedge \omega,$$

($h = 0$ on E_r) with the property that

$$(19)\qquad \begin{cases} \kappa h = \text{ identity on } \mathcal{A}_0^q(E_0) \\ \kappa h + h\kappa = \text{ identity on } \mathcal{A}_0^q(E_p), p > 0. \end{cases}$$

(See §3).

Let $(\Phi_1, \ldots, \Phi_r)$ be as in Lemma 4.2. Define

$$(20)\qquad \Phi^{(1)} = \sum_j \Phi_j e_j \in \mathcal{A}^0(E_1).$$

We have $\kappa(\Phi^{(1)}) = \sum \Phi_j p_j = F$. Consider $\bar{\partial}\Phi^{(1)} \in \mathcal{A}^1(E_1)$. By property (i) of the $\Phi_j, \bar{\partial}\Phi^{(1)} \in \mathcal{A}_0^1(E_1)$ (i.e. $\operatorname{supp}(\bar{\partial}\Phi^{(1)}) \cap \mathcal{Z}(P) = \emptyset$). Using the fact that $\bar{\partial}$ and h do not increase supports, we define, successively,

$$\begin{aligned} \Phi^{(2)} &= h(\bar{\partial}\Phi^{(1)}) \in \mathcal{A}_0^1(E_2), \\ \Phi^{(3)} &= h(\bar{\partial}\Phi^{(2)}) \in \mathcal{A}_0^2(E_3), \\ &\vdots \\ \Phi^{(r)} &= h(\bar{\partial}\Phi^{(r-1)}) \in \mathcal{A}_0^{r-1}(E_r). \end{aligned}$$

We have $\kappa(\bar{\partial}\Phi^{(1)}) = \bar{\partial}\kappa(\Phi^{(1)}) = \bar{\partial}F = 0$ so that $\kappa(\Phi^{(2)}) = \kappa h(\bar{\partial}\Phi^{(1)}) = (\kappa h + h\kappa)(\bar{\partial}\Phi^{(1)}) = \bar{\partial}\Phi^{(1)}$. We obtain, by induction:

$$(21)\qquad \begin{cases} \kappa(\bar{\partial}\Phi^{(p)}) = 0 \ \text{ for } \ 1 \le p \le r \\ \kappa(\Phi^{(p+1)}) = \bar{\partial}\Phi^{(p)} \ \text{ for } \ 1 \le p \le r-1. \end{cases}$$

By Lemma 3.5 and Lemma 4.2, Part (iii), we have the following:

(22) For $k \ge 0, d \ge 1$, there exists a constant $C_k(d, \mathcal{W})$ depending only on k, d and the constants in $\mathcal{W}$, and a constant $N_3 \ge 1$ depending only on the constants in $\mathcal{W}$, such that, for $1 \le p \le r$,

$$|D^{(k)}\Phi^{(p)}(z)|^2 \le C_k(d, \mathcal{W}) \|F\|^2 (1 + |z|^2)^{d+(k+1)N_3}, \ k \ge 0, \ z \in \mathbb{C}^n,$$

$D^{(k)}$ being, as usual, any differentiation of order k.

In the rest of the proof, we denote by Γ_* a constant depending only on those in $\mathcal{W}$, and by $C_*(d)$, a constant depending only on d and the constants in $\mathcal{W}$; they are not necessarily the same at each occurrence.

Let N_3 be as in (22), and define $\mu_{r-1} = d+N_3+n+1$, $\mu_q = \mu_{q+1}+D+2$ for $r-2 \geq q \geq 1$.

We have $\Phi^{(r)} \in \mathcal{A}_0^{r-1}(E_r)$, $\kappa(\overline{\partial}\Phi^{(r)}) = 0$ (by (21)). Hence, since κ is injective on $\mathcal{A}_0^q(E_r)$ (see §3, (17)), we have $\overline{\partial}\Phi^{(r)} = 0$. Moreover, by (22),

$$\int_{\mathbb{C}^n} |\Phi^{(r)}|^2(1+|z|^2)^{-\mu_{r-1}}d\lambda \leq C_*(d)\|F\|^2.$$

Hence, by Hörmander's theorem (Cor. 3.4), there is $\Psi^{(r-1)} \in \mathcal{A}^{r-2}(E_r)$ such that

$$\overline{\partial}\Psi^{(r-1)} = \Phi^{(r)}, \int_{\mathbb{C}^n} |\Psi^{(r-1)}|^2(1+|z|^2)^{-\mu_{r-1}-2}d\lambda \leq C_*(d)\|F\|^2.$$

Since, for any $\Psi \in \mathcal{A}^q(E_p)$, we have, by §3, (18),

$$|\kappa(\Psi)(z)|^2 \leq \Gamma_*(1+|z|^2)^D|\Psi(z)|^2$$

we have also

$$\int_{\mathbb{C}^n} |\Phi^{(r-1)} - \kappa(\Psi^{(r-1)})|^2(1+|z|^2)^{-\mu_{r-1}-2-D}d\lambda \leq C_*(d)\|F\|^2.$$

Moreover, $\Phi^{(r-1)} - \kappa(\Psi^{(r-1)}) \in \mathcal{A}^{r-2}(E_{r-1})$ and $\overline{\partial}(\Phi^{(r-1)} - \kappa(\Psi^{(r-1)})) = \overline{\partial}\Phi^{(r-1)} - \kappa\overline{\partial}\Psi^{(r-1)} = \overline{\partial}\Phi^{(r-1)} - \kappa(\Phi^{(r)}) = 0$ by (21); hence we can find $\Psi^{(r-2)} \in \mathcal{A}^{r-3}(E_{r-1})$ so that $\overline{\partial}\Psi^{(r-2)}) = \Phi^{(r-1)} - \kappa(\Psi^{(r-1)})$ and satisfying

$$\int_{\mathbb{C}^n} |\Psi^{(r-2)}|^2(1+|z|^2)^{-\mu_{r-2}-2}d\lambda \leq C_*(d)\|F\|^2.$$

Thus, we solve, successively, the equations

$$\begin{aligned}\overline{\partial}\Psi^{(r-1)} &= \Phi^{(r)}\\ \overline{\partial}\Psi^{(r-2)} &= \Phi^{(r-1)} - \kappa(\Psi^{(r-1)})\\ &\vdots\\ \overline{\partial}\Psi^{(1)} &= \Phi^{(2)} - \kappa(\Psi^{(2)}),\end{aligned}$$

where $\Psi^{(q)} \in \mathcal{A}^{q-1}(E_{q+1})$ $(r-1 \geq q \geq 1)$ and

$$\int_{\mathbb{C}^n} |\Psi^{(q)}|^2(1+|z|^2)^{-\mu_q-2}d\lambda \leq C_*(d)\|F\|^2. \tag{23}$$

Consider now $G = \Phi^{(1)} - \kappa(\Psi^{(1)}) \in \mathcal{A}^0(E_1)$. We have

$$\begin{aligned}\overline{\partial}G = \overline{\partial}\Phi^{(1)} - \kappa(\overline{\partial}\Psi^{(1)}) &= \overline{\partial}\Phi^{(1)} - \kappa(\Phi^{(2)} - \kappa(\Psi^{(2)}))\\ &= \overline{\partial}\Phi^{(1)} - \kappa(\Phi^{(2)}) = 0 \text{ by (21).}\end{aligned}$$

Thus G is holomorphic. Further $\kappa(G) = \kappa(\Phi^{(1)}) = F$, so that, if $G = \sum G_j e_j$, we have $F = \sum G_j p_j$. The estimate (23) for $\Psi^{(1)}$, and (22) and §3, (18), imply that

$$\int_{\mathbb{C}^n} |G_j|^2(1+|z|^2)^{-\mu_1-2-D} d\lambda \leq C_*(d)\|F\|^2.$$

Using the fact that $|G_j|^2$ is subharmonic as at the end of the proof of Theorem 3.1, we conclude that the G_j are polynomials of degree $\leq \mu_1 + 2 + D - n \leq d + D'$ (with D' depending only on D, n and N_3 in (22)) and that

$$\|G_j\|^2 \leq C_*(d)\|F\|^2.$$

This completes the proof of Theorem 4.1.

5. The auxiliary function.

Let $n \geq 2$, $D \geq 1$, $1 \leq r \leq n-1, C_1 > 0$ be given. Let $\mathcal{W}_0$ be the space of r-tuples $P = (p_1, \ldots, p_r)$ with $p_j \in \mathcal{H}^D$ (i.e. $p_j \in \mathbb{C}[z_1, \ldots, z_n]$, $\deg p_j \leq D$) such that: $P(0) = 0, \|P\| \leq C_1$, $J_P(0) = 1$, where, as before, $J_P(z) = \det\left(\dfrac{\partial p_j}{\partial z_k}\right)_{1\leq j,k\leq r}$.

THEOREM 5.1. — *There exist constants $D' \geq 1$, $C' > 0$ depending only on n, D, C_1, and, for $d \geq 1$, a constant $C(d)$ depending only on d, n, D, C_1 such that the following holds.*

Let $P \in \mathcal{W}_0$. There exists $Q \in \mathcal{H}^{D'}$ with $Q(0) = 1$, $\|Q\| \leq C'$ such that if $F \in \mathcal{H}^d$ and F vanishes on $\mathcal{Z}(P) \cap \{z \in \mathbb{C}^n \mid |z| < \epsilon\}$ for some $\epsilon > 0$, then we can find polynomials $F_1, \ldots, F_r$ having the following properties:

(i) $J_P^d Q F = F_1 p_1 + \cdots + F_r p_r$.

(ii) $\deg F_j \leq nD(d + D')$.

(iii) $\|F_j\| \leq C(d)\|F\|$.

Proof. — As at the beginning of §2, we introduce a new variable z_0 and a polynomial $f_0(z_0, \ldots, z_n) = (z_0 + 1)J_P(z) - 1$, $z = (z_1, \ldots, z_n)$. We write $w = (z_0, z) = (z_0, z_1, \ldots, z_n) \in \mathbb{C}^{n+1}$ and $f_j(w) = p_j(z)$ for $j \geq 1, f_0(w) = f_0(z_0, \ldots, z_n)$.

We have

$$J_f(w) = \det\left(\frac{\partial f_i}{\partial w_j}\right)_{0\leq i,j\leq r} = (J_p(z))^2.$$

(1) There exist constants $c_0 > 0$, $N \geq 1$, $C_2 \geq 1$,$D_1 \geq 1$ depending only on n, D, C_1, such that if $\mathcal{W}$ is the space of $(r+1)$-tuples $q = (q_0, \ldots, q_r)$ of polynomials in $(n+1)$ variables $w_0, \ldots, w_n$ satisfying: $\deg q_j \leq D_1$, $q(0) = 0$, $J_q(0) = 1$, $\|q\| \leq C_2$ and

$$|J_q(w)|^2 + |q_0(w)|^2 + \cdots + |q_r(w)|^2 \geq c_0(1 + |w|^2)^{-N}, w \in \mathbb{C}^{n+1},$$

then, the following holds.

If $P = (p_1, \ldots, p_r) \in \mathcal{W}_0$, the $(r+1)$-tuple $f = (f_0, \ldots, f_r)$ defined by

$$f_0(w) = (z_0 + 1)J_P(z) - 1,\ f_j(w) = p_j(z),\ j \geq 1,\ w = (z_0, \ldots, z_n),$$

belongs to $\mathcal{W}$. [We can take $D_1 = D + n(D-1)$ since $\deg f_0 \leq 1 + r(D-1)$.]

This was noted in §2, and follows easily from the relation

$$(1 + (z_0 + 1)J_P(z))f_0(w) - (z_0 + 1)^2 J_f(w) = -1.$$

Let $X_1, \ldots, X_t$ be the connected components of $\mathcal{Z}(f) \subset \mathbb{C}^{n+1}$, X_1 being the component containing 0. By Theorem 3.1 (applied with n, r, D, C_1, c_0, N replaced by $n+1, r+1, D_1, C_2, c_0, N$) there exists a polynomial $\Phi \in \mathbb{C}[w]$ such that

(2) $\quad \Phi | X_1 = 1,\ \Phi | X_j = 0 \ \text{ for } \ j > 1,\ \deg \Phi \leq D_2,\ \|\Phi\| \leq C_3$

where D_2, C_3 depend only on the constants defining $\mathcal{W}$ above, hence only on n, D, C_1.

Let $F \in \mathbb{C}[z_1, \ldots, z_n]$ and suppose that $\deg F \leq d$ and that F vanishes on $\mathcal{Z}(P) \cap \{z \in \mathbb{C}^n \mid |z| < \epsilon\}$ for some $\epsilon > 0$. Then F, considered as a function of $w = (z_0, z)$ vanishes on a neighborhood of 0 on X_1, so that, by the principle of analytic continuation, $F \mid X_1 \equiv 0$. Thus $\Phi \cdot F \equiv 0$ on $\mathcal{Z}(f)$. By Theorem 4.1, we obtain the following:

(3) There exist polynomials $G_0(w), \ldots, G_r(w)$ with the following properties:

(a) $\Phi F = G_0 f_0 + \cdots + G_r f_r$

(b) $\deg G_j \leq d + D_3$

(c) $\|G_j\| \leq C_1(d) \|\Phi F\|$;

here D_3 depends only on n, D, C_1, and $C_1(d)$ on d, n, D, C_1.

Since $\|\Phi\| \leq C_3$, (c) above implies

(c′) $\quad \|G_j\| \leq C_2(d) \|F\|, C_2(d)$ depending only on d, n, D, C_1.

Let $D'' = \max(D_2, D_3)$ with D_2 as in (2), D_3 as in (3, b), and set

$$(4)\quad \begin{cases} Q(z_1, \dots, z_n) = (J_P(z))^{D''}\Phi(-1 + \frac{1}{J_P(z)}, z_1, \dots, z_n), \\ F_j(z_1, \dots, z_n) = (J_P(z))^{d+D''}G_j(-1 + \frac{1}{J_P(z)}, z_1, \dots, z_n),\ j \geq 1. \end{cases}$$

Since $\deg G_j \leq d + D'', F_j$ is a polynomial of degree $\leq d + D'' + (d + D'')\deg J_P \leq (d + D'') + (1 + r(D-1)) \leq nD(d + D'')$. Moreover, if $G_j(w) = \sum_{|\alpha| \leq d+D''} c_\alpha w^\alpha$, then $c_\alpha (J_P(z))^{d+D''}\left(-1 + \frac{1}{J_P(z)}\right)^{\alpha_0} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ clearly has norm at most $|c_\alpha| C_3(d), C_3(d)$ depending only on d, n, D, C_1. Thus

$$(5)\qquad \deg F_j \leq nD(d + D''),\ \|F_j\| \leq C_4(d)\|G_j\| \leq C(d)\|F\|$$

(by (c′) above). In the same way

$$(6)\qquad \begin{cases} \deg Q \leq D_2 + D'' \cdot n(D-1) = D' \text{ (say)}, \\ \|Q\| \leq C', C' \text{ depending only on } n, D, C_1. \end{cases}$$

Further $Q(0) = J_P(0)^{D''}\Phi(0) = 1$ (since $-1 + \frac{1}{J_P(0)} = 0$). Finally, $f_0(-1 + \frac{1}{J_P(z)}, z_1, \dots, z_n) \equiv 0$, so that

$$(7)\quad (J_P(z))^d Q(z) F(z) = (J_P(z))^{d+D''}\sum_{j=1}^{r} G_j(-1 + \frac{1}{J_P(z)}, z_1, \dots, z_n)p_j(z_1, \dots, z_n) = \sum_{j=1}^{r} F_j(z)p_j(z).$$

Theorem 5.1 follows from (5), (6), (7).

Theorem 5.1 has a real analogue which is, in fact, what we shall need.

THEOREM 5.2. — *Let $n \geq 2$, $D \geq 1$, $1 \leq r \leq n-1$, and $C_1 > 0$ be given. Let W be the space of r-tuples $P = (p_1, \dots, p_r)$, where $p_j \in \mathbb{R}[x_1, \dots, x_n], \deg p_j \leq D$ (i.e. $p_j \in H^D$), satisfying $P(0) = 0$, $\|P\| \leq C_1$, $J_P(0) = 1$, with $J_P(x) = \det\left(\frac{\partial p_j}{\partial x_k}\right)_{1 \leq j,k \leq r}$.*

There exist constants $D' \geq 1$, $C' > 0$ depending only on n, D, C_1 and, for $d \geq 1$ a function $C(d)$ of d, n, D, C_1 such that the following holds.

Let $P \in W$ and $Z(P) = \{z \in \mathbb{R}^n \mid p_1(x) = \cdots p_r(z) = 0\}$. There exists $Q \in H^{D'}$ with $Q(0) = 1$ and $\|Q\| \leq C'$ such that: if $F \in H^d$ and

$F = 0$ on $Z(P) \cap \{x \in \mathbb{R}^n \mid |x| < \epsilon\}$ for some $\epsilon > 0$, then we can find $F_j \in \mathbb{R}[x_1, \ldots, x_n]$, $j = 1, \ldots, r$ having the following properties:

(i) $(J_P(x))^d Q(x) F(x) = \sum_{j=1}^{r} F_j(x) p_j(x), \ x \in \mathbb{R}^n.$

(ii) $\deg F_j \leq nD(d + D')$.

(iii) $\|F_j\| \leq C(d)\|F\|$.

Proof. — The implicit function theorem implies that there are holomorphic functions $h_j(\zeta)$, $j = 1, \ldots, r$, $\zeta = (z_{r+1}, \ldots, z_n)$ defined in a ball $B(0, \rho) \subset \mathbb{C}^{n-r}$, which are real for real values of ζ such that for arbitrarily small neighborhoods U of 0 in $\mathbb{C}^n$, we have

$$\begin{aligned} \mathcal{Z}(P) \cap U &= \{(h_1(\zeta), \ldots, h_r(\zeta), \zeta) \mid \zeta \in \mathbb{C}^{n-r}, |\zeta| < \rho\} \cap U \\ Z(P) \cap U &= \mathcal{Z}(P) \cap \mathbb{R}^n \cap U \\ &= \{(h_1(\zeta), \ldots, h_r(\zeta), \zeta) \mid \zeta \in \mathbb{R}^{n-r}, |\zeta| < \rho\} \cap U. \end{aligned}$$

Hence, if $F \in \mathbb{R}[x_1, \ldots, x_n]$ and $F = 0$ on $Z(P) \cap U$, then $F(h_1(\zeta), \ldots, h_r(\zeta), \zeta)$ vanishes for real ζ sufficiently close to 0, hence also for all small complex ζ. Hence $F|\mathcal{Z}(P)$ is zero in some neighborhood of 0. Theorem 5.1 gives us polynomials Q, F_j with *complex* coefficients satisfying (i), (ii), (iii) above. Replacing them by the real polynomials whose coefficients are the real parts of the Q, F_j, we obtain the desired polynomials.

Remark. — For the application of these results to the proof of the extension theorem, we need the bounds given on Q, and the bounds on the *degrees* of the F_j (not those on the norms). These bounds on the degrees of the F_j can be obtained by using a purely algebraic theorem, although the construction of Q with the bounds given above seems to necessitate analytic methods.

We are grateful to Burt Totaro who told us that the next theorem was known. The reference to work of Bayer and Stillman that he gave us led us to the paper [He] of Grete Hermann in which it is proved. We formulate this theorem, and indicate its proof, although our argument is not very different from that of Hermann. The theorem and its proof remain valid if $\mathbb{R}$ is replaced by $\mathbb{C}$.

THEOREM 5.4. — *Let n, r, s, D, d be integers ≥ 1. There is an integer $D' \geq 1$ depending only on n, r, D, d such that the following holds.*

Let $p_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$, $1 \le i \le r$, $1 \le j \le s$, *be polynomials with* $\deg p_{ij} \le D$. *Let* $F_j \in \mathbb{R}[x_1, \ldots, x_n]$, $j = 1, \ldots, s$, *and suppose that* $\deg F_j \le d$.

Assume that the system of equations

$$F_j = \sum_{i=1}^{r} g_i p_{ij},\ j = 1, \ldots, s \tag{8}$$

has a solution $g = (g_1, \ldots, g_r)$ *with* $g_i \in \mathbb{R}[x_1, \ldots, x_n]$.

Then, there is a solution of (8) with $\deg g_i \le D'$.

Outline of Proof. — The proof is by induction on n; the result is trivial for $n = 0$. Assume the theorem proved for $\mathbb{R}[x_1, \ldots, x_{n-1}]$.

We may assume that the vectors $P_j = (p_{1j}, \ldots, p_{rj})$, $j = 1, \ldots, s$ are linearly independent over the field $\mathbb{R}(x)$ of rational functions. In fact, if $P_1, \ldots, P_{s'}$, $s' \le s$, are independent, and P_j, $j > s'$, are linear combinations of these, then it suffices to solve the equations

$$F_j = \sum_{i=1}^{r} g_i p_{ij},\ j = 1, \ldots, s';$$

the other equations in (8) necessarily follow from these. In particular, we may assume that $s \le r$.

By a change of notation, we may assume that the matrix

$$P = (p_{ij})_{1 \le i \le s, 1 \le j \le s}$$

has a non-zero determinant $\Delta \in \mathbb{R}[x_1, \ldots, x_n]$. Its degree is $\le sD, \Delta \not\equiv 0$. The matrix ΔP^{-1} is a matrix of polynomials of degree $\le (s-1)D$ (its entries are, up to sign, the determinants of the matrices obtained from P by deleting one row and one column).

Multiplying the equation

$$(F_1, \ldots, F_s) = (g_1, \ldots, g_r)(p_{ij})$$

on the right by ΔP^{-1}, we see that, with our assumptions, (8) is equivalent to a system

$$\begin{aligned} G_1 &= g_1 \Delta + \sum_{s < i \le r} g_i q_{i1} \\ &\cdots\cdots\cdots\cdots \\ G_s &= g_s \Delta + \sum_{s < i \le r} g_i q_{is}, \end{aligned} \tag{9}$$

where the q_{ij} are linear combinations of the p_{ij}, the coefficients being entries of ΔP^{-1}. Hence $\deg q_{ij} \leq sD$. The G_j are linear combinations of the F_j with coefficients entries of ΔP^{-1}, so that $\deg G_j \leq d + (s-1)D$.

