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SMOOTHABILITY OF PROPER FOLIATIONS

by J. CANTWELL[(1)] and L. CONLON[(2)]

Introduction.

Let $(M, \mathcal{F})$ be a foliated manifold. If $\partial M \neq \emptyset$, we usually assume that ∂M is $\mathcal{F}$-saturated. The manifold M is assumed to be of classe C^∞, but the foliation $\mathcal{F}$ may only be of class C^0.

DEFINITION. — *If $\mathcal{F}$ is given by a C^n atlas of foliation charts, $0 \leq n \leq \infty$, that are C^n-compatible with the C^∞ structure of M, then $(M, \mathcal{F})$ is a C^n-foliated manifold.*

DEFINITION. — *Let $1 \leq n \leq \infty$. The foliated manifold $(M, \mathcal{F})$ is C^n-smoothable if there exists a C^n-foliated manifold $(M', \mathcal{F}')$ that is homeomorphic to $(M, \mathcal{F})$.*

Let N be a q-manifold. If $q \neq 1, 4$ and if $n \geq 0$, there exists a C^n diffeomorphism $f : N \to N$ that is not topologically conjugate to one of class C^{n+1} [Har1], [Har2]. For $q = 1$ and $n \geq 2$, this is false by a theorem of Denjoy [De]. For $q = 4$, it seems to be unknown. Using the suspension of f, one shows that, in all codimensions other than 1 and 4, and for each integer $n \geq 0$, there are purely topological distinctions between foliations of class C^n and those of class C^{n+1}.

In codimension one, however, there is an interesting class of foliations for which there is no qualitative distinction between smoothness of class C^2 and that of class C^∞. This can be viewed as a partial generalization of Denjoy's theorem.

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DEFINITION. — *A foliated manifold $(M, \mathcal{F})$ is proper if each leaf L of $\mathcal{F}$ is proper. That is $\overline{L} \setminus L$ is a closed subset of M.*

MAIN THEOREM. — *Let M be a compact, orientable manifold, $\mathcal{F}$ a transversely orientable C^2 foliation of M of codimension one and tangent to ∂M. If $(M, \mathcal{F})$ is proper, then $(M, \mathcal{F})$ is C^∞-smoothable and the C^∞ foliation can be required to be C^∞-flat at the boundary.*

Here, C^∞-flat at the boundary means that all holonomy around loops in ∂M is C^∞-tangent to the identity there.

Examples described in §5 show the necessity of requiring that all leaves be proper and that $(M, \mathcal{F})$ be of class C^2.

Our principal tools will be the Epstein-Millett hierarchy [M] for proper C^0-foliated manifolds of arbitrary codimension and the Poincaré-Bendixson theory for totally proper leaves in compact, C^2-foliated manifolds of codimension one [C.C2]. We will construct the smoothing homeomorphism by transfinite induction on the hierarchy. Examples show (§5) that the Epstein-Millett hierarchy for foliated manifolds as in the Main Theorem can be order-isomorphic to any countable, non-limit ordinal.

The hypotheses of orientability and transverse orientability are in the Main Theorem so that we might use the Poincaré-Bendixson theory of [C.C2] without change. There would seem to be no serious obstruction to modifying that theory so as to avoid these restrictions.

There are other ways in which it is natural to try to sharpen the Main Theorem. For instance, can one find a C^∞ subatlas of the C^2 atlas of foliation charts? We think that this is generally impossible. Does our C^∞ smoothing of the foliated manifold leave unchanged the C^∞ diffeomorphism class of the underlying manifold M? We do not know.

1. The Epstein-Millett hierarchy.

Let $(M, \mathcal{F})$ be a proper, C^0-foliated manifold of codimension q. It is not required that M be compact. We sketch here the main facts from [M] that will be needed subsequently.

A C^0 atlas of foliation charts provides an imbedded q-manifold $T \subset M$ that is transverse to $\mathcal{F}$ and meets each leaf and has at most countably many components. Let $d : T \times T \to [0, 1]$ be a metric, compatible with the topology of T and such that $d(x,y) = 1$ precisely when x and y lie in distinct components of T. We can assume that T meets each leaf at least twice.

For $x \in T$, let L_x denote the leaf of $\mathcal{F}$ through x. Define $\sigma : T \to \mathbf{R}$ by

$$\sigma(x) = \inf\{d(x,y) | x \neq y \in L_x \cap T\}.$$

This function is upper semicontinuous.

LEMMA 1.1. — *Let $X \subseteq T$ and $x \in X$. If $\sigma|X$ is continuous at x, then there exists $\varepsilon > 0$ such that the ε-neighborhood of x in X meets each leaf of $\mathcal{F}$ at most once.*

Proof. — Since L_x is proper, $\sigma(x) > 0$. By continuity, find $\varepsilon > 0$ such that $y \in X$ and $d(x,y) < \varepsilon$ imply that $\sigma(y) > 0$. □

DEFINITION. — *If $Y \subseteq M$ is an $\mathcal{F}$-saturated subset and $L \subseteq Y$ is a leaf, then L has locally trivial holonomy pseudogroup relative to Y if, for each $x \in L \cap T$, there exists a neighborhood of x in $Y \cap T$ that meets each leaf at most once.*

COROLLARY 1.2. — *Let $Y \subseteq M$ be a closed, $\mathcal{F}$-saturated set. Then the union of leaves having locally trivial holonomy pseudogroup relative to Y is relatively open and dense in Y.*

Proof. — The set of points of continuity of the upper semicontinuous function $\sigma|(Y \cap T)$ is dense in the locally compact, second countable metric space $Y \cap T$ (in fact, it contains a dense G_δ [F, p. 39, Lemma 1.28]). Now apply (1.1). □

THEOREM 1.3 (Millett). — *There is a unique countable ordinal $\gamma = \gamma(M, \mathcal{F})$ and a unique filtration*

$$\emptyset = M_0 \subset M_1 \subset \ldots \subset M_\alpha \subset \ldots \subset M_\gamma = M$$

by open, $\mathcal{F}$-saturated subsets, order-isomorphic to the ordinals $0 \leq \alpha \leq \gamma$, such that

(1) *M_α is dense in $M, 0 < \alpha \leq \gamma$;*

(2) *if α is a limit ordinal, then $M_\alpha = U_{0 \leq \beta < \alpha} M_\beta$;*

(3) *$M_{\alpha+1} \backslash M_\alpha \neq \emptyset$ and is the union of all leaves that have locally trivial holonomy pseudogroup relative to $M \backslash M_\alpha$, $0 \leq \alpha < \gamma$.*

Proof. — The conditions dictate the transfinite inductive construction. Indeed, $M_0 = \emptyset$ is given and, if M_β has been defined, $0 \leq \beta < \alpha$, then either α is a limit ordinal and M_α is defined by (2) or $\alpha = \beta + 1$ and $M_\alpha \backslash M_\beta = M_{\beta+1} \backslash M_\beta$ is defined by (3). By (1.2), $M_{\beta+1} \backslash M_\beta$ is open and dense in $M \backslash M_\beta$, hence $M_{\beta+1} = M_\alpha$ will be open and dense in M.

Define $\gamma = \gamma(M, \mathcal{F})$ to be the least ordinal for which $M_\gamma = M$. Then γ is least such that $M_\gamma = M_{\gamma+1}$. Let $\{U_i\}_{i=1}^{\infty}$ be an enumeration of a countable base for the topology of M. For $0 \leq \alpha < \gamma$, let $i(\alpha)$ be the least integer such that $U_{i(\alpha)} \subseteq M_{\alpha+1}$, but $U_{i(\alpha)} \not\subseteq M_\alpha$. Then $0 \leq \alpha < \beta < \gamma$ implies that $i(\alpha) \neq i(\beta)$, so γ is countable. □

Remark. — If M and all leaves of $\mathcal{F}$ are compact, this filtration of M coincides with the Epstein filtration [E], [E.M.S].

DEFINITION. — *Let $Y \subseteq M$ be an $\mathcal{F}$-saturated subset, $L \subseteq Y$ a leaf. Then L is stable relative to Y if, given $x \in L \cap T$, there exists a neighborhood V_x of x in $Y \cap T$ and an imbedding $i : V_x \times L \to M$ such that $i(\{y\} \times L) = L_y$, for each $y \in V_x$.*

COROLLARY 1.4. — *If M is compact, $(M, \mathcal{F})$ is proper,* $\operatorname{codim}(\mathcal{F}) = 1$, *and $\mathcal{F}$ is tangent to ∂M, then the filtration $\{M_\alpha\}_{0 \leq \alpha \leq \gamma}$ has the property that each leaf of $M_{\alpha+1} \backslash M_\alpha$ is stable relative to $M \backslash M_\alpha$, $0 \leq \alpha < \gamma$.*

Indeed, for $\alpha = 0$, this is Inaba's stability theorem [I]. In general, it is the essential content of Dippolito's semi-stability theorem (see (2.3)).

DEFINITION. — *Let L be a leaf of $\mathcal{F}$. Then $\gamma(L) \leq \gamma(M, \mathcal{F})$ is the least ordinal such that $L \subseteq M_{\gamma(L)}$.*

LEMMA 1.5. — *For each leaf L of $\mathcal{F}$, $\gamma(L)$ has an immediate predecessor $\gamma(L) - 1$.* □

LEMMA 1.6. — *Let* $\operatorname{codim}(\mathcal{F}) = 1$ *and let M be compact. Let L and L' be leaves of $\mathcal{F}$ with $L' \subseteq \overline{L} \backslash L$. Then $\gamma(L) \leq \gamma(L') - 1$.*

Proof. — Since L is asymptotic to L' and L' is stable relative to $M \backslash M_{\gamma(L')-1}$ (1.4), it follows that $L \subseteq M_{\gamma(L')-1}$. □

2. Open, saturated sets in codimension one.

Assume that M is a compact n-manifold, that $\mathcal{F}$ is transversely oriented, that $\operatorname{codim}(\mathcal{F}) = 1$, and that $\mathcal{F}$ is tangent to ∂M. Let $O(\mathcal{F})$ denote the family of open, $\mathcal{F}$-saturated subsets of M. We review basic facts about the members of $O(\mathcal{F})$. Few proofs will be given since details (modulo notation) are readily available elsewhere [Di], [C.C2], [H.H], [G].

