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WHEN IS A RIESZ DISTRIBUTION A COMPLEX MEASURE?

by Alan D. Sokal

Abstract. — Let $\mathscr{R}_\alpha$ be the Riesz distribution on a simple Euclidean Jordan algebra, parametrized by $\alpha \in \mathbb{C}$. I give an elementary proof of the necessary and sufficient condition for $\mathscr{R}_\alpha$ to be a locally finite complex measure (= complex Radon measure).

Résumé (*Une distribution de Riesz, quand est-elle mesure complexe ?*)

Soit $\mathscr{R}_\alpha$ la distribution de Riesz sur une algèbre de Jordan euclidienne simple, paramétrisée par $\alpha \in \mathbb{C}$. Je donne une démonstration élémentaire de la condition nécessaire et suffisante pour que $\mathscr{R}_\alpha$ soit une mesure complexe localement finie (= mesure de Radon complexe).

1. Introduction

In the theory of harmonic analysis on Euclidean Jordan algebras (or equivalently on symmetric cones) [12], a central role is played by the *Riesz distributions* $\mathscr{R}_\alpha$, which are tempered distributions that depend analytically on a

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parameter $\alpha \in \mathbb{C}$. One important fact about the Riesz distributions is the necessary and sufficient condition for positivity, due to Gindikin [13]:

THEOREM 1.1. — [12, Theorem VII.3.1] *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n = r + \frac{d}{2}r(r-1)$. Then the Riesz distribution $\mathcal{R}_\alpha$ on V is a positive measure if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}$ or $\alpha > (r-1)\frac{d}{2}$.*

The "if" part is fairly easy, but the "only if" part is reputed to be deep [13, 12, 20].(1)

The purpose of this note is to give a completely elementary proof of the "only if" part of Theorem 1.1, and indeed of the following strengthening:

THEOREM 1.2. — *Let V be a simple Euclidean Jordan algebra of dimension n and rank r, with $n = r + \frac{d}{2}r(r-1)$. Then the Riesz distribution $\mathcal{R}_\alpha$ on V is a locally finite complex measure [= complex Radon measure] if and only if $\alpha = 0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}$ or $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$.*

This latter result is also essentially known [18, Lemma 3.3], but the proof given there requires some nontrivial group theory.

The idea of the proof of Theorem 1.2 is very simple: A distribution defined on an open subset $\Omega \subset \mathbb{R}^n$ by a function $f \in L^1_{\mathrm{loc}}(\Omega)$ can be extended to all of $\mathbb{R}^n$ as a locally finite complex measure only if the function f is locally integrable also at the boundary of Ω (Lemma 2.1); furthermore, this fact survives analytic continuation in a parameter (Proposition 2.3). In the case of the Riesz distribution $\mathcal{R}_\alpha$, a simple computation using its Laplace transform (Lemma 3.4) plus a bit of extra work (Lemma 3.5) allows us to determine the allowed set of α, thereby proving Theorem 1.2.

Theorem 1.2 thus states a necessary and sufficient condition for $\mathcal{R}_\alpha$ to be a distribution of order 0. It would be interesting, more generally, to determine the order of the Riesz distribution $\mathcal{R}_\alpha$ for each $\alpha \in \mathbb{C}$.

It would also be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions $\mathcal{R}_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{C}^r$ [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric (i.e. not self-dual) and hence do not arise from a Euclidean Jordan algebra [13, 20].

In an Appendix I comment on a beautiful but little-known elementary proof of Theorem 1.1 — which does not extend, however, to Theorem 1.2 — due to Shanbhag [27] and Casalis and Letac [9].

(1) The set of values of α described in Theorem 1.1 is the so-called *Wallach set* [29, 30, 21, 10, 11, 12].

2. A general theorem on distributions

We assume a basic familiarity with the theory of distributions [26, 19] and recall some key notations and facts.

For each open set $\Omega \subseteq \mathbb{R}^n$, we define the space $\mathcal{D}(\Omega)$ of C^∞ functions having compact support in Ω, the corresponding space $\mathcal{D}'(\Omega)$ of distributions, and the space $\mathcal{D}'^k(\Omega)$ of distributions of order $\leq k$. In particular, the space $\mathcal{D}'^0(\Omega)$ consists of the distributions that are given locally (i.e. on every compact subset of Ω) by a finite complex measure.

Let $f\colon\ \Omega \to \mathbb{C}$ be a measurable function, and extend it to all of $\mathbb{R}^n$ by setting $f \equiv 0$ outside Ω. We say that $f \in L^1_{\rm loc}(\Omega)$ if, for every $x \in \Omega$, f is (absolutely) integrable on some neighborhood of x. Any $f \in L^1_{\rm loc}(\Omega)$ defines a distribution $T_f \in \mathcal{D}'^0(\Omega)$ by

$$T_f(\varphi) = \int \varphi(x)\, f(x)\, dx \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)\ . \tag{1}$$

We are interested in knowing under what circumstances the distribution $T_f \in \mathcal{D}'^0(\Omega)$ can be extended to a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$, i.e. one that is locally everywhere on $\mathbb{R}^n$ a finite complex measure.

