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Pietro Corvaja & Umberto Zannier

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AN *abcd* THEOREM OVER FUNCTION FIELDS AND APPLICATIONS

BY PIETRO CORVAJA & UMBERTO ZANNIER

ABSTRACT. — We provide a lower bound for the number of distinct zeros of a sum $1+u+v$ for two rational functions u, v, in term of the degree of u, v, which is sharp whenever u, v have few distinct zeros and poles compared to their degree. This sharpens the "*abcd*-theorem" of Brownawell-Masser and Voloch in some cases which are sufficient to obtain new finiteness results on diophantine equations over function fields. For instance, we show that the Fermat-type surface $x^a + y^a + z^c = 1$ contains only finitely many rational or elliptic curves, provided $a \geq 10^4$ and $c \geq 2$; this provides special cases of a known conjecture of Bogomolov.

RÉSUMÉ (*Un théorème abcd sur les corps de fonctions et applications*)

Nous démontrons une minoration pour le nombre de zéros distincts d'une somme $1+u+v$, u, v étant deux fonctions rationnelles, en fonction du degré de u et v; cette minoration est forte si le nombre de zéros et poles de u, v est suffisament petit par rapport à leur degré. Dans certains cas, on obtient une amélioration de l'inégalité de Voloch et Brownawell-Masser, qui entraîne des nouveaux résultats de finitude sur les équations diophantiennes sur les corps de fonctions.

Par exemple, nous démontrons que la surface de type Fermat définie par l'équation $x^a+y^a+z^c = 1$ ne contient qu'un nombre fini de courbes rationnelles ou elliptiques, dès que $a \geq 10^4$ et $c \geq 2$. Ce résultat constitue un cas particulier d'une célèbre conjecture de Bogomolov.

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PIETRO CORVAJA, Dip. di Matematica e Informatica, Via delle Scienze, 206, 33100 Udine, Italy • *E-mail :* `pietro.corvaja@dimi.uniud.it`

UMBERTO ZANNIER, Scuola Normale Superiore, Piazza dei Cavalieri, 7, 56100 Pisa, Italy • *E-mail :* `u.zannier@sns.it`

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1. Introduction

We shall be interested in diophantine equations over function fields, involving S-units. We start by recalling a few definitions. Let κ be an algebraically closed field of characteristic zero, $\mathscr{C}$ a smooth complete curve of genus g defined over κ and S be a finite set of points of $\mathscr{C}$. By S-unit we mean a rational function $u \in \kappa(\mathscr{C})$ having all its poles and zeros on S. The group of S-units will be denoted by $\mathscr{O}_S^*$. Let χ be the Euler characteristic of $\mathscr{C} \setminus S$, i.e.

$$\chi = 2g - 2 + \sharp(S).$$

In the sequel we shall suppose $\sharp(S) \geq 2$, otherwise there would be no nonconstant S-units; note that this yields $\chi \geq 0$. We shall use the following notion of height relative to the function field $\kappa(\mathscr{C})$: if $(x_0 : \cdots : x_n) \in \mathbf{P}_n(\kappa(\mathscr{C}))$, we put

$$H(x_0 : \cdots : x_n) = -\sum_{\nu \in \mathscr{C}} \min\{\nu(x_0), \ldots, \nu(x_n)\},$$

where, for each $\nu \in \mathscr{C}(\kappa)$, we denote by the same letter $\nu : \kappa(\mathscr{C})^* \to \mathbf{Z}$ the order function corresponding to such point. So $H(1 : x)$ is the degree of the function x, which will also be written as $H(x)$.

Let u, v be S-units. We shall put

$$z := u + v + 1. \tag{1.1}$$

Whenever z is assumed to be an S-unit, (1.1) is the S-unit equation in three variables; the case of two variables gives rise to the so called *abc* theorem of Mason and Stothers, stating that *if* $z = 0$ *in* (1.1) *and* u, v *are not both constant, we have* $H(1 : u : v) \leq \chi$. This is best possible.