A. The C^0 case.

We assume that $\mathcal{F}$ is integral to a $C^0(n-1)$-plane field and we fix a choice of one dimensional C^∞ foliation $\mathcal{F}^\perp$, everywhere transverse to $\mathcal{F}$. In standard fashion, $\mathcal{F}^\perp$ induces a C^∞ structure on each leaf of $\mathcal{F}$ relative to which the leaf is C^∞-immersed in M. Indeed, let $\{U_\alpha, x_\alpha^1, x_\alpha^2, \dots, x_\alpha^n\}_{\alpha\in\mathfrak{A}}$ be a C^∞ foliation atlas for $\mathcal{F}^\perp$, so that the change of coordinates on overlaps is of the form

$$x_\alpha^i = x_\alpha^i(x_\beta^1, \dots, x_\beta^{n-1}), 1 \le i \le n-1,$$
$$x_\alpha^n = x_\alpha^n(x_\beta^1, \dots, x_\beta^n).$$

Thus, the first $n-1$ coordinates restrict to define coordinate charts on leaves of $\mathcal{F}$, the resulting atlas on each leaf of $\mathcal{F}$ being C^∞. Each leaf of $\mathcal{F}^\perp$ is oriented by the transverse orientation of $\mathcal{F}$, hence we can use interval notation $[x,y]$, $[x,y[$, etc., for subarcs of leaves of $\mathcal{F}^\perp$, and we do so wherever convenient. By parametrizing $\mathcal{F}^\perp$ as a nonsingular, local C^∞ flow, we can regard these as actual subintervals of $\mathbf{R}$.

For $U \in O(\mathcal{F})$, the metric completion $\widehat{U}$ [Di], [C.C2], [H.H], [G] is a manifold with boundary, generally noncompact. The inclusion $i : U \to M$ extends to an immersion $\widehat{\imath} : \widehat{U} \to M$ that carries each component of $\partial\widehat{U}$ diffeomorphically onto a leaf of $\mathcal{F}$, but may identify some of these components pairwise.

DEFINITION. — *The border of U is $\delta U = \widehat{\imath}(\partial\widehat{U})$.*

DEFINITION. — *The foliation $\widehat{\mathcal{F}}$, tangent to $\partial\widehat{U}$, is the pullback $\widehat{\imath}^{-1}(\mathcal{F})$. Similarly, $\widehat{\mathcal{F}}^\perp = \widehat{\imath}^{-1}(\mathcal{F}^\perp)$.*

DEFINITION. — *The foliated manifold $(\widehat{U}, \widehat{\mathcal{F}})$ is a foliated product if there is a homeomorphism $\widehat{U} \cong L \times [0,1]$ that identifies the leaves $[x,y]$ of $\widehat{\mathcal{F}}^\perp$ with the factors $\{x\} \times [0,1], x \in L$.*

If $(\widehat{U},\widehat{\mathcal{F}})$ is a foliated product, we say also that $\widehat{U}$ or U is a foliated product. The most important property of a foliated product is that $(\widehat{U},\widehat{\mathcal{F}})$ is completely described by the total holonomy homomorphism $\varphi : \pi_1(L) \to \mathrm{Homeo}_+[0,1]$. In particular, if we take into account the C^∞ structure on L induced by $\widehat{\mathcal{F}}^\perp$, the following is clear.

LEMMA 2.1. — *Let $U \in O(\mathcal{F})$ be a foliated product, $\widehat{U} \cong L \times [0,1]$ as above. Then the foliated manifold $(\widehat{U},\widehat{\mathcal{F}})$ is C^n-smoothable, $1 \le n \le \infty$, if and only if there exists $h \in \mathrm{Homeo}_+[0,1]$ such that $h(\varphi(\pi_1(L))h^{-1} \subset \mathrm{Diff}^n_+[0,1]$.*

If $U \in O(\mathcal{F})$ is connected, but $\overline{U}$ is not a foliated product, the structure is somewhat more complicated. There is a decomposition $\widehat{U} = K \cup V_1 \cup \ldots \cup V_r$, $r \ge 0$, with the following properties :

(1) The "nucleus" K is a compact, connected n-manifold, with corners if $r \ge 1$. The corners divide ∂K into submanifolds $\partial_\tau K$, to which $\mathcal{F}$ is tangent, and $\partial_{\mathrm{tr}} K$, to which $\widehat{\mathcal{F}}$ is transverse. The corners are "convex" relative to $\widehat{\mathcal{F}}$, meaning that the local model is $[0,\infty[\times[0,\infty[\times\mathbf{R}^{n-2}$ with leaves $\{t\} \times [0,\infty[\times\mathbf{R}^{n-2}$. Each component of $\partial_\tau K$ lies in a component of $\partial\widehat{U}$ and the components of these manifolds thereby correspond one-one. The manifold $\partial_{\mathrm{tr}} K$ is $\widehat{\mathcal{F}}^\perp$-saturated. Of course, if $r = 0$, then $K = \widehat{U}$ and there are no corners, perhaps even no boundary.

(2) Each "arm" V_i is homeomorphic to $B^i \times [0,1]$ with $B^i \times \{0\}$ and $B^i \times \{1\}$ each a complete, noncompact, connected submanifold of $\partial\widehat{U}$. Again, $(V_i, \widehat{\mathcal{F}}|V_i)$ has convex corners. The leaves of $\widehat{\mathcal{F}}^\perp$ are the compact intervals $\{x\} \times [0,1], x \in B^i$. If $i \neq j$, then $V_i \cap V_j = \emptyset$. Finally, ∂B^i is compact and connected, and the components of $\partial_{\mathrm{tr}} K$ are exactly the manifolds $\partial_{\mathrm{tr}} V_i = K \cap V_i = (\partial B^i) \times [0,1], 1 \le i \le r$.

Thus, when $U \in O(\mathcal{F})$ is connected, $\widehat{U}$ is a kind of "octopus" with r arms and $\partial\widehat{U}$ has only finitely many components. The induced foliation $\widehat{\mathcal{F}}|V_i$ is completely described by the total holonomy homomorphism $\varphi_i : \pi_1(B^i) \to \mathrm{Homeo}_+[0,1]$ and the analogue of (2.1) holds.

DEFINITION. — *A decomposition $\widehat{U} = K \cup V_1 \cup \ldots \cup V_r$ as above is called a Dippolito decomposition.*

We will find the following fact to be useful [Di, Theorem 1], [H.H, V.3.2.6].

LEMMA 2.2. — *If $U \in O(\mathcal{F})$, then all but finitely many components* *of U are foliated products.*

DEFINITION. — *Let $U \in O(\mathcal{F})$ and let L be a component of $\partial\widehat{U}$. Let $[x,y[$ be a subarc of a leaf of $\widehat{\mathcal{F}}^{\perp}$ such that $x \in L$. Let* $\mathrm{Fix}(L) = \{z \in]x,y[| L_z \cap]x,y[= \{z\}\}$. *If, for y sufficiently near x,* $\mathrm{Fix}(L)$ *clusters at x, L is said to be semistable. Otherwise, L is said to have unbounded holonomy.*

This definition of unbounded holonomy is equivalent to that of J. Plante [P, Lemma 5.1]. The term "semistable" is justified by the following result of Dippolito which, in fact, is a sharper version of [S.S, Theorem 1].

THEOREM 2.3 [Di, Theorem 3]. — *If L is semistable, then there exists $x \in \mathrm{Fix}(L)$ and an imbedding $j : L \times [x,z] \to \widehat{U}$ such that*

(1) $j|(L \times \{x\}) = \mathrm{id}_L$;

(2) $j|(\{x\} \times [x,z]) = \mathrm{id}_{[x,z]}$;

(3) $j(\{y\} \times [x,z])$ *is a subarc of a leaf of $\widehat{\mathcal{F}}^{\perp}$, for each $y \in L$;*

(4) $w \in \mathrm{Fix}(L) iff$ $j(L \times \{w\})$ *is the leaf of $\widehat{\mathcal{F}}$ through $w, x < w \leq z$.*

A short proof of (2.3) will also be found in [C.C2].

DEFINITION. — *If L is a component of $\partial\widehat{U}$ and $j : L \times [x,z] \to \widehat{U}$ is an imbedding satisfying (1), (2), (3) of (2.3), along with the property that $j(L \times \{z\})$ is the leaf of $\widehat{\mathcal{F}}$ through z, then* $\mathrm{im}(j)$ *is a foliated collar of L.*

PROPOSITION 2.4. — *If the component L of $\partial\widehat{U}$ has a foliated collar, it has a maximal one.*

Proof. — The foliated collars of L form a linearly ordered set under inclusion. We can find a possibly infinite sequence of such collars,

$$L \times [x, z_1] \subset \ldots \subset L \times [x, z_k] \subset \ldots,$$

such that each foliated collar is contained in some $L \times [x, z_k]$. There is a maximal foliated collar if and only if this sequence is finite, so we suppose it to be infinite.

Let $\widehat{U} = K \cup V_1 \cup \ldots \cup V_r$, $V_i = B^i \times [0,1]$, $1 \leq i \leq r$, be a Dippolito decomposition and assume, without loss of generality, that $x \in \partial B^1$. Thus, $[x, z_k]$ can be identified as $[0, t_k]$, $0 < t_k < 1$, where $\{t_k\}_{k=1}^{\infty}$ is a strictly increasing sequence. Choose the indices $i = 1, \ldots r$ so that $L = A \cup B^1 \cup \ldots \cup B^s$, some $s \leq r$, where $A = K \cap L \subseteq \partial_{\tau} K$.

