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BOUNDED ANALYTIC SETS IN BANACH SPACES

by Volker AURICH

0. Introduction.

If $f: \mathrm{E} \to \mathrm{F}$ is a holomorphic Fredholm map between Banach spaces then it is well known that the fibers are finite dimensional analytic subsets. Assume that f satisfies some a priori estimates such that the fibers are bounded. In the case $\mathrm{E} = \mathbf{C}^n$ the maximum principe implies the finiteness of each fiber. If E is infinite dimensional the fibers need not be compact and therefore we cannot use the same reasoning. And indeed, there are Banach spaces which contain bounded closed complex submanifolds isomorphic to the open unit disk [2].

In the present paper we investigate the more general question :

If X is a non discrete reduced complex space and E is a complex Banach space when does there exist a holomorphic embedding $\Phi: \mathrm{X} \to \mathrm{E}$ with bounded image $\Phi(\mathrm{X})$?

This question can be considered under various aspects which involve on the one hand the linear geometry of E and on the other hand intrinsic properties of the complex structure of X which are stronger than Steinness. Our results are the following :

a) If E has the Radon-Nikodym property then there does not exist a bounded holomorphic embedding $\mathrm{X} \to \mathrm{E}$ of a reduced complex space X with positive dimension into E.

b) A reduced complex space X can be embedded holomorphically and boundedly into some Banach space if and only if its Carathéodory pseudometric c_{X} is a complete metric which induces the topology of X and if X has local coordinates by globally holomorphic bounded functions.

Key-words : Bounded SF-analytic subsets of Banach spaces - Radom-Nikodym property - Carathéodory metric - H-completeness.

c) Certain combined conditions on X and E guaranteee that there is a bounded holomorphic embedding $X \to E$. They are certainly too restrictive to be necessary, but using *a*) one can conclude that the dual of a closed subalgebra A of $H^\infty(X)$ (the bounded holomorphic functions on X) does not have the Radon-Nikodym property whenever X has positive dimension and a modified Carathéodory metric c_X^A is a complete metric inducing the topology of X.

These results hold partially also for certain infinite dimensional analytic spaces X instead of complex spaces X, namely the SF-analytic spaces (see section 5. for a definition and its motivation). In particular the results hold for complex Banach manifolds X.

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1. Preliminaries.

An *analytic* subset X of a complex Banach manifold Ω is a closed subset such that for every $x \in X$ there exists a neighborhood U_x in Ω and a holomorphic map $f_x : U_x \to F_x$ into a Banach space F_x with $X \cap U_x = f_x^{-1}(0)$. An analytic set X will be called *finite dimensional* if it lies locally in a finite dimensional complex submanifold. Observe that in case Ω is a Banach space E this does not mean that X lies locally in a finite dimensional linear subspace of E, for example the image of the map Φ considered at the beginning of the third section is a onedimensional complex submanifold of c_0 which meets every finite dimensional linear subspace in only finitely many points. An analytic set is finite dimensional if and only if it is locally a fiber of a holomorphic Fredholm map (this is a map whose differentials are Fredholm operators). Henceforth a finite dimensional analytic set will always be considered as a reduced complex space by endowing it with the reduced structure.

In the following four sections all complex spaces considered will be reduced and locally finite dimensional.

A map $f : X \to E$ from a complex space X into a complex Banach space E is called *holomorphic* iff it has locally a holomorphic extension to any manifold in which X can locally be embedded. According to [4] f is already holomorphic if $\mu \circ f$ is holomorphic for every μ in a $\sigma(E',E)$ – dense subspace of E'; in particular if E is a dual F' then it is enough to check that $\mu \circ f$ is holomorphic for every $\mu \in F$.

We call a map f *proper* if it is continuous and if the inverse image of every compact set is compact; a continuous map $f: X \to Y$ between metrizable spaces is proper iff for every sequence in X without cluster point the image sequence does not converge. A holomorphic map $f: X \to E$ is called an *immersion* if its differentials are injective and an *embedding* if $f(X)$ is a finite dimensional analytic subset and if f maps X biholomorphically onto the complex space $f(X)$. The implicite function theorem implies that f is an embedding if and only if it is an injective proper holomorphic immersion.