Lemma 2.1. — *Let $f\colon\ \Omega \to \mathbb{C}$ be in $L^1_{\rm loc}(\Omega)$, and let $T_f \in \mathcal{D}'^0(\Omega)$ be the corresponding distribution. Then the following are equivalent:*

(a) *$f \in L^1_{\rm loc}(\overline{\Omega})$, i.e. for every $x \in \overline{\Omega}$, f is integrable on some neighborhood of x.*[2]

(b) *There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends T_f and is supported on $\overline{\Omega}$.*

(c) *There exists a distribution $\widetilde{T}_f \in \mathcal{D}'^0(\mathbb{R}^n)$ that extends T_f.*

Proof. — (a) $\Longrightarrow$ (b): It suffices to define $\widetilde{T}_f(\varphi) = \int_\Omega \varphi(x)\, f(x)\, dx$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

(b) $\Longrightarrow$ (c) is trivial.

(c) $\Longrightarrow$ (a): By hypothesis, for every $x \in \partial\Omega$ and every compact neighborhood $K \ni x$, there exists a finite complex measure μ_K supported on K such that $\widetilde{T}_f(\varphi) = \int \varphi\, d\mu_K$ for every $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with support in K. But since $\widetilde{T}_f$ extends T_f, the restriction of μ_K to every compact subset of $K \cap \Omega$ must coincide with the measure $f(x)\, dx$. Since $K \cap \Omega$ is σ-compact, this implies that $\int\limits_{K\cap\Omega} |f(x)|\, dx = |\mu_K|(K \cap \Omega) < \infty$, so that f is integrable in a neighborhood of x. □

[2] Since this has already been assumed for $x \in \Omega$, the content of hypothesis (a) is that it should hold also for $x \in \partial\Omega$.

We now extend this idea to allow for analytic dependence on a parameter. Let Ω be an open set in $\mathbb{R}^n$, let D be a connected open set in $\mathbb{C}^m$, and let $F\colon \Omega \times D \to \mathbb{C}$ be a continuous function such that $F(x,\,\cdot\,)$ is analytic on D for each $x \in \Omega$. Then, for each $\lambda \in D$, define

$$T_\lambda(\varphi) \;=\; \int \varphi(x)\, F(x,\lambda)\, dx \qquad \text{for all } \varphi \in \mathcal{D}(\Omega)\ . \tag{2}$$

Lemma 2.2. — *With F as above, the map $\lambda \mapsto T_\lambda$ is analytic from D into $\mathcal{D}'(\Omega)$ in the sense that $\lambda \mapsto T_\lambda(\varphi)$ is analytic for all $\varphi \in \mathcal{D}(\Omega)$.*

Proof. — This is an immediate consequence of the hypotheses on F together with standard facts about scalar-valued analytic functions in $\mathbb{C}$ (either Morera's theorem or the Cauchy integral formula) and $\mathbb{C}^m$ (e.g. the weak form of Hartogs' theorem). □

Remark. — Weak analyticity in the sense used here is actually *equivalent* to strong analyticity: see e.g. [15, pp. 37–39, Théorème 1 and Remarque 1] [5, Theorems 3.1 and 3.2] [14, Theorem 1]. Indeed, our hypothesis on F is equivalent to the even stronger statement that the map $\lambda \mapsto F(\,\cdot\,,\lambda)$ is analytic from D into the space $C^0(\Omega)$ of continuous functions on Ω, equipped with the topology of uniform convergence on compact subsets [15, p. 41, example (a)]. But we do not need any of these facts; weak analyticity is enough for our purposes.

Putting together these two lemmas, we obtain:

Proposition 2.3. — *Let F be as above, let $D_0 \subseteq D$ be a nonempty open set, and let $\lambda \mapsto \widetilde{T}_\lambda$ be a (weakly) analytic map of D into $\mathcal{D}'(\mathbb{R}^n)$ such that $\widetilde{T}_\lambda$ extends T_λ for each $\lambda \in D_0$. Then, for each $\lambda \in D$, we have:*

(a) *$\widetilde{T}_\lambda$ extends T_λ.*
(b) *If $\widetilde{T}_\lambda \in \mathcal{D}'^0(\mathbb{R}^n)$, then $F(\,\cdot\,,\lambda) \in L^1_{\mathrm{loc}}(\overline{\Omega})$.*

Proof. — (a) This is immediate by analytic continuation: for each $\varphi \in \mathcal{D}(\Omega)$, both $\widetilde{T}_\lambda(\varphi)$ and $T_\lambda(\varphi)$ are (by hypothesis and Lemma 2.2, respectively) analytic functions of λ on D that coincide on D_0, therefore they must coincide on all of D.

(b) This is immediate from (a) together with Lemma 2.1. □

We shall apply this setup with $F(x,\lambda) = f(x)^\lambda$ where $f\colon \Omega \to (0,\infty)$ is a continuous function; in fact, we shall take f to be a polynomial.