Let $t_* = \lim_{k\to\infty}(t_k) \leq 1$. If we show that the leaf L_* throught t_* bounds a foliated collar, we will have the desired contradiction.

Since t_k is fixed by $\varphi_1(\pi_1(B^1))$, for each $k \geq 1$, so is t_*. Thus, the connected component of $L_* \cap V_1$ that meets t_* projects homeomorphically onto B^1 along the fibers of $\widehat{\mathcal{F}}^\perp|V_1$.

Realize $\pi_1(A)$ as $\pi_1(A,x)$. Then each element of $\pi_1(A)$ has holonomy defined on $[0,t_*[$ and this holonomy fixes each t_k. Thus, $\widehat{\mathcal{F}}|K$ has a compact leaf $A_k \cong A$ that meets t_k, for each $k \geq 1$. We need to show that this foliation has a compact leaf A_* meeting t_* that is the uniform limit of $\{A_k\}_{k=1}^\infty$. After doubling $(K, \widehat{\mathcal{F}}|K)$ along $\partial_{\mathrm{tr}}K$, then doubling the resulting foliated manifold along its (tangential) boundary, one obtains a closed, foliated manifold and a sequence $\{2A_k\}_{k=1}^\infty$ of closed leaves. An application of [Hae, (3.2)] then yields the desired result.

Finally, the argument given for V_1 above can now be used in each V_i, using basepoints $x_i \in \partial B^i$ in place of $x, 2 \leq i \leq r$, to obtain the fact that the leaf L_* through t_* bounds a foliated collar of L. □

DEFINITION. — *Let $U \in O(\mathcal{F})$ be connected. Then the core $U_* \in O(\mathcal{F})$ of U is the complement in U of the union of the maximal foliated collars of those components of $\partial\widehat{U}$ that admit foliated collars.*

Remarks. — The core U_* is empty if and only if U is a foliated product. Furthermore, U_* is connected and, by (2.3), each component of $\partial\widehat{U}_*$ has unbounded holonomy.

DEFINITION. — *A connected element $U \in O(\mathcal{F})$ is irreducible if no component of $\partial\widehat{U}$ has a foliated collar other than $\widehat{U}$ itself.*

Remarks. — If U is not a foliated product, it is irreducible precisely when $U = U_*$. In any case, each component of $\partial\widehat{U}$ has unbounded holonomy whenever U is irreducible.

B. The C^2 case.

In addition to the properties in part A, we assume that $(M, \mathcal{F})$ is smooth of class C^2.

THEOREM 2.5 (Hector). — *Let $U \in O(\mathcal{F})$ and let L be a leaf of $\mathcal{F}|U$. Then $\overline{L} \cap U$ contains a minimal set of $\mathcal{F}|U$. The union X of all minimal sets of $\mathcal{F}|U$ is a relatively closed subset of U.*

COROLLARY 2.6. — *Suppose that $(M,\mathcal{F})$ is proper, $U \in O(\mathcal{F})$, and let L be a leaf of $\mathcal{F}|U$. Then $\overline{L} \cap U$ contains a leaf L_0 that is relatively closed in U. The union X of all leaves that are relatively closed in U is itself relatively closed in U.*

Remarks. — Smoothness of class C^2 is essential. A proof of (2.5) will be found in [C.C2]. This result essentially goes back to Hector's thesis.

Let $\emptyset \neq U \in O(\mathcal{F})$, assume that $(M,\mathcal{F})$ is proper, and set $U_\alpha = U \cap M_\alpha$, $0 \leq \alpha \leq \gamma(M,\mathcal{F})$. If $\gamma(U)$ denotes the least ordinal such that $U_{\gamma(U)} = U_{\gamma(U)+1}$, then $\{U_\alpha\}_{0 \leq \alpha \leq \gamma(U)}$, is the Epstein-Millett filtration of $(U,\mathcal{F}|U)$. We set $\gamma(\emptyset) = -1$.

PROPOSITION 2.7. — *Let $(M,\mathcal{F})$ be proper, let $\emptyset \neq U \in O(\mathcal{F})$, and let X be the union of the leaves of $\mathcal{F}|U$ that are relatively closed in U. Then each component W of $U \backslash X$ has $\gamma(W) < \gamma(U)$.*

Proof. — If $U = X$, then $\gamma(W) = \gamma(\emptyset) = -1$ and we are done. Thus, assume that $W \neq \emptyset$ and remark that, by (2.6), $W \in O(\mathcal{F})$ and $W \neq U$.

Let $L \subset W$ be a leaf. Then $L \not\subseteq X$, but there exists a leaf $L_0 \subseteq \overline{L} \cap X$ (2.6). By (1.6), it follows that $\gamma(L) \leq \gamma(L_0) - 1 < \gamma(U)$.

But $L_0 \subseteq \delta W$. Otherwise, the fact that L is asymptotic to L_0 would force some leaf of δW to be asymptotic to L_0, contradicting the fact that $\delta W \subseteq X \cup \delta U$.

Since W is connected, δW contains only finitely many leaves and there are only finitely many possibilities for $\gamma(L_0)$. It follows that there is an ordinal $\alpha < \gamma(U)$ with $\gamma(L) \leq \alpha$, for each leaf L of $\mathcal{F}|W$. □

3. The Poincaré-Bendixson theory of totally proper leaves.

Fix the hypotheses that $(M,\mathcal{F})$ is a compact, C^2-foliated n-manifold of codimension one, that M is oriented, and that $\mathcal{F}$ is transversely oriented and tangent to ∂M. Thus, each leaf of $\mathcal{F}$ is oriented, as is each leaf of $\mathcal{F}^\perp$.

DEFINITION. — *A leaf L of $\mathcal{F}$ is totally proper if $\overline{L}$ is a union of proper leaves.*

LEMMA 3.1 (A special case of [C.C2, (4.7)].). — *If L is totally proper, then $\overline{L}$ is a finite union of totally proper leaves.*

By contrast, if L is not totally proper, the results of [C.C2] imply that $\overline{L}$ is the union of uncountably many leaves.

By (3.1), the *level* of a totally proper leaf is an integer $k \geq 0$, well defined as follows.

DEFINITION. — *Let L be a totally proper leaf of $\mathcal{F}$. If L is compact, then* level$(L) = 0$. *If the maximum level of any leaf in $\overline{L}\backslash L$ is $k \geq 0$, then* level$(L) = k+1$.

For L totally proper, the Poincaré-Bendixson theory of [C.C2,§6] gives a detailed description of how L winds in on the finitely many leaves of $\overline{L}\backslash L$. We review this carefully since it is crucial for our construction of the smoothing homeomorphism.

DEFINITION. — *Let $[x', x]$ be a subarc of a leaf of $\mathcal{F}^\perp$ and let L' be the leaf of $\mathcal{F}$ through x'. If $]x', x] \cap L' = \emptyset$, we say that x projects (in the negative direction) to L' and we write $p(x) = x'$. Projection in the positive direction is defined analogously by using a subarc $[x, x']$.*

Let L and L' be leaves of $\mathcal{F}$ and let $B \subset L$ be a complete, noncompact, connected submanifold of dimension $n-1$, with $N_0 = \partial B$ compact and connected. Suppose that each point of B projects to L'. Remark that $p : B \to L'$ is smooth and locally a diffeomorphism. Let $N = p(N_0)$.

The following generalizes the usual Poincaré-Bendixson picture of a flow line in $\mathbb{R}^2$ winding in on a periodic orbit.

DEFINITION. — *The projection $p : B \to L'$ in the negative direction (respectively, in the positive direction), as above, is a spiral on the positive side (respectively, on the negative side) of L' if*

(a) $B = \bigcup_{i=0}^{\infty} B_i$, *where each B_i is a complete, connected, $(n-1)$-manifold;*

(b) $B_i \cap B_{i+1} = N_{i+1}$ *is a common boundary component of B_i and B_{i+1}, for each $i \geq 0$, $N_0 = \partial B$ is a component of ∂B_0, and $j \geq 2$ implies that $B_i \cap B_{i+j} = \emptyset$, for each $i \geq 0$;*

(c) $p|N_i$ *carries N_i diffeomorphically onto N, for each $i \geq 0$;*

(d) *$y \in L'$ implies that $p^{-1}(y) \cap (B_i \backslash N_{i+1})$ is a singleton $\{y_i\}$, for each $i \geq 0$;*

(e) *for each $y \in L'$, $p^{-1}(y) = \{y_i\}_{i=0}^{\infty}$ converges monotonically to y in $[y, y_0]$ (respectively, in $[y_0, y]$).*

As an example, take L' to be the toral boundary leaf in the Reeb foliation of $S^1 \times D^2$. If L is one of the planar leaves, then $B \cong S^1 \times [0, \infty[$ is the complement of an open disk in L and spirals on L'. One can take N to be a meridian circle $\{z\} \times \partial D^2$ on L' and each N_i to be $S^1 \times \{i\}$. The projection $p : B \to L'$ carries $B_i = S^1 \times [i, i+1]$ onto L', identifying both boundary components to N.

For a picture in which L' is not a compact leaf, see [C.C2, p. 201].

DEFINITION. — *The manifold $N \subset L'$ is called the juncture of the spiral $p : B \to L'$.*

Remark that the juncture is a compact, connected, oriented, non-separating submanifold of L' of codimension one and that each B_i is diffeomorphic, via p, to the manifold obtained by cutting L' along the juncture.