2. Conditions for E not containing bounded analytic sets of positive dimension.

The key to nonexistence criteria is the maximum principle. Applying it to the continuous linear functionals one concludes that every compact finite dimensional analytic subset of a Banach space is finite. In order to exclude the existence of non compact but bounded finite dimensional analytic subsets of positive dimension it would be enough to know that sufficiently many holomorphic functions attain a maximum on such subsets. A statement of this kind can be derived from a theorem proved by Huff and Morris [11] in a completely different context.

2.1. THEOREM (Huff-Morris). — *A real Banach space* E *has the Radon-Nikodym property* (RNP *for short*) *if and only if for every closed bounded nonempty subset* $A \subset E$ *the set of all continuous linear forms attaining a maximum on* A *is norm-dense in* E'.

The RNP has been studied intensively; in [6] a list of equivalent properties is compiled. Among the Banach spaces with RNP there are the reflexive spaces, the separable duals, $\ell_1(\Gamma)$ and the Hardy spaces $H^p(D)$ for $1 \leqslant p < \infty$, but not the following ones: $L_1(\mu)$ (μ not purely atomic), c_0, c, ℓ_∞, $L_\infty[0,1]$, $H^\infty(D)$ and $\mathscr{C}(K)$ (K infinite compact) [6, p. 218].

2.2. THEOREM. — *If a complex Banach space* E *has the* RNP *then there does not exist a non constant proper holomorphic* map $\Phi: X \to E$ *from an irreducible complex space* X *of positive dimension into* E such that $\Phi(X)$ *is bounded.*

Proof. — Suppose that we could find such a map Φ and assume $0 \in \Phi(X)$. Let G be the closure of the complex linear subspace spanned by $\Phi(X)$. Then G has also the RNP since the RNP is inherited by closed linear subspaces. According to 2.1. we can find a continuous $\mathbf{R}$-linear form $\mu : G \to \mathbf{R}$, $\mu \neq 0$, which attains its maximum on $\Phi(X)$. Let μ^* be the complexification. Then $\mu^* \circ \Phi$ is holomorphic on X and its real part attains its maximum. Hence $\mu^* \circ \Phi$ is constant and $\Phi(X)$ is contained in the proper subspace $\operatorname{Ker} \mu^*$ of G. This contradicts the definition of G. q.e.d.

2.3. COROLLARY. – *Every bounded closed finite dimensional analytic subset of a Banach space with* RNP *is discrete* (*but of course not always finite*).

The Banach spaces which do not contain bounded finite dimensional subsets of positive dimension are not yet characterized. It is unlikely that the RNP is characteristic. A strictly weaker condition, the analytic Radon-Nikodym property, guarantees already that there are no bounded finite dimensional subsets isomorphic to the open unit disk [2].

2.4. COROLLARY. – *Let* E *be a complex Banach space with* RNP, Y *a complex Banach manifold, and* $f : E \to Y$ *a holomorphic Fredholm map with bounded fibers. If* f *is surjective or if the index of* f *is* 0 *then* f *is open.*

Proof. – By 2.3. the fibers are discrete, and by lemma 3.4. in [1] $\operatorname{ind} f \leqslant 0$. If $\operatorname{ind} f < 0$ then $f(E)$ is the set of critical values of f and the Sard-Smale theorem [20] implies that $f(E)$ is meager, hence f is not surjective. Therefore $\operatorname{ind} f = 0$. The openness of f follows now from theorem 3.5. in [1]. q.e.d.

3. Conditions for X being embedded boundedly.

A simple example for a bounded embedding is the map

$$\Phi : D \to c_0, \qquad z \mapsto (z^n)_{n \in \mathbf{N}}$$

where D is the open unit disk in $\mathbf{C}$ and c_0 is the Banach space of all complex null sequences endowed with the supremum norm [2]. Whereas Φ depends on the special shape of the unit disk there is for every complex space X a canonical map χ which is a candidate for a bounded embedding. Denote by $H^\infty(X)$ the Banach space of all bounded holomorphic functions on X with the supremum norm and let $\chi : X \to H^\infty(X)'$

be the map which assigns to every $x \in X$ the evaluation homomorphism $\hat{x}$ defined by $\hat{x}(h) := h(x)$ for $h \in H^\infty(X)$. Then χ is holomorphic and $\chi(X)$ is contained in the unit sphere of $H^\infty(X)'$.

Remember the definition of the Carathéodory pseudometric

$$c_X(x,y) := \sup \{\rho(h(x),h(y)) : h \in H^\infty(X), \|h\| \leqslant 1\}, \qquad x, y \in X,$$

where ρ is the Poincaré metric in the unit disk D i.e. the integrated form of the hyperbolic differential metric $(1-|z|^2)^{-2}\, dz\, d\bar{z}$.