By a linear change of coordinates, we may assume that if $\deg \Delta = N$ $(\leq sD)$, then the coefficient of x_n^N is a non-zero constant (i.e. the coefficient of the monomial $x_1^0 \cdots x_{n-1}^0 x_n^N$ is $\neq 0$).

If $(g_1, \ldots, g_r)$ is a solution of (9) and $h \in \mathbb{R}[x_1, \ldots, x_n]$, the vector

$$(g_1 + hq_{s+1,1}, g_2 + hq_{s+1,2}, \ldots, g_s + hq_{s+1,s}, g_{s+1} - h\Delta, g_{s+2}, \ldots, g_r)$$

is also a solution of (9). By our assumption on the coefficient of x_n^N in Δ, we can choose h so that $\deg_{x_n}(g_{s+1} - h\Delta) < N$ ($\deg_{x_n}$ being the degree in x_n). Repeating the process with $g_{s+2}, \ldots, g_r$, we obtain:

If (9) has a solution, it has one in which

$$\deg_{x_n}(g_i) < N \quad \text{for} \quad s < i \leq r.$$

Now, if $(g_1, \ldots, g_r)$ satisfies this condition, we can use (9) to conclude that for $1 \leq j \leq s$,

$$\begin{aligned}\deg_{x_n}(g_j\Delta) &\leq \max(\deg_{x_n} G_j, \deg_{x_n} q_{ij} + N - 1)\\ &\leq \max(d + (s-1)D, 2sD - 1) < d + 2sD.\end{aligned}$$

Thus, if (9) has a solution, it has one, $(g_1, \ldots, g_r)$, with $\deg_{x_n} g_i < d + 2sD$ for $i = 1, \ldots, r$.

We now simply write G_j, g_i, q_{ij}, Δ, in the form

$$\sum_{0 \leq \nu < d+2sD} a_\nu(x_1, \ldots, x_{n-1})x_n^\nu, \; a_\nu \in \mathbb{R}[x_1, \ldots, x_{n-1}];$$

equating coefficients of powers of x_n in the system (9), we obtain a system of equations of the same form as (8), but in the variables $x_1, \ldots, x_{n-1}$, and we can proceed by induction.

Theorem 5.2 can be improved in one respect at the cost of losing the bounds in Theorem 5.2 (iii); we do not, however lose the bounds on Q (which are essential for us in what follows). Since this provides what appears to be the right analogue of the factorization into the *main factor* and the *other factor* that we used in [FN], we shall now discuss this.

The following result is what we called Theorem 3 in the Introduction.

THEOREM 5.5. — *Let n, D, r, C_1 and W be as in Theorem 5.2. There exist constants $D' \geq 1$, $C' > 0$ depending only on these data, and, for $d \geq 1$, a function $\bar{d}$ of d, n, D, r and C_1 such that the following holds.*

Given $P = (p_1, \dots, p_r) \in W$, there exists $Q \in \mathbb{R}[x_1, \dots, x_n]$ (depending only on P) with

(a) $Q(0) = 1$, $\deg Q \le D'$, $\|Q\| \le C'$

such that

(b) *If $F \in H^d$ and $F = 0$ as a germ in $\mathcal{G}(P)$ (i.e. F vanishes on some neighborhood of 0 in $V(P)$), then there exist $F_1, \dots, F_r \in \mathbb{R}[x_1, \dots, x_n]$ with*

$$\deg F_j \le \overline{d} \text{ and } Q \cdot F = \sum_{j=1}^{r} F_j \cdot p_j.$$

The proof is based on Theorem 5.6′ below (which is simply the real analogue of the following theorem).

THEOREM 5.6. — *Let $n \ge 2$, $D \ge 1$, $1 \le r \le n-1$ be given integers. Denote by $\mathcal{W}_1$ the space of r-tuples $P = (p_1, \dots, p_r)$, $p_j \in \mathcal{H}^D$, with $P(0) = 0$, $J_P(0) = 1$.*

There exist integers $D_*, r_* \ge 1$ depending only on n, D, r, and for $d \ge 1$, a function d_* of d, n, D, r such that:

Given $P \in \mathcal{W}_1$, there exist $f_1, \dots, f_{r_*} \in \mathcal{H}^{D_*}$ with the following properties:

(i) *$f_j = 0$ on some neighborhood of 0 in $\mathcal{Z}(P)$, $1 \le j \le r_*$; and*

(ii) *if $f \in \mathcal{H}^d$ and $f = 0$ on some neighborhood of 0 in $\mathcal{Z}(P)$, then, we can find $g_j \in \mathcal{H}^{d_*}$, $j = 1, \dots, r_*$, such that*

$$f = \sum_{1 \le j \le r_*} g_j f_j.$$

The real analogue is

THEOREM 5.6′. — *Let $n \ge 2$, $D \ge 1$, $1 \le r \le n-1$, and let W_1 be the space of r-tuples $P = (p_1, \dots, p_r)$ of polynomials $p_j \in \mathbb{R}[x_1, \dots, x_n]$ with $\deg p_j \le D$, $P(0) = 0$, $J_P(0) = 1$.*

There exist integers $D_*, r_* \ge 1$ depending only on n, D, r and for $d \ge 1$, a function d_* of d, n, D, r such that, given $P \in W_1$, we can find polynomials $f_j \in \mathbb{R}[x_1, \dots, x_n]$, $1 \le j \le r_*$ with $\deg f_j \le D_*$ which vanish on some neighborhood of 0 in $V(P)$ and such that if $f \in H^d$ and $f = 0$ as a germ in $\mathcal{G}(P)$, then, there exist $g_j \in H^{d_*}$, $1 \le j \le r_*$ with $f = \sum_{1 \le j \le r_*} g_j f_j$.

We first remark that Theorem 5.6′ follows from Theorem 5.6. In fact, if $P = (p_1, \dots, p_r) \in W_1 \subset \mathcal{W}_1$,and $f \in \mathbb{C}[z_1, \dots, z_n]$ is such that it vanishes on $\{x \in \mathbb{R}^n \mid P(x) = 0\}$ near 0, then it vanishes on $\{z \in \mathbb{C}^n \mid P(z) = 0\}$ near 0 (see proof of Theorem 5.2). Theorem 5.6 gives us polynomials $f_1, \dots, f_{r_*} \in \mathcal{H}^{D_*}$ vanishing on $\mathcal{Z}(P)$ near 0 and generating all such f; we have only to replace f_j by the two polynomials f_{1j}, f_{2j} whose coefficients are the real and imaginary parts of those of f_j. [Note that if $x \in \mathbb{R}^n$ and $f_j(x) = 0$, then $f_{1j}(x) = f_{2j}(x) = 0$; further, if $f = \sum g_j f_j$ with $g_j \in \mathbb{C}[z_1, \dots, z_n]$ and we write $g_j = g_{1j} + ig_{2j}$ where g_{1j}, g_{2j} are polynomials with real coefficients, and if f also has real coefficients, we have $f = \sum g_{1j} f_{1j} - \sum g_{2j} f_{2j}$.]

Next, we remark that Theorem 5.6′ and Theorem 5.2 imply Theorem 5.5. To see this, if $P \in W \subset W_1$, Theorem 5.2 implies that there is $Q_0 \in H^{D'}$, $Q_0(0) = 1$, $\|Q_0\| \leq C'$ such that, with the f_j as in Theorem 5.6′, we have

$$J_P^{D_*} Q_0 f_j = \sum_{k=1}^{r} f_{jk} p_k,\ f_{jk} \in H^{D''},\ 1 \leq j \leq r_*;$$

here D', C', D'', D_* depend only on n, D, r, C_1. If $F \in H^d$ and vanishes on some neighborhood of 0 in $V(P)$, then

$$F = \sum_{1 \leq j \leq r_*} g_j f_j \text{ with } g_j \in H^{d_*},$$

and we obtain

$$J_P^{D_*} Q_0 F = \sum_{k=1}^{r} F_k p_k \text{ with } F_k = \sum_{1 \leq j \leq r_*} f_{jk} g_j,$$

so that $\deg F_k \leq d_* + D''$. We have only to take $Q = J_P^{D_*} Q_0$ to complete Theorem 5.5.

Thus, to prove Theorem 5.5, we have only to prove Theorem 5.6. Before starting on the proof of this theorem, we make some preliminary remarks.

Consider $\mathbb{C}^n$ as an open subset of projective space $\mathbb{P}^n$ with homogeneous coordinates $(z_0 : \cdots : z_n)$, $\mathbb{C}^n$ being defined by $z_0 \neq 0$. The hyperplane at $\infty : H = \{(z_0 : \cdots : z_n) \in \mathbb{P}^n \mid z_0 = 0\}$ can be naturally identified with $\mathbb{P}^{n-1}$.

Let X be an affine algebraic variety, $X \subset \mathbb{C}^n$, of pure dimension k. Its closure $\overline{X}$ in $\mathbb{P}^n$ is a projective variety of pure dimension k, and $\dim(\overline{X} \cap H) = k - 1$.

Any linear subspace of dimension $n-k$ in $\mathbb{P}^n$ meets $\overline{X}$, but there is a linear subspace $L \subset H$ of dimension $n-k-1$ such that $L \cap \overline{X} = \emptyset$.

(These are standard facts; see e.g. Mumford's book [M].)

A linear space $L \subset H$ as above defines a projection $\varphi : \mathbb{P}^n \backslash L \to \mathbb{P}^k$ as follows.

Fix a linear subspace of dimension k in $\mathbb{P}^n$ disjoint from L; we shall simply call it $\mathbb{P}^k$. If $w \in \mathbb{P}^n \backslash L, w$ and L span an $(n-k)$-dimensional linear subspace which meets $\mathbb{P}^k$ in a unique point $\varphi(w)$. The map φ sends w to $\varphi(w)$. Now, $\mathbb{C}^n \subset \mathbb{P}^n \backslash L$, and, with a suitable choice of linear coordinates, $\varphi | \mathbb{C}^n$ is just the standard projection of $\mathbb{C}^n$ onto $\mathbb{C}^k$.

We set $H_0 = \mathbb{P}^k \cap H; H_0$ is a hyperplane in $\mathbb{P}^k$, and $\mathbb{P}^k \backslash H_0 = \mathbb{C}^k$. We have $\varphi(H \backslash L) \subset H_0$ and $\varphi^{-1}(H_0) \subset H$ (since, if $w \in H \backslash L$, the span of w and L lies in H).

(10) Let $X \subset \mathbb{C}^n$ be an affine variety of pure dimension k, and let $\pi = \varphi | X$. Then $\pi(X) \subset \mathbb{C}^k = \mathbb{P}^k \backslash H_0; \pi : X \to \mathbb{C}^k$ is a proper map and its fibres $\pi^{-1}\pi(x), x \in X$, are finite.

In fact, if $K \subset \mathbb{C}^k$ is compact, then $\pi^{-1}(K) = (\varphi | \overline{X})^{-1}(K)$ (since $\varphi^{-1}(H_0) \cap \overline{X} = \overline{X} \cap H$) and so is compact since $\overline{X}$ is compact. The fibres $\pi^{-1}\pi(x)$ are compact analytic sets in $X \subset \mathbb{C}^n$ and so are finite sets.

(11) There exists an algebraic variety $B \subset \mathbb{C}^k$ of dimension $k-1$ (the branch locus) such that, if $\tilde{B} = \pi^{-1}(B)$, then $\tilde{B}$ has dimension $k-1$ and $\pi \mid X \backslash \tilde{B} \to \mathbb{C}^k \backslash B$ is a finite unramified covering; $X \backslash \tilde{B}$ is smooth, and π is of maximal rank k at every point of $X \backslash \tilde{B}$.

This again is standard; for a proof, see e.g. [M].

The number of points in $\pi^{-1}(y), y \in \mathbb{C}^k \backslash B$, is independent of y (since $\mathbb{C}^k \backslash B$ is connected). We call it the degree of π and denote it by μ.

Let $P = (p_1, \ldots, p_r) \in \mathcal{W}_1$ (so that $p_j \in \mathcal{H}^D$, $P(0) = 0$, $\det\left(\frac{\partial p_j}{\partial z_k}(0)\right) = 1$). Let $\mathcal{V}(P)$ be the irreducible component of $\mathcal{Z}(P) = \{z \in \mathbb{C}^n \mid P(z) = 0\}$ containing the origin (there is only one such component since $\mathcal{Z}(P)$ is smooth at 0 because $J_P(0) = 1$).

We take $X = \mathcal{V}(P)$ ($\dim X = n-r$) and project from a linear subspace $L \subset H$ of dimension $n - (n-r) - 1 = r - 1$ not meeting $\overline{X}$. We use the notation introduced above.

Let $\tilde{S}$ be the union of $\tilde{B}$ and the intersection of $\mathcal{V}(P)$ with all other irreducible components of $\mathcal{Z}(P)$, and let $S = \pi(\tilde{S})$; S and $\tilde{S}$ are algebraic varieties of dimension $\leq n-r-1$.

We shall need the following:

LEMMA 5.7. — *The degree μ of $\pi : \mathcal{V}(P) \to \mathbb{C}^{n-r}$ satisfies $\mu \leq D^r$.*

The proof uses Bezout's Theorem stated below. For a proof, see van der Waerden, *Algebra*, vol. 2, Chap. XI, §83.

BEZOUT'S THEOREM. — *Let $f_1, \dots, f_n$ be homogeneous polynomials of degrees $d_1, \dots, d_n$ respectively in $n+1$ variables $z_0, \dots, z_n$. Assume that the set*

$$\{z = (z_0 : \cdots : z_n) \in \mathbb{P}^n \mid f_1(z) = \cdots = f_n(z) = 0\}$$

is finite. Then, the number of points in this set is $\leq d_1 \cdots d_n$.

In fact, counted with the proper multiplicities (which are ≥ 1), the sum of the multiplicities of these points equals $d_1 \cdots d_n$.

Proof of Lemma 5.7. — If $p \in \mathbb{C}[z_1, \dots, z_n], p = \sum c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ has degree d, we denote by $\tilde{p}$ the "polynomial made homogeneous", viz. $\tilde{p}(z_0, \dots, z_n) = \sum c_\alpha z_0^{d-|\alpha|} z_1^{\alpha_1} \cdots z_n^{\alpha_n}$. If $P = (p_1, \dots, p_r) \in W$, the fibre $\pi^{-1}(c)$ of $\pi : \mathcal{V}(P) \to \mathbb{C}^{n-r}$ over $c \in \mathbb{C}^{n-r}, c = (c_1, \dots, c_{n-r})$ is the intersection of $\mathcal{V}(P)$ with the affine subspace $\varphi(z_1, \dots, z_n) = c$; if the coordinates are chosen so that $\varphi|\mathbb{C}^n$ is the standard projection $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_{n-r})$, $\pi^{-1}(c) = \{z \in \mathcal{V}(P) \mid z_\nu - c_\nu = 0, \nu = 1, \dots, n-r\}$. Let $\ell_\nu(z_0, \dots, z_n) = z_\nu - c_\nu z_0$. Then $\pi^{-1}(c) \subset \{z \in \mathbb{C}^n \mid \tilde{p}_j(1, z_1, \dots, z_n) = 0 = \ell_\nu(1, z_1, \dots, z_n)\}$.

If $c \in \mathbb{C}^{n-r} \backslash S$, the map $(z_1, \dots, z_n) \mapsto (p_1(z), \dots, p_r(z), \ell_1(1, z), \dots, \ell_{n-r}(1, z))$ is of maximal rank n at every point $z^{(\alpha)}$ of $\pi^{-1}(c)$, $\alpha = 1, \dots, \mu$. Let U_α be a small neighborhood of $z^{(\alpha)}$, $U_\alpha \cap U_\beta = \emptyset$ if $\alpha \neq \beta$. There exists $\epsilon > 0$ such that if q_j is a homogeneous polynomial in $z_0, \dots, z_n$ of degree $d_j = \deg p_j$, if λ_ν is a homogeneous linear form, and if $\|\tilde{p}_j - q_j\| < \epsilon$, $\|\ell_\nu - \lambda_\nu\| < \epsilon$ (the norms being for polynomials in $n+1$ variables), then each U_α contains a point at which $q_j(1, z_1, \dots, z_n) = 0 = \lambda_\nu(1, z_1, \dots, z_n)$ $(1 \leq j \leq r, 1 \leq \nu \leq n-r)$. Hence:

$$(12) \quad \mu \leq \#\{z = (z_0 : \cdots : z_n) \in \mathbb{P}^n \mid q_1(z) = \cdots = q_r(z) = 0 = \lambda_1(z) = \cdots = \lambda_{n-r}(z)\}$$

whenever $\deg q_j = d_j$, $\|q_j - \tilde{p}_j\| < \epsilon$, $\|\lambda_\nu - \ell_\nu\| < \epsilon$.

We now remark that for any $\epsilon > 0$, we can choose q_j, λ_ν close to $\tilde{p}_j, \ell_\nu$ so that the set

$$(13) \qquad \{z \in \mathbb{P}^n \mid q_1(z) = \cdots = q_r(z) = 0 = \lambda_1(z) = \cdots = \lambda_{n-r}(z)\}$$

is finite. In fact, take $q_1 = \tilde{p}_1$ and choose $q_2, \|q_2 - \tilde{p}_2\| < \epsilon$, such that q_2 does not vanish identically on any irreducible component of the set $\{z \in \mathbb{P}^n \mid q_1(z) = 0\}$. The variety $\{z \in \mathbb{P}^n \mid q_1(z) = q_2(z) = 0\}$ then has dimension $\leq n-2$. Proceeding in this way, we find q_j, λ_ν close to $\tilde{p}_j, \ell_\nu$, so that the set $\{q_j(z) = 0 = \lambda_\nu(z)\}$ has dimension 0, i.e. is finite.

Bezout's Theorem implies that the number of points in (13) is $\leq d_1, \cdots d_r 1 \cdots 1 \leq D^r$. This and (12) prove Lemma 5.7.

Remark. — If we apply this argument to the whole variety $\mathcal{Z}(P)$ with $P \in \mathcal{W}$, we find that the number t of connected components of $\mathcal{Z}(P)$ is $\leq D^r$ as stated in the proof of Theorem 3.1.

Proof of Theorem 5.6. — We consider the affine variety $\mathcal{V}(P) \subset \mathbb{C}^n$, $P \in \mathcal{W}$, and project onto $\mathbb{C}^{n-r} \subset \mathbb{P}^{n-r}$ from a linear subspace $L \subset H$ (of dimension $r-1$), $L \cap \overline{\mathcal{V}(P)} = \emptyset$ as above. We retain the notation above. Suppose linear coordinates $(x_1, \ldots, x_{n-r}, y_1, \ldots, y_r)$ chosen on $\mathbb{C}^n$ so that the map $\varphi : \mathbb{P}^n \backslash L \to \mathbb{P}^{n-r}$, restricted to $\mathbb{C}^n$, is simply $\varphi(x,y) = x$.

We first prove

LEMMA 5.8. — *Let $\ell(z) = \ell(x,y)$, $z = (x,y)$ be a linear function on $\mathbb{C}^n$ (i.e. a polynomial of degree ≤ 1). For $1 \leq i \leq \mu$ (μ = degree of $\pi : \mathcal{V}(P) \to \mathbb{C}^{n-r}$), there are polynomials $b_i(x)$ in x of degree $\leq i$ such that*

(i) $(\ell(z))^\mu + \sum\limits_{1 \leq i \leq \mu} b_i(\pi(z))(\ell(z))^{\mu-i} = 0$ *for any $z \in \mathcal{V}(P)$.*

(ii) *If $x \in \mathbb{C}^{n-r}\backslash S$ (notation as above), then $b_i(x)$ is the i-th elementary symmetric function in $\ell(z^{(1)}), \ldots, \ell(z^{(\mu)})$, where $\pi^{-1}(z) = (z^{(1)}, \ldots, z^{(\mu)})$.*

[Note: the elementary symmetric functions b_i in $\xi_1, \ldots, \xi_\mu$ are defined by $\prod\limits_{\nu=1}^{\mu} (T - \xi_\nu) = T^\mu + b_1 T^{\mu-1} + \cdots + b_\mu$, T being an indeterminate.]

Proof of Lemma 5.8. — For $x \in \mathbb{C}^{n-r}\backslash S$, define $b_i(x)$ by (ii) above. The b_i are holomorphic on $\mathbb{C}^{n-r}\backslash S$. If $x_0 \in S$ and U is a bounded open set in $\mathbb{C}^{n-r}$ with $x_0 \in U$, then $\pi^{-1}(\overline{U})$ is compact in $\mathcal{V}(P)$, so that ℓ is bounded on $\pi^{-1}(U)$. Hence $b_i \mid U\backslash S$ is bounded, so that it extends to a holomorphic function on U (by the Riemann extension theorem). Since $x_0 \in S$, is arbitrary this gives us holomorphic functions b_i on $\mathbb{C}^{n-r}$.

The equation (i) holds on $\mathcal{V}(P)\backslash\pi^{-1}(S)$ by definition, hence on all $\mathcal{V}(P)$ since $\pi^{-1}(S)$ has dimension $< n-r$ and so is nowhere dense in $\mathcal{V}(P)$.