The juncture N is Poincaré dual to a class $\varphi_N \in H^1_c(L'; \mathbf{Z})$. If σ is a loop on L' based at y, the holonomy h_σ is defined on $[y, y_k]$ with image in $[y, y_0]$, k sufficiently large, and $h_\sigma(y_j) = y_{j+\varphi_N(\sigma)}$, for each $j \geq k$. In particular, we can fix a loop τ_0 on L' that crosses N just once and such that τ_0 lifts to a path on B starting at y_0. Hence, $h_{\tau_0} : [y, y_0] \to [y, y_0]$ is defined and $h_{\tau_0}(y_j) = y_{j+1}$, for each $j \geq 0$.

Compactness of the juncture will be essential. We remark that there is a C^1 foliation of T^3 having a totally proper leaf diffeomorphic to $\mathbf{R}^2$ that "spirals" on a leaf diffeomorphic to $S^1 \times \mathbf{R}$, but with noncompact juncture $N = \{z\} \times \mathbf{R} \subset S^1 \times \mathbf{R}$ [C.C1, pp. 248-249]. In this case, the Poincaré dual $\varphi_N \in H^1(S^1 \times \mathbf{R}; \mathbf{Z})$ is not compactly supported. Such behavior is excluded by our definitions, hence the following result requires the C^2 hypothesis.

PROPOSITION 3.2 [C.C2, (6.3)]. — *Let L be a totally proper leaf at level $k \geq 1$, $X \subset L$ a compact subset. Then there exists a decomposition $L = A \cup B^1 \cup \ldots \cup B^r$, where A is a compact, connected $(n-1)$-manifold, $X \subset \mathrm{int}(A)$, each B^j is a complete, noncompact, connected $(n-1)$-manifold with compact, connected boundary and*

(a) *$A \cap B^j = \partial B^j$ is a component of $\partial A, l \leq j \leq r$;*

(b) *there is a spiral* $p^j : B^j \to L^j$, *where* L^j *is a leaf in* $\overline{L}\setminus L, 1 \leq j \leq r$;

(c) *for at least one value of* j, $\text{level}(L^j) = k - 1$.

COROLLARY 3.3 [C.C2,§6]. — *Let L be a totally proper leaf at level k. Then L has exactly polynomial growth of degree k.*

Our only interest in 3.3 will be in §5, where we use (3.3) to show that certain proper foliations are not C^2-smoothable. On the other hand, (3.2) itself is fundamental to the proof of the Main Theorem. The following is proven in [C.C2,§6] in the course of proving (3.2).

PROPOSITION 3.4. — *If L and L' are proper leaves of $\mathcal{F}$ and L approaches L' from (say) the positive side, then there exists a spiral $p : B \to L'$ on the positive side of L'.*

Let $p : B \to L'$ be a spiral on (say) the positive side of L'. If $x \in B_0\setminus N_1$, then $B \cap [p(x), x] = \{x_i\}_{i=0}^{\infty}$, each $x_i \in B_i\setminus N_{i+1}$ and $x_0 = x$. Let

$$\pi : B_i\setminus N_{i+1} \to B_{i+1}\setminus N_{i+2}$$

be the diffeomorphism defined by $\pi(x_i) = x_{i+1}$. Then $\pi : B \to B$ is a C^∞ imbedding with $\pi(B) = \bigcup_{i\geq 1} B_i$. Here, as always, we are using the C^∞ structures on the leaves of $\mathcal{F}$ that are induced by $\mathcal{F}^\perp$.

We obtain a "collar" C of L' in M by setting $C = \bigcup_{x\in B_0\setminus N_1} [p(x), x]$. This is a manifold with corners, the connected components of the corners being N_0 and N_1. Projection $p : C \to L'$ is well defined and can be viewed as a "fibering" of C over L'. The fiber $p^{-1}(y)$ is the leaf of $\mathcal{F}^\perp|C$ that meets y. The induced foliation $\mathcal{F}|C$ decomposes ∂C into $\partial_\tau C = B_0 \cup L'$ and $\partial_{\text{tr}} C = \bigcup_{x\in N_0} [\pi(x), x]$. The corner N_0 is convex and the corner N_1 is concave.

DEFINITION. — *The foliated manifold $(C, \mathcal{F}|C)$ with corners, together with $p : C \to L'$, is called a spiral collar of the leaf L'.*

Loops σ on L', based at y, define holonomy h_σ on subarcs $[y, z] \subseteq p^{-1}(y)$. Generally, the maximal such subarc depends on the loop σ, so the situation is less comfortable than in the case of an honest foliated bundle. With care, however, one can mimic much of the theory of foliated products.

DEFINITION. — *Let L be a connected, oriented manifold and let $N \subset L$ be a (possibly empty) compact, connected, oriented, nonseparating* *submanifold of codimension one. Let $y \in L\backslash N$. A collection G of oriented loops $\{\tau_i\}_{i\geq 0}$ on L, based at y, is a generating system for $\pi_1(L,y)$ that respects N if*

(1) *G is a system of generators for $\pi_1(L,y)$;*

(2) *if $N \neq \emptyset$, τ_0 meets N exactly once and has homological intersection number $\tau_0 * N = 1$;*

(3) *τ_i lies in $L\backslash N$, for each $i \geq 1$.*

PROPOSITION 3.5. — *Let $(C, \mathcal{F}|C)$ be a spiral collar of the leaf L' and let $N \subset L'$ be the juncture of the spiral $p : B \to L'$. Let $y \in L'\backslash N$ and let $G = \{\tau_i\}_{i\geq 0}$ be a generating system for $\pi_1(L',y)$ that respects N. Then $h_{\tau_i} : p^{-1}(y) \to p^{-1}(y)$ is defined, for each $i \geq 0$, h_{τ_0} is a C^2 contraction of $p^{-1}(y)$ to y, and $h_{\tau_i} \in \mathrm{Diff}^2_+(p^{-1}(y))$, for each $i \geq 1$. The spiral collar is completely determined (up to a C^2 diffeomorphism that preserves all relevant structure) by $\{h_{\tau_i}\}_{i\geq 0}$. The foliated manifold $(C, \mathcal{F}|C)$ is C^r-smoothable, $2 < r \leq \infty$, if and only if there exists $h \in \mathrm{Homeo}_+(p^{-1}(y))$ and a C^r contraction f of $p^{-1}(y)$ to y such that, for each $i \geq 1$, $hh_{\tau_i}h^{-1} \in \mathrm{Diff}^r_+(p^{-1}(y))$ and $hh_{\tau_0} = fh$.*

The proof, which is completely analogous to that of the corresponding theorem about foliated bundles, is left to the reader.

DEFINITION. — *Let $p : B \to L'$ be a spiral with juncture N, $y \in L'\backslash N$, $p^{-1}(y) = [y, y_0]$, and let $G = \{\tau_i\}_{i\geq 0}$ be a generating system for $\pi_1(L',y)$ that respects N. The p-lift of G is the generating system $G(B)$ for $\pi_1(B,y_0)$ consisting of the p-lifts $\tau_{i,k}$ of the loops $\tau_0^{-k}\tau_i\tau_0^k$ to loops on B at y_0, $i \geq 1$, $k \geq 0$.*

Let $W \in O(\mathcal{F}|C)$ be the $\mathcal{F}|C$-saturation of the open arc $]y_1, y_0[=]\pi(y_0), y_0[$. Then $\widehat{W} \cong B \times [y_1, y_0]$ and $\widehat{i} : \widehat{W} \to C$ carries $\widehat{W}$ onto $C\backslash L'$. If $x \in B$, then, relative to the identification $\widehat{W} = B \times [y_1, y_0]$, $\widehat{i}(x, y_0) = x$ and $\widehat{i}(x, y_1) = \pi(x)$. If σ is a loop on B based at y_0, view σ as a loop on $B \times \{y_0\}$ based at (y_0, y_0), and let $\overline{h}_\sigma \in \mathrm{Diff}^2_+[y_1, y_0]$ denote the corresponding total holonomy for $\widehat{W}$.

LEMMA 3.6. — *If there exists $\overline{h} \in \mathrm{Homeo}_+[y_1, y_0]$ such that $\overline{h}\overline{h}_\sigma\overline{h}^{-1}$ is an element of $\mathrm{Diff}^\infty_+[y_1, y_0]$ and is C^∞-flat at y_1 and y_0, for each $\sigma \in G(B)$, then $\overline{h}$ defines a C^∞-smoothing of $(C\backslash L', \mathcal{F}|(C\backslash L'))$.*

Proof. — By (2.1), $\overline{h}$ defines a C^∞-smoothing of $(\widehat{W}, \widehat{\mathcal{F}}|\widehat{W})$ so that the smoothed foliation is C^∞-flat at $\partial_\tau\widehat{W}$. By the above remarks, $C\backslash L'$

is obtained by gluing $B \times \{y_1\}$ to $\pi(B) \times \{y_0\}$ via the diffeomorphism that takes (x, y_1) to $(\pi(x), y_0)$, for each $x \in B$. The C^∞ structure on the resulting manifold is provided by [W,(2.2)]. □

PROPOSITION 3.7. — *In order that $(C, \mathcal{F}|C)$ be C^∞-smoothable so that the smoothed foliation is C^∞-flat at $\partial_\tau(C)$, it is sufficient that, for suitably chosen functions $\varepsilon : G(B) \to]0,\infty[$ and $\nu : G(B) \to \mathbb{Z}^+$, there exists a smoothing homeomorphism $\overline{h} \in \mathrm{Homeo}_+[y_1, y_0]$, as in (3.6), with the property that $\overline{h}\overline{h}_\sigma\overline{h}^{-1}$ and its first $\nu(\sigma)$ derivatives are $\varepsilon(\sigma)$-close to the identity and the corresponding derivatives of the identity, uniformly on $[y_1, y_0]$, for each $\sigma \in G(B)$. This extends the smoothing induced on $(C\backslash L', \mathcal{F}|(C\backslash L'))$ by $\overline{h}$, as in (3.6).*

Proof. — Let f be an orientation preserving C^∞ diffeomorphism of $[y, y_0]$ onto $[y, y_1]$, C^∞-flat at y and such that $f(z) < z$ for $y < z \leq y_0$. For arbitrary ε, ν, $\overline{h}$ as in the hypothesis, define $h \in \mathrm{Homeo}_+[y, y_0]$ by $h|[y_{k+1}, y_k] = f^k\overline{h}\overline{h}_{\tau_0}^{-k}$, for each $k \geq 0$, and $h(y) = y$. Then $hh_{\tau_0} = fh$ on $[y, y_0]$. If we set $\overline{y}_k = f^k(y_0)$, then $h[y_{k+1}, y_k] = [\overline{y}_{k+1}, \overline{y}_k]$.