3.1. THEOREM. — *For every connected complex space* X *the following properties are equivalent :*

(i) *There exists a Banach space* E *and an injective proper holomorphic map* $\Phi : X \to E$ *with bounded image* $\Phi(X)$.

(ii) *The canonical map* $\chi : X \to H^\infty(X)'$ *is injective and proper.*

(iii) *The Carathéodory pseudometric* c_X *is a sequentially complete metric which induces the topology of* X.

3.2. *Supplement.* — The equivalences remain true if we replace in (i) and (ii) the term « injective proper holomorphic map » by « embedding » and if we add in (iii) that X is $H^\infty(X)$-regular i.e. for each $x \in X$ there exists a holomorphic map $f : X \to F$ into some Banach space F such that $f(X)$ is bounded and $f | U \to V$ is an embedding of a suitable neighborhood U of x into a domain V of F.

Before going into the proof let us recall some properties of c_X [8, 9, 13, 19]. The Poincaré metric ρ can be expressed as

$$\rho(z,w) = \tau\left(\left|\frac{w-z}{1-w\bar{z}}\right|\right)$$

with $\tau(t) := \dfrac{1}{2}\log\dfrac{1+t}{1-t} = \tanh^{-1}(t)$ for $t \in [0,1[$, in particular $\rho(0,w) = \tau(|w|) \geqslant |w|$, and $\tau : [0,1[\to [0,\infty[$ is strictly increasing. By the Schwarz lemma every holomorphic map $f : D \to D$ is a contraction with respect to ρ. Hence the automorphisms are isometries and $c_D = \rho$. Because Aut D acts transitively on D we get

$$(1) \quad c_X(x,y) = \sup \rho(0,h(y)) = \sup \tau(|h(y)|) = \tau(\sup |h(y)|)$$

where all the suprema are taken for $h \in H^\infty(X)$, $\|h\| \leqslant 1$, $h(x) = 0$. For every connected complex space X the Carathéodory pseudometric c_X is finite and continuous [12, 19]. Any holomorphic map between complex spaces is a contraction with respect to the Carathéodory pseudometrics.

Proof of 3.1. — (ii) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (iii): Since $\Phi(X)$ is bounded there is $r > 0$ such that $\|\Phi(x) - \Phi(y)\| \leqslant r$ for all $x, y \in X$. Equation (1) implies

$$(2)\quad c_X(x,y) \geqslant \sup\{|h(y)| : h \in H^\infty(X), \|h\| \leqslant 1, h(x)=0\}$$
$$\geqslant \sup\left\{|\mu(\Phi(y)-\Phi(x))| : \mu \in E', \|\mu\| \leqslant \frac{1}{r}\right\}$$
$$= \frac{1}{r}\|\Phi(y)-\Phi(x)\|.$$

Hence the injectivity of Φ implies that c_X is a metric. Because $\Phi|X \to \Phi(X)$ is topological (2) shows also that the topology induced by c_X is finer than the original one of X. On the other hand c_X is continuous, hence both topologies coincide.

In order to prove the sequential completeness of c_X let $(x_n)_n$ be a Cauchy sequence for c_X. The inequality (2) implies that $(\Phi(x_n))_n$ is a Cauchy sequence in the norm topology of E and therefore converges to a point $a \in E$. The set $K := \{a\} \cup \{\Phi(x_n) : n \in \mathbf{N}\}$ is compact, and since Φ is proper $\Phi^{-1}(K)$ is also compact. Hence $(x_n)_n$ contains a convergent subsequence $(z_k)_k$, $z_k \to z$. Because c_X is continuous $(z_k)_k$ converges also in the c_X-topology towards z, and because $(x_n)_n$ is a Cauchy sequence it converges towards z as well.