In (1.1), to avoid consideration of "trivial" cases, we shall assume that no subsum on the right vanishes. Under this condition, we have the inequalities of Brownawell-Masser, generalising the *abc* theorem; the case of four terms a, b, c, d implies for (1.1) that $H(1 : u : v : z) \leq 3\chi$, where the coefficient 3 replaces the previous coefficient 1. The constant 3 is sharp, in view of an example of Browkin-Brzezinski: $\kappa(\mathscr{C}) = \kappa(t)$, $u = -t^3, v = (t-1)^3, z = 3t(t-1)$. However, one expects improvements on the coefficient 3 under supplementary conditions; in fact a conjecture of Vojta predicts an estimate $H(1 : u : v : z) \leq (1 + \epsilon)\chi$ for every $\epsilon > 0$, provided the point $(1 : u : v)$ does not lie on a curve on $\mathbf{P}_2$, depending only on ϵ. See the discussion at §14.5.26 of [4].

Any improvement on the coefficient 3 may be crucial for applications to diophantine equations over function fields: for instance the equation $y^2 = 1 + u + v$ in S-units u, v and S-integer y escapes from the Brownawell-Masser estimates (which cover e.g. $y^3 = 1+u+v$), but could be treated by the analogue inequality with any coefficient < 2.

To improve on the methods of the paper [5] seems to lead to delicate problems. In the paper [6], by means of different methods, among other things we succeded to treat completely the said equation $y^2 = 1 + u + v$. A crucial ingredient was a bound for the gcd of $u - 1$, $v - 1$, for S-units u, v, which we recall below as Theorem CZ.

In the present paper, we develop certain applications of those results and methods especially to present a kind of general "*abcd* theorem" and some corollaries. Roughly speaking, we obtain the coefficient $1 + \epsilon$, whenever two of the S-units are multiplicatively independent modulo constants and the set of their zeros and poles has cardinality $< \delta\sharp(S)$, for a suitably small function $\delta = \delta(\epsilon, g)$ (see the Corollary to Theorem 1.1). One can easily show that in fact the same estimate holds also if, say, u, v are multiplicatively dependent modulo constants, unless they satisfy a multiplicative dependence relation of degree bounded in terms of ϵ. In other words, we obtain the estimate of Vojta's conjecture, although under a further condition, on the number of zeros of u, v.

This will be done in Theorem 1.1. Then we shall present some other applications, where we preferred simplicity to generality, to better illustrate the methods.

A first application concerns Fermat-type equations of the shape $x^a + y^b + 1 = z^c$, to be solved in non constant polynomials or rational functions $x, y, z \in \kappa(t)$. The quoted inequalities of Brownawell-Masser for instance imply that there are no non trivial polynomial solutions if $1/a + 1/b + 1/c < 1/3$. This condition in particular requires $\min(a, b, c) \geq 4$. Our methods also cover the cases when the minimal exponent is 2 or 3, supposing the remaining two are large enough. The conclusion is that the solutions fall within certain explicitly described families, of which the typical example comes from an identity $(1 + x^a)^2 = 1 + 2x^a + x^{2a}$. This will be done in Theorem 1.2. We remark that the method also works for rational functions x, y, z on higher genus curves; in that case one can obtain bounds for the degrees of the solutions in terms of the genus.

This application can be viewed as a non-existence or finiteness statement for rational curves on a fixed surface. In Theorem 1.2 bis we also consider genus one curves with a finiteness statement.

A natural extension concerns the study of sections of a fibration whose fibers are surfaces. For instance we can consider the case of a fibration of the form $X \to \mathscr{C}$, for a threefold X and a curve $\mathscr{C}$, where the generic fiber is a Fermat-like surface. This amounts for instance to an equation of the form $f(t)x^a + g(t)y^b + h(t) = z^c$, for coefficients $f(t), g(t), h(t) \in \kappa(t)$. We can obtain a bound for the degree of the solutions; again, we do not state it explicitly, and leave it to the interested reader.

Going back to (1.1), we shall assume that

$$z \neq 0, 1, u, v \tag{1.2}$$

and let S_z be the minimal set containing S such that z is an S_z-unit: it is the union

$$S_z := S \cup z^{-1}(0).$$

We also put

$$\tilde{H} := H(1 : u : v) = H(1 : u : v : z), \qquad H^* := \tilde{H} + \chi + \sharp S.$$

Since we are assuming $\sharp(S) \geq 2$, we have

$$H^* > \tilde{H} = H(1 : u : v : z) \geq \max\{H(u), H(v), H(z)\}.$$

Our first result is a lower bound for the number of zeros of z; this is sharp when χ is fixed, or small compared to $\tilde{H}$:

THEOREM 1.1. — *Let u, v be S-units, not both constant, let $z = u+v+1$ satisfy (1.2). Then if u, v are multiplicatively independent modulo κ^* the number of zeros of z outside S satisfies the lower bound*

$$\sharp(S_z \setminus S) = \sum_{\nu \notin S;\, \nu(z)>0} 1 \geq \tilde{H} - 15\chi - 6 \cdot {H^*}^{2/3} \chi^{1/3}.$$

If instead u, v are multiplicatively dependent modulo constants, there is a relation $u^r = \lambda v^s$ with $\lambda \in \kappa^*$, r, s coprime integers, and

$$\sum_{\nu \notin S;\, \nu(z)>0} 1 \geq \tilde{H}\left(1 - \frac{1}{\max(|r|, |s|)}\right) - 15\chi.$$

REMARK. — Since $\tilde{H} \geq \deg(z)$, the above estimates can also be viewed as upper bounds for the number of multiple zeros of z. Note that when z is a square, the left-hand side above is $\leq H(z)/2$. In this case we obtain a bound for its height in term only of χ, unless u, v are multiplicatively dependent modulo κ^*, satisfying a relation with exponents ≤ 2. In that case, however, z can be a square as follows from the identity: $z = (w+w^{-1}/2)^2 = w^2+w^{-2}/4+1$. We also note that qualitative estimates for the number of multiple zeros of polynomial expressions $P(u, v)$ for S-units u, v appear in [C-Z, Theorem 1.3]. The above result provides a completely explicit estimate in the case $P(u, v) = 1 + u + v$; this may also be seen as a special case of an *abcd* theorem.

It will be convenient also to give an alternative statement, of the first case, in terms of an upper bound for H^*:

THEOREM 1.1*. — *In the above notation, when u, v are multiplicatively independent modulo κ^* the following inequality holds*

$$H^* \leq \left(2^{14/3}\chi^{1/3} + \left(\sharp(S_z) + 16\chi\right)^{1/3}\right)^3;$$

in particular

$$\tilde{H}^{1/3} < H^{*1/3} \leq \left(\sharp(S_z) + 16\chi\right)^{1/3} + (2^{14}\chi)^{1/3}.$$

As a consequence of any of the above theorems one can immediately obtain a result in the direction of Vojta's conjecture (which we give without proof):

COROLLARY. — *For every positive ϵ there exists a number $\delta = \delta(\epsilon, g) > 0$ such that if $\sharp(S) \leq \delta H^*$ and u, v are multiplicatively independent modulo κ^*,*

$$H^* < (1+\epsilon)\sharp(S_z).$$

The second part of Theorem 1.1 allows to add further precision in this corollary: the conclusion holds also in case u, v are multiplicatively dependent modulo κ^*, but with minimal relation $u^r = \lambda v^s$, $\lambda \in \kappa^*$ and r, s coprime integers sufficiently large with respect to ϵ. (For the remaining u, v, the relation $u^r = \lambda v^s$ has bounded exponents; it leads to $u = t^s, v = \mu t^r$, for a certain $\mu \in \kappa^*, t \in \kappa(\mathscr{C})$. At his point, on factoring the polynomial $1 + t^s + \mu t^r$, by means of the *abc* inequality one can easily prove the same conclusion, unless μ lies in a finite set depending on ϵ. This provides the finitely many curves predicted by Vojta's conjecture.)

A natural application of Theorem 1.1 concerns diophantine equations in rational functions; for instance we can deduce

THEOREM 1.2. — *Let a, b be positive integers with*

$$\frac{1}{a} + \frac{1}{b} < 2.5 \cdot 10^{-4}. \qquad (*)$$

Let $x(t), y(t) \in \kappa(t)$ be non constant rational functions. If the function $x(t)^a + y(t)^b + 1$ is a perfect power in $\kappa(t) \setminus \kappa$ then it is a square. Also, in this case, either $a = 2b$ and $y^{2b} = 4x^a$, or $b = 2a$ and $4y^b = x^{2a}$.

This result can be stated geometrically by saying that certain surfaces of Fermat type, namely $z^c = 1 - x^a - y^b$, contain only finitely many rational curves. This last fact can be extended to genus one curves; our general statement is:

THEOREM 1.2 BIS. — *Let a, c be positive integers with $c \geq 2$ and $a \geq 10^4$. Let $\mathscr{S} \subset \mathbf{A}^3$ be the surface defined by the equation*

$$\mathscr{S}: \quad x^a + y^a + z^c = 1.$$

Then $\mathscr{S}$ contains only finitely many (affine) curves of geometric genus ≤ 1.