Remark. — Let P be a polynomial that is strictly positive on Ω *and vanishes on* $\partial\Omega$, and define for $\operatorname{Re}\lambda > 0$ a tempered distribution $\mathscr{P}_\Omega^\lambda \in \mathscr{S}'(\mathbb{R}^n)$ by the formula

$$\mathscr{P}_\Omega^\lambda(\varphi) \;=\; \int\limits_\Omega P(x)^\lambda\, \varphi(x)\, dx \qquad \text{for } \varphi \in \mathscr{S}(\mathbb{R}^n)\,. \tag{3}$$

Then $\mathscr{P}_\Omega^\lambda$ is a tempered-distribution-valued analytic function of λ on the right half-plane, and it is a deep result of Atiyah, Bernstein and S.I. Gelfand [3, 1, 2, 4] that $\mathscr{P}_\Omega^\lambda$ can be analytically continued to the whole complex plane as a meromorphic function of λ with poles on a finite number of arithmetic progressions. It is important to note that our Proposition 2.3 does *not* rely on this deep result; rather, it says that *whenever* such an analytic continuation exists (however it may be constructed), the analytically-continued distribution $\mathscr{P}_\Omega^\lambda$ can be a complex measure only if $P^\lambda \in L^1_{\rm loc}(\overline{\Omega})$.

3. Application to Riesz distributions

We refer to the book of Faraut and Korányi [12] for basic facts about symmetric cones and Jordan algebras. Let V be a simple Euclidean (real) Jordan algebra of dimension n and rank r, with Peirce subspaces V_{ij} of dimension d; recall that $n = r + \frac{d}{2}r(r-1)$. We denote by $(x|y) = \operatorname{tr}(xy)$ the inner product on V, where tr is the Jordan trace and xy is the Jordan product. Let $\Omega \subset V$ be the positive cone (i.e. the interior of the set of squares in V, or equivalently the set of invertible squares in V); it is self-dual, i.e. $\Omega^* = \Omega$. We denote by $\Delta(x) = \det(x)$ the Jordan determinant on V: it is a homogeneous polynomial of degree r on V, which is strictly positive on Ω and vanishes on $\partial\Omega$, and which satisfies [12, Proposition III.4.3]

$$\Delta(gx) \;=\; \operatorname{Det}(g)^{r/n}\, \Delta(x) \qquad \text{for } g \in G,\ x \in V\,, \tag{4}$$

where G denotes the identity component of the linear automorphism group of Ω [it is a subgroup of $GL(V)$] and Det denotes the determinant of an endomorphism. We then have the following fundamental Laplace-transform formula:

THEOREM 3.1. — [12, Corollary VII.1.3] *For $y \in \Omega$ and $\operatorname{Re}\alpha > (r-1)\frac{d}{2} = \frac{n}{r} - 1$, we have*

$$\int\limits_\Omega e^{-(x|y)}\, \Delta(x)^{\alpha - \frac{n}{r}}\, dx \;=\; \Gamma_\Omega(\alpha)\, \Delta(y)^{-\alpha} \tag{5}$$

where

$$\Gamma_\Omega(\alpha) \;=\; (2\pi)^{(n-r)/2} \prod_{j=0}^{r-1} \Gamma\Big(\alpha - j\frac{d}{2}\Big)\,. \tag{6}$$

Thus, for $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$, the function $\Delta(x)^{\alpha-\frac{n}{r}}/\Gamma_\Omega(\alpha)$ is locally integrable on $\overline{\Omega}$ and polynomially bounded, and so defines a tempered distribution $\mathscr{R}_\alpha$ on V by the usual formula

$$\mathscr{R}_\alpha(\varphi) \;=\; \frac{1}{\Gamma_\Omega(\alpha)} \int\limits_\Omega \varphi(x)\,\Delta(x)^{\alpha-\frac{n}{r}}\,dx \qquad \text{for } \varphi\in\mathcal{S}(V)\,. \tag{7}$$

Using (5), a beautiful argument — which is a special case of Bernstein's general method for analytically continuing distributions of the form $\mathscr{P}_\Omega^\lambda$ [2, 4] — shows that the Riesz distributions $\mathscr{R}_\alpha$ can be analytically continued to the whole complex α-plane:

THEOREM 3.2. — [12, Theorem VII.2.2 et seq.] *The distributions $\mathscr{R}_\alpha$ can be analytically continued to the whole complex α-plane as a tempered-distribution-valued entire function of α. Furthermore, the distributions $\mathscr{R}_\alpha$ have the following properties:*

$$\mathscr{R}_0 = \delta \tag{8a}$$

$$\mathscr{R}_\alpha * \mathscr{R}_\beta = \mathscr{R}_{\alpha+\beta} \tag{8b}$$

$$\Delta(\partial/\partial x)\,\mathscr{R}_\alpha = \mathscr{R}_{\alpha-1} \tag{8c}$$

$$\Delta(x)\,\mathscr{R}_\alpha = \left(\prod_{j=0}^{r-1}\Big(\alpha - j\frac{d}{2}\Big)\right)\mathscr{R}_{\alpha+1} \tag{8d}$$