(iii) $\Rightarrow$ (ii): To prove the injectivity let $x, y \in X$, $x \neq y$. Because c_X is a metric $c_X(x,y)$ does not vanish and there must exist an $h \in H^\infty(X)$ with $h(x) \neq h(y)$. Hence $\chi(x) \neq \chi(y)$. To show the properness of χ suppose that $(x_n)_n$ is a sequence without cluster points in X. Because c_X induces the topology of X the sequence $(x_n)_n$ cannot be a Cauchy sequence for c_X. Therefore we may assume that $c_X(x_n, x_{n+1}) \geqslant \varepsilon$ for a suitable $\varepsilon > 0$ and all $n \in \mathbf{N}$. With the use of equation (1) we obtain

$$\begin{aligned}\|\chi(x_n)-\chi(x_{n+1})\| &= \sup\{|h(x_n)-h(x_{n+1})| : h \in H^\infty(X), \|h\| \leqslant 1\}\\ &\geqslant \sup\{|h(x_n)| : h \in H^\infty(X), \|h\| \leqslant 1, h(x_{n+1})=0\}\\ &= \tau^{-1}(c_X(x_n, x_{n+1}))\\ &\geqslant \tau^{-1}(\varepsilon)\\ &> 0.\end{aligned}$$

Hence $(\Phi(x_n))_n$ does not converge. q.e.d.

Proof of 3.2. — (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are trivial.

To prove (iii) $\Rightarrow$ (ii) we have to show that χ is an immersion. Let $x \in X$ and ξ be a tangent vector in x. The derivation $\eta := D\chi(x)\xi$ is the derivative in direction of a vector $e \in H^\infty(X)'$. If $u \in H^\infty(X)''$ is represented by $h \in H^\infty(X)$ then $u(e) = \eta(u) = \xi(u \circ \chi) = \xi(h)$ since $u \circ \chi(y) = h(y)$ for every $y \in X$. Now suppose $\xi \in \operatorname{Ker} D\chi(x)$ and let $f : X \to F$ be a bounded holomorphic map which is an embedding near x. For each $v \in F'$ putting $h := v \circ f$ we get $h \in H^\infty(X)$ and $\xi(v \circ f) = \eta(u) = 0$, hence $\xi = 0$. q.e.d.

3.3. *Remarks.* — *a) Concerning property* (iii) *in* 3.1.

Obviously c_X does not always induce the topology of X e.g. c_C vanishes. Even if c_X is a metric this is not always true [19, p. 255, Rem. 5]. There are, however, some useful sufficient conditions. c_X induces the topology of X if X is a relatively compact domain in a Stein space [19, p. 222, Prop. 6] or if c_X is a strongly complete metric in the sense that every closed ball $\{y \in X : c_X(x,y) \leqslant r\}$ with $x \in X$, $r > 0$, is compact in the original topology of X [19, Cor. 4]. Examples for the second case are generalized analytic polyhedra [13], in particular all domains whose boundary points have peak functions.

b) Concerning property (ii) *in* 3.1.

A complex space is holomorphically convex (holomorphically separable) iff the canonical map $\psi : X \to \mathbf{C}^{\mathcal{O}(X)}$, $x \mapsto (f(x))_{f \in \mathcal{O}(X)}$ is proper (injective) where $\mathbf{C}^{\mathcal{O}(X)}$ carries the product topology. $H^\infty(X)'$ endowed with the weak-*-topology σ can be considered as a linear subspace of $\mathbf{C}^{\mathcal{O}(X)}$. Therefore it is seductive to conjecture that $\chi : X \to (H^\infty(X)',\sigma)$ is proper iff X is H^∞-convex i.e. iff the H^∞-hull $\hat{H}^\infty(K) := \{x \in X : |h(x)| \leqslant \|h\|_K , h \in H^\infty(X)\}$ of each compact $K \subset X$ is again compact. But this is wrong! $\chi : X \to (H^\infty(X)',\sigma)$ cannot be proper whenever X is not compact, for the spectrum of $H^\infty(X)$ is a weakly compact subset of $H^\infty(X)'$ and contains $\chi(X)$. Also H^∞-convexity and strong completeness of c_X do not coincide [19]. But if we define H^∞-completeness in analogy to holomorphic completeness by $\chi : X \to (H^\infty(X)'$, norm top.) being injective and proper then 3.1. states that X is H^∞-complete if and only if the uniformity induced by c_X is sequentially complete and compatible with the topology of X. This statement corresponds to the fact that X is holomorphically complete if and only if the $\mathcal{O}$-uniformity is sequentially complete and compatible with the topology of X [10, p. 104; 18]. The $\mathcal{O}$-uniformity is the coarsest uniformity such that all holomorphic functions are uniformly continuous.