This confirms a conjecture of Bogomolov for the surface in question. (This conjecture predicts the finiteness of curves of geometric genus ≤ 1 on every surface of general type. By means of a different approach, Bogomolov himself settled a weaker form of the conjecture in [3], covering remarkably general cases of it).

Instead of a finiteness conclusion, under the same assumptions one can bound, by the same method of proof, the degree (in $\mathbf{P}_3$) of the curves on $\mathcal{S}$ of any fixed genus. The proof below, especially inequality (5), implies the bound $10^4(2g-1)$ for curves of genus g.

We conclude by a further simple application of Theorem CZ below to the so-called 'diophantine k-tuples'; there are many variants of this notion; here we deal with the k-tuples of distinct polynomials $a_1, \dots, a_k$ such that all the expressions $1 + a_i a_j$, $i \neq j$, are perfect powers. Recently this polynomial case has been studied, for instance in the paper [8]; Theorem 2 therein states:

There do not exist five distinct polynomials $a_1, \dots, a_5 \in K[t]$ not all constant and such that for $1 \leq i < j \leq 5$, $1 + a_i a_j = x_{ij}^{n_{ij}}$, with polynomials x_{ij} and integers $n_{ij} \geq 7$.

The authors' method relies on the *abc*-theorem for polynomials and does not yield any conclusion for a similar question for three or even four polynomials, no matter how large the lower bound imposed on the exponents n_{ij}, unless further conditions are imposed. Here we show that Theorem CZ yields in a very simple way an impossibility conclusion for diophantine triples of polynomials, up to a well-described exception, provided the n_{ij} are sufficiently large. Namely, we have the following

THEOREM 1.3. — *Let a, b, c be three distinct nonzero complex polynomials a, b, c, not all constant and such that $1 + ab = x^p, 1 + ac = y^q, 1 + bc = z^r$ for complex polynomials x, y, z and integers $p, q, r \geq 864$. Then, after permuting a, b, c, we have $c^2 + 1 = 0$ and $a + b = 2c$.*

Note that the case $c^2 + 1 = 0, b = 2c - a$ gives rise to solutions, on putting $a := -c(W^e - 1)$ for an arbitrary polynomial W and integer e: in this case in fact we have $1 + ab = 1 + 2ca - a^2 = (1 + ac)^2 = W^{2e}$, and $1 + ac = W^e$, $1 + bc = 1 - 2 - ac = (\theta W)^e$ where $\theta^e = -1$.

2. Proofs

We start with some lemmas on S-units:

LEMMA 2.1. — *With the above notation and if* $z = 1 + u + v \neq u, v, 1$, *the height of* z *satisfies*

$$H(z) \geq \tilde{H} - 3\chi.$$

Also

$$\sum_{\nu \in S} \max(0, \nu(z)) \leq 3\chi.$$

Proof. — The present assumptions on z, u, v allow us to apply Theorem 1 in [9], with $\sigma = z$, $u_1 = 1, u_2 = u, u_3 = v$; this gives

$$\sum_{\nu \in S} (\nu(z) - \min\{0, \nu(u), \nu(v)\}) \leq 3\chi.$$

Now, to obtain the first inequality, it suffices to note that $\sum_{\nu \in S} \nu(z) \geq -H(z)$, while, by definition, $-\sum_{\nu \in S} \min\{0, \nu(u), \nu(v)\} = \tilde{H}$. As to the second one, note that every term in the last displayed sum is non negative and that $\min\{0, \nu(u), \nu(v)\} \leq 0$. Hence, for every subset $S' \subset S$, we have $\sum_{\nu \in S'} \nu(z) \leq 3\chi$; in particular this holds when S' is the complement of the set of poles of z in S. □

LEMMA 2.2. — *Let* w *be a non constant* R*-unit for a certain finite set* R. *The number of zeros with multiplicity of the differential* $\mathrm{d}w$ *outside* R *is at most* $2g - 2 + \sharp R$.