Let $i \geq 1$ and let $h_i = hh_{\tau_i}h^{-1} \in \mathrm{Homeo}_+[y, y_0]$. Then the element $h_i|[\overline{y}_{k+1}, \overline{y}_k] = f^k(\overline{h}\overline{h}_{\tau_{i,k}}\overline{h}^{-1})f^{-k} \in \mathrm{Diff}^\infty_+[\overline{y}_{k+1}, \overline{y}_k]$ is C^∞-flat at $\overline{y}_{k+1}$ and $\overline{y}_k$. Thus, h_i is C^∞ on $]y, y_0]$. So far, this amounts to a smoothing of the foliated manifold $(C\backslash L', \mathcal{F}|(C\backslash L'))$, diffeomorphic to the one defined as in (3.6) by this same $\overline{h}$.

We can now specify how the functions ε, ν are to be "suitably chosen". The choices should be such that $h_i|[\overline{y}_{k+1}, \overline{y}_k]$ is $(1/k)$-close to the identity in the C^k-topology on $\mathrm{Diff}^\infty_+[\overline{y}_{k+1}, \overline{y}_k]$, for each $k \geq 0$, for each $i \geq 1$. Then $h_i \in \mathrm{Diff}^\infty_+[y, y_0]$ and this deformation is C^∞-flat at y and y_0, for each $i \geq 1$. Now apply (3.5). □

The vagueness about "suitably chosen" in the statement of (3.7) will cause no logical problems. The proof of the Main Theorem, by induction on Epstein-Millett hierarchies, will have an inductive hypothesis that $\overline{h}$ can be chosen as in (3.7) for each *arbitrary* choice of ε, ν. A concept needed for the appropriate formulation of this inductive hypothesis is the following.

DEFINITION. — *Let L be a totally proper leaf at level k and let $N \subset L$ be a (possibly empty) compact, connected, oriented, nonseparating submanifold of codimension one. Let $x_0 \in L\backslash N$ and let $L = A \cup B^1 \cup \ldots \cup B^r$ be as in (3.2), with $N \cup \{x_0\} \subset \mathrm{int}(A)$. An admissible generating system $G(L)$ for $\pi_1(L, x_0)$, respecting N and the decomposition of L, is defined by* *induction on k as follows :*

(1) *If $k = 0$, then $L = A$ and $G(L)$ is any finite generating system respecting the submanifold N.*

(2) *If $k - 1 \geq 0$ and level $(L) = k$, let N_j be the juncture of $p^j : B^j \to L^j$, as in (3.2), $1 \leq j \leq r$. Since level$(L^j) \geq k - 1$, we choose an admissible generating system $G(L^j)$ for $\pi_1(L^j, x^j)$ that respects both N^j and any appropriate decomposition of L^j. Let $G(B^j)$ be the p^j-lift of $G(L^j)$, based at $x_0^j \in B_0^j$, and let σ_j be a path on $L \backslash N$ from x_0 to x_0^j, $1 \leq j \leq r$. Let $G(A)$ be a finite generating system for $\pi_1(A, x_0)$ that respects N. Then set*

$$G(L) = G(A) \cup \sigma_1^{-1} G(B^1) \sigma_1 \cup \ldots \cup \sigma_r^{-1} G(B^r) \sigma_r.$$

LEMMA 3.8. — *Let $G(L)$ be an admissible generating system for $\pi_1(L, x_0)$ respecting the decomposition $L = A \cup B^1 \cup \ldots \cup B^r$. Let $X \subset L$ be compact. Then $G(L)$ also respects a decomposition $L = A_* \cup B_*^1 \cup \ldots \cup B_*^q$, as in (3.2), such that $X \subset \text{int}(A_*)$.*

The proof of (3.8) is a straightforward induction on level(L). The inductive hypothesis is applied to the compact sets $p^j(X \cap B^j) \subset L^j, 1 \leq j \leq r$.

If $(C, \mathcal{F}|C)$ is a spiral collar of L', and $p : B \to L'$ is the spiral, then the general leaf of $\mathcal{F}|C$ need not itself be a spiral on L'. Those that are spirals must be parallel to B in a fairly obvious sense. These parallel spirals exactly correspond to the fixed points in $[y_1, y_0]$ of the total holonomy group of the arm $(\widehat{W}, \mathcal{F}|\widehat{W})$.

LEMMA 3.9. — *Let $(M, \mathcal{F})$ be proper. Let $U \in O(\mathcal{F})$ be connected and irreducible, L' a component of $\partial \widehat{U}$. Then L' has a spiral collar $C \subset \widehat{U}$ with associated spiral $p : B \to L'$. If L is a leaf of $\mathcal{F}|U$ that is closed in U, then $L \cap C$ is a finite union of spirals parallel to B.*

Proof. — Existence of the spiral collar is by (3.4). If $L \cap C \neq \emptyset$, then L meets $[y_1, y_0]$ in a finite set of points. This is because L is closed in U. It follows that each component of $L \cap C$ meets $[y_1, y_0]$ in points fixed by the total holonomy group of $(\widehat{W}, \widehat{\mathcal{F}}|\widehat{W})$. Such points correspond one-one to spirals parallel to B, so each component of $L \cap C$ is such a spiral. □

DEFINITION. — *A Dippolito decomposition $\widehat{U} = K \cup V_1 \cup \ldots \cup V_m$ is admissible if, for each $j \in \{1, \ldots, m\}$, $\widehat{i}(\partial_\tau(V_j))$ consists of one or two parallel spirals. If $p^j : B^j \to L^j$ is one of these spirals with juncture N^j,* *then $G(V_j)$ denotes the p^j-lift $G(B^j)$ of a generating system $G(L^j)$ for $\pi_1(L^j)$ that respects N^j.*

Applying (3.2) to the components of δU, one easily proves the following.

LEMMA 3.10. — *If the leaves of δU are totally proper, then $\widehat{U}$ has an admissible Dippolito decomposition.*

LEMMA 3.11. — *Let $(M, \mathcal{F})$ be proper and let $U \in O(\mathcal{F})$ be connected and irreducible. Let $\widehat{U} = K \cup V_1 \cup \ldots \cup V_m$ be an admissible Dippolito decomposition. Let X denote the union of the leaves of $\mathcal{F}|U$ that are closed in U and let Y be a component of $U \backslash X$. Then there is an admissible Dippolito decomposition $\widehat{Y} = Q \cup Y_1 \cup \ldots \cup Y_q$ such that each arm Y_j is of one of the following two types :*

(1) for some $i(j) \in \{1, \ldots, m\}$, $V_{i(j)} \cong B^{i(j)} \times [a, b]$ and Y_j is of the form $B^{i(j)} \times [\bar{a}, \bar{b}] \subseteq V_{i(j)}$;

(2) for some component L' of $\partial\widehat{U}$, there is a spiral collar C of L' with $p : B \to L'$ the corresponding spiral, $Y_j \subset C$, and $\partial_\tau(Y_j)$ consists of spirals parallel to B.

Proof. — Clearly $\delta Y \subseteq X \cup \delta U$. We view components of $\partial\widehat{Y}$ as leaves in δY. If such a component clusters on a component L' of $\partial\widehat{U}$, then (3.9) implies the existence of finitely many arms of $\widehat{Y}$ satisfying (2). If V_i is an arm of $\widehat{U}$ and $V_i \cap L' \neq \emptyset$, L' a component of $\partial\widehat{U}$, then each component of $\delta Y \cap V_i$ must either be part of a spiral onto L', as above, or it must border an arm Y_j of $\widehat{Y}$ as in (1). Evidently, $\widehat{Y}$ contains only finitely many arms of these two types and, since $\delta Y \subseteq X \cup \delta U$, these arms exhaust all but a bounded portion Q_0 of $\widehat{Y}$. The closure Q of Q_0 is the desired nucleus. □

When L' is the boundary leaf of a foliated product, all of the above simplifies considerably.

DEFINITION. — *Let L and L' be totally proper leaves and suppose that L projects to L'. If $L = \bigcup_{i \in \mathbb{Z}} B_i$ in such a way that $p| \bigcup_{i \geq n} B_i$ is a spiral onto L', for each $n \in \mathbb{Z}$, we say that $p : L \to L'$ is a doubly infinite spiral.*

In particular, a doubly infinite spiral $p : L \to L'$ is an infinite cyclic covering.

LEMMA 3.12. — *Let $U \in O(\mathcal{F})$ be a foliated product and let L' be a component of $\partial\widehat{U}$. If U is irreducible and if L is a leaf of $\mathcal{F}|U$ that is closed in U, then projection along the leaves of $\widehat{\mathcal{F}}^{\perp}$ defines a doubly infinite spiral $p : L \to L'$ and all leaves of $\mathcal{F}|U$ that are closed in U are parallel to this doubly infinite spiral.*

This is an easy consequence of (3.9). In fact, one can see that every doubly infinite spiral arises in this way.