To prove that b_i is a polynomial of degree $\leq i$, let $a \in H_0 = H \cap \mathbb{P}^{n-r}$; and let U_0 be a small neighborhood of a in $\mathbb{P}^{n-r}$. We may suppose that there are homogeneous coordinates $(w_0 : \cdots : w_{n-r})$ on $\mathbb{P}^{n-r}$ so that $w_1 \neq 0$ on U_0 and $U_0 \cap H_0 = \{w \in U_0 \mid w_0 = 0\}$. Let h be the holomorphic function w_0/w_1 on U_0. Since ℓ is linear (so has only simple poles on H), the function $z \mapsto \ell(z)h(\varphi(z))$ is holomorphic on $\varphi^{-1}(U_0) \subset \mathbb{P}^n \setminus L$. In particular, if K_0 is a compact neighborhood of a, $K_0 \subset U_0$, there is a constant $C > 0$ with $|\ell(z)h(\pi(z))| < C$ if $z \in \pi^{-1}(K_0 \setminus H_0)$. Moreover, $\dfrac{w_1}{w_0}, \ldots, \dfrac{w_{n-r}}{w_0}$ form linear coordinates in $\mathbb{C}^{n-r}$. Hence, if $x \in K_0 \setminus H_0$, $|h(x)| = 1/|\dfrac{w_1}{w_0}(x)| \geq \dfrac{C'}{1+|x|}$, $C' > 0$ being a constant. Thus $|\ell(z)| \leq C''(1 + |\pi(z)|)$, $z \in \pi^{-1}$ $(K_0 \setminus H_0)$, C'' being a constant. Since H_0 is compact, it follows that there is a constant $\overline{C} > 0$ so that

$$|\ell(z)| \leq \overline{C}(1 + |\pi(z)|), \ z \in \mathcal{V}(P). \tag{14}$$

Hence, if $x \in \mathbb{C}^{n-r} \setminus S$ and $\pi^{-1}(x) = (z^{(1)}, \ldots, z^{(\mu)})$, we have $|b_i(x)| = |(-1)^i \sum_{1 \leq \nu_1 < \cdots < \nu_i \leq \mu} \ell(z^{(\nu_1)}) \cdots \ell(z^{(\nu_i)})| \leq$ const. $(1 + |z|)^i$ since $\pi(z^{(\nu)}) = x$ for $1 \leq \nu \leq \mu$. Since b_i is holomorphic on $\mathbb{C}^{n-r}$, this inequality holds everywhere on $\mathbb{C}^{n-r}$ and shows that b_i is a polynomial of degree $\leq i$.

Choose an $r \times r$ invertible complex matrix A such that if $(u_1, \ldots, u_r) = (y_1, \ldots, y_r)\, A$, then u_1 separates the points of $\pi^{-1}(x_0) = (z_0^{(1)}, \ldots, z_0^{(\mu)})$ for some $x_0 \in \mathbb{C}^{n-r} \setminus S_1$ (possible since the r-tuple $(y_1, \ldots, y_r)$ takes distinct values at the points of $\pi^{-1}(x_0)$).

Let $F_j(x, u_j) = u_j^\mu + \sum_{\nu=1}^{\mu} b_{j\nu}(x) y^{\mu-\nu}$ $(1 \leq j \leq r)$ be the polynomial constructed in Lemma 5.8 for the linear form $\ell = u_j$. We shall simply write b_ν for $b_{1\nu}$. We have $F_j(x, u_j) \mid \mathcal{V}(P) \equiv 0$. Moreover, if $G(x, u_1)$ is a polynomial in x, u_1 alone with $\deg_{u_1} G < \mu$, then if $G \mid \mathcal{V}(P) \equiv 0$, G must be the zero polynomial (because $u_1(z^{(\nu)})$, $1 \leq \nu \leq \mu$, where $\pi^{-1}(x) = (z^{(1)}, \ldots, z^{(\mu)})$ would be μ distinct roots of $G(x, u_1)$ for x near x_0).

Let $\delta = \delta(x)$ be the discriminant of the polynomial $F_1(x, u_1)$. Then $\delta \not\equiv 0$; in fact $\delta(x_0) \neq 0$ since $F_1(x_0, u_1)$ has μ distinct roots. If $S_1 = S \cup \{x \in \mathbb{C}^{n-r} \mid \delta(x) = 0\}$, then $\pi^{-1}(S_1)$ has dimension $< n - r$ and, for $x \in \mathbb{C}^{n-r} \setminus S_1$, the polynomial $F_1(x, u_1)$ has μ distinct roots, and u_1 takes distinct values on $\pi^{-1}(x)$. The discriminant δ can be defined as the determinant of a $(2\mu - 1) \times (2\mu - 1)$ matrix whose entries are the $b_\nu(x)$ and $(\mu - \nu) \cdot b_\nu(x)$ (the coefficients of F_1 and $\partial F_1/\partial u_1$). Hence the degree of δ is $\leq \mu(2\mu - 1)$ since $\deg b_i \leq i \leq \mu$.

We now use $(x_1, \dots, x_{n-r}, u_1, \dots, u_r) = (x, u)$ as coordinates on $\mathbb{C}^n$.

(15) Let $d \geq 1$. There is a constant $C(d, \mu)$ depending only on d and μ such that the following holds.

If $f \in \mathbb{C}[x, u], \deg f \leq d$, there exist $g_0, \dots, g_{\mu-1} \in \mathbb{C}[x]$, such that $\deg g_\nu \leq C(d, \mu)$ and

$$\delta(x)f(x,u) - \sum_{\nu=0}^{\mu-1} g_\nu(x)u_1^\nu \equiv 0 \text{ on } \mathcal{V}(P).$$

Proof of (15). — We write $F_1(x, u_1) = \sum_{i=0}^{\mu} b_i(x)u_1^{\mu-i}$ with $b_0 \equiv 1$. Let $x \in \mathbb{C}^{n-r} \backslash S_1, \pi^{-1}(x) = (z^{(1)}, \dots, z^{(\mu)})$. Since $\delta(x) \neq 0$, the values $u_1(z^{(\nu)})$, $1 \leq \nu \leq \mu$ are distinct. Consider the sum

$$\begin{aligned}\sum_{\nu=1}^{\mu} \frac{F_1(x,u_1)}{u_1 - u_1(z^{(\nu)})} f(z^{(\nu)}) &= \sum_{\nu=1}^{\mu} \frac{F_1(x,u_1) - F_1(x, u_1(z^{(\nu)}))}{u_1 - u_1(z^{(\nu)})} f(z^{(\nu)}) \\ &= \sum_{\nu=1}^{\mu} f(z^{(\nu)}) \sum_{i=0}^{\mu} b_i(x) \frac{u_1^{\mu-i} - (u_1(z^{(\nu)}))^{\mu-i}}{u_1 - u_1(z^{(\nu)})} \\ &= \sum_{k=0}^{\mu-1} g'_k(x) u_1^k,\end{aligned}$$

where $g'_k(x) = \sum_{k<i\leq\mu} b_{\mu-i}(x) \sum_{1\leq\nu\leq\mu} (u_1(z^{(\nu)}))^{i-k-1} f(z^{(\nu)})$. Since $b_{\mu-i}(x)$ has degree $\leq \mu - i$ and $|z^{(\nu)}| \leq \text{const.}(1 + |x|)$ by (14) we obtain, if we note that u_1 is linear and f has degree $\leq d$,

$$\begin{aligned}|g'_k(x)| &\leq \text{const.} \max_{0<i\leq\mu} [(1+|x|)^{\mu-i}(1+|x|)^{i-k-1}(1+|x|)^d] \\ &\leq \text{const.}(1+|x|)^{d+\mu-k-1}, \ x \in \mathbb{C}^{n-r} \backslash S_1.\end{aligned}$$

This implies that g'_k extends holomorphically to $\mathbb{C}^{n-r}$ and is a polynomial of degree $\leq d + \mu - k - 1$.

If we substitute $u_1 = u_1(z^{(j)})$, $1 \leq j \leq \mu$, $(x, u) = z^{(j)}$, in the above equation, we obtain, for $x \notin S_1$,

$$\frac{\partial F_1}{\partial u_1}(x, u_1(z^{(j)})) f(x, u(z^{(j)})) = \sum_{k=0}^{\mu-1} g'_k(x)(u_1(z^{(j)}))^k, \ 1 \leq j \leq \mu.$$

Since $1 \leq j \leq \mu$ is arbitrary, this gives:

$$\frac{\partial F_1}{\partial u_1}(x, u_1) f(x,u) - \sum_{k=0}^{\mu-1} g'_k(x) u_1^k = 0 \ \text{ on } \ \mathcal{V}(P) \tag{16}$$

if $x \notin S_1$; since $\pi^{-1}(S_1)$ is nowhere dense in $\mathcal{V}(P)$, (16) *holds for all* $(x, u) \in \mathcal{V}(P)$. Here g'_k is a polynomial of degree $\leq d + \mu - k - 1$.

We now use the following fact about discriminants (see van der Waerden, *Algebra*, vol. 1).

(17) Let $F_1(x, u_1) = u_1^\mu + b_1(x)u_1^{\mu-1} + \cdots + b_\mu(x)$. There exist polynomials $A(x, u_1)$ and $B(x, u_1)$ with the following properties:

(i) $\deg_{u_1} A < \mu - 1, \deg_{u_1} B < \mu$.

(ii) $\delta(x) = A(x, u_1)F_1(x, u_1) + B(x, u_1)\dfrac{\partial F_1}{\partial u_1}(x, u_1)$.

(iii) The coefficients (of the powers of u_1) in $A(x, u_1), B(x, u_1)$ are polynomials, with integer coefficients depending only on μ, in $b_1, \ldots, b_\mu$ of total degree $\leq 2(\mu - 1)$.

Note. — The existence of A, B satisfying (1) and (ii) is well known. For (iii), if δ is the determinant of the $(2\mu - 1) \times (2\mu - 1)$ matrix Δ with entries b_ν, $(\mu-\nu)b_\nu$ referred to earlier, the coefficients of A, B are, up to sign, the determinants of the $(2\mu - 2) \times (2\mu - 2)$ matrices obtained from Δ by deleting one row and one column; see the book of van der Waerden cited above.

Since $\deg b_i \leq i \leq \mu$, A and B have total degree $\leq 2\mu^2 - \mu$.

From (16) and (17), we obtain

$$\begin{aligned}
\delta(x)f(x, u) - B(x, u_1)\sum_{k=0}^{\mu-1} g'_k(x)u_1^k
&= A(x, u_1)F_1(x, u_1)f(x, u) \\
&\quad + B(x, u_1)\Big(\frac{\partial F_1}{\partial u_1}(x, u_1)f(x, u) - \sum_{k=0}^{\mu-1} g'_k(x)u_1^k\Big) \\
&\equiv 0 \text{ on } \mathcal{V}(P).
\end{aligned}$$

We now make an algebraic division of $B(x, u_1)\sum_{k=0}^{\mu-1} g'_k(x)u_1^k$ by the monic polynomial $F_1(x, u_1)$ and write

$$B(x, u_1)\sum_{k=0}^{\mu-1} g'_k(x)u_1^k = \varphi(x, u_1)F_1(x, u_1) + \sum_{\nu=0}^{\mu-1} g_\nu(x)u_1^\nu. \tag{19}$$

Let $G(x, u_1) = a_0(x)u_1^N + a_1(x)u_1^{N-1} + \cdots + a_N(x)$ be a polynomial in x, u_1 of total degree $\leq d$. Then $\deg a_0(x) \leq d - N$. We claim that if we

write
$$G(x,u_1) = q(x,u_1)F_1(x,u_1) + h(x,u_1)$$
with $q,h \in \mathbb{C}[x,u_1]$ and $\deg_{u_1} h < N$, then the total degree, $\deg h$, of h is $\leq d$. To see this, note first that q and h are uniquely determined by G, so that, if $N < \mu$, we have $h = G$. If $N \geq \mu$, and we set $G_1(x,u_1) = G(x,u_1) - a_0(x)u_1^{N-\mu}F_1(x,u_1)$, then $\deg_{u_1} G_1 < N$. Further, F_1 has total degree $\leq d$ (since $\deg b_i \leq i$) and $\deg a_0 \leq d - N$. Hence G_1 has total degree $\leq d$, and $\deg_{u_1} G_1 \leq N-1$. Our claim now follows by induction on N since the remainder on division of G_1 by F_1 is the same as for division of G by F_1.

Thus, in (19),
$$\begin{aligned}\deg\Big(\sum_{\nu=0}^{\mu-1} g_\nu(x)u_1^\nu\Big) &\leq \deg B + \deg \sum_{k=0}^{\mu-1} g'_k(x)u_1^k \\ &\leq C(d,\mu).\end{aligned}$$

Since $F_1(x,u_1) \equiv 0$ on $\mathcal{V}(P)$, (18) and (19) imply that
$$\delta(x)f(x,u) - \sum_{\nu=0}^{\mu-1} g_\nu(x)u_1^\nu \equiv 0 \text{ on } \mathcal{V}(P),$$
which proves (15).

If $F_j(x,u_j)$ is the polynomial constructed in Lemma 5.8 for the linear form $\ell = u_j$ and we apply (15) to the polynomials $f(x,u) = u_\nu$ $(\nu = 2,\dots,\mu)$, we obtain:

(20) There exist monic polynomials
$$F_j(x,u_j) = u_j^\mu + \sum_{i=1}^{\mu} b_{ji}(x)u_j^{\mu-i},\ 1 \leq j \leq r,$$
where $\deg b_{ji} \leq i$, $\deg F_j = \mu$, and polynomials $G_\nu(x,u_1)$, $\nu = 2,\dots,r$ with $\deg_{u_1} G_\nu < \mu$, $\deg G_\nu \leq C(\mu)$ such that
$$F_j(x,u_j), j = 1,\dots,r,\ \ \delta(x)u_\nu - G_\nu(x,u_1),\ \ \nu = 2,\dots,r,$$
all vanish on $\mathcal{V}(P)$.

LEMMA 5.9. — *Let $N = \max(\mu,(\mu-1)(r-1))$. We have the following:*

(21) *If $f \in \mathbb{C}[x,u]$, then $f \equiv 0$ on $\mathcal{V}(P)$ if and only if there exist polynomials $A_j(x,u)$, $j = 1,\dots,r$, such that*
$$(\delta(x))^N f(x,u) = A_1(x,u)F_1(x,u_1) + \sum_{\nu=2}^{r} A_\nu(x,u)(\delta(x)u_\nu - G_\nu(x,u_1)).$$

Proof of Lemma 5.9. — If f satisfies this equation, then $\delta^N f \equiv 0$ on $\mathcal{V}(P)$, hence also $f \equiv 0$ on $\mathcal{V}(P)$ since $\{(x,u) \in \mathcal{V}(P) \mid \delta(x) \neq 0\}$ is dense in $\mathcal{V}(P)$.

To prove the converse, let $f(x,u) \in \mathbb{C}[x,u]$. We make an algebraic division of f by the monic polynomial $F_1(x,u_1)$ to write

$$f(x,u) = A_1'(x,u)F_1(x,u_1) + \sum_{\nu=0}^{\mu-1} f_\nu(x,u_2,\ldots,u_r)u_1^\nu, A_1' \in \mathbb{C}[x,u].$$

Dividing the coefficients f_ν by $F_2(x,u_2)$ and repeating this process with $F_3(x,u_3),\ldots,F_r(x,u_r)$, we can write

$$f(x,u) = \sum_{j=1}^{r} A_j'(x,u)F_j(x,u_j) + \sum_{0\leq\nu_1<\mu} f_{\nu_1\cdots\nu_r}(x)u_1^{\nu_1}\cdots u_r^{\nu_r},$$

where $A_j' \in \mathbb{C}[x,u]$ and $f_{\nu_1\cdots\nu_r} \in \mathbb{C}[x]$.

We remark that for $j = 2,\ldots,r$, if we substitute $G_j(x,u_1)$ for $\delta(x)u_j$, we find that

$$(\delta(x))^\mu F_j(x,u_j) \equiv R_j'(x,u_1) \operatorname{mod}(\delta(x)u_j - G_j(x,u_1))$$

where $R_j' \in \mathbb{C}[x,u_1]$. Dividing $R_j'(x,u_1)$ by $F_1(x,u_1)$ in $\mathbb{C}[x,u_1]$, we see that

$$(\delta(x))^\mu F_j(x,u_j) \equiv R_j(x,u_1) \operatorname{mod}(F_1(x,u_1), \delta(x)u_j - G_j(x,u_1))$$

where $\deg_{u_1} R_j(x,u_1) < \mu$. Since $F_j(x,u_j), F_1(x,u_1), \delta(x)u_j - G_j(x,u_1)$ all vanish on $\mathcal{V}(P)$, so also does $R_j(x,u_1)$. But, since $\deg_{u_1} R_j(x,u_1) < \mu$, this implies that $R_j \equiv 0$ as remarked earlier. Hence:

(23) $(\delta(x))^\mu F_j(x,u_j) \equiv 0 \operatorname{mod}(F_1(x,u_1), \delta(x)u_j - G_j(x,u_1)), j = 2,\ldots,r.$

Now, for $k = 2,\ldots,r$, $0 \leq \nu < \mu$,

$$(\delta(x))^{\mu-1}u_k^\nu \equiv \delta(x)^{\mu-\nu-1}(G_k(x,u_1))^\nu \operatorname{mod}(\delta(x)u_k - G_k(x,u_1)).$$

Hence

$$\delta(x)^{(\mu-1)(r-1)} \sum_{0\leq\nu_i<\mu} f_{\nu_1\cdots\nu_r}(x)u_1^{\nu_1}\cdots u_r^{\nu_2} \equiv B'(x,u_1)$$
$$\operatorname{mod}(\delta(x)u_2 - G_2(x,u_1),\ldots,\delta(x)u_r - G_r(x,u_1)),$$

where $B'(x,u_1)$ is a polynomial in x, u_1 alone. Dividing $B'(x,u_1)$ by $F_1(x,u_1)$ in $\mathbb{C}[x,u_1]$, we see that $B'(x,u_1) \equiv B(x,u_1) \operatorname{mod} F_1(x,u_1)$, where $\deg_{u_1} B(x,u_1) < \mu$. Multiplying (22) by $\delta(x)^N$, $N = \max(\mu, (\mu-1)(r-1))$ and using the above facts together with (23), we have:

Given $f \in \mathbb{C}[x,u]$, there exists a polynomial $B(x,u_1)$ in x and u_1 alone, with $\deg_{u_1} B < \mu$, such that

$$(\delta(x))^N f(x,u) \equiv B(x,u_1) \operatorname{mod}(F_1(x,u_1), \delta(x)u_\nu - G_\nu(x,u_1), 2 \leq \nu \leq r).$$

Now, if $f \equiv 0$ on $\mathcal{V}(P)$, then $B(x,u_1) \equiv 0$ on $\mathcal{V}(P)$. But again, as remarked earlier, a polynomial in x, u_1 alone, of degree $< \mu$ in u_1 and vanishing on $\mathcal{V}(P)$ must be the zero polynomial. Thus, $f \equiv 0$ on $\mathcal{V}(P)$ if and only if $B(x,u_1) \equiv 0$, i.e. if and only if $(\delta(x))^N f(x,u) \equiv 0$ $\operatorname{mod}(F_1(x,u_1), \delta(x)u_\nu - G_\nu(x,u_1)$, $2 \leq \nu \leq r)$, which is Lemma 5.9.

To complete the proof of Theorem 5.6, we invoke the following theorem, which is also proved in the paper [He] of Grete Hermann cited earlier. It can be proved in exactly the same way as Theorem 5.4.

THEOREM. — *Let n, D, r, s be integers ≥ 1. There are integers $D' \geq 1$ and $r' \geq 1$ depending only on n, D, r such that the following holds.*

Let $(p_{ij})_{1\leq i\leq r, 1\leq j\leq s}$ be a matrix of polynomials $p_{ij} \in \mathbb{C}[z_1,\ldots,z_n]$ with $\deg p_{ij} \leq D$. Consider the system of homogeneous equations

$$\sum_{i=1}^{r} g_i p_{ij} = 0,\ j = 1,\ldots,s,\ g_i \in \mathbb{C}[z_1,\ldots,z_n]. \tag{24}$$

There exist r' solutions $g^{(\alpha)} = (g_{\alpha 1},\ldots,g_{\alpha r})$, $\alpha = 1,\ldots,r'$ of (24) where $g_{\alpha i} \in \mathbb{C}[z_1,\ldots,z_n]$, $\deg g_{\alpha i} \leq D'$, such that

(i) $\sum_i g_{\alpha i} p_{ij} = 0$, $j = 1,\ldots,s$, $\alpha = 1,\ldots,r'$.

(ii) If $g = (g_1,\ldots,g_r)$ satisfies (24), then we can find $h_\alpha \in \mathbb{C}[z_1,\ldots,z_n]$, $\alpha = 1,\ldots,r'$, with $g = \sum_\alpha h_\alpha g^{(\alpha)}$.

Applying this to the equation in Lemma 5.9, (21), we obtain the following:

There exist constants $\overline{D}, \overline{r} \geq 1$ depending only on μ, n, r, D, N hence only on n, r, D (since $\deg \delta \leq \mu(2\mu - 1), \mu \leq D^r$ and $N \leq \mu r$) and a set of $\overline{r}\,(r+1)$-tuples

$$(f_\alpha, A_1^{(\alpha)},\ldots,A_r^{(\alpha)}),\ \alpha = 1,\ldots,\overline{r}$$

of polynomials of degree $\leq \overline{D}$ in (x,u) such that

(a) $(\delta(x))^N f_\alpha(x,u) = A_1^{(\alpha)}(x,u)F_1(x,u_1) + \sum_{\nu=2}^{r} A_\nu^{(\alpha)}(x,u)(\delta(x)u_\nu - G_\nu(x,u_1))$

and

(b) if (f, A_j) satisfies the equation in (21), then, there exist $h_\alpha \in \mathbb{C}[x,u]$ with

$$(f, A_1, \ldots, A_r) = \sum_\alpha h_\alpha (f_\alpha, A_1^{(\alpha)}, \ldots, A_r^{(\alpha)}).$$

It follows from Lemma 5.9 that the ideal of $f \in \mathbb{C}[x,u]$ which vanish on $\mathcal{V}(P)$ is generated by the $f_\alpha, \alpha = 1, \ldots, \overline{r}$. Further, if $f \in \mathbb{C}[x,u]$ vanishes in some neighborhood of 0 in $\mathcal{V}(P)$, then $f \mid \mathcal{V}(P) \equiv 0$ since the set of regular points of $\mathcal{V}(P)$ is connected and dense.

Theorem 5.6 follows from this and Theorem 5.4.

Note. — Madhav Nori has shown us a proof (using generic flatness) that if I is an ideal in $\mathbb{C}[z_1, \ldots, z_n]$ generated by r polynomials of degree $\leq D$, then the radical of I is generated by r' polynomials of degree $\leq D'$ (r', D' depending only on n, D, r). Nori ascribes the method to Grothendieck. We could use this theorem instead of Theorem 5.6 (because of Theorem 3.1).

The proof given above seems to us more elementary, and picks out the radical of one primary component in the decomposition of I.

6. Semi-algebraic sets and maps: general properties.

In this section, we collect together the general properties of semi-algebraic sets and maps that we shall need. Although several of these results are in the literature (see e.g. [BCR]), we have given direct proofs based on the two basic structure theorems of the subject.

DEFINITION 6.1.