Proof. — We can clearly suppose that R is exactly the set of zeros and poles of w. Let $A = w^{-1}(0), B = w^{-1}(\infty)$ so $R = A \cup B$. Then the number of poles with multiplicity of the differential $\mathrm{d}w$ is $H(w) + \sharp(B)$. The number of zeros (with multiplicity) in R is $H(w) - \sharp(A)$, because every zero of $\mathrm{d}w$ in R must be a zero of w (since it cannot be a pole). Since the difference between the total number of zeros and of poles is $2g - 2$, and since there is no pole outside R, there are $\sharp(A) + \sharp(B) + 2g - 2$ zeros with multiplicity outside R. Since $\sharp(A) + \sharp(B) = \sharp(R)$ we obtain the upper bound $\sharp(R) + 2g - 2$ for the number of zeros with multiplicity. □

Our main tool will be the following result drawn from [6], Corollary 2.3, where by "generating relation" we mean a multiplicative dependence relation modulo κ^* with coprime exponents:

THEOREM CZ. — *Let* $u, v \in \mathcal{O}_S^*$ *be* S*-units, not both constant.*

(i) *If* u, v *are multiplicatively independent, we have*

$$\sum_{\nu \notin S} \min\{\nu(1 - u), \nu(1 - v)\} \leq 3\sqrt[3]{2}(H(u)H(v)\chi)^{\frac{1}{3}} \leq 3\sqrt[3]{2}(\tilde{H}^2\chi)^{\frac{1}{3}}.$$

(ii) *If u, v are multiplicatively dependent, let $u^r = \lambda v^s$ be a generating relation. Then either $\lambda \neq 1$ and $\sum_{\nu \notin S} \min\{\nu(1-u), \nu(1-v)\} = 0$, or $\lambda = 1$ and*

$$\sum_{\nu \notin S} \min\{\nu(1-u), \nu(1-v)\} \leq \min\left\{\frac{H(u)}{|s|}, \frac{H(v)}{|r|}\right\} \leq \frac{\tilde{H}}{\max\{|r|, |s|\}}.$$

Proof of Theorem 1.1. — If either of u, v or u/v is constant, there is a multiplicative relation $u^r = \lambda v^s$, with $r, s \in \mathbf{Z}, \lambda \in \kappa^*$ and $\max(|r|, |s|) = 1$. We fall in the second case, and the relevant inequality is trivial. So we shall suppose that $u, v, u/v$ are non-constant. Let us denote by δ the additive operator on $\kappa(\mathscr{C})^*$

$$\delta(f) = \frac{df}{f}.$$

Letting $z = u + v + 1$, for non costant S-units u, v as above, and putting $\phi := \delta u/\delta v$ (note that ϕ is well-defined and nonzero), and is a constant if u, v are multiplicatively dependent modulo constants. We have the easily proved identities

$$\begin{cases} \frac{vdz - zdv}{dv} & = u\,(\phi - 1) - 1 \\ \frac{udz - zdu}{du} & = v\left(\phi^{-1} - 1\right) - 1 \end{cases}$$

We are going to apply Theorem CZ to the functions

$$u_1 := u(\phi - 1), \qquad v_1 = v(\phi^{-1} - 1).$$

We let then S_1 be the minimal set containing S and the zeros of $du, dv, d(u/v)$. We note that

$$\sharp(S_1) \leq \sharp(S) + 3\chi \tag{2.1}$$

To justify this formula, it suffices to apply three times Lemma 2.2, with $w = u, v, u/v$ and $R = S$. From the identities $\phi = \frac{(du)v}{(dv)u}$ and $\phi - 1 = \frac{v^2 d(u/v)}{(dv)u}$ it follows immediately that $\phi, \phi - 1$, hence u_1, v_1, are S_1-units.

Next, observe that if a point $P \in \mathscr{C}$ does not lie in S_1 and is a zero of z of order l, then it is a zero of both $u_1 - 1, v_1 - 1$ of order at least $l - 1$. Therefore

$$\sum_{\nu \notin S_1,\, \nu(z) > 0} (\nu(z) - 1) \leq \sum_{\nu \notin S_1} \min(\nu(u_1 - 1), \nu(v_1 - 1)). \tag{2.2}$$

To go on, we distinguish two cases:

First case. — u_1, v_1 are multiplicatively independent modulo constants. This implies that u, v too are multiplicatively independent modulo constant. In this case, part *(i)* of Theorem CZ, applied with S_1 instead of S and u_1, v_1 instead of u, v, gives the bound

$$\sum_{\nu \notin S_1} \min(\nu(u_1 - 1), \nu(v_1 - 1)) \leq 3\sqrt[3]{2}(H(u_1)H(v_1)\chi_1)^{1/3},$$

where $\chi_1 = 2g - 2 + \sharp(S_1)$. To estimate $H(u_1), H(v_1)$ we note that $\max(H(u_1), H(v_1)) \leq \tilde{H} + \deg\phi$. Moreover, looking at the poles, we see that $\deg\phi$ is at most the sum of number of poles of δu and the number of zeros of δv, both counted with multiplicity. But δu has only simple poles, contained in S, while the number of zeros of δv is bounded by χ in view of Lemma 2.2 (applied with $R=$ set of zeroes and poles of v). Hence $\max(H(u_1), H(v_1)) \leq \tilde{H} + \sharp(S) + \chi = H^*$. We have reached the following inequality

$$\sum_{\nu\notin S_1:\nu(z)>0} (\nu(z)-1) \leq 3\cdot 2^{1/3} H^{*2/3}\chi_1^{1/3}.$$

As to χ_1 we have, taking into account (2.1), the upper bound $\chi_1 \leq 4\chi$, so

$$\sum_{\nu\notin S_1:\nu(z)>0} (\nu(z)-1) \leq 6 H^{*2/3}\chi^{1/3}.$$

To estimate the number of zeros of z inside S_1 we just use the second part of Lemma 2.1, with S_1 in place of S (which is legitimate). This yields

$$\sum_{\nu\in S_1} \max(0,\nu(z)) \leq 3\chi_1 \leq 12\chi.$$

We express $H(z)$ as $\sum_\nu \max(0,\nu(z))$; this sum can be decomposed into three parts:

$$\sum_{\nu}\max(0,\nu(z)) = \sum_{\nu\in S_1}\max(0,\nu(z)) + \sum_{\nu\notin S_1:\nu(z)>0} 1 + \sum_{\nu\notin S_1:\nu(z)>0}(\nu(z)-1). \quad (4)$$

Taking into account the previous result we obtain

$$H(z) \leq 12\chi + \sum_{\nu\notin S_1:\nu(z)>0} 1 + 6H^{*2/3}\chi^{1/3},$$

whence by the first part of Lemma 2.1

$$\tilde{H} \leq \sum_{\nu\notin S_1:\nu(z)>0} 1 + 15\chi + 6H^{*2/3}\chi^{1/3}.$$

whence the inequality of Theorem 1.1, taking into account that $\sum_{\nu\notin S_1:\nu(z)>0} 1 = \sharp(S_z\setminus S_1) \leq \sharp(S_z\setminus S)$.

Second case. — u_1, v_1 are multiplicatively dependent modulo constants. We show that the same holds for u, v, by following [6, Lemma 3.14]. We first note that the multiplicative dependence of u_1, v_1 modulo constants amounts to the linear dependence of $\delta u_1, \delta v_1$ over the rationals. We now recall that, putting $\phi = \delta u/\delta v$, we have

$$\alpha = \frac{1}{\phi - 1},\ \beta = \frac{-\phi}{\varphi - 1},\ \phi = \frac{\delta u}{\delta v},$$

so $\alpha+\beta+1=0$. By taking differentials,

$$\alpha\delta\alpha+\beta\delta\beta=0.$$

On the other hand, from the definition of φ,

$$\alpha\delta u+\beta\delta v=0.$$

From the last two displayed identities it follows that the matrix

$$\begin{pmatrix}\delta\alpha & \delta\beta\\ \delta u & \delta v\end{pmatrix}$$

has rank ≤ 1. From the definitions of u_1, v_1, namely $u_1=u/\alpha, v_1=v/\beta$ we obtain

$$\delta u=\delta\alpha+\delta u_1$$
$$\delta v=\delta\beta+\delta v_1.$$

Replacing in the above matrix $\delta\alpha,\delta\beta$ by their expressions in term of $\delta u,\delta u_1,\delta v,\delta v_1$ we obtain

$$\operatorname{rank}\begin{pmatrix}\delta\alpha & \delta\beta\\ \delta u & \delta v\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\delta u_1-\delta u & \delta v_1-\delta v\\ \delta u & \delta v\end{pmatrix}=\operatorname{rank}\begin{pmatrix}\delta u_1 & \delta v_1\\ \delta u & \delta v\end{pmatrix}\le 1$$

Then the ratio $\delta u/\delta v$ must be equal to the ratio $\delta u_1/\delta v_1$, which is a rational number by the multiplicative dependence of u_1, v_1. Hence u, v are also multiplicatively dependent modulo constants and satisfy the same multiplicative relation modulo κ^* as u_1, v_1. By taking a suitable root, we can obtain a relation of the kind $u^r=\lambda v^s$ with $\lambda\in\kappa^*$, r, s coprime integers, and an analogous relation $u_1^r=\lambda_1 v_1^s$.