(here δ denotes the Dirac measure at 0) and

$$\mathscr{R}_\alpha(\varphi\circ g^{-1}) \;=\; \operatorname{Det}(g)^{\alpha r/n}\,\mathscr{R}_\alpha(\varphi) \qquad \textit{for } g\in G,\ \varphi\in\mathcal{S}(V) \tag{9}$$

(in particular, $\mathscr{R}_\alpha$ is homogeneous of degree $\alpha r - n$). Finally, the Laplace transform of $\mathscr{R}_\alpha$ is

$$(\mathscr{L}\mathscr{R}_\alpha)(y) \;=\; \Delta(y)^{-\alpha} \tag{10}$$

for y in the complex tube $\Omega + iV$.

The property (8d) is not explicitly stated in [12], but for $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$ it is an immediate consequence of (6)/(7), and then for other values of α it follows by analytic continuation (see also [18, Proposition 3.1(iii) and Remark 3.2]). Likewise, the property (9) is not explicitly stated in [12], but for $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$ it is an immediate consequence of (4)/(7), and then for other values of α it follows by analytic continuation (see also [18, Proposition 3.1(i)]). It follows from (8a,b) that the distributions $\mathscr{R}_\alpha$ are all nonzero; and it follows from this and (9) that $\mathscr{R}_\alpha \neq \mathscr{R}_\beta$ whenever $\alpha \neq \beta$.

It is fairly easy to find a *sufficient* condition for the Riesz distributions to be a positive measure:

Proposition 3.3 ([12, Proposition VII.2.3], see also [18, Section 3.2], [21, 6])

(a) *For $\alpha = k\frac{d}{2}$ with $k = 0, 1, \ldots, r-1$, the Riesz distribution $\mathscr{R}_\alpha$ is a positive measure that is supported on the set of elements of $\overline{\Omega}$ of rank exactly k (which is a subset of $\partial\Omega$).*

(b) *For $\alpha > (r-1)\frac{d}{2}$, the Riesz distribution $\mathscr{R}_\alpha$ is a positive measure that is supported on $\overline{\Omega}$ and given there by a density (with respect to Lebesgue measure) that lies in $L^1_{\mathrm{loc}}(\overline{\Omega})$.*

The interesting and nontrivial fact (Theorem 1.1 above) is that the converse of Proposition 3.3 is also true: the foregoing values of α are the *only* ones for which $\mathscr{R}_\alpha$ is a positive measure. Here I shall use Proposition 2.3 together with the Laplace-transform formula (5)/(10) to provide an alternate and extremely elementary proof of the stronger converse result stated in Theorem 1.2.

Lemma 3.4. — *$\Delta^\lambda \in L^1_{\mathrm{loc}}(\overline{\Omega})$ if and only if $\operatorname{Re}\lambda > -1$; or in other words, $\Delta^{\alpha - \frac{n}{r}} \in L^1_{\mathrm{loc}}(\overline{\Omega})$ if and only if $\operatorname{Re}\alpha > (r-1)\frac{d}{2} = \frac{n}{r} - 1$.*

Proof. — Since $|\Delta(x)|^\lambda = \Delta(x)^{\operatorname{Re}\lambda}$, it suffices to consider real values of λ.

For $\lambda > -1$ [i.e. $\alpha > (r-1)\frac{d}{2}$], fix any $y \in \Omega$: the fact that the integral (5) is convergent, together with the fact that $x \mapsto e^{+(x|y)}$ is locally bounded, implies that $\Delta^\lambda \in L^1_{\mathrm{loc}}(\overline{\Omega})$.

Now consider $\lambda = -1$: again fix any $y \in \Omega$, and let $\mu = \inf\limits_{\substack{x \in \overline{C} \\ \|x\| = 1}} (x|y) > 0$ where $\|\cdot\|$ is any norm on V. Choose $R > 0$ such that $|\Delta(x)| \leq 1$ whenever $\|x\| \leq R$. Then

$$\int\limits_{\substack{x \in \Omega \\ \|x\| \leq R}} e^{-(x|y)}\, \Delta(x)^{-1}\, dx \;=\; \lim_{\lambda \downarrow -1} \int\limits_{\substack{x \in \Omega \\ \|x\| \leq R}} e^{-(x|y)}\, \Delta(x)^{\lambda}\, dx \tag{11}$$

by the monotone convergence theorem. We now proceed to obtain a lower bound on

$$M_\lambda \;:=\; \int\limits_{\substack{x \in \Omega \\ \|x\| \leq R}} e^{-(x|y)}\, \Delta(x)^{\lambda}\, dx\,. \tag{12}$$