4. Conditions for (X, E) admitting a bounded embedding X → E.

Let X be a complex space and $\mathrm{F} \subset \mathrm{H}^\infty(\mathrm{X})$ a linear subspace. Define

$$c_\mathrm{X}^\mathrm{F}(x, y) := \sup \{\rho(h(x), h(y)) : h \in \mathrm{F}, \|h\| \leqslant 1\} \quad \text{for} \quad x, y \in \mathrm{X}.$$

c_X^F is a pseudometric which is coarser than c_X, hence it is finite and continuous in the topology of X.

F is called *Aut* D-*invariant* iff Aut D acts from left on the unit ball of F i.e. iff for every $h \in \mathrm{F}$, $\|h\| \leqslant 1$, and every $\gamma \in \mathrm{Aut\,D}$ also $\gamma \circ h \in \mathrm{F}$. For Aut D-invariant subspaces F equality (1) generalizes to

$$(3) \quad c_\mathrm{X}^\mathrm{F}(x, y) = \tau(\sup \{|h(y)| : h \in \mathrm{F}, \|h\| \leqslant 1, h(x) = 0\}).$$

P. Woitasczcyk informed us of the following result.

4.1. Proposition. — *A closed linear subspace* F *of* $\mathrm{H}^\infty(\mathrm{X})$ *is* Aut D-*invariant if and only if it is a subalgebra.*

Proof (due to Woitasczcyk). — Let F be Aut D-invariant. Because $ab = \frac{1}{2}((a+b)^2 - a^2 - b^2)$ it is enough to show that $f^2 \in \mathrm{F}$ for each $f \in \mathrm{F}$. Define $\mathrm{M} := \{\varphi \in \mathrm{A(D)} : \varphi \circ f \in \mathrm{F}$ for every $f \in \mathrm{F}\}$ where A(D) is the disk algebra. M is a closed linear subspace of A(D) and contains $\psi := \mathrm{id_D}$. The Aut D-invariance of F implies that M is Moebius-invariant in the sense that $\varphi \circ \gamma \in \mathrm{M}$ for every $\varphi \in \mathrm{M}$ and every $\gamma \in \mathrm{Aut\,D}$. The classification of Moebius-invariant subspaces of $\mathscr{C}(\bar{\mathrm{D}})$ in [14, 16] shows that $\mathrm{M} = \mathrm{A(D)}$. Hence $\psi^2 \in \mathrm{M}$ and therefore $f^2 \in \mathrm{F}$ for each $f \in \mathrm{F}$.

To prove the converse implication observe that each $\gamma \in \mathrm{Aut\,D}$ is holomorphic in a neighborhood of $\bar{\mathrm{D}}$, hence the power series expansion $\gamma(z) = \sum_{n=0}^{\infty} a_n z^n$ converges uniformly in $\bar{\mathrm{D}}$. If F is a subalgebra then $\sum_{n=0}^{k} a_n (f(z))^n \in \mathrm{F}$ for every $k \in \mathbf{N}$ and $f \in \mathrm{F}$, and since F is closed one obtains $\gamma \circ f \in \mathrm{F}$ for every $f \in \mathrm{F}$.

4.2. PROPOSITION. — *Let* X *be a complex space,* G *a complex Banach space, and* $T : G \to H^\infty(X)$ *a continuous linear map. Suppose that* $F := T(G)$ *is* Aut D-*invariant and that* c_X^F *is a complete metric inducing the topology of* X. *Then the map* $\Phi := T^* \circ \chi : X \to H^\infty(X)' \to G'$ *is injective, proper, and holomorphic, and* $\Phi(X)$ *is bounded.*

Proof. — For $x \in X$ and $g \in G$ one gets $\Phi(x)(g) = \chi(x)(T(g)) = T(g)(x)$. Φ is holomorphic since χ and T^* are holomorphic. $\Phi(X)$ is bounded because

$$\begin{aligned}\|\Phi(x)\| &= \sup\{|\Phi(x)(g)| : g \in G, \|g\| \leqslant 1\}\\ &= \sup\{|T(g)(x)| : g \in G, \|g\| \leqslant 1\}\\ &\leqslant \sup\{\|T(g)\| : g \in G, \|g\| \leqslant 1\}\\ &= \|T\|.\end{aligned}$$