We use again (4) and the same estimates as above for all terms but the last one, for which we now apply (ii) of Theorem CZ, obtaining

$$\sum_{\nu\notin S_1;\,\nu(z)>0}(\nu(z)-1)\le\frac{\tilde{H}}{\max(|r|,|s|)}.$$

Arguing exactly as before, and using the above estimate, we arrive at the second inequality of Theorem 1.1. □

Proof of Theorem 1.2. — In the first part of the proof we shall argue, more generally, with non-constant rational functions $x, y\in\kappa(\mathscr{C})$, on an arbitrary curve $\mathscr{C}$ of genus g. This shall be useful also in the proof of Theorem 1.2 bis. At the end of this proof, we shall specialize to $\mathscr{C}=\mathbf{P}_1$, $g=0$.

We the suppose that x^a+y^b+1 is a perfect power in $\kappa(\mathscr{C})\setminus\kappa$. We apply Theorem 1.1 with $u=x^a, v=y^b, z=1+u+v$ and S the set consisting of the zeros and poles of u, v. The assumptions $z\ne 0,1,u,v$ are plainly satisfied by the present hypotheses that $x^a, y^b, x^a + y^b + 1$ are non-constant. We have $\chi = \sharp(S) - 2 + 2g$ and

$$\tilde{H} \geq \max(a \deg x, b \deg y), \qquad H^* = \tilde{H} + 2\sharp(S) - 2 + 2g$$

We are supposing that z is a perfect power, so we write $z = w^m$ with $m \geq 2$ an integer and a rational function $w \in \kappa(\mathscr{C})$; then z has at most $H(z)/m \leq H(z)/2 \leq \tilde{H}/2$ distinct zeros, therefore

$$\sharp(S_z \setminus S) \leq \frac{\tilde{H}}{2}.$$

Suppose first u, v are multiplicatively independent modulo constants, which amounts to the same condition on x, y. Then, taking into account the last displayed inequality, the first inequality of Theorem 1.1 gives

$$\frac{\tilde{H}}{2} \leq 15(\sharp(S) - 2 + 2g) + 6(\tilde{H} + 2\sharp(S) - 2 + 2g)^{2/3}(\sharp(S) - 2 + 2g)^{1/3}.$$

Let us put $\xi := (\sharp(S) + 2g - 1)/\tilde{H}$; then the last displayed formula yields:

$$\frac{1}{2} \leq 15\xi + 6(1 + 2\xi)^{2/3}\xi^{1/3}$$

one may check that this inequality entails $\xi > 5 \cdot 10^{-4}$. On the other hand the definition of S yields

$$\sharp(S) \leq 2\tilde{H}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Comparing with the bound for ξ we have

$$\left(5 \cdot 10^{-4} - \frac{2}{a} - \frac{2}{b}\right)\tilde{H} \leq 2g - 1. \tag{5}$$

Note that if we assume that a, b are large enough, this inequality gives a bound for the height in terms of the genus.

In the case $g = 0$ of Theorem 1.2 the assumption on a, b gives a contradiction.

Let us now treat the case where u, v, hence x, y, are multiplicatively dependent modulo constants, again for arbitrary genus. In this case we may write $u^r = \lambda v^s$, with $\lambda \in \kappa^*$ and r, s coprime integers. This time we apply the second part of Theorem 1.1 obtaining

$$\frac{\tilde{H}}{m} \geq \tilde{H}\left(1 - \frac{1}{\max(|r|, |s|)}\right) - 15(\sharp(S) - 2 + 2g).$$

Using that $\sharp(S) - 2 < 2\tilde{H}(1/a + 1/b)$, we obtain after dividing both sides by $\tilde{H}$

$$\frac{1}{m} + \frac{30}{a} + \frac{30}{b} + \frac{1}{\max(|r|, |s|)} + \frac{30g}{\tilde{H}} > 1. \tag{6}$$

Back again to the case $g = 0$ of Theorem 1.2, if $m \geq 3$, this forces $|r| = |s| = 1$, so x^a is a constant multiple of y^b. Then $x^a + y^b$ would be a perfect a-th power, non zero by hypothesis. Since the curve $\gamma X^a + 1 = Z^m$ has positive genus, we have a contradiction. Then $m = 2$ and $\max(r, s) \leq 2$, which forces as before $\max(|r|, |s|) = 2$, so $\{r, s\} = \{\pm 1, \pm 2\}$.