For any $\beta \geq 1$, we have

$$\int\limits_{\substack{x \in \Omega \\ \frac{\beta}{2}R \leq \|x\| \leq \beta R}} e^{-(x|y)}\, \Delta(x)^\lambda\, dx = \beta^{n+r\lambda} \int\limits_{\substack{x \in \Omega \\ \frac{R}{2} \leq \|x\| \leq R}} e^{-\beta(x|y)}\, \Delta(x)^\lambda\, dx \tag{13a}$$

$$\leq \beta^{n+r\lambda} e^{-(\beta-1)\frac{R}{2}\mu} \int\limits_{\substack{x \in \Omega \\ \frac{R}{2} \leq \|x\| \leq R}} e^{-(x|y)}\, \Delta(x)^\lambda\, dx \tag{13b}$$

$$\leq \beta^{n+r\lambda} e^{-(\beta-1)\frac{R}{2}\mu} M_\lambda \tag{13c}$$

where the first equality used the homogeneity of Δ. Now sum this over $\beta = 2^k$ ($k = 1, 2, 3, \ldots$); the sum is convergent, and we conclude that

$$\int\limits_{x \in \Omega} e^{-(x|y)}\, \Delta(x)^\lambda\, dx \;\leq\; C M_\lambda \tag{14}$$

for a universal constant $C < \infty$ that is independent of λ for $-1 < \lambda \leq 0$. Since (5) tells us that

$$\lim_{\lambda \downarrow -1} \int\limits_{x \in \Omega} e^{-(x|y)}\, \Delta(x)^\lambda\, dx \;=\; +\infty \tag{15}$$

due to the pole of the gamma function at $\alpha = (r-1)\frac{d}{2}$, we conclude that $\lim\limits_{\lambda \downarrow -1} M_\lambda = +\infty$ as well. Therefore

$$\int\limits_{\substack{x \in \Omega \\ \|x\| \leq R}} e^{-(x|y)}\, \Delta(x)^{-1}\, dx \;=\; +\infty\,, \tag{16}$$

which proves that $\Delta^{-1} \notin L^1_{\mathrm{loc}}(\overline{\Omega})$.

Since Δ is locally bounded, it also follows that $\Delta^\lambda \notin L^1_{\mathrm{loc}}(\overline{\Omega})$ for $\lambda < -1$. □

We shall also need a uniqueness result related to Proposition 3.3(a). If μ is a locally finite complex measure on V, we say that μ is *G-relatively invariant with exponent κ* in case

$$\mu(gA) \;=\; \mathrm{Det}(g)^\kappa\, \mu(A) \qquad \text{for } g \in G,\ A \text{ compact} \subseteq V\,. \tag{17}$$

In particular, every such μ is *$G \cap SL(V)$-invariant*, i.e.

$$\mu(gA) \;=\; \mu(A) \qquad \text{for } g \in G \cap SL(V),\ A \text{ compact} \subseteq V\,. \tag{18}$$

Now define $\Omega_k = \{x \in \overline{\Omega}\colon\ \mathrm{rank}(x) = k\}$, so that $\partial\Omega = \bigcup_{k=0}^{r-1} \Omega_k$ and $\Omega = \Omega_r$. We then have the following result, which seems to be of some interest in its own right:

LEMMA 3.5. — (a) *The group $G \cap SL(V)$ acts transitively on each set Ω_k $(0 \le k \le r-1)$.*

(b) *Let μ be a locally finite complex measure that is supported on Ω_k $(0 \le k \le r-1)$ and is $G \cap SL(V)$-invariant. Then μ is a multiple of $\mathcal{R}_{kd/2}$.*

(c) *Let μ be a locally finite complex measure that is supported on $\partial\Omega$ and is G-relatively invariant with some exponent κ. Then there exists $k \in \{0, 1, \ldots, r-1\}$ such that μ is a multiple of $\mathcal{R}_{kd/2}$ (and hence $\kappa = kdr/2n$ if $\mu \ne 0$).*

Proof. — (a) Fix a Jordan frame $c_1, \ldots, c_r$, and let $V = \bigoplus_{1\le i\le j\le r} V_{ij}$ be the corresponding orthogonal Peirce decomposition [12, Theorem IV.2.1]. Then, for $\lambda > 0$, let $M_\lambda = P(c_1 + \cdots + c_{r-1} + \lambda c_r) \in GL(V)$, where P is the quadratic representation [12, p. 32]. From [12, p. 32 and Theorem IV.2.1(ii)] we see that M_λ acts as multiplication by λ^2 on the space V_{rr}, as multiplication by λ on the spaces V_{ir} with $1 \le i \le r-1$, and as the identity on the other subspaces.(3) We have $M_\lambda \in G$ [12, Proposition III.2.2] and $\mathrm{Det}(M_\lambda) = \lambda^{(r-1)d+2} = \lambda^{2n/r}$.

Now write $e_k = c_1 + \cdots + c_k$. By construction we have $M_\lambda e_k = e_k$ for $0 \le k \le r-1$. Now, we know [12, Proposition IV.3.1] that $\Omega_k = Ge_k$, so that for any $x \in \Omega_k$ there exists $g \in G$ such that $x = ge_k$. Therefore, if we set $\lambda = \mathrm{Det}(g)^{-r/2n}$, we have $x = gM_\lambda e_k$ with $gM_\lambda \in G \cap SL(V)$.