Φ is injective: Let $x, y \in X$, $x \neq y$. Because c_X^F is a metric there are $f \in F$, $g \in G$ with $f = T(g)$ and $f(x) \neq f(y)$. This implies

$$\Phi(x)(g) = T(g)(x) = f(x) \neq f(y) = T(g)(y) = \Phi(y)(g).$$

Φ is proper: Let $(x_n)_n$ be a sequence without cluster points in X. Then $(x_n)_n$ does not contain a Cauchy sequence and we may assume $c_X^F(x_n, x_{n+1}) \geqslant \varepsilon$ for a suitable $\varepsilon > 0$ and every $n \in \mathbf{N}$. With the use of (3) we obtain

$$\begin{aligned}&\|\Phi(x_n) - \Phi(x_{n+1})\| \, \|T\|^{-1}\\ &\qquad \geqslant \sup\{|T(g)(x_n) - T(g)(x_{n+1})| : g \in G, \|g\| \leqslant \|T\|^{-1}\}\\ &\qquad \geqslant \sup\{|T(g)(x_n)| : g \in G, \|g\| \leqslant \|T\|^{-1}, T(g)(x_{n+1}) = 0\}\\ &\qquad = \tau^{-1}(c_X^F(x_n, x_{n+1}))\\ &\qquad \geqslant \tau^{-1}(\varepsilon)\\ &\qquad > 0.\end{aligned}$$

Hence $(\Phi(x_n))_n$ does not converge. q.e.d.

Notice that the Aut D-invariance of F may not be omitted. Put $X := D$, $F := G := \operatorname{span}\{\mathrm{id}_D\}$, $T :=$ inclusion. Then c_X^F coincides with the Poincaré metric ρ and consequently all assumptions of 4.2. are satisfied. But Φ is the inclusion of D into $G' \cong \mathbf{C}$ and hence Φ is not proper.

4.3. COROLLARY. — *Let* X *be a complex space of positive dimension and* F *a closed subalgebra of* $H^\infty(X)$ *such that* c_X^F *is a complete metric inducing the topology of* X. *Then* F' *does not have the Radon-Nikodym property.*

Proof. — Because of 4.1. the assumptions in 4.2. are fulfilled, and 4.2. yields a proper holomorphic mapping $\Phi : X \to F'$ with bounded image. 2.2. excludes that F' has the RNP.

5. Infinite dimensional X.

For infinite dimensional Banach spaces E it is natural to ask when they contain infinite dimensional bounded analytic subsets X. In this generality the question is as difficult as uninteresting because arbitrary analytic sets can be very pathological. We recall Douady's construction in [7] which yields compact analytic sets which are finite dimensional topological manifolds but do not possess any complex manifold point. Thus, in order to obtain some kind of analytic geometry in Banach spaces one has to impose regularity conditions on the analytic sets. In [1, 3] we introduced the SF-analytic sets. An *SF-analytic* subset of a Banach manifold is a closed subset which is locally a fiber of a holomorphic semi-Fredholm map; this is a map whose differentials are semi-Fredholm operators i.e. kernel and image are complemented and kernel or cokernel are finite dimensional. Such mappings occur e.g. in bifurcation problems of Fredholm maps. The SF-analytic sets are precisely the analytic sets which are locally contained in some complex submanifold (with complemented tangent spaces) where they are finitely defined in the sense of Ramis [15] i.e. they are locally the common zero set of finitely many holomorphic functions. Hence SF-analytic sets have the same good local properties as the finitely defined analytic sets [15] but the global behavior can differ. Finite dimensional analytic sets are SF-analytic, hence SF-analytic sets can be bounded and non discrete whereas finitely defined analytic subsets of an infinite dimensional Banach space are never bounded [15, p. 73]. We list some essential properties of SF-analytic sets.

5.1. LEMMA. — *Let* X *be an* SF-*analytic subset of a complex Banach manifold* Ω.

a) X *is locally an analytically ramified finitely sheeted covering of a domain in a Banach space. In particular the regular points are dense and* X *has the usual decomposition in irreducible components.*

b) X *is locally connected by complex arcs i.e. for every* $x \in X$ *there are arbitrarily small neighborhoods* U *such that for every* $y \in U$ *there exists a holomorphic map* γ *from the open unit disk* D *into* Ω *with* $x, y \in \gamma(D) \subset X$.

c) *If* X *is irreducible then every non constant holomorphic function on* X *is open.*

d) *Every bounded family of holomorphic functions on* X *is equicontinuous.*

Proof. – a) [15, 1, 3], b) follows from a) [1, 3] and c) from b). According to a hint of W. Kaup d) can be derived from a); by using symmetric functions one reduces d) to the well-known analogous statement where X is a domain in a Banach space.