(a) *The family of semi-algebraic sets in* $\mathbb{R}^n$ *is the smallest class of sets containing all sets of the form* $\{x \in \mathbb{R}^n \mid P(x) > 0\}$ *where* $P \in \mathbb{R}[x_1, \ldots, x_n]$, *and which is closed under the operations of finite unions, finite intersections and complementation.*

The class of semi-algebraic sets is invariant under linear isomorphisms of $\mathbb{R}^n$, so that we may speak of semi-algebraic sets in any finite dimensional vector space over $\mathbb{R}$.

(b) *If $E \subset \mathbb{R}^n$, a map (or function) $f : E \to \mathbb{R}^M$ is called semi-algebraic if its graph $\{(x, f(x)) \mid x \in E\}$ is a semi-algebraic subset of $\mathbb{R}^{m+n}$.*

(c) *A function $f : E \to \mathbb{R} \cup \{-\infty, +\infty\}$ is called an extended semi-algebraic function if the sets $E_- = f^{-1}(-\infty)$,$E_+ = f^{-1}(+\infty)$, $E_0 = f^{-1}(\mathbb{R})$ are all semi-algebraic and $f|E_0 \to \mathbb{R}$ is a semi-algebraic function.*

(d) *If $E \subset \mathbb{R}^n$ is semi-algebraic, a semi-algebraic partition of E is a finite collection $\{E_\nu\}$ of semi-algebraic sets which are pointwise disjoint and whose union is E.*

A partition $\{F_\mu\}$ of E is a *refinement* of the partition $\{E_\nu\}$ of E if, for each ν, E_ν is the union of those F_μ which meet it: $E_\nu = \bigcup_{F_\mu \cap E_\mu \neq \emptyset} F_\mu$.

We note, explicitly, that we are not assuming that a semi-algebraic map is continuous. If E is semi-algebraic and $\{E_\nu\}$ is a semi-algebraic partition of E, then a function $f : E \to \mathbb{R}^m$ is semi-algebraic if and only if $f|E_\nu$ is semi-algebraic for all ν.

The basic structure theorem, which enables one to reduce the study of semi-algebraic sets in $\mathbb{R}^{n+1}$ to that of sets in $\mathbb{R}^n$ is the following. For a proof see Cohen [C]; see also the book of Bochnak, Coste, Roy [BCR].

THE STRUCTURE THEOREM. — *Let E be a semi-algebraic set in $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. There is a (finite) semi-algebraic partition $\{A_\nu\}$ of $\mathbb{R}^n$ and, for each ν, a finite family of functions $t_{\nu j}$ on A_ν, $0 \leq j \leq r_\nu + 1$, with values in $\mathbb{R} \cup \{-\infty, \infty\}$ such that*

$$-\infty \equiv t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,r_\nu} < t_{\nu,r_\nu+1} \equiv +\infty,$$

$t_{\nu,j}$ is continuous for $1 \leq j \leq r_\nu$, and having the following properties:

(a) *Each set $\{(x, t) \in \mathbb{R}^{n+1} \mid x \in A_\nu,\ t_{\nu,j}(x) < t < t_{\nu,j+1}(x)\}$, $0 \leq j \leq r_\nu$, is semi-algebraic.*

(b) *Each set $\{(x, t) \in \mathbb{R}^{n+1} \mid x \in A_\nu,\ t = t_{\nu,j}(x)\}$, $0 < j \leq r_\nu$, is semi-algebraic.*

(c) *E is a finite disjoint union of sets of the form* (a) *or* (b).

By considering the graph of a semi-algebraic map $f : \mathbb{R}^n \to \mathbb{R}^m$ and using the structure theorem, one obtains the central theorem about semi-algebraic sets.

TARSKI-SEIDENBERG THEOREM. — *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a semi-algebraic map and let $E \subset \mathbb{R}^n$ be a semi-algebraic set. Then $f(E) \subset \mathbb{R}^m$ is again semi-algebraic.*

We list some basic properties in the next two results. They are easy consequences of the Tarski-Seidenberg Theorem (see [BCR]).

PROPOSITION 6.2.

(a) *Let $E \subset \mathbb{R}^n$ be semi-algebraic, and let $f : E \to \mathbb{R}^m$ be a semi-algebraic map. Then, for any semi-algebraic set $F \subset \mathbb{R}^m$, the set $f^{-1}(F)$ is again semi-algebraic.*

(b) *Let $E \subset \mathbb{R}^n, F \subset \mathbb{R}^m$ be semi-algebraic, let $f : E \to \mathbb{R}^m$ be a semi-algebraic map with $f(E) \subset F$ and let $g : F \to \mathbb{R}^k$ be semi-algebraic. Then $g \circ f : E \to \mathbb{R}^k$ is semi-algebraic.*

(c) *Let $E \subset \mathbb{R}^n \times \mathbb{R}^m$ be semi-algebraic. Then, the set*

$$\{x \in \mathbb{R}^n \mid \exists\, y \in \mathbb{R}^m \text{ with } (x, y) \in E\}$$

is semi-algebraic.

(d) *Let $E \subset \mathbb{R}^n \times \mathbb{R}^m$ be semi-algebraic, let $F \subset \mathbb{R}^m$ be semi-algebraic. Then, the set*

$$\{x \in \mathbb{R}^n \mid (x, y) \in E,\ \forall u \in F\}$$

is semi-algebraic.

LEMMA 6.3. — *The closure $\overline{E}$ of a semi-algebraic set $E \subset \mathbb{R}^n$ is again semi-algebraic.*

We shall also need a second fundamental theorem on semi-algebraic sets which goes back to Whitney's work on real algebraic varieties [W]. The theorem is proved in [BCR].

Before giving the definition needed to state this theorem, we remark that by a smooth (real analytic) submanifold M of $\mathbb{R}^n$, we mean a subset $M \subset \mathbb{R}^n$ with the following property: for any $a \in M$, we can find an open neighborhood U of a in $\mathbb{R}^n$, an integer $d \geq 0$, and smooth (real analytic) coordinates $x_1, \ldots, x_n$ on U such that $M \cap U = \{x \in U \mid x_{d+1} = \cdots = x_n = 0\}$.

DEFINITION 6.4. — *Let $E \subset \mathbb{R}^n$. A stratification $\{S_i\}$ of E is a finite family of pairwise disjoint subsets $S_i \subset E$ with $E = \cup S_i$ and having the following properties:*

(a) *Each S_i is a real analytic submanifold of $\mathbb{R}^n$.*

(b) *If $i \neq j$ and $\overline{S}_i \cap S_j \neq 0$, then $S_j \subset \overline{S}_i$ and $\dim S_j < \dim S_i$.*

The $\{S_i\}$ are called the strata of the stratification.

If E is semi-algebraic, the stratification $\{S_i\}$ is called semi-algebraic if each stratum S_i is semi-algebraic. In particular, $\{S_i\}$ is a semi-algebraic partition of E.

THE STRATIFICATION THEOREM. — *Let $E \subset \mathbb{R}^n$ be semi-algebraic and let $\{E_\nu\}$ be a semi-algebraic partition of E.*

There exists a semi-algebraic stratification $\{S_i\}$ of E which is also a refinement of E_ν, i.e. each E_ν is the union of those S_i which meet it.

DEFINITION 6.5. — *Let $E \subset \mathbb{R}^n$ be semi-algebraic. Let $\{S_i\}$ be a semi-algebraic stratification of E. We define the dimension of E, $\dim E$, by:*

$$\dim E = -1 \text{ if } E = \emptyset, \ \dim E = \max_i \dim S_i \ \text{ if } E \neq \emptyset.$$

Remark. — Given a semi-algebraic set $E \subset \mathbb{R}^n$, let t be the largest integer such that E contains the image of the unit ball B_t in $\mathbb{R}^t$ under a C^∞ map $\varphi : B_t \to \mathbb{R}^n$ of rank t everywhere on B_t. Then $t = \dim E$; in fact if $d = \max \dim S_i$ ($\{S_i\}$ a semi-algebraic stratification of E), then clearly a submanifold of dimension d of $\mathbb{R}^n$ contains the image of a ball of dimension d under a smooth map of maximal rank, so that $t \geq d$. If $t > d$ and $\varphi : B_t \to \mathbb{R}^n$ has rank t, then $\varphi^{-1}(S_i)$ has measure 0 in B_t for each i and we cannot have $\varphi(B_t) \subset \cup S_i$.

We shall say that a set $E \subset \mathbb{R}^n$ contains the diffeomorphic image of a ball of dimension t if there exists a C^∞ embedding $\varphi : B_t \to \mathbb{R}^n$ (i.e. φ is injective of maximal rank) so that $\varphi(B_t) \subset E$. We may formulate our remark above as follows:

If $E \subset \mathbb{R}^n$ is semi-algebraic, then $\dim E$ is the largest integer t such that E contains the diffeomorphic image of a ball of dimension t.

In particular, $\dim E$ is independent of the stratification $\{S_i\}$ used in Definition 6.5.

LEMMA 6.6. — *Let $E_1, \dots, E_N$ be semi-algebraic sets in $\mathbb{R}^n$. Then*

$$\dim(E_1 \cup \cdots \cup E_N) = \max_{1 \leq \nu \leq N} \dim E_\nu.$$

Proof. — If $E = \bigcup_{1\le\nu\le N} E_\nu$, then clearly $\dim E \ge \max_\nu \dim E_\nu$ (since if E_ν contains the diffeomorphic image of a ball of dimension t, so does E).

Let now $t > \max_\nu \dim E_\nu$ and $\varphi : B_t \to \mathbb{R}^n$ be a C^∞ embedding. If $\{S_{i,\nu}\}$ is a stratification of E_ν, then $\dim S_{i,\nu} < t$ (for all i,ν) so that $\varphi^{-1}(S_{i,\nu})$ has measure 0 in B_t. Hence $\varphi^{-1}(E)$ has measure 0 in B_t, so that $\varphi(B_t) \not\subset E$. Hence $t > \dim E$, which shows that $\dim E \le \max_\nu \dim E_\nu$.

LEMMA 6.7. — *Let $E \subset \mathbb{R}^n$ be semi-algebraic and let $\overline{E}$ be the closure. Then,*

$$\dim \overline{E} = \dim E \textit{ and } \dim(\overline{E}\backslash E) < \dim E.$$

Proof. — Let $\{S_i\}$ be a semi-algebraic stratification of $\overline{E}$ refining the partition $\{E,\overline{E}\backslash E\}$ of $\overline{E}$. Let $J = \{i \mid S_i \cap E \neq \emptyset\}$. Then $E = \bigcup_{i\in J} S_i$, $\overline{E}\backslash E = \bigcup_{j\notin J} S_j$. We have $\overline{E} = \bigcup_{i\in J} \overline{S}_i$, so that, if $j \notin J$, then $S_j \cap \overline{S}_i \notin \emptyset$ for some $i \in J$. By condition (b) in Definition 6.4, this implies that $S_j \subset \overline{S}_i$ and $\dim S_j < \dim S_i \le \dim E$. Both statements in the lemma follow from this.

We now make a simple remark which we shall use in Lemma 6.9 below and also later.

Remark 6.8. — Let X, Y be connected C^∞ manifolds and $f : X \to Y$, a smooth map. Assume that for any $y \in Y$, the fibre $f^{-1}(y)$ is discrete (consists of isolated points). Then, the set $\{x \in X \mid \text{rank } (df_x) = \dim X\}$, df_x being the differential of f at x, is (open and) dense in X.

In fact, let $U \subset X$ be open, $U \neq \emptyset$. Let $p = \max_{x\in U} \text{rank } (df_x)$, and let $x_0 \in U$ be such that rank $(df_{x_0}) = p$. Then, there is a connected open set $U_0 \subset U$, $x_0 \in U_0$, so that rank $(df_x) = p$ for all $x \in U_0$. By the rank theorem, the sets $f^{-1}f(x) \cap U_0$, $x \in U_0$, are submanifolds of U_0 of dimension $\dim X - p$; since the fibres are discrete, we must have $p = \dim X$.

LEMMA 6.9. — *Let $E \subset \mathbb{R}^n$ be a semi-algebraic set and let $f : E \to \mathbb{R}^m$ be a semi-algebraic map. Let $\Gamma = \{(x, f(x)) \mid x \in E\}$ be the graph of f. Then:*

(a) $\dim E = \dim \Gamma$

(b) $\dim f(E) \le \dim E$.

(c) *If, moreover, the fibres of* f *are finite, then* $\dim f(E) = \dim E$.

Proof.

(a) Let $S \subset \Gamma$ be a submanifold of $\mathbb{R}^n \times \mathbb{R}^m$ of dimension d. The projection $\pi : \Gamma \to E$, $\pi(x, f(x)) = x$ is bijective, so that by 4.8, there is an open set $U \subset S$ such that $\pi|U$ has rank $= \dim S$ at every point of U. Hence E contains the image of a ball of dimension d, under a map of maximal rank, so that $d \leq \dim E$; thus $\dim \Gamma \leq \dim E$.

On the other hand, if $\varphi : B_t \to \mathbb{R}^n$ is a C^∞ embedding with $t > \dim \Gamma$, and $\{S_1, \ldots, S_N\}$ is a semi-algebraic stratification of Γ, then $\dim S_i < t$ and $\pi(S_i) \subset \mathbb{R}^n$ has t-dimensional measure 0. Hence $\varphi(B_t) \not\subset \pi(S_1 \cup \cdots \cup S_N) = \pi(\Gamma) = E$. Hence $\dim E < t$, i.e. $\dim E \leq \dim \Gamma$.

(b) By (a), it is enough to show that if $\alpha : \Gamma \to \mathbb{R}^m$ is the projection $(x, y) \mapsto y$ restricted to Γ, then $\dim \alpha(\Gamma) \leq \dim \Gamma$.

If $t = \dim \Gamma = \dim E$ and $\Gamma = \cup S_i$ is a semi-algebraic stratification of Γ, the $(t+1)$-dimensional measure of $\alpha(S_i)$ is 0 for any i, so that $\alpha(\Gamma)$ cannot contain the diffeomorphic image of a ball of dimension $\geq t + 1$.

(c) If $f : E \to \mathbb{R}^m$ has finite fibres and α is the restriction to Γ of the projection $(x, y) \mapsto y$, then α has finite fibres. If $S \subset \Gamma$ is a smooth submanifold with $\dim S = \dim \Gamma$, Remark 6.8 implies that there is an open set $U \subset S$, $U \neq \emptyset$, so that $\alpha|U$ has rank $\dim S$ at every point of U. Since $\alpha(U) \subset \alpha(\Gamma)$, we must have $\dim \alpha(\Gamma) \geq \dim S = \dim \Gamma$. This, together with (a) and (b), show that $f(E) = \alpha(\Gamma)$ has the same dimension as E.

Remark 6.10. — We shall use the following simple remark. If $A \subset \mathbb{R}^n$, denote by $\delta(A)$ the closure in $\mathbb{R}^n$ of the set $\overline{A} \backslash A$ ($\overline{A}$ being the closure of A). Then $A \backslash \delta(A)$ is closed in $\mathbb{R}^n \backslash \delta(A)$.

In fact, if $\{x_\nu\}_{\nu \geq 1}$ is a sequence of points in $A \backslash \delta(A)$ converging to $x_0 \in \mathbb{R}^n \backslash \delta(A)$, then $x_0 \in \overline{A}$. If $x_0 \notin A$, we would have $x_0 \in \overline{A} \backslash A \subset \delta(A)$. Thus $x_0 \in A \backslash \delta(A)$, and $A \backslash \delta(A)$ is closed in $\mathbb{R}^n \backslash \delta(A)$.

LEMMA 6.11. — *Let* $E \subset \mathbb{R}^n$ *be a semi-algebraic set of dimension* d *and let* $f : E \to \mathbb{R}^m$ *be a semi-algebraic map.*

There exists a closed semi-algebraic set $E' \subset \mathbb{R}^n$ *of dimension* $< d$ *such that* $E \backslash E'$ *is a closed real analytic submanifold of* $\mathbb{R}^n \backslash E'$ *and* $f \mid E \backslash E'$ *is real analytic.*

Proof. — Let $\{S_i\}$, $1 \leq i \leq p$, be a semi-algebraic stratification of E, and suppose that $\dim S_i = d$ for $1 \leq i \leq q(\leq p)$, $\dim S_i < d$ for $q+1 \leq i \leq p$. Let E_0 be the closure in $\mathbb{R}^n$ of $\bigcup\limits_{q<i\leq p} S_i \cup \bigcup\limits_{1\leq j\leq p} \delta(S_j)$ ($\delta(S_j)$ =closure of $\overline{S}_j\backslash S_j$; see remark 6.10). Then E_0 is a closed semi-algebraic set of dimension $< d$ (by Lemmas 6.3, 6.6, 6.7), and $E\backslash E_0$ is a closed real analytic submanifold of $\mathbb{R}^n\backslash E_0$ of pure dimension d since $\dim E = d = \max(\dim E_0, \dim(E\backslash E_0))$ by Lemma 6.6.

Let $\Gamma = \{(x, f(x)) \in \mathbb{R}^n \times \mathbb{R}^m \mid x \in E\}$ be the graph of f; we have $\dim \Gamma = d$ by Lemma 6.9. Moreover, by Lemma 6.9, (c), if $F \subset \mathbb{R}^n$ is a semi-algebraic set, then $\dim((F \times \mathbb{R}^m) \cap \Gamma) = \dim(E \cap F) \leq \dim F$.

Let $\{T_j\}$, $1 \leq j \leq k$, be a semi-algebraic stratification of Γ; let $\dim T_j = d$ for $1 \leq j \leq \ell$, $\dim T_j < d$ for $\ell+1 \leq j \leq k$. If $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is the projection, let E_1 be the closure in $\mathbb{R}^n$ of $\pi\Big(\bigcup\limits_{\ell+1\leq j\leq k} T_j\Big)$; E_1 is a closed semi-algebraic set of dimension $< d$. Let E_2 be the closure in $\mathbb{R}^n$ of $\pi\Big(\bigcup\limits_{1\leq j\leq k} \delta(T_j)\Big)$ $[\delta(T_j) = \text{closure of } \overline{T}_j\backslash T_j]$ and let $E_3 = E_0 \cup E_1 \cup E_2$; E_3 is a closed semi-algebraic set of dimension $< d$ (by Lemmas 6.6, 6.7, 6.9).

We have $\tilde{\Gamma} = \Gamma \cap (\mathbb{R}^n \times \mathbb{R}^m\backslash\pi^{-1}(E_3)) = \bigcup\limits_{1\leq j\leq l} (T_j\backslash\pi^{-1}(E_3))$, $\tilde{\Gamma}$ is a closed real analytic submanifold of $\mathbb{R}^n \times \mathbb{R}^m\backslash\pi^{-1}(E_3)$ of pure dimension $d, \pi(\tilde{\Gamma}) = E\backslash E_3$ and $\tilde{\Gamma}$ is the graph of $f \mid E\backslash E_3$. Further, $E\backslash E_3$ is a closed real analytic submanifold of $\mathbb{R}^n\backslash E_3$ of pure dimension d.

We now make the following remark. Let $M \subset \mathbb{R}^n \times \mathbb{R}^m$ be a semi-algebraic set which is a smooth submanifold of $\mathbb{R}^n \times \mathbb{R}^m$. Let $\varphi : M \to \mathbb{R}^n$ be the restriction to M of the projection $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Then, the set $C = \{x \in M \mid \text{the differential } d\varphi_x \text{ of } \varphi \text{ at } x \text{ is not injective}\}$ is semi-algebraic.

To see this, let

$$\begin{aligned}\mathcal{S} = \{(x,y,v) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \mid x, y \in M,\ x \neq y, \text{ and}\\ \exists\, \lambda \in \mathbb{R} \text{ so that } v = \lambda x + (1-\lambda)y\}\end{aligned}$$

(we have written $\mathbb{R}^{n+m}$ for $\mathbb{R}^n \times \mathbb{R}^m$). $\mathcal{S}$ is the family of all secants of M at distinct points and is semi-algebraic. Hence so it its closure $\overline{\mathcal{S}}$. Let

$$\mathcal{T} = \{(x,v) \in \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \mid (x,x,v) \in \overline{\mathcal{S}},\ x \in M\}.$$

Then $\mathcal{T}$ is semi-algebraic, and consists of pairs (x, v) where v lies on a tangent line to M at $x \in M$. [Since M is smooth, tangent lines are precisely limits of secant lines at distinct points.] The kernel of $d\varphi_x$, $x \in M$, consists of those $v \in \mathbb{R}^{n+m}$ such that $(x, x+v) \in \mathcal{T}$ and v projects to 0 in $\mathbb{R}^n$, (i.e. has first n coordinates 0). Thus

$$C = \{x \in M \mid \exists w \neq x,\ w \in \mathbb{R}^{n+m} \ \text{ with } \ \pi(w-x) = 0,\ (x,w) \in \mathcal{T}\}.$$

This is clearly semi-algebraic because of the Tarski-Seidenberg theorem.

We return to the proof of Lemma 6.11 and the notation introduced earlier. Let $\varphi = \pi \mid \tilde{\Gamma} : \tilde{\Gamma} \to E \backslash E_3$; φ is a bijection. Let $C = \{(x,y) \in \tilde{\Gamma} \mid d\varphi_{(x,y)}$ is not injective$\}$. Then C is semi-algebraic and $\tilde{\Gamma}\backslash C$ is open and dense in $\tilde{\Gamma}$ by remark 6.8. We have $\dim C < \dim \tilde{\Gamma} = d$ (because the image in $\tilde{\Gamma}$ of a ball of dimension d under a map of maximal rank contains an open subset of $\tilde{\Gamma}$, hence cannot be contained in C).

Let E' be the union of E_3 and the closure in $\mathbb{R}^n$ of $\pi(C) = \varphi(C)$. E' is a closed semi-algebraic set of dimension $< d$ (since $\dim C < d$). By construction $\varphi \mid \tilde{\Gamma}\backslash\pi^{-1}(E') \to E\backslash E'$ is a real analytic bijection between manifolds of pure dimension d and has maximal rank, so that its inverse is real analytic. Since $f \mid E\backslash E'$ is the composite of this inverse with the projection $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, the lemma is proved.