(b) follows from (a) and Proposition 3.3(a) together with a standard result about the uniqueness of invariant measures: see e.g. [7, Chapitre 7, sec. 2.6, Théorème 3], [24, p. 138, Theorem 1] or [31, Theorem 7.4.1 and Corollary 7.4.2].

(c) is now an easy consequence, as we can write (uniquely) $\mu = \sum_{k=0}^{r-1} \mu_k$ with μ_k supported on Ω_k, and each μ_k is G-relatively invariant with exponent κ [since each set Ω_k is a separate G-orbit]. But in at most one case can κ take the correct value $kdr/2n$; so all but one of the measures μ_k must be zero. □

Remarks. — 1. Assertions (a) and (b) are false when $k = r$: the determinant $\Delta(x)$ is invariant under the action of $G \cap SL(V)$ [cf. (4)], so $G \cap SL(V)$ cannot act transitively on Ω_r; and all the measures $\mathcal{R}_\alpha$ with $\mathrm{Re}\,\alpha > (r-1)\frac{d}{2}$ are G-relatively invariant [hence $G \cap SL(V)$-invariant] and supported on Ω_r.

2. A slight weakening of Lemma 3.5(b) — in which "$G \cap SL(V)$-invariant" is replaced by "G-relatively invariant with some exponent κ" — is asserted in [21, p. 391, Remarque 3], but the proof given there is insufficient (if it were valid, it

(3) More generally, we see that $P(\sum \lambda_i c_i)$ acts as multiplication by $\lambda_i \lambda_j$ on V_{ij}.

would apply also to $k = r$). However, Michel Lassalle has kindly communicated to me a simple alternative proof of this result, based on [21, Théorème 3 and Proposition 11(b)].

3. Further information on the Riesz measures $\mathscr{R}_{kd/2}$ for $0 \le k \le r-1$ can be found in [21, 6].

Proof of Theorem 1.2. — We already know from Proposition 3.3(b) that $\mathscr{R}_\alpha$ is a locally finite complex measure for $\operatorname{Re}\alpha > (r-1)\frac{d}{2}$. On the other hand, by applying Proposition 2.3 to $F(x,\alpha) = \Delta(x)^{\alpha-\frac{n}{r}}/\Gamma_\Omega(\alpha)$ and using Lemma 3.4, we deduce that $\mathscr{R}_\alpha$ is *not* a locally finite complex measure whenever $\operatorname{Re}\alpha \le (r-1)\frac{d}{2}$ and $\Gamma_\Omega(\alpha) \ne \infty$. So it remains only to study the values of α for which $\Gamma_\Omega(\alpha) = \infty$, namely $\alpha \in \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\} - \mathbb{N}$. For $\alpha \in \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\}$ we know from Proposition 3.3(a) that $\mathscr{R}_\alpha$ is a positive measure. For $\alpha \in \big(\{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\} - \mathbb{N}\big) \setminus \{0, \frac{d}{2}, \ldots, (r-1)\frac{d}{2}\}$, we know from Proposition 3.3(a) and (8c) that $\mathscr{R}_\alpha$ is a distribution supported on $\partial\Omega$; and by (9) and Lemma 3.5(b) we conclude that it cannot be a locally finite complex measure (here we use the fact that $\mathscr{R}_\alpha \ne \mathscr{R}_\beta$ when $\alpha \ne \beta$). □

Remark. — For $\operatorname{Re}\alpha < 0$, an alternate proof that $\mathscr{R}_\alpha$ is not a complex measure can be based on the following fact, which is a special case of the $N = 0$ case of [19, Theorem 7.4.3] (compare [19, Theorem 7.3.1]) but can also easily be proven by direct computation:

LEMMA 3.6. — *Let Ω be a proper open convex cone in a real vector space V, and let $\Omega^* \subset V^*$ be the open dual cone. Let $T \in \mathscr{S}'(V) \cap \mathscr{D}'^0(V)$ be a tempered distribution of order 0 (i.e. a polynomially bounded complex measure) that is supported in $\overline{\Omega}$. Then the Laplace transform $\mathscr{L}T$ is analytic in the complex tube $\Omega^* + iV^*$ and is bounded in every set $K + \Omega^* + iV^*$ where K is a compact subset of Ω^*.*

It then follows from (10) that $\mathscr{R}_\alpha$ cannot be a locally finite complex measure when $\operatorname{Re}\alpha < 0$, because $\Delta(y)^{-\alpha}$ is unbounded at infinity. This argument handles (without the need for Lemma 3.5) the cases $d = 1$ (real symmetric matrices) and $d = 2$ (complex hermitian matrices) in Theorem 1.2.