A map $f : X \to Y$ between SF-analytic subsets of Banach manifolds Ω and Ξ is called *holomorphic* if it has locally holomorphic extensions f^* to open subsets of the embedding manifolds. Hence its differentials can be defined as the restrictions $Df(x) := Df^*(x) | T_xX \to T_yY$ where the tangent spaces are defined by $T_xX := \{u \in T_x\Omega : u \in \operatorname{Ker} Dh(x)$ for every holomorphic function germ h which vanishes on $X\}$.

An SF-analytic subset of a Banach manifold with the reduced structure and the above defined holomorphic maps will be called an *SF÷analytic space*. It is not known whether every complex space is SF-analytic, but at least every Stein space is SF-analytic even if its global dimension is not finite [17]. Define an embedding as in the first section and call a holomorphic map an *immersion* iff its differentials are injective and have complemented images. Then a map $f : X \to E$ from an SF-analytic space X into a Banach space E is an embedding if and only if it is an injective proper immersion.

Because of 5.1.c) theorem 2.2. holds for SF-analytic spaces X instead of complex spaces X. In particular one obtains

5.2. COROLLARY. – *Every bounded* SF-*analytic subset of a Banach space with Radon-Nikodym property is discrete.*

Whereas every real differentiable Hilbert manifold can be embedded boundedly into ℓ_2 [5, th. 2.7.] corollary 5.2. implies that no complex Hilbert manifold at all is realizable as a bounded closed complex submanifold of ℓ_2. In contrast to that we have the following example.

5.3. *Example.* – The open unit ball B in ℓ_∞ is isomorphic to a bounded closed complex submanifold in ℓ_∞.

Proof. – Let $E := \{(a_n)_n \in \ell_\infty^{\mathbf{N}} : \sup \{\|a_n\|_\infty : n \in \mathbf{N}\} < \infty\}$ be the ℓ_∞-sum of countably many copies of ℓ_∞. Choose a bijection $\sigma : \mathbf{N} \to \mathbf{N}^2$. σ

induces an isometry $T : E \to \ell_\infty$. The map $\Phi : D \to \ell_\infty$, $z \to (z^n)_n$ is an embedding and $\Phi(D)$ lies in the open unit ball of ℓ_∞ (cf. section 3.). Define $g : B \to E$ by $g(x) := (\Phi(x_n))_{n \in N}$ and put $\psi := T \circ g$. Then ψ is holomorphic, injective and proper, and the differentials of ψ are injective. Thus it remains to show that the differentials have complemented images. $\operatorname{Im} D\psi(x)$ is the ℓ_∞-sum of $I_n := \operatorname{Im} D\Phi(x_n)$. Because each I_n is one dimensional there exist continuous projections $\pi_n : \ell_\infty \to I_n$ with $\|\pi_n\| = 1$. The product of these projections can be restricted to E and yields a continuous projection $E \to \operatorname{Im} D\psi(x)$. q.e.d.

In order to extend the result of the last two sections to SF$-$analytic spaces X define the Carathéodory pseudometrics c_X and c_X^F in the same way as above. They have the same properties as for complex spaces X. In particular equation (1) holds as well. Let us mention explicitly only two properties.

5.4. Lemma. — *For every connected* SF$-$*analytic space* X *the Carathéodory pseudometric* c_X *is finite and continuous.*

Proof. — Let $x, y \in X$. Apply 5.1.b) to obtain $x_0, \ldots, x_{n+1} \in X$, $a_0, \ldots, a_n \in D$, and holomorphic maps $\gamma_k : D \to X$ such that $\gamma_k(0) = x_k$, $\gamma_k(a_k) = x_{k+1}$, $x_0 = x$, $x_n = y$. Then

$$\begin{aligned} c_X(x, y) &\leqslant \sum_{k=0}^{n} c_X(x_k, x_{k+1}) \leqslant \sum_{k=0}^{n} c_X(\gamma_k(0), \gamma_k(a_k)) \\ &\leqslant \sum_{k=0}^{n} c_D(0, a_k) \quad \leqslant \sum_{k=0}^{n} \rho(0, a_k) < \infty . \end{aligned}$$

Hence c_X is finite.