Note. — For the application we have in mind, it would suffice to show the existence of a semi-algebraic $E' \subset \mathbb{R}^n$ with $\dim E' < d$ such that $f \mid E\backslash E'$ is continuous. This can be done more simply, using the structure theorem directly. If $\Gamma \subset \mathbb{R}^{n+1}$ is the graph of a semi-algebraic function $f : E \to \mathbb{R}$, we have only to make sure that E' contains the sets $\overline{A}_\nu \backslash A_\nu$ for the sets of the form (b) in the structure theorem; sets of the form (a) do not occur if Γ is a graph.

LEMMA 6.12. — *Let $E \subset \mathbb{R}^n$ and let $f : E \to \mathbb{R}^m$ be a semi-algebraic map. Set $f(E) = F$. Then, there exists a semi-algebraic section $\sigma : F \to E$; in other words, σ is a semi-algebraic map $F \to E$ such that $f(\sigma(y)) = y$ for all $y \in F$.*

Proof. — Replacing E by the graph of f, we may assume that $E \subset \mathbb{R}^n \times \mathbb{R}^m$ and that f is the projection $\pi : E \to \mathbb{R}^m$, $\pi(x,y) = y$ for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. If we write $y = (y_1, \dots, y_m)$ and $\pi_1 : E \to \mathbb{R}^n \times \mathbb{R}^{m-1}$ is the projection $(x, (y_1, \dots, y_m)) \to (x, (y_2, \dots, y_m))$, then, by induction on m, it suffices to prove the lemma for π_1 (because of Prop. 6.2 (b)). Thus, by a change of notation, it is sufficient to prove the following statement.

Let $E \subset \mathbb{R}^{N+1}$ be semi-algebraic, and let $F = \pi(E)$, $\pi : \mathbb{R}^{N+1} \to \mathbb{R}^N$ being the projection $(x_1, \dots, x_{N+1}) \mapsto (x_1, \dots, x_N)$. Then, there exists a semi-algebraic map $\sigma : F \to \mathbb{R}^{N+1}$ of the form $\sigma(x_1, \ldots, x_N) = (x_1, \ldots, x_N, s(x_1, \ldots, x_N))$ with the property that $\sigma(x) \in E$ for all $x \in F$.

We use the structure theorem.

There exists a semi-algebraic partition $\{A_\nu\}$ of F, and, for each ν, a finite family $\{t_{\nu,j}\}_{0 \le j \le r_\nu + 1}$ of functions on A_ν such that $-\infty \equiv t_{\nu,0} < t_{\nu,1} < \cdots < t_{\nu,r_\nu} < t_{\nu,r_\nu+1} \equiv +\infty$, $t_{\nu,j}$ is continuous and semi-algebraic on A_ν for $1 \le j \le r_\nu$ and such that E is a finite disjoint union of sets of the form

(a) $\{(x,t) \in \mathbb{R}^{N+1} \mid x \in A_\nu,\ t_{\nu,j}(x) < t < t_{\nu,j+1}(x)\}$, $0 \le j \le r_\nu$

or of the form

(b) $\{(x,t) \in \mathbb{R}^{N+1} \mid x \in A_\nu,\ t = t_{\nu,j}(x)\}$, $1 \le j \le r_\nu$.

If E contains a set of the form (b) for a given ν, we define $\sigma|A_\nu$ by $\sigma(x_1, \ldots, x_N) = (x_1, \ldots, x_N, t_{\nu,j}(x))$, $x = (x_1, \ldots, x_N) \in A_\nu$. If, for a given ν, E contains no set of the form (b), it contains one of the form (a), and we define $\sigma|A_\nu$ by $\sigma(x_1, \ldots, x_N) = (x_1, \ldots, x_N, \tau(x))$, $x \in A_\nu$, where τ is a semi-algebraic function on A_ν with

$$t_{\nu,j}(x) < \tau(x) < t_{\nu,j+1}(x), \quad x \in A_\nu.$$

[If $r_\nu = 0$, so that we are dealing with the set $\{(x,t) \mid x \in A_\nu, -\infty < t < +\infty\}$, we take $\tau \equiv 0$ on A_ν; if $r_\nu > 0$ and $j = 0$, take $\tau(x) = t_{\nu,1}(x) - 1$; if $r_\nu > 0$ and $j = r_\nu$, take $\tau(x) = t_{\nu,r_\nu}(x) + 1$; if $1 \le j \le r_\nu - 1$, take $\tau(x) = \frac{1}{2}(t_{\nu,j}(x) + t_{\nu,j+1}(x))$.]

Since $\sigma|A_\nu$ is semi-algebraic for each ν, σ is semi-algebraic on $\cup A_\nu = F$. Clearly it has the form required.

LEMMA 6.13. — *Let $E \subset \mathbb{R}^n \times \mathbb{R}^m$ be a semi-algebraic set, and let $\pi : E \to \mathbb{R}^n$ be the restriction to E of the projection $\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Let $F = \pi(E)$.*

Assume that for each $x \in F$, the set $E_x = \{y \in \mathbb{R}^m \mid (x,y) \in E\}$ is a vector subspace of $\mathbb{R}^m$.

Then, we have

(a) *If $k \ge 0$ is an integer, the set*

$$F_k \subset F,\ F_k = \{x \in F \mid \dim E_x = k\}$$

is semi-algebraic. ($F_k = \emptyset$ if $k > m$).

(b) *For each k, there are semi-algebraic maps $v_1, \dots, v_k : F_k \to \mathbb{R}^m$ such that, if $x \in F_k$, the vectors $v_1(x), \dots, v_k(x)$ lie in E_x and form a basis of E_x.*

Note. — If we define $v_\alpha(x) = 0$ for $k < \alpha \leq m$, $x \in F_k$, we can formulate (b) in the following equivalent form:

(b′) There exist semi-algebraic maps $v_1, \dots, v_m : F \to \mathbb{R}^m$ such that, for any x, the non-zero vectors among $v_1(x), \dots, v_m(x)$ form a basis of E_x.

Proof.

(a) It is sufficient to show that for any integer $k \geq 0$, the set

$$F'_k = \{x \in F \mid \dim E_x \geq k\}$$

is semi-algebraic. We have $F'_0 = F$. Let $k \geq 1$.

Let $(\mathbb{R}^m)^k = \mathbb{R}^m \times \cdots \times \mathbb{R}^m$ (k-times) and let $S_k \subset (\mathbb{R}^m)^k$ be the set

$$S_k = \{(v_1, \dots, v_k) \in (\mathbb{R}^m)^k \mid v_1 \wedge \cdots \wedge v_k \neq 0 \text{ in } \overset{k}{\wedge} \mathbb{R}^m\}.$$

Clearly, S_k is semi-algebraic.

Consider now the set

$$V_k \subset \mathbb{R}^n \times (\mathbb{R}^m)^k : \{(x, v_1, \dots, v_k) \mid (x, v_j) \in E \text{ for } 1 \leq j \leq k\}.$$

Again, V_k is a semi-algebraic set. Now $\dim E_k \geq k$ if and only if there exist $v_1, \dots, v_k \in E_x$ with $v_1 \wedge \cdots \wedge v_k \neq 0$. Hence

$$F'_k = \{x \in F \mid \dim E_x \geq k\} = \{x \in F \mid \exists\, (x, v_1, \dots, v_k) \in V_k \text{ such that } (v_1, \dots, v_k) \in S_k\}.$$

Hence F'_k is the image under the projection of $\mathbb{R}^n \times (\mathbb{R}^m)^k$ onto $\mathbb{R}^n$, of the semi-algebraic set $V_k \cap (\mathbb{R}^n \times S_k)$, and so is semi-algebraic (by the Tarski-Seidenberg theorem).

(b) We have $\dim E_x = k$ for $x \in F_k$. Consider the set $A_k \subset \mathbb{R}^n \times (\mathbb{R}^m)^k$ defined by

$$A_k = \{(x, v_1, \dots, v_k) \mid x \in F_k,\ (x, v_j) \in E \text{ for } 1 \leq j \leq k, \text{ and } v_1 \wedge \cdots \wedge v_k \neq 0\}.$$

Clearly A_k is semi-algebraic and $(x, v_1, \dots, v_k) \in A_k$ if and only if $(v_1, \dots, v_k)$ is a basis of E_x.

Now, the projection $\varphi : A_k \to \mathbb{R}^n$, $(x, v_1, \dots, v_k) \mapsto x$, maps A_k onto F_k. By Lemma 6.12, φ has a semi-algebraic section $\sigma : F_k \to A_k$ (i.e. σ is semi-algebraic and $\varphi(\sigma(x)) = x$ for $x \in F_k$). If we write

$$\sigma(x) = (x,\ v_1(x), \dots, v_k(x)),\ x \in F_k,$$

then the $v_\alpha : F_k \to \mathbb{R}^m$, $1 \le \alpha \le k$, are semi-algebraic and $(v_1(x), \dots, v_k(x))$ forms a basis of E_x for $x \in F_k$ (since $(x, v_1(x), \dots, v_k(x)) \in A_k$).

LEMMA 6.14. — *Let E be a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^m$ and define $E^\perp = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid \langle y, u\rangle = 0$ for all $u \in \mathbb{R}^m$ with $(x,u) \in E\}$; here $\langle y, u\rangle = \sum y_j u_j$, $y = (y_1, \dots, y_m)$, $u = (u_1, \dots, u_m)$, is the usual inner product on $\mathbb{R}^m$.*

Then $E^\perp$ is again semi-algebraic.

Proof. — Let $F \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ be the set

$$F = \{(x,y,u) \mid (x,u) \in E \text{ and } \langle y, u\rangle = 0\}.$$

Then F is semi-algebraic and we have

$$E^\perp = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid (x,y,u) \in F \text{ for all } u \text{ with } (x,u) \in E\}.$$

This latter set is semi-algebraic since its complement is the projection in $\mathbb{R}^n \times \mathbb{R}^m$ of the semi-algebraic set

$$\{(x,y,u) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \mid (x,u) \in E,\ (x,y,u) \notin F\}.$$

Finally, we note two results proved in [FN] (Lemma 5.2 and Cor. 3 to Theorem 5.1 in that paper).

LEMMA 6.15. — *Let S be a compact semi-algebraic set in $\mathbb{R}^n$ and let $E \subset S$ be a semi-algebraic subset. Let $f : E \to \mathbb{R}$ be a semi-algebraic function which is locally bounded on E.*

Then, there exist constants $m, C > 0$ such that

$$|f(x)| \le C\,(\mathrm{dist}(x, S\backslash E))^{-m} \quad \text{for } x \in E.$$

($\mathrm{dist}(x, S\backslash E)$ is, of course, the distance of x from $S\backslash E$.)

LEMMA 6.16. — *Let $E \subset \mathbb{R}^n \times \mathbb{R}^m$ be semi-algebraic, and let $f : E \to \mathbb{R}$ be a semi-algebraic function. For $x \in \mathbb{R}^n$, define $\tilde f(x) = \sup\{f(x,y) \mid (x,y) \in E\}$ (the sup over the empty set being defined to be $-\infty$).*

Then $\tilde f$ is an extended semi-algebraic function.

7. Some special semi-algebraic sets and maps.

LEMMA 7.1. — *Let $E_1, E_2 \subset \mathbb{R}^n \times \mathbb{R}^m$ be semi-algebraic sets, and let*

$$E = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y = y_1 + y_2 \text{ where } (x,y_1) \in E_1,\ (x,y_2) \in E_2\}.$$

Then E is semi-algebraic.

Proof. — The set $\{(x,y_1,y_2) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \mid (x,y_1) \in E_1, (x,y_2) \in E_2\}$ is semi-algebraic and E is the image of this set under the map $(x,y_1,y_2) \mapsto (x, y_1+y_2)$.

For our next lemma, recall that $W \subset H^D \times \cdots \times H^D$ (r times) is the space of r-tuples $P = (p_1,\dots,p_r)$ of polynomials $p_j \in \mathbb{R}[x_1,\dots,x_n]$, $\deg p_j \le D$, $||P|| \le C_1$, $P(0) = 0$, $J_P(0) = 1$ (where $J_P(x) = \det\left(\frac{\partial p_j}{\partial x_k}\right)_{1\le j,k\le r}$). Clearly W is a semi-algebraic subset of $H^D \times \cdots \times H^D$.

We have seen that there is $\rho_1 > 0$ such that whenever $P \in W$, $f \in \mathbb{R}[x_1,\dots,x_n]$ and $f = 0$ on some neighborhood of 0 in $Z(P) = \{x \in \mathbb{R}^n \mid P(x) = 0\}$ (i.e. $f = 0$ as a germ in $\mathcal{G}(P)$), then $f(x) = 0$ for any x with $|x_j| \le \rho_1$, $P(x) = 0$.

LEMMA 7.2. — *Let $d \ge 1$ be a given integer. The set*

$$Z_d = \{(P,f) \in W \times H^d \mid f = 0 \text{ as a germ in } \mathcal{G}(P)\}$$

is semi-algebraic.

Proof. — Let $\rho_1 > 0$ be as above. We first remark that the set

$$X = \{(P,f,x) \in W \times H^d \times Q_{\rho_1} \mid f(x) = 0 \text{ or } P(x) \ne 0\}$$

is semi-algebraic. In fact, X is the inverse image of the semi-algebraic set

$$S = \{(x_1,\dots,x_n,t) \in \mathbb{R}^n \times \mathbb{R} \mid t = 0\} \cup \{(x_1,\dots,x_n,t) \in \mathbb{R}^n \times \mathbb{R} \mid x_j \ne 0 \text{ for some } j,\ 1 \le j \le n\}$$

under the semi-algebraic map $W \times H^d \times Q_{\rho_1} \to \mathbb{R}^r \times \mathbb{R}$ given by $(P,f,x) \mapsto (P(x), f(x))$ and so is semi-algebraic (Prop. 6.2 (a)). Since $Z_d = \{(P,f) \in W \times H^d \mid (P,f,x) \in X,\ \forall x \in Q_{\rho_1}\}$, Z_d is semi-algebraic by Proposition 6.2 (d).

We next define two norms.

DEFINITION 7.3.

(a) *Let $d \geq 1$ be an integer, let $P \in W$, $q \in H^d$, $0 < \rho \leq \rho_1$. We define the norm N by $N(P, d, q, \rho)$ = infimum of the real numbers $C \geq 0$ such that, for any $f \in H^D$, there exists $F \in H^d$ for which $F - qf = 0$ as a germ in $\mathcal{G}(P)$ and $\|F\| \leq C \sup_{V_\rho(P)} |f|$; if no such C exists, we set $N(P, d, q, \rho) = +\infty$.*

(b) *Let $d \geq 1$ be an integer, let $C \geq 0$ and $0 < \rho \leq \rho_1$. If $P \in W$, we define the norm N_* by*

$$N_*(P, d, C, \rho) = \inf\{N(P, d, q, \rho) \mid q \in H^d,\ q(0) = 1, \|q\| \leq C\}.$$

LEMMA 7.4. — *The two norms defined above are extended semi-algebraic functions of their arguments. More precisely, given $d \geq 1$, the maps*

$$\text{(a)} \qquad \begin{array}{rcl} N : W \times H^d \times (0, \rho_1] & \to & \mathbb{R} \cup \{\infty\}, \\ (P, q, \rho) & \mapsto & N(P, d, q, \rho) \end{array}$$

and

$$\text{(b)} \qquad \begin{array}{rcl} N_* : W \times \mathbb{R}^+ \times (0, \rho_1] & \to & \mathbb{R} \cup \{\infty\},\ \mathbb{R}^+ = \{C \in \mathbb{R} \mid C \geq 0\}, \\ (P, C, \rho) & \mapsto & N_*(P, d, C, \rho) \end{array}$$

are extended semi-algebraic functions.

Proof.

(a) We first note that the set

$$(1) \qquad E_1 = \{(P, F, f, q) \in W \times H^d \times H^D \times H^d \mid F - qf = 0 \text{ as a germ in } \mathcal{G}(P)\}$$

is semi-algebraic. In fact, it is the inverse image of the semi-algebraic set $Z_{d+D} \subset W \times H^{d+D}$ (Lemma 7.2) under the semi-algebraic map $(P, F, f, q) \mapsto (P, F - qf)$ of $W \times H^d \times H^D \times H^d$ into $W \times H^{d+D}$.

Next, we show that the map

$$(2) \qquad \varphi : W \times H^D \times (0, \rho_1] \to \mathbb{R},\ (P, f, \rho) \mapsto \sup_{V_\rho(P)} |f|$$

is semi-algebraic. To see this, remark that the set

$$E_2 = \{(P, f, x, \rho) \in W \times H^D \times Q_{\rho_1} \times (0, \rho_1] \mid P(x) = 0, |x_j| \leq \rho,\ j = 1, \ldots, n\}$$

is clearly semi-algebraic, as is the map $\psi : W \times H^D \times Q_{\rho_1} \times (0, \rho_1] \to \mathbb{R}$ given by

$$\psi(P, f, x, \rho) = |f(x)|.$$

Now, given (P, f, ρ), we have

$$\sup_{V_\rho(P)} |f| = \sup\{\psi(P, f, x, \rho) \text{ over all } x \text{ such that } (P, f, x, \rho) \in E_2\}.$$

Further, since $V_\rho(P)$ is compact, this sup is finite. Hence (2) follows from Lemma 6.16.

It follows from (2) that the set

$$E_3 = \{(P, F, f, \rho, C) \in W \times H^d \times H^D \times (0, \rho_1] \times \mathbb{R}^+ \mid \|F\| \leq C \sup_{V_\rho(P)} |f|\}$$

is semi-algebraic, so that (see (1)), so is the set

$$E_4 = \{(P, F, f, \rho, C, q) \in W \times H^d \times H^D \times (0, \rho_1] \times \mathbb{R}^+ \times H^d \mid (P, F, f, \rho, C) \in E_3 \text{ and } (P, F, f, q) \in E_1\}.$$

Hence, by Proposition 6.2 (c), (d), the set

$$E_5 = \{(P, \rho, C, q) \in W \times (0, \rho_1] \times \mathbb{R}^+ \times H^d \mid \forall f \in H^D, \exists F \in H^d \text{ such that } (P, F, f, \rho, C, q) \in E_4\}$$

is semi-algebraic. Finally,

$$N(P, d, q, \rho) = \inf\{C \in \mathbb{R}^+ \mid (P, \rho, C, q) \in E_5\},$$

so that N is an extended semi-algebraic function by Lemma 6.16.

(b) The set

$$E_6 = \{(P, C, \rho, q) \in W \times \mathbb{R}^+ \times (0, \rho_1] \times H^d \mid q \in H^d, q(0) = 1, \|q\| \leq C\}$$

is clearly semi-algebraic. By (a), so is the set

$$E_7 = \{(P, C, \rho, q) \in E_6 \mid N(P, d, q, \rho) < \infty\}.$$

If E_0 is the image of E_7 under the projection $(P, C, \rho, q) \mapsto (P, C, \rho)$, then, for $(P, C, \rho) \in E_0$,

$$N_*(P, d, C, \rho) = \inf\{N(P, d, q, \rho) \mid (P, C, \rho, q) \in E_7\},$$

while $N_*(P, d, C, \rho) = \infty$ if $(P, C, \rho) \notin E_0$. Hence N_* is an extended semi-algebraic function by Lemma 6.16.

8. The induction scheme and some technical lemmas.

We recall the notation that we have been using. Let $n \geq 2$, $1 \leq r \leq n-1$, $D \geq 1$ be given integers. Let $C_1 > 0$. We have set:

$$H^D = \{p \in \mathbb{R}[x_1, \dots, x_n] \mid \deg p \leq D\},$$

$$W = \Big\{P = (p_1, \dots, p_r) \mid p_j \in H^D,\ P(0) = 0,\ \|P\| \leq C_1,\ \det\Big(\frac{\partial p_j}{\partial x_k}(0)\Big)_{1 \leq j,k \leq r} = 1\Big\}$$

where, if $p = \sum p_\alpha x^\alpha \in H^d$, $d \geq 1$, $\|p\|^2 = \sum |p_\alpha|^2$ and $\|P\|^2 = \sum_j \|p_j\|^2$.

Fix $\rho_1 > 0$ such that, for any $P \in W$, the set

$$\{x \in \mathbb{R}^n \mid |x_j| \leq \rho_1,\ p_1(x) = \dots = p_r(x) = 0\}$$

is contained in the connected component of $Z(P) \cap \{x \in \mathbb{R}^n \mid J_P(x) \neq 0\} = \{x \in \mathbb{R}^n \mid P(x) = 0,\ \det\Big(\dfrac{\partial p_j}{\partial x_k}\Big) \neq 0\}$ which contains the origin (see §1, (4) and the remark following Lemma 2.1).

For $0 < \rho \leq \rho_1$, and $P \in W$, we define

$$V_\rho(P) = Q_\rho \cap Z(P) = \{x \in \mathbb{R}^n \mid |x_j| \leq \rho,\ p_1(x) = \dots = p_r(x) = 0\}.$$

Recall that $\mathcal{G}(P)$ is the space of germs at 0 of functions defined on some neighborhood of 0 in $Z(P)$ [or $V(P)$ or $V_{\rho_1}(P)$].

We shall use, both in this section and the next, the following lemma proved in [FN], and so state it here.

LEMMA 8.1. — *Let $d \geq 1$, $E \subset \mathbb{R}^n$ and $\varphi_\alpha \in H^d, \alpha = 1, \dots, s$. Assume that $\varphi_1|E, \dots, \varphi_s|E$ are linearly independent over $\mathbb{R}$.*

Then, there exist constants $\epsilon > 0$, $K > 0$, $K' > 0$ (depending only on E and the φ_α) and points $x_\beta \in E$, $\beta = 1, \dots, s$, such that the following holds.