Appendix A
Remarks on an elementary proof of Theorem 1.1

Casalis and Letac [9, Proposition 5.1] have given an elementary proof of Theorem 1.1 that deserves to be more widely known than it apparently is.(4)

(4) Science Citation Index shows only ten publications citing [9], and six of these have an author in common with [9].

They employ a method due to Shanbhag [27, p. 279, Remark 3] — who proved Theorem 1.1 for the cases of real symmetric and complex hermitian matrices — which they abstract as a general "Shanbhag principle" [9, Proposition 3.1]. Here I would like to abstract their method even further, with the aim of revealing its utter simplicity and beauty.

Let V be a finite-dimensional real vector space, and let V^* be its dual space. We then make the following trivial observations:

(a) If μ is a positive (i.e. nonnegative) measure on V, then its Laplace transform

$$\mathscr{L}(\mu)(y) \;=\; \int e^{-\langle y,x\rangle}\, d\mu(x) \tag{19}$$

is nonnegative on any subset of V^* where it is well-defined (i.e. where the integral is convergent).

(b) If μ is a positive measure on V, then so is $f\mu$ for every continuous (or even bounded measurable) function f on V that is nonnegative on $\operatorname{supp}\mu$.

(c) If μ is a (positive or signed) measure on V whose Laplace transform is well-defined (and finite) on a nonempty open set $\Theta \subseteq V^*$, then the same is true for $P\mu$, where P is any polynomial on V; furthermore, $\mathscr{L}(P\mu) = P(-\partial)\mathscr{L}(\mu)$.(5)

Putting together these observations, we conclude:

Proposition A.1 (Shanbhag–Casalis–Letac principle)

If μ is a positive measure on V whose Laplace transform is well-defined (and finite) on a nonempty open set $\Theta \subseteq V^*$, and P is a polynomial on V that is nonnegative on $\operatorname{supp}\mu$*, then $P(-\partial)\mathscr{L}(\mu) \geq 0$ everywhere on Θ.*

Remark. — Proposition A.1 also has a strong converse, which we shall state and prove at the end of this appendix.

Using Proposition A.1, we can give the following slightly simplified version of the Shanbhag–Casalis–Letac argument:

Proof of Theorem 1.1. — (Based on [9, Proposition 5.1].) In view of Proposition 3.3, it suffices to prove the converse statement. So let $\alpha \in \mathbb{R}$ and suppose that $\mathscr{R}_\alpha$ is a positive measure. Using Proposition A.1 with $P = \Delta$ together with the Laplace-transform formula (10), we conclude that

$$\Delta(-\partial/\partial y)\, \Delta(y)^{-\alpha} \;\geq\; 0 \qquad \text{for all } y \in \Omega\,. \tag{20}$$

(5) Indeed, the same holds if the measure μ is replaced by a distribution $T \in \mathscr{D}'(V)$. See [26, Chapitre VIII] or [19, Section 7.4] for the theory of the Laplace transform on $\mathscr{D}'(V)$.

But the "Cayley" identity [12, Proposition VII.1.4] tells us that

$$\Delta(\partial/\partial y)\,\Delta(y)^{\lambda} \;=\; \Delta(y)^{\lambda-1}\prod_{j=0}^{r-1}\Bigl(\lambda+j\frac{d}{2}\Bigr)\,, \tag{21}$$

hence (since Δ is homogeneous of degree r)

$$\Delta(-\partial/\partial y)\,\Delta(y)^{-\alpha} \;=\; \Delta(y)^{-\alpha-1}\prod_{j=0}^{r-1}\Bigl(\alpha-j\frac{d}{2}\Bigr)\,. \tag{22}$$

It follows from (20) and (22) that $\mathscr{R}_\alpha$ is *not* a positive measure when $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. But using the convolution equation (8b) with $\beta = d/2$ together with the fact that $\mathscr{R}_{d/2}$ is a positive measure [Proposition 3.3(a)], we conclude successively that $\mathscr{R}_\alpha$ is not a positive measure when $(k-1)\frac{d}{2} < \alpha < k\frac{d}{2}$ for *any* integer $k \le r-1$. This leaves only negative multiples of $d/2$; and the argument given after Lemma 3.6 shows that $\mathscr{R}_\alpha$ is not a positive measure whenever $\alpha < 0$.(6) □

Remarks. — 1. This method has been used recently by Letac and Massam [22, proof of Proposition 2.3] to determine the set of acceptable powers p for the noncentral Wishart distribution, generalizing the earlier proof of Shanbhag [27] and Casalis and Letac [9] for the ordinary Wishart distribution (which is essentially Theorem 1.1).

2. A very different proof of Theorem 1.1 for the cases $d = 1, 2$, using zonal polynomials, was given by Peddada and Richards [25, Theorems 1 and 3].