DEFINITION. — *Let U be an irreducible foliated product, L' a component of $\partial\widehat{U}$, and $p : \widehat{U} \to L'$ projection along the leaves of $\widehat{\mathcal{F}}^{\perp}$. Let L be a leaf of $\mathcal{F}|U$ that is closed in U. Let $N \subset L'$ be the juncture of the doubly infinite spiral $p : L \to L'$, let $y \in L'\backslash N$, and let $y_0 \in p^{-1}(y) \cap L$. If $G = \{\tau_i\}_{i\geq 0}$ is a generating system for $\pi_1(L', y)$ that respects N, then the p-lift of G to y_0 is the set of lifts $\tau_{i,k}$ of $\tau_0^{-k}\tau_i\tau_0^k$ to loops on L at y_0, for each $i \geq 1$, for each $k \in \mathbb{Z}$.*

LEMMA 3.13. — *Let U, $p : L \to L'$, and N be as above. If $G(L')$ is an admissible generating system for $\pi_1(L'y)$ that respects N, then the p-lift of $G(L')$ is an admissible generating system $G(L)$ for $\pi_1(L, y_0)$.*

Proof. — Index $L = \bigcup_{i\in\mathbb{Z}} B_i$ so that $y_0 \in \operatorname{int}(B_0)$. Let $G(L')$ respect a decomposition $L' = A' \cup B^1 \cup \ldots \cup B^q$ with $N \subset \operatorname{int}(A')$. Denote by A the manifold obtained by cutting A' along N and note that p determines a natural identification $B_0 = A \cup B^1 \cup \ldots \cup B^q$. Let $B^{q+1} = \bigcup_{i\geq 1} B_i$ and $B^{q+2} = \bigcup_{i\leq -1} B_i$. Then $L = A \cup B^1 \cup \ldots \cup B^{q+1} \cup B^{q+2}$ is an in (3.2) and, clearly, $G(L)$ is admissible, respecting this decomposition. □

4. Construction of the smoothing homeomorphism.

Fix the hypotheses of the Main Theorem.

A. Smoothing foliated products.

Let $\widehat{U}$ be a foliated product and let L' be a component of $\partial\widehat{U}$. Let $G(L')$ be an admissible generating system for $\pi_1(L', x_0)$ and let $[x_0, \overline{x}_0]$ be the leaf of $\widehat{\mathcal{F}}^{\perp}$ issuing from the basepoint x_0. We view $[x_0, \overline{x}_0]$ as a subinterval of $\mathbf{R}$ via a parametrization of $\mathcal{F}^{\perp}$ as a local C^∞ flow on M.

For each $\tau \in G(L')$, the total holonomy $h_\tau \in \operatorname{Diff}^2_+[x_0, \overline{x}_0]$ is defined.

Finally, let $\varepsilon : G(L') \to]0,\infty[$ and $\nu : G(L') \to \mathbf{Z}^+$ be arbitrary.

PROPOSITION 4.1. — *For each choice of the functions* ε, ν, *there exists* $h \in \mathrm{Homeo}^+[x_0,\overline{x}_0]$ *such that, for each* $\tau \in G(L')$,

(1) $\widetilde{h}_\tau = hh_\tau h^{-1} \in \mathrm{Diff}^\infty_+[x_0,\overline{x}_0]$ *and is* C^∞*-flat at* x_0 *and* $\overline{x}_0$;

(2) $\widetilde{h}_\tau$ *and its first* $\nu(\tau)$ *derivatives are* $\varepsilon(\tau)$*-close to the identity and its first* $\nu(\tau)$ *derivatives respectively, uniformly on* $[x_0,\overline{x}_0]$.

Proof. — (1) If the assertions have been proven for the case in which U is irreducible, they follow in general. Indeed, let $\emptyset \neq Z = \mathrm{Fix}(L') \subseteq]x_0,\overline{x}_0[$. The $\mathcal{F}$-saturation of Z is the union X of all leaves of $\mathcal{F}|U$ that are closed in U. If $X = U$, then $\widetilde{\mathcal{F}}$ is the product foliation and all assertions hold trivially. Assume that $X \neq U$ and let $\{W_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of components of $U\backslash X$. By second countability, we can take $\mathcal{A} \subseteq \mathbf{Z}^+$. Each W_α is a foliated product and is irreducible, with total holonomy group defined on $[a_\alpha,b_\alpha] = [x_0,\overline{x}_0] \cap \widehat{W}_\alpha$. Via the projection $p : \widehat{U} \to L'$ along the leaves of $\widehat{\mathcal{F}}^\perp$, the admissible generating system $G(L')$ lifts one-one to admissible generating systems on the components of $\partial\widehat{W}_\alpha$, for each $\alpha \in \mathcal{A}$. Identify these with $G(L')$. Choose $\varepsilon_\alpha : G(L') \to]0,\infty[$ so that $\varepsilon_\alpha(\tau) < \varepsilon(\tau)$ and, if $\mathcal{A}$ is infinite, $\lim_{\alpha\to\infty}\varepsilon_\alpha(\tau) = 0$, for each $\tau \in G(L')$. Similarly, choose $\nu_\alpha : G(L') \to \mathbf{Z}^+$ so that $\nu_\alpha(\tau) \geq \nu(\tau)$ and $\lim_{\alpha\to\infty}\nu_\alpha(\tau) = \infty$ (if $\mathcal{A}$ is infinite). If $\alpha \in \mathcal{A}$, apply the assertions to $\widehat{W}_\alpha$, using the functions ε_α and ν_α, to produce the smoothing homeomorphism $h_\alpha \in \mathrm{Homeo}_+[a_\alpha,b_\alpha]$. Define $h|(Z \cup \{x_0,\overline{x}_0\})$ to be the identity and $h|[a_\alpha,b_\alpha]$ to be h_α, for each $\alpha \in \mathcal{A}$.

(2) If U is irreducible, let X denote the union of the leaves of $\mathcal{F}|U$ that are closed in U. By (2.6), X is closed in U and is nonempty. The leaves $L \subseteq X$ are mutually parallel, doubly infinite spirals $p : L \to L'$ (3.12). Let $N \subset L'$ be the common juncture of these spirals. Since only the homology class of N really matters, we assume, without loss of generality, that $x_0 \in L'\backslash N$.

By (3.8), $G(L')$ respects a decomposition $L' = A \cup B^1 \cup \ldots \cup B^r$ such that $N \subset \mathrm{int}(A)$. If $G(L') = G(A) \cup \sigma_1^{-1}G(B^1)\sigma_1 \cup \ldots \cup \sigma_r^{-1}G(B^r)\sigma_r$, then, by a different choice of the finite generating set $G(A)$ and of $\sigma_1,\ldots,\sigma_r$, we obtain an admissible generating system $\{\tau_i\}_{i\geq 0}$ that respects N. The assertions to be proven hold relative to the one system and arbitrary choices of ε and ν if and only if they hold relative to the other system and arbitrary choices of ε and ν. Thus, without loss of generality, assume that $G(L') = \{\tau_i\}_{i\geq 0}$ respects N. In particular, $h_{\tau_0} \in \mathrm{Diff}^2_+[x_0,\overline{x}_0]$ is a contraction of $[x_0, \overline{x}_0[$ to x_0. By (3.13), the p-lifts of $G(L')$ to leaves $L \subseteq X$ are admissible generating systems $G(L)$.

We will prove the assertions by induction on $\gamma(U)$.

(3) Let U be irreducible and $\gamma(U) = 1$. Then $h_{\tau_i} = \mathrm{id}$, for each $i \geq l$. But h_{τ_0} is topologically conjugate to every $f \in \mathrm{Homeo}_+[x_0, \overline{x}_0]$ that is a contraction of $[x_0, \overline{x}_0[$ to x_0, so all assertions follow.

(4) Let U be irreducible and $\gamma(U) = \alpha > 1$. Assume, inductively, that the proposition holds whenever $\gamma(U) < \alpha$.

Let $L \subseteq X$ be a leaf. Let $W = U \backslash L \in O(\mathcal{F})$ and remark that W is a foliated product, irreducible if and only if $L = X$. If W is irreducible, then $\gamma(W) < \gamma(U)$ by (2.7). Alternatively, let V be any component of $W \backslash X$ and use (2.7) to get $\gamma(V) < \gamma(W) \leq \gamma(U)$.

If W is irreducible, we apply the inductive hypothesis so as to suitably smooth $(\widehat{W}, \widehat{\mathcal{F}}|\widehat{W})$.

If W is not irreducible, we apply the hypothesis to each component V of $W \backslash X$ and argue as in step (1), again suitably smoothing $(\widehat{W}, \widehat{\mathcal{F}}|\widehat{W})$.

Since $\widehat{\mathcal{F}}|\widehat{W}$ is now C^∞-flat at $\partial\widehat{W}$, we obtain a C^∞ smoothing of $(U, \mathcal{F}|U)$ by the argument in (3.6). To complete the inductive step, we must extend the smoothing to $(\widehat{U}, \widehat{\mathcal{F}})$. But the choice of ε and ν has been arbitrary, so we choose ε smaller, if necessary, and ν larger, if necessary, so as to apply (3.7) to spiral neighborhoods C of each component L' of $\partial\widehat{U}$. Since the smoothing induced on $C \backslash L' = C \cap U$ by the smoothing of C is diffeomorphic to the one induced from U, these smoothed collars can be attached to U via C^∞ diffeomorphisms, achieving the desired smoothing of $(\widehat{U}, \widehat{\mathcal{F}})$. □

B. Smoothing the general $\widehat{U}$.

Let $U \in O(\mathcal{F})$ be connected and choose an admissible Dippolito decomposition $\widehat{U} = K \cup V_1 \cup \ldots \cup V_m$ (3.10). Let $G(\widehat{U}) = \bigcup_{j=1}^{m} G(V_j)$ and let $\varepsilon : G(\widehat{U}) \to]0, \infty[$, $\nu : G(\widehat{U}) \to \mathbf{Z}^+$ be arbitrary. Let I_j denote the leaf of $\widehat{\mathcal{F}}^\perp$, a fiber of V_j, on which the total holonomy h_σ is defined, for each $\sigma \in G(V_j)$, $1 \leq j \leq m$. As usual, we have an identification of I_j as a compact subinterval of $\mathbf{R}$.