If $\tilde{x}_\beta \in \mathbb{R}^n$, $|\tilde{x}_\beta - x_\beta| < \epsilon$, $\beta = 1, \dots, s$, and $\tilde{\varphi}_\alpha \in H^d$, $\|\tilde{\varphi}_\alpha - \varphi_\alpha\| < \epsilon$, $\alpha = 1, \dots, s$,then, for any $A_\alpha \in \mathbb{R}$, $\alpha = 1, \dots, s$, we have

$$\sum_{1 \leq j \leq s} |A_j| \leq K \max_\beta \Big|\sum_{\alpha=1}^s A_\alpha \tilde{\varphi}_\alpha(\tilde{x}_\beta)\Big|.$$

In particular,

$$\Big\|\sum_{1 \leq j \leq s} A_j \tilde{\varphi}_j\Big\| \leq K' \max_\beta \Big|\sum_{\alpha=1}^s A_\alpha \tilde{\varphi}_\alpha(\tilde{x}_\beta)\Big|.$$

The extension theorem stated in §1 will be proved by proving the following:

MAIN LEMMA 8.2. — *Fix an integer t with $-1 \leq t \leq \dim W$.*

There exist constants $C_t, D_t > 0$ depending only on t and the given constants n, r, D, C_1, and a closed semi-algebraic set $W^t \subset W$ with the following properties:

(A) $\dim W^t \leq t$ *(so that $W^{-1} = \emptyset$).*

(B) *Given $P_0 \in W \backslash W^t$ and $\rho > 0, \rho \leq \rho_1$, there exist constants $\delta > 0$ and $K > 0$ depending only on P_0 and ρ such that: for any $P \in W$ with $\|P - P_0\| < \delta$, we can find $q \in H^{D_t}$ satisfying the following requirements:*

(i) $q(0) = 1,\ \|q\| \leq C_t$;

(ii) *For any $f \in H^D$, there exists $F \in H^{D_t}$ so that $qf = F$ as germs in $\mathcal{G}(P)$ and $\|F\| \leq K \sup\limits_{V_\rho(P)} |f|$.*

If $t = \dim W$, this statement is trivial; we have only to take $W^t = W$.

In the rest of this section and in all of §9, we assume that t is fixed and that the main lemma holds for the number t.

Thus, we assume that the set W^t and the constants C_t, D_t are given.

We use the following:

Convention. — Constants written $C, C', C_{\#}, \underline{C}, \ldots, D, D'$ (not involving auxiliary parameters) will be understood to depend only on n, r, D, C_1 unless otherwise stated. Constants written as $C(d), \overline{C}(m), \ldots$ will depend only on n, r, D, C_1 and the auxiliary parameters indicated (such as d, m). Constants written as $\delta, \tilde{\delta}_1, \ldots, K, K', \ldots$ may depend on other data which will be indicated explicitly.

For $d \geq 1$, consider the set

(1) $$Z_d = \{(P, f) \in W \times H^d \mid f = 0 \text{ as a germ in } \mathcal{G}(P)\}.$$

The set Z_d is semi-algebraic by Lemma 7.2.

For $C > 0$, $d \geq 1$, define

(2) $$S(C, d) = \{q \in H^d \mid q(0) = 1,\ \|q\| \leq C\}.$$

This set is clearly semi-algebraic for any C, d.

LEMMA 8.3. — *There exist constants $C_* > 0, D_* \geq 1$, and, for $d \geq 1$, a function $d_* \geq 1$ of d, n, r, D, C_1 such that the following holds.*

There exists a semi-algebraic map

$$Q : W \to S(C_*, D_*)$$

(independent of d) and, for $d \geq 1$, semi-algebraic maps

$$g_j : Z_d \to H^{d_*},\ j = 1, \ldots, r,$$

such that

$$Q(P)f = \sum_{j=1}^{r} g_j(P, f)p_j \text{ (in } \mathbb{R}[x_1, \ldots, x_n]) \text{ for all } (P, f) \in Z_d.$$

Proof. — Let $D_0 > 0$. Let E be the following subset of $W \times H^{D_0}$.

(3) E is the set of pairs $(P, q) \in W \times H^{D_0}$ such that for any $d \geq 1$, if $f \in H^d$ and $f = 0$ as a germ in $\mathcal{G}(P)$, then $qf = \sum g_j p_j$ for some $g_1, \ldots, g_r \in \mathbb{R}[x_1, \ldots, x_n]$.

We claim that E is semi-algebraic. To prove this claim, we use Theorem 5.6′, which shows that if $P \in W$, the ideal in $\mathbb{R}[x_1, \ldots, x_n]$ of those f which $= 0$ as a germ in $\mathcal{G}(P)$ is generated by $f_1, \ldots, f_{r_*}$ with $\deg f_\nu \leq D'$, $1 \leq \nu \leq r_*$, where D' depends only on n, r, D. Clearly, E is the set of pairs (P, q) such that $qf_\nu = \sum\limits_j g_j^{(\nu)} p_j$, $\nu = 1, \ldots, r_*, g_j^{(\nu)} \in \mathbb{R}[x_1, \ldots, x_n]$. By Theorem 5.4 if the $g_j^{(\nu)}$ exist, we may assume that $\deg(g_j^{(\nu)}) \leq D''$, where D'' depends only on D_0, D', n, r, D.

Hence, we have:

(4) $E = \{(P, q) \in W \times H^{D_0} \mid \text{if } f \in H^{D'} \text{ and } f = 0 \text{ as a germ in } \mathcal{G}(P),$ then there exist $g_1, \ldots, g_r \in H^{D''}$ such that $qf = \sum g_j p_j\}$.

The set $E_1 \subset W \times H^{D_0} \times H^{D'} \times (H^{D''})^r$ defined by

$$E_1 = \{(P, q, f, (g_1, \ldots, g_r)) \mid qf = \sum g_j p_j\}$$

is clearly semi-algebraic. Hence its projection on $W \times H^{D_0} \times H^{D'}$: $E_2 = \{(P, q, f) \in W \times H^{D_0} \times H^{D'} \mid \exists g \in (H^{D''})^r \text{ with } qf = \sum g_j p_j\}$ is semi-algebraic.

We have:

$$E = \{(P, q) \in W \times H^{D_0} \mid \forall f \text{ with } (P, f) \in Z_{D'}, \text{ we have } (P, q, f) \in E_2\}.$$

This set is semi-algebraic by Prop. 6.2, (d), thus proving our claim.

Now, by Theorem 5.5, there exist $C_*, D_* \geq 1$ such that if E is the set (3) defined with $D_0 = D_*$, then the projection

$$E \cap W \times S(C_*, D_*) \to W$$

is *onto* W. By Lemma 6.12, we can find a semi-algebraic section of this projection, say $\widetilde{Q} : W \to E \cap W \times S(C_*, D_*)$; set $\widetilde{Q}(P) = (P, Q(P))$. By the defining property of E, we have:

(5) If $P \in W$, $f \in H^d$ and $f = 0$ as a germ in $\mathcal{G}(P)$, there exist $g_1, \dots, g_r \in \mathbb{R}[x_1, \dots, x_n]$ such that $Q(P)f = \sum g_j p_j$.

By Theorem 5.4, there is a function d_* of d, n, r, D, C_1 such that:

(5′) If $P \in W$, $f \in H^d$ and $(P, f) \in Z_d$, then there exist $g_1, \dots, g_r \in H^{d_*}$ so that $Q(P)f = \sum g_j p_j$.

In other words, if $\tilde{E} \subset W \times H^d \times (H^{d_*})^r$ is the semi-algebraic set defined by

$$\tilde{E} = \{(P, f, (g_1, \dots, g_r)) \mid Q(P)f = \sum g_j p_j\},$$

then projection onto $W \times H^d$ maps $\tilde{E}$ onto Z_d. Again by Lemma 6.12, this map has a semi-algebraic section. This means precisely that there are semi-algebraic maps $g_j : Z_d \to H^{d_*}, j = 1, \dots, r$, such that $(P, f, (g_1(P, f), \dots, g_r(P, f))) \in \tilde{E}$ for any $(P, f) \in Z_d$, i.e. such that $Q(P)f = \sum\limits_j g_j(P, f)p_j$. This is Lemma 8.3.

We now proceed to the principal technical step which enables us to prove the Main Lemma 8.2.

Technical Lemma 8.4. — *Let W^t, C_t, D_t be as in the Main Lemma, and let $Q : W \to S(C_*, D_*)$ be the semi-algebraic map constructed in Lemma* 8.3

Let $m \geq 0$ be a given integer.

We can find a semi-algebraic partition $\{Y_\nu\}$ of W^t, a closed semi-algebraic set $Z \subset W^t$ of dimension $\leq t-1$ ($Z = \emptyset$ if $t = 0$), a constant $D(m) \geq 1$, and, for each ν, a finite number of semi-algebraic maps $\Phi_\alpha : Y_\nu \to H^{D(m)}$ (we do not indicate their dependence on ν in the notation) with the following properties:

(I_m) If $P \in Y_\nu$, the polynomials $\Phi_\alpha(P)$ are linearly independent as elements of $\mathcal{G}(P)$.

(II_m) Let $P_0 \in Y_\nu \backslash Z$. Then, there exist constants $\delta > 0$, $K > 0$ (depending only on P_0 and the above data) such that whenever we are

given:

$P \in Y_\nu$ with $\|P - P_0\| < \delta; g_1, \ldots, g_r \in H^D$ with $\|g_i\| \leq 1$;

a real number $\tau, 0 < \tau < 1$; and $f \in H^{D_t}$,

then, we can find real numbers A_α and a polynomial $\tilde{f} \in H^{D(m)}$ for which we have

(i) $$(Q(P))^m f = \sum_\alpha A_\alpha \Phi_\alpha(P) + \tau^m \tilde{f} \operatorname{mod}(p_1 + \tau g_1, \ldots, p_r + \tau g_r), \quad P = (p_1, \ldots, p_r),$$

and

(ii) $$\|\tilde{f}\| \leq K \|f\|.$$

Proof of the Technical Lemma. — The proof is by induction on m. For $m = 0$, the result is obvious: take $\{Y_\nu\}$ to consist of W^t alone, $f = \tilde{f}$, and $\{\Phi_\alpha\}$ to be the empty family.

Assume therefore that $m, \{Y_\nu\}, Z, \Phi_\alpha : Y_\nu \to H^{D(m)}$ are all given with the properties (I_m) and (II_m) above.

For $P \in Y_\nu$, let $V'_P = \{f \in H^{D(m)} \mid f = 0$ as a germ in $\mathcal{G}(P)\} = \{f \in H^{D(m)} \mid (P, f) \in Z_{D(m)}\}$, and let V''_P be the orthogonal complement of $V'_P \oplus \bigoplus_\alpha \mathbb{R}\Phi_\alpha(P)$ [$\bigoplus_\alpha$ is the linear span of the $\Phi_\alpha(P)$]. The set

$$\{(P, f) \in Y_\nu \times H^{D(m)} \mid f \in V'_P\} = Z_{D(m)} \cap (Y_\nu \times H^{D(m)})$$

is semi-algebraic. By Lemma 7.2 and Lemma 6.14, the set

$$\{(P, f) \in Y_\nu \times H^{D(m)} \mid f \in V''_P\}$$

is also semi-algebraic. We apply Lemma 6.13 to these sets to obtain the following:

(6) There is a semi-algebraic partition $\{Y'_\mu\}$ of W^t, refining the partition $\{Y_\nu\}$, such that, for each μ, we can find finitely many semi-algebraic mappings $h_\beta : Y'_\mu \to H^{D(m)}$, $\psi_\gamma : Y'_\mu \to H^{D(m)}$ with the property that for $P \in Y'_\mu$, the $\{h_\beta(P)\}$ form a basis of V'_P, the $\{\psi_\gamma(P)\}$ form a basis of V''_P.

Moreover, if we restrict the Φ_α from Y_ν to the Y'_μ contained in Y_ν, properties (I_m) and (II_m) continue to hold.

Note that for $P \in Y'_\mu$, the polynomials $\{h_\beta(P), \psi_\gamma(P), \Phi_\alpha(P)\}$ form a basis of $H^{D(m)}$. In fact, since $h_\beta(P) = 0$ in $\mathcal{G}(P)$, and the images of the $\Phi_\alpha(P)$ are linearly independent in $\mathcal{G}(P)$, $\{h_\beta(P), \Phi_\alpha(P)\}$ form a basis of $V'_P \oplus \bigoplus_\alpha \mathbb{R}\,\Phi_\alpha(P)$, while the $\psi_\gamma(P)$ form a basis of the orthogonal complement of this space in $H^{D(m)}$.

Since $(P, h_\beta(P)) \in Z_{D(m)}$ for $P \in Y'_\mu$ and any β, Lemma 8.3 can be applied, and provides semi-algebraic maps

$$g^j_\beta : Y'_\mu \to H^{d(m)}$$

[where $d(m)$ depends only on $D(m)$ and the constants defining W] such that

$$Q(P)h_\beta = \sum_{j=1}^{r} g^j_\beta(P)p_j,\ P \in Y'_\mu, \tag{7}$$

(Q being again the map constructed in Lemma 8.3).

Given $P \in Y'_\mu$ and $F \in H^{D(m)}$, we can write

$$F = \sum_\alpha a_\alpha \Phi_\alpha(P) + \sum_\beta b_\beta h_\beta(P) + \sum_\gamma e_\gamma \psi_\gamma(P), \tag{8}$$

where $a_\alpha, b_\beta, e_\gamma$ are real numbers depending on P and F (they are semi-algebraic functions of P and F, as is easily proved, but we do not need this fact).

Multiplying (8) by $Q(P)$ and using (7), we obtain

$$\begin{aligned} Q(P)F = \sum_\alpha a_\alpha Q(P)\Phi_\alpha(P) + \sum_\gamma e_\gamma Q(P)\psi_\gamma(P) \\ + \sum_{j=1}^{r} \big(\sum_\beta b_\beta g^j_\beta(P)\big)p_j. \end{aligned} \tag{9}$$

Suppose that in addition to $P \in Y'_\mu$ and $F \in H^{D(m)}$, we are given $\tau \in (0,1)$ and $g_1, \ldots, g_r \in H^D$ with $\|g_j\| \le 1$. Then (9) implies

$$\begin{aligned} Q(P)F \equiv \sum_\alpha a_\alpha Q(P)\Phi_\alpha(P) + \sum_\gamma e_\gamma Q(P)\psi_\gamma(P) \\ + \tau \underline{F} \operatorname{mod}(p_1 + \tau g_1, \ldots, p_r + \tau g_r) \end{aligned} \tag{10}$$

where

$$\underline{F} = -\sum_{j=1}^{r} \big(\sum_\beta b_\beta g^j_\beta(P)\big) g_j.$$

We can now define sets $\{\underline{Y_\nu}\}, \underline{Z}$, a constant $D(m+1)$, and, for each ν, maps $\underline{\Phi_\alpha} : \underline{Y_\nu} \to H^{D(m+1)}$ which have properties (I_{m+1}) and (II_{m+1}) of Lemma 8.4.

(a) We take the partition $\{\underline{Y_\nu}\}$ of W^t be the partition $\{Y'_\mu\}$ above.

(b) $\underline{Z}$ will be any closed semi-algebraic set of dimension $\leq t-1$ such that $\underline{Z} \supset Z$ and the maps $Q, \Phi_\alpha, h_\beta, \psi_\gamma, g^j_\beta$ are continuous on $\underline{Y_\nu} \backslash \underline{Z}$.

Note that we can find such a $\underline{Z}$ by Lemma 6.11 and Lemmas 6.6, 6.7 since $\dim \underline{Y}_\nu \leq \dim W^t \leq t$ and $\dim Z \leq t-1$ by assumption.

(c) $D(m+1) = \max(D(m) + D_*, d(m) + D)$ with D_* as in Lemma 8.3, $\deg g^j_\beta(P) \leq d(m)$ and $\{\underline{\Phi}_\alpha\}$ is an enumeration of the maps

$$P \mapsto Q(P)\Phi_\alpha(P) \text{ and } P \mapsto Q(P)\psi_\gamma(P).$$

This completes our choices. It remains to show that with these choices, properties (I_{m+1}) and (II_{m+1}) hold.

Verification of Property (I_{m+1}). — We have to check that the polynomials $Q(P)\Phi_\alpha(P)$, $Q(P)\psi_\gamma(P)$ are linearly independent in $\mathcal{G}(P)$ for $P \in \underline{Y}_\nu$. Since $Q(P) = 1$ at the origin, it is enough to show that $\Phi_\alpha(P), \psi_\gamma(P)$ are linearly independent in $\mathcal{G}(P)$. If $\sum u_\alpha \Phi_\alpha(P) + \sum w_\gamma \psi_\gamma(P) = 0$ in $\mathcal{G}(P)$, this sum lies in V'_P by definition of V'_P, so that there are constants $v_\beta \in \mathbb{R}$ so that $\sum u_\alpha \Phi_\alpha(P) + \sum w_\gamma \psi_\gamma(P) = \sum v_\beta h_\beta(P)$ (since the $h_\beta(P)$ span V'_P). But this can only happen if $u_\alpha = v_\beta = w_\gamma = 0$ since, as noted earlier, $\Phi_\alpha(P), h_\beta(P), \psi_\gamma(P)$ form a basis of $H^{D(m)}$ for $P \in Y'_\mu = \underline{Y_\nu}$.

Verification of Property (II_{m+1}). — Recalling the choices made in (a), (b), (c) above, we can formulate property (II_{m+1}) as follows.

(11) Let $P_0 \in Y'_\mu \backslash \underline{Z}$. There exist $\underline{\delta} > 0$ and $\underline{K} > 0$ (depending only on P_0 and $m+1$) such that, whenever we are given:

$$P \in Y'_\mu \text{ with } \|P - P_0\| < \underline{\delta};$$

$$g_1, \ldots, g_r \in H^D \text{ with } \|g_j\| \leq 1;$$

$$\tau \in (0,1); \text{ and } f \in H^{D_t},$$

we can find real numbers $\underline{A_\alpha}, \underline{E_\gamma}$ and a polynomial $\underline{f} \in H^{D(m+1)}$ such that

$$(\mathrm{i}, m{+}1) \quad (Q(P))^{m+1} f \equiv \sum_\alpha \underline{A_\alpha} Q(P)\Phi_\alpha(P) + \sum_\gamma \underline{E_\gamma} Q(P)\psi_\gamma(P) + \tau^{m+1}\underline{f} \mod(p_1 + \tau g_1, \ldots, p_r + \tau g_r)$$

and

$$(\mathrm{ii}, m{+}1) \qquad \|\underline{f}\| \leq \underline{K}\|f\|.$$

To verify this, recall that by inductive hypothesis (Technical Lemma for the value m), we have the following:

Given $P_0 \in Y'_\mu \backslash \underline{Z}$, there exist $\delta, K > 0$ such that whenever we are given: $P \in Y'_\mu$ with $\|P - P_0\| < \delta$; $g_1, \ldots, g_r \in H^D$ with $\|g_j\| \leq 1$; $\tau \in (0,1)$; and $f \in H^{D_t}$; then there exist $A_\alpha \in \mathbb{R}$ and $F \in H^{D(m)}$ such that

$$\text{(a)} \quad (Q(P))^m f \equiv \sum_\alpha A_\alpha \Phi_\alpha(P) + \tau^m F \operatorname{mod}(p_1 + \tau g_1, \ldots, p_r + \tau g_r),$$

and

$$\text{(b)} \quad \|F\| \leq K\|f\|.$$

To prove (11), given P and f, let F be as in (a) above; we have $F \in H^{D(m)}$ and we can apply (10). We multiply (a) by $Q(P)$ and substitute into (10). This gives

$$\begin{aligned}(12) \quad (Q(P))^{m+1} f \equiv & \sum_\alpha A_\alpha Q(P) \Phi_\alpha(P) \\ & + \tau^m \{\sum_\alpha a_\alpha Q(P)\Phi_\alpha(P) + \sum_\gamma e_\gamma Q(P)\psi_\gamma(P)\} \\ & + \tau^{m+1} \underline{F} \operatorname{mod}(p_1 + \tau g_1, \ldots, p_r + \tau g_r),\end{aligned}$$

where

$$(13) \qquad \underline{F} = -\sum_{j=1}^r \Big(\sum_\beta b_\beta g^j_\beta(P)\Big) g_j.$$

Equation (12) is of the form (i, $m+1$) if we set

$$\underline{A_\alpha} = A_\alpha + \tau^m a_\alpha, \; \underline{E_\gamma} = \tau^m e_\gamma, \underline{f} = \underline{F}.$$

Thus, to complete the proof of the technical lemma, it remains to prove the estimate (ii, $m+1$) for $\underline{F} = \underline{f}$. To do this, we have only to prove the following:

(14) Given $P_0 \in Y'_\mu \backslash \underline{Z}$, there exist $\overline{\delta}, \overline{K} > 0$ (depending only on P_0) such that: if $P \in Y'_\mu$ and $\|P - P_0\| < \overline{\delta}$; if

$$(15) \qquad F = \sum a_\alpha \Phi_\alpha(P) + \sum b_\beta h_\beta(P) + \sum_\gamma e_\gamma \psi_\gamma(P);$$

and we are given $g_1, \ldots, g_r \in H^D$ with $\|g_j\| \leq 1$; then we have

$$(16) \qquad \Big\|\sum_{j=1}^r \sum_\beta b_\beta g^j_\beta(P) g_j\Big\| \leq \overline{K}\|F\|.$$

Now, by the choice of $\underline{Z}$, the map $P \mapsto (\Phi_\alpha(P), h_\beta(P), \psi_\gamma(P))$ is continuous on Y'_μ at P_0 (since its restriction to $Y'_\mu \backslash \underline{Z}$ is continuous and $\underline{Z}$

is closed). Moreover, the triple in brackets is a basis of $H^{D(m)}$ close to the basis $(\Phi_\alpha(P_0), h_\beta(P_0), \psi_\gamma(P_0))$ if P is close to P_0. Hence, if $\|P - P_0\| \leq \tilde{\delta}$, $P \in Y'_\mu$($\tilde{\delta}$ depending only on P_0 and $D(m)$), there is $\tilde{K}_1 > 0$ (depending only on P_0 and $D(m)$) so that

$$|a_\alpha|, |b_\beta|, |e_\gamma| \leq \tilde{K}_1 \|F\| \tag{17}$$

whenever (15) holds.

In addition, the maps $g^j_\beta(P)$ are continuous on Y'_μ at P_0 (again because $\underline{Z}$ is closed). Thus, there exist $\tilde{\delta}_2, \tilde{K}_2 > 0$ (depending only on P_0 and $D(m)$) such that

$$\|g^j_\beta(P)\| \leq \tilde{K}_2 \text{ if } \|P - P_0\| < \tilde{\delta}_2,\ P \in Y'_\mu. \tag{18}$$

Since $\|g_j\| \leq 1$ by assumption, (17) and (18) show that whenever (15) holds, we have, for $\|P - P_0\| < \overline{\delta} = \min(\tilde{\delta}_1, \tilde{\delta}_2)$, $P \in Y'_\mu$,

$$\Big\| \sum_j \sum_\beta b_\beta g^j_\beta(P) g_j \Big\| \leq \overline{K} \|F\|.$$

This completes the verification of property (II_{m+1}).