But this is not yet the end of the story; the proof can be further simplified. The use of the Laplace transform in the foregoing proof is in reality a red herring, as it is used *twice* in opposite directions: once in the proof of Proposition A.1, and once again in the proof of (21).(7) We can therefore give a direct proof that makes almost no reference to the Laplace transform:

Second proof of Theorem 1.1. — Consider first $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. If $\mathscr{R}_\alpha$ is a positive measure, then so is $\Delta(x)\,\mathscr{R}_\alpha$, which by (8d) equals $C_\alpha \mathscr{R}_{\alpha+1}$,

(6) *Alternate argument:* For $k = 1, 2, 3, \ldots$ we know from Proposition 3.3(a,b) and (9) that $\mathscr{R}_{kd/2}$ is a positive measure that is not supported on a single point. If $\mathscr{R}_{-kd/2}$ were a positive measure (recall that we know it is nonzero), then $\mathscr{R}_{kd/2} * \mathscr{R}_{-kd/2}$ could not be supported on a single point, contrary to the fact that $\mathscr{R}_{kd/2} * \mathscr{R}_{-kd/2} = \delta$ [cf. (8a,b)].

(7) The simplest proof of (21) is probably the one given in [12, Proposition VII.1.4], using Laplace transforms. However, direct combinatorial proofs are also possible: see [8] for a detailed discussion in the cases of real symmetric and complex hermitian matrices.

where

$$C_\alpha = \prod_{j=0}^{r-1}\Big(\alpha - j\frac{d}{2}\Big) < 0\,. \tag{23}$$

It follows that $\mathscr{R}_{\alpha+1}$ must be a negative (i.e. nonpositive) measure. But this is surely not the case, as the Laplace-transform formula (10) immediately implies that *no* $\mathscr{R}_\beta$ can be a negative measure.[8] This shows that $\mathscr{R}_\alpha$ is not a positive measure when $(r-2)\frac{d}{2} < \alpha < (r-1)\frac{d}{2}$. The proof is then completed as before.[9] □

It would be interesting to know whether this approach is powerful enough to handle the multiparameter Riesz distributions [12, Theorem VII.3.2] and/or the Riesz distributions on homogeneous cones that are not symmetric and hence do not arise from a Euclidean Jordan algebra [13, 20].

To conclude, let us give the promised strong converse to Proposition A.1:

Proposition A.2. — *Let $T \in \mathscr{D}'(V)$ be a distribution whose Laplace transform is well-defined on a nonempty open set $\Theta \subseteq V^*$. Let $S \subseteq V$ be a closed set, and suppose that there exists $y_0 \in \Theta$ such that $[P(-\partial)\mathscr{L}(T)](y_0) \geq 0$ for all polynomials P on V that are nonnegative on S. Then T is in fact a positive measure that is supported on S.*

Proof. — By replacing $T(x)$ by $e^{-\langle y_0, x\rangle}T(x)$, we can assume without loss of generality that $y_0 = 0$. Then the derivatives of $\mathscr{L}(T)$ at the origin give us the moments of T; and the hypothesis $[P(-\partial)\mathscr{L}(T)](y_0) \geq 0$ implies, by Haviland's theorem [16, 17] [23, Theorem 3.1.2], that there exists a positive measure μ supported on S that has these moments. Furthermore, the analyticity of $\mathscr{L}(T)$ in the open set $\Theta + iV^*$ implies that these moments satisfy a bound of the form $|c_{\mathbf{n}}| \leq AB^{|\mathbf{n}|}\mathbf{n}!$, so that $\int e^{\epsilon|x|}\, d\mu(x) < \infty$ for some $\epsilon > 0$. It follows that the Laplace transform $\mathscr{L}(\mu)$ is well-defined and analytic in a neighborhood of the origin; and since its derivatives at the origin agree with those of $\mathscr{L}(T)$, we must have $\mathscr{L}(\mu) = \mathscr{L}(T)$. But by the injectivity of the distributional Laplace transform [26, p. 306, Proposition 6], it follows that $\mu = T$. □

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(8) It would be interesting to know whether this residual use of the Laplace transform can be avoided. For $d \leq 2$ it can definitely be avoided, as $\alpha + 1 > (r-1)\frac{d}{2}$, so that $\mathscr{R}_{\alpha+1}$ is a nonzero positive measure by Proposition 3.3(b); but for $d > 2$ I do not know.

(9) The argument given after Lemma 3.6 explicitly uses the Laplace transform. But the alternate argument given in footnote 6 does not.

In Proposition A.2 it is essential that the Laplace transform of T be well-defined on a nonempty open set $\Theta \ni y_0$, or in other words (when $y_0 = 0$) that T have some exponential decay at infinity [in the sense that $\cosh(\epsilon|x|)T \in \mathscr{S}'(V)$ for some $\epsilon > 0$]. It is *not* sufficient for T to have finite moments of all orders satisfying $T(P) \geq 0$ for all polynomials P on V that are nonnegative on S. Indeed, Stieltjes' [28] famous example

$$f(x) = \begin{cases} e^{-\log^2 x} \sin(2\pi \log x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \tag{24}$$

belongs to $\mathscr{S}(\mathbb{R})$ and has zero moments of all orders [i.e. $T(P) = 0$ for all polynomials P] but is not nonnegative.

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