PROPOSITION 4.2. — *The C^2-foliated manifold $(\widehat{U}, \widehat{\mathcal{F}})$ is homeomorphic to a C^∞-foliated manifold $(\widetilde{U}, \widetilde{\mathcal{F}})$ such that*

(1) $\widetilde{\mathcal{F}}$ *is* C^∞*-flat at* $\partial\widetilde{U}$*;*

(2) *the smoothing homeomorphism* $h : \widehat{U} \to \widetilde{U}$ *carries* I_j *onto itself,* $1 \le j \le r$*, so let* $h_j = h|I_j \in \mathrm{Homeo}_+(I_j)$*;*

(3) *the total holonomy* $\widetilde{h}_\sigma = h_j h_\sigma h_j^{-1} (\in \mathrm{Diff}_+^\infty(I_j))$ *and its first* $\nu(\sigma)$ *derivatives are* $\varepsilon(\sigma)$*-close to the identity and its first* $\nu(\sigma)$ *derivatives respectively, uniformly on* I_j*, for each* $\sigma \in G(V_j)$*,* $1 \le j \le m$*.*

Proof. — (1) By (4.1) and (2.4), we can assume that U is irreducible. In particular, each component of $\partial\widehat{U}$ has a spiral collar in $\widehat{U}$ (3.9).

(2) Suppose that $\gamma(U) = 1$. That is, $\mathcal{F}|U$ is without holonomy. If $U = M = \widehat{U}$ and $\partial M = \emptyset$, then $\mathcal{F}$ fibers M over S^1 (Reeb stability) and C^∞-smoothability is trivial. If $\partial\widehat{U} \ne \emptyset$, the parallel spirals onto any component of $\partial\widehat{U}$ can be used to construct, in standard fashion, a closed transversal Σ to $\mathcal{F}|U$ that meets each leaf at most once. The $\mathcal{F}$-saturation of Σ, being open and closed in U, is exactly U. The natural projection $\pi : U \to \Sigma$ is a locally trivial fibration, hence $(U, \mathcal{F}|U)$ is C^∞-smoothable. Since the holonomy of each component of $\partial\widehat{U}$ is generated by the single contraction producing the spirals, an easy application of (3.7) produces the desired C^∞-smoothing of $(\widehat{U}, \widehat{\mathcal{F}})$. Since $h_\sigma = \mathrm{id}_{I_j}$, for each $\sigma \in G(V_j)$, $1 \le j \le m$, properties (1), (2), and (3) are trivial.

(3) Inductively, let $\gamma(U) = \alpha$ and assume the assertions for all $W \in O(\mathcal{F})$, connected and irreducible, with $\gamma(W) < \alpha$. Let $X \subset U$ be the union of all leaves of $\mathcal{F}|U$ that are closed in U. Since X is closed in U, each component of $U \backslash X$ belongs to $O(\mathcal{F})$ and at most finitely many of these, $U_1, \ldots, U_q$, fail to be foliated products (2.2). It is clear that U_k is irreducible, $1 \le k \le q$.

Let $X' = U \cap \bigcup_{k=1}^{q} \delta(U_k)$, a closed subset of X. A component of $\delta(U_k)$ fails to lie in X' if and only if it is also a component of $\delta(U)$, $1 \le k \le q$. By an easy application of (2.4), we see that the components of $U \backslash X'$ consist of $U_1, \ldots, U_q$, together with foliated products $U_{q+1}, \ldots, U_r$. These latter are generally not irreducible.

By (2.7), $\gamma(U_k) < \alpha$, $1 \le k \le q$, so the inductive hypothesis applies. The assertions of the proposition also hold for $U_{q+1}, \ldots, U_r$ by (4.1).

If $L_1, \ldots, L_s$ are the components of $\partial\widehat{U}$ not approached by any leaf in X, then the manifolds $U \cup L_1 \cup \ldots \cup L_s = U_0 \subset \widehat{U}$ is obtained from the *disjoint* union $\bigcup_{k=1}^{r} \widehat{U}_k$ by pairwise identifications of some boundary components. Since $\widehat{\mathcal{F}}|\widehat{U}_k$ is C^∞-flat at the boundary, this gives a C^∞-smoothing of $(U_0, \widehat{\mathcal{F}}|U_0)$ by [W,(2,2)]. By the previous paragraph, the hypotheses of (3.7) are guaranteed at each component of $\partial\widehat{U}\backslash\partial U_0$, hence we obtain a C^∞-smoothing $(\widetilde{U}, \widetilde{\mathcal{F}})$ of $(\widehat{U}, \widehat{\mathcal{F}})$ that is C^∞-flat at the boundary. Also, by the previous paragraph and (3.11), it is easy to guarantee properties (2) and (3). □

C. Completion of the proof of the Main Theorem.

If $\partial M \neq \emptyset$, apply (4.2) to $U = \mathrm{int}(M)$, $\widehat{U} = M$. If $\partial M = \emptyset$, then the hypotheses that $(M, \mathcal{F})$ is proper and M is compact imply that there is a compact leaf L of $\mathcal{F}$. By cutting M along L, we produce a compact, C^2-foliated manifold, with one or two components, and we apply (4.2) to these components. The C^∞-smoothed foliation is C^∞-flat at the boundary, so we can reglue along the two copies of L to obtain a C^∞-smoothing of $(M, \mathcal{F})$. □

5. Examples.

We illustrate one or another aspect of the Main Theorem. The examples will be of the form $(\Sigma_2 \times S^1, \mathcal{F}(f,g))$, where Σ_2 is the 2-holed torus, $f, g \in \mathrm{Diff}^r_+(S^1)$, $0 \le r \le \infty$, and $\mathcal{F}(f,g)$ is obtained by suspension. With this understood, set $M = \Sigma_2 \times S^1$.

A. Nonsmoothable, proper foliations.

We show the necessity of assuming C^2-smoothness.

Realize S^1 as $[-1,1]/\{-1 \equiv 1\}$ and let

$$G = \{h \in \mathrm{Homeo}_+[-1,1] | h(-1) = -1, h(1) = 1\},$$

a subgroup of $\mathrm{Homeo}_+(S^1)$.

Let $f \in G$, $f(x) > x$, $-1 < x < 1$, and set $x_p = f^p(0)$, for each $p \in \mathbb{Z}$. Let $c : \mathbb{Z} \to \mathbb{Z}$ be any map such that $c(0) = 1$. Let $g_0 \in \mathrm{Homeo}_+[x_0, x_1]$, $g_0(x) > x$, $x_0 < x < x_1$, and define $g_c \in G$ by requiring that $g_c|[x_p, x_{p+1}] = f^p g_0^{c(p)} f^{-p}$, for each $p \in \mathbb{Z}$, and $g_c(\pm 1) = \pm 1$. Let $\mathcal{F}_c = \mathcal{F}(f, g_c)$.

Then, $(M, \mathcal{F}_c)$ is a proper, C^0-foliated manifold. There is one compact leaf, there is one proper leaf at level 1 (corresponding to $\{x_p\}_{p \in \mathbb{Z}}$), and the remaining leaves are at level 2, mutually homeomorphic, and with a common growth type γ_c.

PROPOSITION 5.1. — *Let $c(k) = 0$, for each $k < 0$, and $c(0) = 1$. If $c(k)$ is nondecreasing and $c(k+1)/c(k)$ is a bounded function of $k \geq 0$, then $\gamma_c = \mathrm{gr}(k^2 c(k))$, the growth type of $k^2 c(k)$.*

Proof. — (1) Let $x(0,0) \in]x_0, x_1[$ and let $f^p g^q(x(0,0)) = x(p,q)$, for each $p,q \in \mathbf{Z}$. Clearly, the G-orbit of $x(0,0)$ is $G(x(0,0)) = \{x(p,q)\}_{p,q\in\mathbf{Z}}$.

(2) Let $\gamma(k)$ denote the number of distinct points of the form $x(p,q) = w(x(0,0))$, where w is a reduced word in f and g of length at most k. For such points, it is elementary that $-k \leq p \leq k$, $-kc(k) \leq q \leq kc(k)$, these being very generous estimates. Thus, $\gamma(k) < (2k+1)(1+2kc(k))$, so $\gamma_c < \mathrm{gr}(k^2 c(k))$.

(3) Choose $N \in \mathbf{Z}^+$ such that $c(k+1)/c(k) \leq N$, for each $k \geq 0$. Let $P_k = \sum_{j=0}^{k} c(j)$. By an elementary induction on k, every integer from 0 to NP_k can be written as $\sum_{j=0}^{k} \varepsilon(j)c(j)$, $0 \leq \varepsilon(j) \leq N$.

(4) Consider the reduced words $w = f^m g^{\varepsilon(2k)} f g^{\varepsilon(2k-1)} \ldots f g^{\varepsilon(0)}$, $\varepsilon(j) \in \{0,1,\ldots,N\}$, $0 \leq j \leq 2k$, $1 \leq m \leq k+1$. These words are of length at most $(2N+3)k$ and $w(x(0,0)) = x(m+2k, \sum_{j=0}^{2K} \varepsilon(j)c(j))$. By step (3), it follows that

$$\begin{aligned}\gamma((2N+3)k) &\geq (k+1)NP_{2k} \\ &\geq kN(c(k+1) + \ldots + c(2K)) \\ &\geq kN(kc(k)) \geq k^2 c(k).\end{aligned}$$

That is, $\gamma_c \geq \mathrm{gr}(k^2 c(k))$. □

In this way, uncountably many distinct growth types γ_c can be obtained. Since γ_c is a topological invariant of $(M, \mathcal{F}_c)$, (5.1) and (3.3) imply the following.