The induction step being complete, Technical Lemma 8.4 is proved.

LEMMA 8.5. — *Let Y be a semi-algebraic subset of W with $\dim Y \leq t$. Then, there exists a closed semi-algebraic subset $Z \subset W$ with $\dim Z \leq t - 1$ (so that $Z = \emptyset$ if $t = 0$) with the following property:*

If $P_0 \in Y \backslash Z$ and $\rho > 0$, there exist constants $\delta, K > 0$ (depending only on P_0 and ρ) such that: given $P \in Y$ with $\|P - P_0\| < \delta$ and $f \in H^D$, there exists $F \in H^D$ with $F = f$ in $\mathcal{G}(P)$ and $\|F\| \leq K \sup_{V_\rho(P)} |f|$.

Proof. — If $P \in Y$, denote by V''_P the orthogonal complement in H^D of the space of polynomials in H^D which are 0 as germs in $\mathcal{G}(P)$. By Lemma 7.2 and Lemmas 6.14, 6.13, we can find a semi-algebraic partition $\{Y_\nu\}$ of Y and, on each Y_ν, semi-algebraic maps $\Phi_\alpha : Y_\nu \to H^D$ such that $\{\Phi_\alpha(P)\}$ is a basis of V''_P for any $P \in Y_\nu$.

Let Z be a closed semi-algebraic set in W such that $\dim Z \leq t - 1$, $Z \supset \overline{Y}_\nu \backslash Y_\nu$ for each ν ($\overline{Y}_\nu$ being the closure of Y_ν), and such that for any ν and any α, $\Phi_\alpha \mid Y_\nu \backslash Z$ is continuous; there is such a Z by Lemmas 6.11, 6.7. Since $Z \supset \overline{Y}_\nu \backslash Y_\nu$ for all ν, if $P_0 \in Y_\nu \backslash Z$, then $P_0 \notin \overline{Y}_\mu$ for $\mu \neq \nu$. Hence, given $P_0 \in Y_\nu \backslash Z$, there is $\delta_1 > 0$ such that if $P \in Y$ and $\|P - P_0\| < \delta_1$, then $P \in Y_\nu$.

Now, if $P \in Y_\nu$ and $f \in H^D$, we can find real numbers A_α such that $f = \sum_\alpha A_\alpha \Phi_\alpha(P)$ as germs in $\mathcal{G}(P)$. Since, for $0 < \rho \leq \rho_1$ (see §1, (4)), $V_\rho(P)$ is contained in the connected component through 0 of $Z(P) \cap \left\{x \in \mathbb{R}^n \mid \det\left(\frac{\partial p_j}{\partial x_k}\right) \neq 0\right\}$ (see remark after Lemma 2.1), we have

$$f = \sum_\alpha A_\alpha \Phi_\alpha(P) \text{ on } V_\rho(P). \tag{19}$$

We now make the following remarks:

(a) If $x_0 \in V_{\rho/2}(P_0)$ and $\epsilon > 0$ are given, there is $\delta_2 > 0$ such that if $P \in W$ and $\|P - P_0\| < \delta_2$, then, there exists $x \in V_\rho(P)$ such that $|x - x_0| < \epsilon$.

This follows from the real version of Lemma 2.1.

(b) Given $P_0 \in Y_\nu \backslash Z$ and $\epsilon > 0$, there is $\delta_3 > 0$ such that if $P \in Y_\nu$ and $\|P - P_0\| < \delta_3$, then $\|\Phi_\alpha(P) - \Phi_\alpha(P_0)\| < \epsilon$ for any α.

This is simply the statement that the Φ_α are continuous on Y_ν at $P_0 \notin Z$.

(c) $\{\Phi_\alpha(P)\}$ are linearly independent functions on $V_{\rho/2}(P_0)$. (They form a basis of V''_{P_0} and so are linearly independent in $\mathcal{G}(P)$).

Because of (c), we can apply Lemma 8.1. Thus, there exist finitely many points $x_\beta \in V_{\rho/2}(P_0), \epsilon > 0$ and K (depending only on the $\Phi_\alpha(P_0)$, thus only on P_0) such that if $\tilde{x}_\beta \in \mathbb{R}^n$, $|\tilde{x}_\beta - x_\beta| < \epsilon$ and $\varphi_\alpha \in H^D$, $\|\varphi_\alpha - \Phi_\alpha(P_0)\| < \epsilon$, then

$$\sum_\alpha |A_\alpha| \leq K \max_\beta \Big| \sum_\alpha A_\alpha \varphi_\alpha(\tilde{x}_\beta) \Big|, \ A_\alpha \in \mathbb{R}.$$

We can apply this with $\varphi_\alpha = \Phi_\alpha(P), \tilde{x}_\beta \in V_\rho(P)$ if $\|P - P_0\| < \min(\delta_2, \delta_3)$ because of (a) and (b) above. This gives

$$\begin{aligned} \|\sum A_\alpha \Phi_\alpha(P)\| &\leq \max_\alpha(\|\Phi_\alpha(P_0)\| + \epsilon) \sum_\alpha |A_\alpha| \\ &\leq K' \max_\beta \Big| \sum_\alpha A_\alpha \Phi_\alpha(P)(\tilde{x}_\beta) \Big| \\ &\leq K' \sup_{V_\rho(P)} \Big| \sum_\alpha A_\alpha \Phi_\alpha(P) \Big|. \end{aligned}$$

Taking $F = \sum_\alpha A_\alpha \Phi_\alpha(P)$ and using (19), we obtain the lemma.

Remark 8.6. — We have used Theorem 5.6 in Lemma 8.3 and thus in the statement and proof of the Technical Lemma 8.4. However, to prove the Main Lemma, a weaker version of the Technical Lemma would be sufficient; this weaker version can be proved using only Theorem 5.2 (not 5.6). This version of the Technical Lemma is the following.

Given W^t, C_t, D_t and $m \geq 0$ as in 8.4, there exist constants $C_0(m), D_0(m)$ (depending only on these data and the constants in W) and a semi-algebraic map $\check{q}$: $W^t \to S(C_0(m), D_0(m))$ (so that $\deg \check{q}\,(P) \leq D_0(m)$, $\check{q}\,(P)(0) = 1$, $\|\check{q}\,(P)\| \leq C_0(m)$ for $P \in W^t$), with the property that the statement of the Technical Lemma holds if, in (II_m), we replace $(Q(P))^m$ by $\check{q}\,(P)$.

The sharper form in the text is the exact analogue of the argument given in [FN]. Moreover, Theorem 5.5 and Theorem 5.6 on which it is based, are clearly of interest in themselves.

9. The induction step in the proof of the Main Lemma.

We are assuming that the Main Lemma has been proved for the value t of the parameter; in this section we carry out the induction step proving it for the value $t-1$ under this assumption, thus completing the proof of the Main Lemma 8.2.

If we recall the definition of the norms N, N_* defined in §7, we can reformulate the Main Lemma for t as follows.

There exists a closed semi-algebraic set $W^t \subset W$ of dimension $\leq t$, and constants $C_t, D_t \geq 1$ such that

Given $P_0 \in W \backslash W^t$ and $\rho > 0$, $\rho \leq \rho_1$, there exist $\delta > 0$, $K_* > 0$ so that

$$N_*(P, D_t, C_t, \rho) \leq K_* \text{ whenever } \|P - P_0\| < \delta, P \in W \backslash W^t.$$

If we note that W^t is closed in W and that, for fixed values of the other parameters, N_* is a decreasing function of ρ, we obtain the following:

(1) With C_t, D_t as above, the function

$$(P, \rho) \mapsto N_*(P, D_t, C_t, \rho)$$

is locally bounded on the set $(W \backslash W^t) \times (0, \rho_1]$.

Since N_* is semi-algebraic where it is finite (Lemma 7.4, (b)) and $W \times [0, \rho_1]$ is compact, we can apply Lemma 6.15 to N_* and the pair $(W \times [0, \rho_1], (W \backslash W^t) \times (0, \rho_1])$.

Note that the distance of $(P, \rho) \in (W \backslash W^t) \times (0, \rho_1]$ from $(W \times [0, \rho_1]) \backslash ((W \backslash W^t) \times (0, \rho_1]) = W^t \times [0, \rho_1] \cup W \times \{0\}$ is at least $\min(\operatorname{dist}(P, W^t), \rho) \geq \rho \operatorname{dist}(P, W^t)$ (since $\rho_1 \leq 1$; see §1, (4)). From this, and Lemma 6.15, we obtain:

LEMMA 9.1 (Quantitative Form of Main Lemma). — *Let W^t, C_t, D_t be as in the Main Lemma 8.2. There exist constants m_t, $K_t > 0$ (depending only on these data) such that the following holds.*

Given $P \in W \backslash W^t$ and $\rho > 0$, $0 < \rho \leq \rho_1$, we can find $q \in H^{D_t}$ with $q(0) = 1$, $\|q\| \leq C_t$ such that, for any $f \in H^D$, there exists $F \in H^{D_t}$ for which

(i) $F = qf$ *on* $V_\rho(P)$

and

(ii) $\|F\| \leq K_t \cdot \rho^{-m_t} (\operatorname{dist}(P, W^t))^{-m_t} \sup\limits_{V_\rho(Z)} |f|$.

We now apply the technical lemma with $m = m_t$, where m_t is as in Lemma 9.1. This gives us: a semi-algebraic partition $\{Y_\nu\}$ of W^t; a closed semi-algebraic set $Z \subset W^t$ with $\dim Z \leq t-1$; a constant $D(m_t) \geq 1$; and, for each ν, semi-algebraic maps $\Phi_\alpha : Y_\nu \to H^{D(m_t)}$, having the properties (I_{m_t}) and (II_{m_t}) of Lemma 8.4 ($Q : W \to S(C_*, D_*)$ is the semi-algebraic map constructed in Lemma 8.3).

Let Z' be a closed semi-algebraic set of dimension $\leq t-1$ such that, for any $\nu, \alpha, Q, \Phi_\alpha$ restricted to $Y_\nu \backslash Z'$ are continuous; there is such a Z' by Lemma 6.11.

Let $Z'' \subset W$ be a closed semi-algebraic set of dimension $\leq t-1$ such that, for any ν, if $P_0 \in Y_\nu \backslash Z''$ and $0 < \rho \leq \rho_1$, if $P \in Y_\nu$ is close to P_0, and if $f \in H^D$, then there exists $F \in H^D$ with $F = f$ on $V_\rho(P)$ and $\|F\| \leq K \sup\limits_{V_\rho(P)} |f|$, K being a constant depending only on P_0 and ρ. The existence of Z'' is guaranteed by Lemma 8.5 (and Lemma 6.6).

We define W^{t-1} by:

(2) $W^{t-1} =$ closure in W of $Z \cup Z' \cup Z'' \cup \bigcup\limits_\nu (\overline{Y}_\nu \backslash Y_\nu)$, $\overline{Y}_\nu$ being the closure of Y_ν.

Since Z, Z', Z'' have dimension $\leq t-1$ by construction, and $\dim Y_\nu \leq \dim W^t \leq t$, we have, by Lemmas 6.6, 6.7,

$$\dim W^{t-1} \leq t-1. \tag{2'}$$

By construction, we have the following properties:

(3) If $P_0 \in Y_\nu \backslash W^{t-1}$ and $0 < \rho \leq \rho_1 (\leq 1)$, there exist $\delta_0, K_0 > 0$ depending only on P_0 and ρ such that the following assertions hold:

(3.i) Q, Φ_α are continuous on Y_ν at P_0.

(3.ii) If $P \in W^t$ and $\|P - P_0\| < \delta_0$, then $P \in Y_\nu$.

(3.iii) If $P \in Y_\nu$ and $\|P - P_0\| < \delta_0$, then given $f \in H^D$, there exists $F \in H^D$ with $F = f$ on $V_\rho(P)$ and $\|F\| \leq K_0 \sup\limits_{V_\rho(P)} |f|$.

To complete the induction step, we have to show that there exist constants $C_{t-1}, D_{t-1} \geq 1$ (depending only on the constants defining W) having the following property:

(4) Let $P_0 \in W \backslash W^{t-1}$ and let $0 < \rho \leq \rho_1$. There exist constants $\underline{\delta}$, $\underline{K} > 0$ (depending only on P_0 and ρ) such that whenever $\hat{P} \in W$ and $\|\hat{P} - P_0\| < \underline{\delta}$, we can find $\hat{q} \in H^{D_{t-1}}$ with $\hat{q}(0) = 1$, $\|\hat{q}\| \leq C_{t-1}$ with the property that for any $f \in H^D$, there is $F \in H^{D_{t-1}}$ for which $F = \hat{q}f$ on $V_\rho(\hat{P})$ and $\|F\| \leq \underline{K} \sup\limits_{V_\rho(\hat{P})} |f|$.

If $P_0 \in W \backslash W^t$, this follows from our inductive hypothesis that the Main Lemma is true for the value t.

If $P_0 \in W^t \backslash W^{t-1}$ and $\hat{P} \in W^t$, then, if ν is such that $P_0 \in Y_\nu$ and $\|\hat{P} - P_0\|$ is small, (3.ii) implies that $\hat{P} \in Y_\nu$ (same ν) and (3.iii) then shows that (4) holds with $\hat{q} \equiv 1$.

Thus, to prove (4) and hence complete the induction step, we have only to prove

LEMMA 9.2. — *Let $\{Y_\nu\}, W^{t-1}$ be as above. There exist constants $C_{t-1} > 0$, $D_{t-1} \geq 1$ (depending only on the constants defining W) for which we have the following:*

Given $P_0 \in Y_\nu \backslash W^{t-1}$ and $0 < \rho \leq \rho_1$, there exist $\underline{\delta}, \underline{K} > 0$ (depending only on P_0 and ρ) such that if $\hat{P} \in W \backslash W^t$ and $\|\hat{P} - P_0\| < \underline{\delta}$, then there is $\underline{q} \in H^{D_{t-1}}$ with the following properties:

(a) $\underline{q}(0) = 1$, $\|\underline{q}\| \leq C_{t-1}$;

(b) *For any* $f \in H^D$, *we can find* $F \in H^{D_{t-1}}$ *so that* $F = \underline{q}f$ *on* $V_\rho(\hat{P})$ *and* $\|F\| \leq \underline{K} \sup_{V_\rho(\hat{P})} |f|$.

Proof. — Let $P_0 \in Y_\nu \backslash W^{t-1}$ and $\rho > 0$ be given. Let $\hat{P} \in W \backslash W^t$ and suppose that $\|\hat{P} - P_0\| < \underline{\delta} < 1/2$; we shall list the conditions that $\underline{\delta}$ must satisfy for the lemma to hold.

Since W^t is closed, we have $\tau = \operatorname{dist}(\hat{P}, W^t) > 0$; further $\tau < \underline{\delta}$ since $P_0 \in Y_\nu \subset W^t$. We choose $P \in W^t$ such that $\|\hat{P} - P\| = \tau$; we then have

$$\|P - P_0\| \leq \|P - \hat{P}\| + \|\hat{P} - P_0\| \leq \tau + \underline{\delta} < 2\underline{\delta}.$$

Hence:

(5) If $\delta_1 > 0$ is the constant in (3.i), (3.ii), (3.iii) above, and if $2\underline{\delta} \leq \delta_1$, then $P \in Y_\nu$.

Write $\hat{P} = P + \tau(g_1, \ldots, g_r)$ with $g_j \in H^D$. Then $\sum \|g_j\|^2 = 1$. We apply Lemma 9.1 (quantitative form of Main Lemma) to $\hat{P} \in W \backslash W^t$. This gives:

(6) Given $\rho > 0$, there is $q \in H^{D_t}$ with $q(0) = 1$, $\|q\| \leq C_t$ such that for any $f \in H^D$, we can find $F_0 \in H^{D_t}$ for which $F_0 = qf$ on $V_\rho(\hat{P})$ and $\|F_0\| \leq K_t \tau^{-m_t} \rho^{-m_t} \sup_{V_\rho(\hat{P})} |f|$.

We wish to apply the Technical Lemma 8.4 to F_0 with $m = m_t$. To do this, we must make sure that

(7) $2\underline{\delta} \leq \delta$, δ being the constant in Property (II_{m_t}) of 8.4.

If (5) and (7) hold, we have $\|P - P_0\| < \delta$, $P_0 \in Y_\nu \backslash W^{t-1}$, $P \in Y_\nu$, $g_1, \ldots, g_r \in H^D$, $\|g_j\| \leq 1$, $0 < \tau < 1$, $F_0 \in H^{D_t}$, so that the conditions imposed in Lemma 8.4 are verified. Hence, if the Φ_α are as in Lemma 8.4 with $m = m_t$, we have:

(8) There exist real numbers A_α such that

$$(Q(P))^{m_t} F_0 \equiv \sum_\alpha A_\alpha \Phi_\alpha(P) + \tau^{m_t} \tilde{F} \ \operatorname{mod}(\hat{p}_1, \ldots, \hat{p}_r), \ \hat{P} = (\hat{p}_1, \ldots, \hat{p}_r)$$

and

$$\|\tilde{F}\| \leq \tilde{K} \|F_0\|;$$

note that $\hat{p}_j = p_j + \tau g_j$ by definition of the g_j. Here $\tilde{K}$ is a constant depending only on τ, P_0 and ρ, and $\Phi_\alpha(P)$, $\tilde{F} \in H^{D(m_t)}$.

From (6) and (8), we obtain:

(9) Set $\underline{q} = q \cdot Q^{m_t}(P)$ (with q as in (6) and Q as in 8.3). Then, if $f \in H^D$, we have

$$\text{(9.i)} \quad \underline{q} \cdot f = \sum_\alpha A_\alpha \Phi_\alpha(P) + \tau^{m_t} \tilde{F} \text{ on } V_\rho(\hat{P}),$$

$$\text{(9.ii)} \quad \|\tau^{m_t} \tilde{F}\| \leq \tilde{K} \cdot K_t \rho^{-m_t} \sup_{V_\rho(\hat{P})} |f|,$$

and, because of Lemma 8.3 and (6) above, we also have

$$\text{(9.iii)} \quad \begin{aligned} &\underline{q}(0) = 1, \ \underline{q} \in H^{D'_{t-1}} \text{ with } D'_{t-1} = D_t + m_t\, D_*, \\ &\|\underline{q}\| \leq \tilde{C} \cdot C_t \cdot C_*^{m_t} = C_{t-1} \end{aligned}$$

where C_* is as in Lemma 8.3, C_t is as in (6) above, and $\tilde{C}$ is a constant depending only on n and the degrees of q, $Q(P)^{m_t}$, i.e. only on n, D_t, D_*, m_t.

Since, by (9.iii), $\sup_{V_\rho(\hat{P})} |\underline{q} \cdot f| \leq K'_{t-1} \sup_{V_\rho(\hat{P})} |f|$, where K'_{t-1} depends only on P_0, ρ and $t-1$, (9.i) and (9.ii) imply that

$$\text{(10)} \qquad \sup_{V_\rho(\hat{P})} |\sum_\alpha A_\alpha \Phi_\alpha(P)| \leq K'' \sup_{V_\rho(\hat{P})} |f|,$$

where K'' depends only on P_0, ρ and t.

Since the functions $\Phi_\alpha(P_0)|V_{\rho/2}(P_0)$ are linearly independent, we can apply Lemma 8.1 to obtain the following:

(11) There are finitely many points $x_\beta \in V_{\rho/2}(P_0)$, and constants $\epsilon > 0$, $K_* > 0$ such that if $\tilde{x}_\beta \in \mathbb{R}^n$, $|\tilde{x}_\beta - x_\beta| < \epsilon$, and $\varphi_\alpha \in H^{D(m_t)}$, $\|\varphi_\alpha - \Phi_\alpha(P_0)\| < \epsilon$, then, for any real numbers A_α, we have

$$\|\sum_\alpha A_\alpha \varphi_\alpha\| \leq K_* \max_\beta |\sum_\alpha A_\alpha \varphi_\alpha(\tilde{x}_\beta)|.$$

Here, ϵ, K_* depend only on P_0, ρ and m_t. With ϵ as in (11), there is $\delta_2 > 0$ depending only on P_0, ρ, t such that if $\|\hat{P} - P_0\| < \delta_2$, then, for any $x_0 \in V_{\rho/2}(P_0)$, there exists $x \in V_\rho(\hat{P})$ such that $|x - x_0| < \epsilon$.

We assume that

$$\text{(12)} \qquad \underline{\delta} < \delta_2.$$

Now note that the maps $P' \mapsto \Phi_\alpha(P')$ are continuous on Y_ν at the point $P' = P_0$ (since $P_0 \in Y_\nu \backslash W^{t-1}$ and W^{t-1} contains the points of discontinuity of the Φ_α on Y_ν by construction). This gives

(13) There is $\delta_3 > 0$ depending only on P_0 and ρ such that if $\underline{\delta} < \delta_3$, then $\|\Phi_\alpha(P') - \Phi_\alpha(P_0)\| < \epsilon$ for $\|P' - P_0\| < 2\underline{\delta}$, $P' \in Y_\nu$.

We now pick $\underline{\delta}$ to satisfy the conditions (5), (7), (12) and (13). We may then apply (11) with $\varphi_\alpha = \Phi_\alpha(P)$ and $\tilde{x}_\beta \in V_\rho(\hat{P})$, $|\tilde{x}_\beta - x_\beta| < \epsilon$, to obtain

$$\begin{aligned} \|\sum_\alpha A_\alpha \Phi_\alpha(P)\| &\le K_* \max_\beta |\sum_\alpha A_\alpha \Phi_\alpha(P)(\tilde{x}_\beta)| \\ &\le K_* \sup_{V_\rho(\hat{P})} |\sum_\alpha A_\alpha \Phi_\alpha(P)| \\ &\le K'' \cdot K_* \sup_{V_\rho(\hat{P})} |f| \text{ by (10).} \end{aligned} \tag{14}$$

Thus, if we define

$$F = \sum_\alpha A_\alpha \Phi_\alpha(P) + \tau^{m_t}\tilde{F}, \text{ with } \tilde{F} \text{ as in (8),} \tag{15}$$

we obtain, from (6), (9) and (14),

$$\begin{aligned} &\underline{q}f = F \text{ on } V_\rho(\hat{P}),\ F \in H^{D(m_t)}, \\ &\|F\| \le \sup_{V_\rho(\hat{P})} |f|, \text{ where } \underline{K} = K''K_* + \tilde{K}\cdot K_t \cdot \rho^{-m_t}. \end{aligned} \tag{16}$$

If we set $D_{t-1} = \max(D(m_t),\ D_t + m_t D_*)$, we also have, by (9.iii), $\underline{q} \in H^{D_{t-1}}$, $\underline{q}(0) = 1$, $\|\underline{q}\| \le C_{t-1}$, $F \in H^{D_{t-1}}$.