Because of 5.1.d) the family $\{h \in H^\infty(X) : \|h\| \leqslant 1, h(x_0) = 0\}$ is equicontinuous in $x_0 \in X$. Equation (1) in the third section implies that the map $x \mapsto c_X(x_0, x)$ is continuous. Because every pseudometric satisfies $|d(x_0, y_0) - d(x, y)| \leqslant d(x_0, x) + d(y_0, y)$ the continuity of c_X follows. q.e.d.

Let us note that c_X induces the topology of X if X is a bounded domain in a Banach space [8, 9] and that because of 5.1.d) the canonical map $\chi : X \to H^\infty(X)'$ is continuous. Now we have all tools to see that the proof of (ii) $\Leftrightarrow$ (iii) in 3.1. holds verbally for SF-analytic spaces X instead of complex spaces X.

5.5. THEOREM. — *Let* X *be a connected* SF-*analytic space. Then* $\chi : X \to H^\infty(X)'$ *is injective and proper if and only if* c_X *is a complete metric which induces the topology of* X.

In order to establish the other results of sections 3 and 4 two essential items are still missing, namely that the canonical map χ is holomorphic and that their differentials have complemented kernels and images. Whereas for finite dimensional X Bungart showed that weak holomorphicity implies holomorphicity [4] an analogous result for SF-analytic spaces X is not known except for Banach manifolds X. And even for Banach manifolds X we do not know whether the images of $D\chi(x)$ are always complemented. Hence we can state only the following proposition.

5.6. PROPOSITION. — *The theorems* 3.1., 4.2., *and* 4.3. *hold also for complex Banach manifolds* X *instead of complex spaces* X.

If, however, a Banach manifold X is modelled in a reflexive Banach space we are able to show that the H^∞-regularity of X implies that $\operatorname{Im} D\chi(x)$ is always complemented. Therefore we obtain

5.7. PROPOSITION. — *The supplement* 3.2. *holds for connected complex Banach manifolds* X *which are modelled in a reflexive Banach space* E.

Proof. — We have only to show that $\operatorname{Im} D\chi(x)$ is complemented if X is H^∞-regular. Let $x \in X$. Then there exists a neighborhood U of x, a biholomorphic map $\varphi : U \to W$ onto a domain W in E, and a bounded holomorphic map $f : X \to F$ into a Banach space F such that $f \mid U \to V$ is an embedding into an open subset V of F. Let $\chi^* := \chi \circ \varphi^{-1}$ and $f^* := f \circ \varphi^{-1}$, $y := \varphi(x)$, and identify the tangent space $T_y W$ with E. Then $D\chi^*(y)$ maps each $e \in E$ onto the derivative η in the direction e at x i.e. $\eta \in H^\infty(X)'$ is defined by $\eta(h) := \frac{d}{dt} h \circ \varphi^{-1}(y+te)|_{t=0}$. Let $\tilde{E} := \operatorname{Im} D\chi(x) = \operatorname{Im} D\chi^*(y)$. Since $f \mid U$ is an embedding $Df^*(y) : E \to F$ maps E isomorphically onto a complemented subspace of F. Hence the adjoint $Df^*(y)^t : F' \to E'$ has a section $\sigma : E' \to F'$. Because $f(X)$ is bounded we get $\tau \circ f \in H^\infty(X)$ for every $\tau \in F'$ and we can define $p : H^\infty(X)' \to E''$ by $p(\mu)(\nu) := \mu(\sigma(\nu) \circ f)$ for $\mu \in H^\infty(X)'$ and $\nu \in E'$. Since E is reflexive the canonical map $J : E \to E''$ is an isomorphism and with $e := J^{-1}(p(\mu))$ we obtain $p(\mu)(\nu) = \nu(e)$. Now define $\pi := D\chi^*(y) \circ J^{-1} \circ p : H^\infty(X)' \to E'' \cong E \to \tilde{E}$.

π is a continuous linear map. In order to prove that π is a projection onto E we show $\pi | \tilde{E} = \mathrm{id}_E$. Let $\eta \in E$ be the derivative in direction $e \in E$ i.e. $\eta = D\chi^*(y)e$. Then

$$p(\eta)(\nu) = \eta(\sigma(\nu) \circ f) = \frac{d}{dt}(\sigma(\nu) \circ f^*(y + te))|_{t=0}$$
$$= \sigma(\nu) \circ Df^*(y)(e) = Df^*(y)^t(\sigma(\nu))(e) = \nu(e)$$

for every $\nu \in E'$, hence $J^{-1}(p(\eta)) = e$ and thus $\pi(\eta) = \eta$. q.e.d.

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