COROLLARY 5.2. — *There are uncountably many topologically distinct foliations of $\Sigma_2 \times S^1$ (of codimension one) that are proper but not C^2-smoothable.*

Uncountably many of the growth types γ_c are *subexponential*, in the sense that

$$\lim_{k\to\infty} (1/k) \log(k^2 c(k)) = 0$$

(this is also called *quasi-polynomial* growth in much of the literature). For instance, take $c(k)$ to be the greatest integer in $k^{\sqrt{k}}, k \geq 1$. Uncountably many others fail to be subexponential, but are *nonexponential*, in the sense that

$$\lim_{k\to\infty} \inf (1/k)\log(k^2 c(k)) = 0.$$

Finally, by taking $c(k) = 2^k$, we produce exponential growth.

If $\mathcal{F}_c$ is interpreted as a foliation of $\Sigma_2 \times [-1,1]$, we can also prove the following.

PROPOSITION 5.3. — *If $c : \mathbb{Z} \to \mathbb{Z}$ is as in (5.1), then $(\Sigma_2 \times [-1,1], \mathcal{F}_c)$ is C^1-smoothable so as to be C^1-flat at the boundary if and only if γ_c is subexponential.*

Thus, if $(M, \mathcal{F}_c)$ is modified by thickening the compact leaf to a continuum of compact leaves, $(M, \mathcal{F}_c)$ will be C^1-smoothable for $c : \mathbb{Z} \to \mathbb{Z}$ as in (5.1), if and only if γ_c is subexponential.

COROLLARY 5.4. — *There are uncountably many topologically distinct foliations of $\Sigma_2 \times S_1$ that are proper but not C^1-smoothable.*

B. Obstructions to higher order smoothing.

There are examples of compact, C^n-foliated manifolds of codimension one that are not proper and not C^{n+1}-smoothable, $2 \leq n < \infty$. Our examples were written up in an earlier preprint (unpublished) but, in the meantime, T. Tsuboi has found easier ones [T]. Nonetheless, we will sketch ours here, without proof, since they have further potentially interesting features.

Let S^1 and G be as above. Let $G_n = G \cap \mathrm{Diff}^n_+(S^1), 1 \leq n \leq \infty$. Let $f \in G_\infty, f(x) < x, -1 < x < 1$. For technical reasons, we also require that f imbed in a C^∞ flow on S^1 that is C^∞-trivial at ± 1.

Fix $s \in]2,\infty[$ and write $s = 2 + (1/r)$. There is an integer $q \geq 1$ such that $2^{rq} > q$ and $[2^{r(k+1)}] > [2^{rk}]$, for each $k \geq q$. Here, $[a]$ denotes the greatest integer $\leq a$. Set $N_s(k) = [2^{rk}]$, for each $k > q$. For $0 \leq k \leq q$, set $N_s(k) = k - 1$. Let $c(k) = 2^{1-k}$, for each $k \geq 1$.

Set $x_p = f^p(0)$, for each $p \in \mathbb{Z}$. Let Φ_t be a C^∞ flow on $[-1,1]$, supported on $[x_1, x_0]$, with $\Phi_1(x) < x$, $x_1 < x < x_0$. For $x_{p+1} \leq x \leq x_p$ and for $N_s(k-1) < p \leq N_s(k)$, $k \geq 1$, set $g_s(x) = f^p(\Phi_{c(k)}(f^{-p}(x))$. For $0 \leq x \leq 1$ and $x = -1$, let $g_s(x) = x$. Thus, $g_s \in G$.

Set $\mathcal{F}_s = \mathcal{F}(f, g_s)$. Then $(M, \mathcal{F}_s)$ has a continuum of compact leaves, each homeomorphic to Σ_2, one proper leaf at level 1 (corresponding to $\{x_p\}_{p\in\mathbf{Z}}$, and the remaining leaves are locally dense without holonomy. These leaves are mutually quasi-isometric with common growth type denoted γ_s.

PROPOSITION 5.5. — *The growth type* γ_s *is* $\mathrm{gr}(k^s)$.

This is proven by the estimates in [C.C3,§7]. We can also prove the following.

THEOREM 5.6. — *Let* $n \geq 1$ *and* $s \in]n+1, n+2]$. *Then the above construction can be carried out so that* $(M, \mathcal{F}_s)$ *is of class* C^n, *but it cannot be carried out so that* $(M, \mathcal{F}_s)$ *is* C^{n+1}*-smoothable.*

Again, the growth type γ_s, $n + 1 < s \leq n + 2$, parametrizes uncountably many topologically distinct examples. It is not true, however, that "fractional growth", by itself, obstructs C^∞ smoothness [C.C4]. It seems reasonable to conjecture that, if $n+1 < s \leq n+2$, then the quasi-isometry type of the nonproper leaves of $\mathcal{F}_s$ cannot be realized in any compact, C^{n+1}-foliated 3-manifold.

C. Examples of Epstein-Millett hierarchies.

When M is compact, $\gamma(M, \mathcal{F})$ cannot be a limit ordinal (easy). This is the only restriction, as we now show.

PROPOSITION 5.7. — *Let* $\alpha \geq 0$ *be a countable ordinal. Then there exists a proper,* C^∞*-foliated manifold* $(M, \mathcal{F}_\alpha)$ *such that* $\gamma(M, \mathcal{F}_\alpha) = \alpha + 1$.

Proof. — (1) First we agree on a notational convention. If $(M, \mathcal{F}(f,g))$ has been constructed, the leaf corresponding to $x \in [-1, 1]$ will be denoted L_x. If $J \subset]-1, 1[$ is a closed interval, then $\bigcup_{x\in\mathrm{int}(J)} L_x \in O(\mathcal{F}(f,g))$ will be denoted U_J, so $\widehat{U}_J = \bigcup_{x\in J} L_x$.

(2) For $\alpha = 0$, take the product foliation. For $\alpha = 1$, let $f_1 \in G_\infty$, $f_1(x) < x$, $-1 < x < 1$. Let $g_1 = \mathrm{id}$ and take $\mathcal{F}_1 = \mathcal{F}(f_1, g_1)$. Then $M_1 = M \backslash (L_1 \cup L_{-1})$ and $\gamma(L_1) = \gamma(L_{-1}) = 2$. Thus, $\gamma(M, \mathcal{F}_1) = 2$.

(3) Inductively, assume for $1 \leq \beta < \alpha$ that there exists $f_\beta, g_\beta \in G_\infty$, as close to the identity as desired, uniformly on $[-1,1]$, in as (finitely) many derivatives as desired, such that $\mathcal{F}_\beta = \mathcal{F}(f_\beta, g_\beta)$ satisfies the condition that $\gamma(M, \mathcal{F}_\beta) = \beta + 1$. We also assume that $f_\beta(x) < x, -1 < x < 1$.

Choose $f_\alpha \in G_\infty$ such that $f_\alpha(x) < x$ for $-1 < x < 1$. Set $x_p = f_\alpha^p(0)$, for each $p \in \mathbb{Z}$. Let $\{J_r\}_{r\geq 0}$ be a family of disjoint, nondegenerate, closed subintervals of $]x_1, x_0[$. Then $I_r = f_\alpha(J_r)$ defines a similar family in $]x_2, x_1[$. Fix orientation preserving, C^∞ diffeomorphisms $h_r : J_r \to [-1,1]$, for each $r \geq 0$.

(4) Suppose that $\alpha = \beta + 1$. Let $g_\alpha|J_0 = h_0^{-1} f_\beta h_0$ and $g_\alpha|I_0 = f_\alpha h_0^{-1} g_\beta h_0 f_\alpha^{-1}|I_0$. Elsewhere, $g_\alpha = \mathrm{id}$. Since f_β and g_β are as C^∞-close to the identity as desired, so is g_α. Let $\widehat{U}_0 = \widehat{U}_{J_0} = \widehat{U}_{I_0}$ and observe that M_1, the union of the stable leaves, contains $M\backslash(\widehat{U}_0 \cup L_1 \cup L_{-1})$. By the inductive hypothesis, it is easy to check that $\gamma(\widehat{U}_0, \widehat{\mathcal{F}}_\alpha) = \beta + 1 = \alpha$. Since $\widehat{U}_0$ spirals on L_1 and L_{-1}, we conclude that $\gamma(L_1) = \gamma(L_{-1}) = \alpha + 1$ and $\gamma(M, \mathcal{F}_\alpha) = \alpha + 1$.

(5) Suppose that α is a limit ordinal and choose a sequence $\{\alpha_r\}_{r\geq 0}$ such that $\alpha_r \uparrow \alpha$ strictly, this being possible by the countability of α. Define $g_\alpha|J_r = h_r^{-1} f_{\alpha_r} h_r$ and $g_\alpha|I_r = f_\alpha h_r^{-1} g_{\alpha_r} h_r f_\alpha^{-1}|I_r$. Elsewhere, $g_\alpha = \mathrm{id}$. Since each f_{α_r} and each g_{α_r} can be taken as close to the identity as desired in as large a (finite) number of derivatives as desired, we can arrange that $g_\alpha \in G_\infty$ and that g_α be as C^∞-close to the identity as desired. Set $\widehat{U}_r = \widehat{U}_{J_r} = \widehat{U}_{I_r}, r \geq 0$. Then M_1 contains the subset $M\backslash(L_1 \cup L_{-1} \cup \widehat{U}_0 \cup \widehat{U}_1 \cup \ldots)$ and $\gamma(\widehat{U}_r, \widehat{\mathcal{F}}_\alpha) = \alpha_r + 1$. Consequently, $M\backslash(L_1 \cup L_2) \subseteq M_\alpha$ and $\alpha \leq \gamma(M, \mathcal{F}_\alpha) \leq \alpha + 1$. But $\gamma(M, \mathcal{F}_\alpha)$ cannot be a limit ordinal. □

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