Thus, if $D_{t-1} = \max(D(m_t), D_t + m_t D_*)$, and $C_{t-1} = \tilde{C}\cdot C_t \cdot C_*^{m_t}$ (as in (9.iii)), then choosing $\underline{\delta} > 0$ to satisfy (5), (7), (12), (13) and $\underline{K} > 0$ as in (16), we have shown that if $\hat{P} \in W\backslash W^t$, $\|\hat{P} - P_0\| < \underline{\delta}$ and $\underline{q}$ is defined as in (9) (where $P \in Y_\nu$ satisfies $\|\hat{P} - P\| = \tau = (\hat{P}, W^t)$), then the conclusions of Lemma 9.2 hold.

This proves the lemma, and with it the Main Lemma 8.2 as remarked earlier.

10. Proof of the Extension Theorem.

Proof of Part 1. — We take $t = -1$ in the Main Lemma 8.2, so that $W^t = \emptyset$. Using the definition of the norm N_* (8.3, (b)), we can formulate the conclusion of the Main Lemma as follows.

(1) There exist constants $C_{-1} > 0$, $D_{-1} \ge 1$ depending only on n, r, D, C_1 such that if $\rho > 0$ and $P_0 \in W$, then there exist $\delta_0 > 0$, $K_0 > 0$ depending only on P_0 and ρ such that

$$N_*(P, D_{-1}, C_{-1}, \rho) \le K_0 \text{ for } P \in W, \|P - P_0\| < \delta_0.$$

Since W is compact, for each $\rho > 0$, there is a constant K_* depending only on ρ and W such that

$$N_*(P, D_{-1}, C_{-1}, \rho) \leq K_* \quad \text{for} \quad P \in W.$$

Since N_* is a decreasing function of ρ $(0 < \rho \leq \rho_1)$, we conclude, in particular, that

(2) The function $(P, \rho) \mapsto N_*(P, D_{-1}, C_{-1}, \rho)$ is locally bounded on $W \times (0, \rho_1]$.

Because of Lemma 7.4, (b), this function is semi-algebraic, and we can apply Lemma 6.15 with $S = W \times [0, \rho_1]$, $E = W \times (0, \rho_1]$. This gives:

(3) There exist constants $m > 0$, $\overline{C} > 0$ (depending only on ρ_1, D_{-1}, C_{-1} and W) such that

$$N_*(P, D_{-1}, C_{-1}, \rho) \leq \overline{C}\rho^{-m} \quad \text{for all} \quad P \in W,\ 0 < \rho \leq \rho_1.$$

If we take $D' = D_{-1}$ and $C' = \max(C_{-1}, \overline{C})$, this is simply a restatement of Part 1 of the Extension Theorem. [Note that if $P \in W$ and a polynomial g is 0 as a germ in $\mathcal{G}(P)$, then $g = 0$ on $V_{\rho_1}(P)$.]

Proof of Part 2. — Choose $\rho_0 > 0$ such that $2\rho_0 \leq \rho_1$ (≤ 1) and such that if $q \in H^{D'}$, $q(0) = 1$, $\|q\| \leq C'$, then

$$\frac{1}{2} < q < 2 \text{ on } Q_{2\rho_0}.$$

By Part 1 of the Extension Theorem, there exists $q \in H^{D'}$, $q(0) = 1$, $\|q\| \leq C'$ (hence $\frac{1}{2} < q < 2$ on $Q_{2\rho_0}$) such that for any $f \in H^D$, we can find $F \in H^{D'}$ with

$$f = F/q \text{ on } V_{2\rho_0}(P)$$

and

$$\|F\| \leq C'\rho_0^{-m} \sup_{V_{\rho_0}(P)} |f|.$$

Since clearly $\sup_{Q_{2\rho_0}} |F| \leq C(D', n)\|F\|$, we obtain Part 2 of the Extension Theorem for the value $\rho = \rho_0$.

If $0 < \rho \leq \rho_0$, given $P = (p_1, \ldots, p_r) \in W$, consider the polynomials $\tilde{p}_j(x) = \dfrac{\rho_0}{\rho} p_j\Big(\dfrac{\rho}{\rho_0}x\Big)$. Since $p_j(0) = 0$, we have $\|\tilde{p}_j\| \leq \|p_j\|$. Moreover,

$\dfrac{\partial \tilde{p}_j}{\partial x_k}(0) = \dfrac{\partial p_j}{\partial x_k}(0)$. Thus we can apply Part 2 for the value ρ_0 to $\tilde{f}(x) = f\Big(\dfrac{\rho}{\rho_0}x\Big)$, $f \in H^D$, to write

$$\tilde{f} = \tilde{F}/\tilde{q} \text{ on } V_{2\rho_0}(\tilde{P}),\ \tilde{P} = (\tilde{p}_1, \dots, \tilde{p}_r),$$
$$\frac{1}{2} < \tilde{q} < 2 \text{ on } Q_{2\rho_0},\ \sup_{Q_{2\rho_0}} |\tilde{F}| \le C'' \sup_{V_{\rho_0}(\tilde{P})} |\tilde{f}|.$$

Writing $\tilde{F}(x) = F\Big(\dfrac{\rho x}{\rho_0}\Big)$, $\tilde{q} = q\Big(\dfrac{\rho x}{\rho_0}\Big)$ gives us Part 2 for the value ρ, $0 < \rho \le \rho_0$.

The proof of the Extension Theorem is complete.

11. Polynomial behaviour of algebraic functions.

In this section, we shall use the extension theorem to prove Theorem 1 stated in the Introduction.

Given $n \ge 2$, $1 \le r \le n-1$, $D \ge 1$ and $C_1 > 0$, we consider our basic space W defined by these constants.

If $\theta > 0$, and $s \ge 1$ is an integer, we denote by $B_s(\theta)$ the ball in $\mathbb{R}^s : B_s(\theta) = \{(x_1, \dots, x_s) \in \mathbb{R}^s \mid \sum x_j^2 < \theta^2\}$ (see §1).

By Lemma 2.1, there exist arbitrarily small constants θ_1, θ_2 with $0 < \theta_1, \theta_2 < 1$ depending only on n, r, D, C_1 such that if $P \in W$ and $\pi = \pi_P$ is the restriction to $Z(P) \cap B_r(\theta_1) \times B_{n-r}(\theta_2)$ of the projection $(x_1, \dots, x_n) \mapsto (x_{r+1}, \dots, x_n)$ ($Z(P)$ is, as usual, the set $\{x \in \mathbb{R}^n \mid P(x) = 0\}$), then we have:

(1) π is a real analytic isomorphism of $Z(P) \cap B_r(\theta_1) \times B_{n-r}(\theta_2)$ onto $B_{n-r}(\theta_2)$. Moreover, there is a real analytic map $\varphi = (\varphi_1, \dots, \varphi_r) : B_{n-r}(\theta_2) \to B_r(\theta_1)$ such that

$$Z(P) \cap B_r(\theta_1) \times B_{n-r}(\theta_2) = \{(\varphi_1(y), \dots, \varphi_r(y), y) \mid y \in B(\theta_2)\};$$

thus π^{-1} is the map $y \mapsto (\varphi(y), y)$, $y \in B(\theta_2)$.

If θ_1, θ_2 are chosen sufficiently small (depending on n, D, r, C_1), we have

(2) There is a constant $C_2 > 0$ (depending only on n, D, r, C_1) such that if $P = (p_1, \dots, p_r) \in W$, then

$$\left|\frac{\partial p_j}{\partial x_k}\right| < C_2,\ 1 \le j \le r,\ 1 \le k \le n,\ x \in B_r(\theta_1) \times B_{n-r}(\theta_2).$$

Further, if $J_P(x) = \det\left(\frac{\partial p_j}{\partial x_k}\right)_{1\le j,k\le r}$, we have

$$J_P(x) \ge \frac{1}{2} \quad \text{for} \quad x \in B_r(\theta_1) \times B_{n-r}(\theta_2).$$

If (1) and (2) hold, we also have the following estimate:

(3) There is a constant $C_3 > 0$ depending only on n, r, D, C_1 such that if $P \in W$ and $\varphi_\nu(x_{r+1}, \ldots, x_n)$, $\nu = 1, \ldots, r$, are the functions defined in (1), then

$$\left|\frac{\partial \varphi_\nu}{\partial x_i}\right| < C_3, r < i \le n,\ 1 \le \nu \le r,\ (x_{r+1}, \ldots, x_n) \in B_{n-r}(\theta_2).$$

In fact, since, if we set $y = (x_{r+1}, \ldots, x_n) \in B_{n-r}(\theta_2)$, we have

$$p_j(\varphi_1(y), \ldots, \varphi_r(y), y) \equiv 0,\ 1 \le j \le r,$$

we see that, for $r < i \le n$, $1 \le j \le r$,

$$\frac{\partial p_j}{\partial x_i} + \sum_{\ell=1}^{r} \frac{\partial p_j}{\partial x_\ell} \cdot \frac{\partial \varphi_\ell}{\partial x_i} \equiv 0, \tag{4}$$

the derivatives $\partial p_j/\partial x_\nu$ being evaluated at $(\varphi_1(y), \ldots, \varphi_r(y), y)$, $y = (x_{r+1}, \ldots, x_n)$.

Now, by (2), the matrix $\left(\frac{\partial p_j}{\partial x_\ell}\right)_{1\le j,\ell\le r}$ is invertible on $B_r(\theta_1) \times B_{n-r}(\theta_2)$ and its inverse, considered as a linear map of $\mathbb{R}^n$ into itself, has norm $\le C_4$, where C_4 depends only on n, r, D, C_1. Hence (3) follows from (4) and (2).

We now formulate Theorem 2 of the Introduction with the present notation.

THEOREM 11.1. — *There exist constants $\delta_*, C_* > 0$ depending only on n, r, D, C_1 with $0 < \delta_* \le \theta_2$ such that, if $P \in W, f \in H^D$ and we set $F = f \circ \pi^{-1} \in C^\infty(B_{n-r}(\theta_2))[F(x_{r+1}, \ldots, x_n) = f(\varphi_1(y), \ldots, \varphi_r(y), y)$ with $y = \pi(x) = (x_{r+1}, \ldots, x_n)]$, then the following inequalities hold:*

(A) *Polynomial Growth. If $0 < \delta \le \delta_*$, then*

$$\sup_{B_{n-r}(2\delta)} |F| \le C_* \sup_{B_{n-r}(\delta)} |F|.$$

(B) *Bernstein's Inequality. If $\nabla F = \left(\frac{\partial F}{\partial x_{r+1}}, \ldots, \frac{\partial F}{\partial x_n}\right)$ is the gradient of F, then, for $0 < \delta \le \delta_*$,*

$$\sup_{B_{n-r}(\delta)} |\nabla F| \le \frac{C_*}{\delta} \cdot \sup_{B_{n-r}(\delta)} |F|.$$

(C) *Equivalence of Norms. If* $0 < \delta \leq \delta_*$, *then*

$$\sup_{B_{n-r}(\delta)} |F| \leq \frac{C_*}{\delta^{n-r}} \int_{B_{n-r}(\delta)} |F| d\lambda,$$

where $d\lambda$ *is Lebesgue measure in* $\mathbb{R}^{n-r}$.

Proof. — Let C', D' be as in the Extension Theorem, Part 1. Choose constants $\rho_*, \delta_* > 0$ and $\rho_0 > 0$ depending only on n, r, D, C_1 such that $0 < \rho_0$, $\rho_* < \rho_1$ and such that

(5.i) If $q \in H^{D'}, q(0) = 1, \|q\| \leq C'$, then $\frac{1}{2} < q < 2$ on Q_{ρ_*}.

(5.ii) If $P \in W$, then $\pi_P^{-1}(B_{n-r}(2\delta_*)) \subset Q_{\rho_*}$.

(5.iii) If $P \in W$, then $\pi_P(Q_{\rho_0}) \subset B_{n-r}(\delta_*)$.

By the extension theorem (Part 1), given $P \in W$ and $f \in H^D$, we can find $G, q \in H^{D'}$ such that

$$q(0) = 1,\ \|q\| \leq C',\ q \cdot f = G \text{ on } V_{\rho_1}(P)$$

and

$$\|G\| \leq C' \rho_0^{-m} \sup_{V_{\rho_0}(P)} |f|. \tag{7}$$

By (5.iii), we have

$$\sup_{V_{\rho_0}(P)} |f| \leq \sup_{B_{n-r}(\delta_*)} |f \circ \pi^{-1}|,\ \pi = \pi_P. \tag{8}$$

Moreover,

$$\sup_{Q_{\rho_*}} |G| \leq \overline{C} \|G\|, \tag{9}$$

where $\overline{C}$ depends only on ρ_*, n, D'. Using (5), (6), (7), (8) and (9), we obtain

$$\begin{aligned} \sup_{B_{n-r}(2\delta_*)} |f \circ \pi^{-1}| &\leq \frac{\sup_{Q_{\rho_*}} |G|}{\inf_{Q_*} |q|} \leq 2\overline{C} \|G\| \\ &\leq 2\overline{C} C' \rho_0^{-m} \sup_{V_{\rho_0}(P)} |f| \\ &\leq 2\overline{C} C' \rho_0^{-m} \sup_{B_{n-r}(\delta_*)} |f \circ \pi^{-1}|. \end{aligned} \tag{10}$$

Next, if $F = f \circ \pi^{-1}$ and $r < \nu \leq n$, and we set $y = (x_{r+1}, \ldots, x_n)$, we have

$$\begin{aligned} \frac{\partial F}{\partial x_\nu} = \frac{\partial}{\partial x_\nu} G(\varphi(y), y)) \cdot (q(\varphi(y), y))^{-1} \\ - (q(\varphi(y), y))^{-2} G(\varphi(y), y) \frac{\partial}{\partial x_\nu} q(\varphi(y), y); \end{aligned}$$

if we remark that $\|\frac{\partial G}{\partial x_k}\| \le D'\|G\|$ and $\|\frac{\partial q}{\partial x_k}\| \le D'\|q\| \le D'C'$ $(1 \le k \le n)$ and use (3) and (5.i), we obtain

$$\sup_{B_{n-r}(\delta_*)} \left|\frac{\partial F}{\partial x_\nu}\right| \le C''\|G\| \le C_{\#} \sup_{B_{n-r}(\delta_*)} |F|$$

by (7) and (8); here $C'', C_{\#}$ depend only on n, r, D, C_1. This gives

$$\sup_{B_{n-r}(\delta_*)} |\nabla F| \le \sqrt{n}\, C_{\#} \sup_{B_{n-r}(\delta_*)} |F|. \tag{11}$$

Next we make the following remark. Let U be a bounded convex open set in $\mathbb{R}^N$, $U \ne \emptyset$. Let $\varphi = C^1(\overline{U})$, $\overline{U}$ being the closure of U. Then

$$\sup_U |\varphi| \le \operatorname{diam}(U) \sup_U |\nabla\varphi| + \frac{1}{\operatorname{vol}(U)} \int_U |\varphi|\, d\,\lambda. \tag{12}$$

To prove this, we first remark that it is sufficient to prove (12) when φ is real-valued. In fact, if $x \in U$ and $\alpha \in \mathbb{R}$ is so chosen that $e^{i\alpha}\varphi(x) \in \mathbb{R}$, we can apply (12) to the function $\varphi_1 = \operatorname{Re}(e^{i\alpha}\varphi)$ to obtain

$$|\varphi(x)| = |\varphi_1(x)| \le \operatorname{diam}(U)|\nabla\varphi_1| + \frac{1}{\operatorname{vol}(U)} \int_U |\varphi_1| d\lambda;$$

since $x \in U$ is arbitrary and $|\varphi_1| \le |\varphi|$, $|\nabla\varphi_1| \le |\nabla\varphi|$, we obtain (12) for φ.

Now, if $\psi \in C^1(\overline{U})$ is real-valued and $\int_U \psi d\lambda = 0$, then, there is a point $a \in U$ with $\psi(a) = 0$. Hence, if $x \in U$, there is a point ξ on the line segment from a to x such that

$$\psi(x) = \psi(x) - \psi(a) = \sum_{\nu=1}^{N} (x_\nu - a_\nu) \frac{\partial\psi}{\partial x_\nu}(\xi),$$

which gives $|\psi(x)| \le |x - a|\, |\nabla\psi(\xi)| \le \operatorname{diam}(U) \sup_U |\nabla\psi|$. If $\varphi \in C^1(\overline{U})$ and we apply this to $\psi(x) = \varphi(x) - \frac{1}{\operatorname{vol}(U)} \int_U \varphi d\lambda$, we obtain (12) when φ is real-valued, hence in general as remarked above.

Let $\epsilon = \frac{1}{4\sqrt{n}C_{\#}}$ where $C_{\#}$ is as in (11) and let $y_0 \in \overline{B_{n-r}(\delta_*)}$ be such that $|F(y_0)| = \sup_{B_{n-r}(\delta_*)} |F|$. We apply (12) with $N = n - r$, $\varphi = F$ and $U = B_{n-r}(\delta_*) \cap \{y \in \mathbb{R}^{n-r} \mid |y - y_0| < \epsilon\}$. If we remark that there is a constant $\gamma > 0$ depending only on ϵ, δ_* and $n - r$ such that $\operatorname{vol}(U) \ge \gamma$, we obtain

$$\begin{aligned}\sup_{B_{n-r}(\delta_*)} |F| = |F(y_0)| = \sup_U |F| &\le 2\epsilon \sup_U |\nabla F| + \frac{1}{\gamma} \int_U |F| d\lambda \\ &\le \frac{1}{2} \sup_{B_{n-r}(\delta_*)} |F| + \frac{1}{\gamma} \int_U |F| d\lambda \quad \text{(by (11))}.\end{aligned}$$

Thus

$$\sup_{B_{n-r}(\delta_*)} |F| \le \frac{2}{\gamma}\int_U |F|d\lambda. \tag{13}$$

If we take $C_* = \max\Big(\frac{2}{\gamma}\delta^{n-r}, \sqrt{n}C_{\#}\delta_*, 2\overline{C}C'\rho_0^{-m}\Big)$, (10), (11), (13) give, respectively, statements (A), (B), (C) of Theorem 11.1 for the value $\delta = \delta_*$.

To prove these statements for $0 < \delta \le \delta_*$, we simply rescale as in the proof of Part 2 of the extension theorem. To do this, given $P = (p_1, \ldots, p_r) \in W$, let $\tilde{P} = (\tilde{p}, \ldots, \tilde{p}_r)$ where $\tilde{p}_j(x) = \frac{\delta_*}{\delta} p_j\Big(\frac{\delta x}{\delta_*}\Big)$. Then $\tilde{P} \in W$. We denote $(x_{r+1}, \ldots, x_n) = \pi_P(x_1, \ldots, x_n)$ by y. Let $\varphi = (\varphi_1, \ldots, \varphi_r) : B_{n-r}(\theta_2) \to B_r(\theta_1)$ be the map given by (1), so that $Z(P) \cap B_r(\theta_1) \times B_{n-r}(\theta_2) = \{(\varphi(y), y) \mid y \in B_{n-r}(\theta_2)\}$. We claim that the map $(\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r) = \tilde{\varphi} : B_{n-r}(\theta_2) \to B_r(\theta_1)$ corresponding to $\tilde{P}$ is given by $\tilde{\varphi}_j(y) = \frac{\delta_*}{\delta}\varphi_j\Big(\frac{\delta y}{\delta_*}\Big)$. In fact, if $|y|$ is sufficiently small, this is clear, since, for $1 \le j \le r$, $\tilde{p}_j(\tilde{\varphi}(y), y) = \tilde{p}_j\Big(\frac{\delta_*}{\delta}\varphi\Big(\frac{\delta y}{\delta_*}\Big), y\Big) = 0$ (by the definition of $\tilde{p}_j$ and φ); the above formula then holds for $|y| < \theta_2$ since both functions are real analytic.

If $f \in H^D$, set $\tilde{f}(x) = f\Big(\frac{\delta x}{\delta_*}\Big)$. Then $\tilde{f} \in H^D$. Let $\tilde{\pi} = \pi_{\tilde{P}}$ be the projection $Z(\tilde{P}) \cap B_r(\theta_1) \times B_{n-r}(\theta_2) \to B_{n-r}(\theta_2)$ corresponding to $\tilde{P}$. We have, for $|y| < \theta_2$,

$$\tilde{F}(y) = \tilde{f} \circ \tilde{\pi}^{-1}(y) = f\left(\varphi\left(\frac{\delta y}{\delta_*}\right), \frac{\delta y}{\delta_*}\right) = (f \circ \pi^{-1})\left(\frac{\delta y}{\delta_*}\right) = F\left(\frac{\delta y}{\delta_*}\right).$$

It follows that

$$\sup_{B_{n-r}(t\delta_*)} |\tilde{F}| = \sup_{B_{n-r}(t\delta)} |F|, \; t = 1, 2,$$

$$(\nabla \tilde{F})(y) = \frac{\delta}{\delta_*}(\nabla F)\left(\frac{\delta y}{\delta_*}\right),$$

and

$$\int_{B_{n-r}(\delta_*)} |\tilde{F}|d\lambda = \left(\frac{\delta_*}{\delta}\right)^{n-r} \int_{B_{n-r}(\delta)} |F|d\lambda.$$

Consequently (A), (B), (C) in Theorem 11.1 for the value δ_* and the pair $(\tilde{F}, \tilde{P})$ are equivalent, respectively, to (A), (B), (C) for the value δ and the pair (F, P).

This proves Theorem 11.1.

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