The multiplicative dependence relation $u^r = \lambda v^s$ leads to the existence of a rational function w and a constant γ such that $x = w^\alpha, y = \gamma w^\beta$, for integers α, β with $r\alpha a = s\beta b$. Then, putting $A = \alpha a$, $B = \beta b$, the curve

$$1 + W^A + \gamma^b W^B = Z^2$$

has a component of genus zero. Note that the ratio A/B is ± 2 or $\pm 1/2$, and by symmetry it is easy to reduce to the case $A = 2B$. If the polynomial $1 + \gamma^b W^B + W^{2B}$ is not a square it must have only simple zeros, but then the above curve would be irreducible of positive genus. Then the polynomial is a square, and this leads to the listed cases. □

Proof of Theorem 1.2 bis. — We start by noting that a curve on the surface $\mathcal{S}$ corresponds to a morphism $\mathcal{C} \to \mathcal{S}$ from an abstract smooth curve $\mathcal{C}$. This, in turn, corresponds to a solution in $\kappa(\mathcal{C})$ to the equation of the surface $\mathcal{S}$. Theorem 1.2 immediately provides the finiteness of the set of curves of geometric genus zero on $\mathcal{S}$. Concerning curves of any fixed genus g, inequality (5) or inequality (6) (depending on whether x, y are multiplicatively dependent or not) provides the bound for the degree depending on g; by degree we mean the degree in the projective completion.

The existence of infinitely many curves of bounded degree implies the existence of an algebraic family of such curves; for large genus such families exist and in some cases even having the same function field: namely our surface may contain a Zariski dense set of rational points over certain function fields of curves. This last fact can be seen for instance by cutting with the plane $z =$ const. In the case of genus one, however, we can prove finiteness. As already noted, we may first obtain a bound on the degree; now, if we had infinitely many pairs (E, ϕ) for an elliptic curve E and a rational map $\phi : E \to \mathcal{S}$, of bounded degree, with a Zariski dense union of images, by looking at the j-invariants we would obtain a dominant morphism $\pi : \mathcal{E} \to \mathcal{S}$ from an (affine) elliptic surface $\mathcal{E}$ to $\mathcal{S}$. This is excluded by the fact that $\mathcal{S}$ is a surface of general type; see the Appendix, Proposition A and its corollary. □

Proof of Theorem 1.3. — Suppose a, b, c be as above and let, without loss of generality, $\delta := \deg a \geq \deg b \geq \deg c$. We shall apply Theorem CZ with the following data:

We choose the curve $\mathcal{C} = \mathbf{P}_1$, and we set $u := 1 + ab, v := 1 + ac$, so u, v are rational functions on $\mathcal{C}$, not both constant. We let S be the union of the infinite point of $\mathcal{C}$ with the set of zeros of u and of v. By assumption we have

$u = x^p, v = y^q$ with polynomials u, v and integers $p, q \geq 864$. We also have $\deg u, \deg v \leq 2\delta$, so $\deg x, \deg y \leq \delta/432$, whence

$$\#S \leq 1 + \frac{\delta}{216}.$$

Suppose first that u, v are multiplicatively dependent, so $1 + ab = w^m, 1 + ac = w^n$ for a suitable polynomial w and coprime integers $n, m \geq 0$, where necessarily $m \geq n > 0$. Plainly the gcd of the polynomials $w^m - 1, w^n - 1$ is $w - 1$, and also a multiple of a, whence

$$\delta \leq \deg w.$$

But $m \deg w = \deg a + \deg b \leq 2\delta \leq 2 \deg w$. So $m \leq 2$ and this implies $n = 1$ and $1 + ab = 1 + 2ac + a^2c^2$, i.e. $b = 2c + ac^2$, forcing c to be constant and $\deg b = \deg a = \delta$.

Now, the equation $z^r = c^2y^q + c^2 + 1$ implies $c^2 + 1 = 0$, because otherwise the curve $Z^r = c^2Y^q + c^2 + 1$ has positive genus. In this case we fall in the final possibility in the statement.

Hence in the sequel we may assume that u, v are multiplicatively independent complex polynomials, not both constant. Observe that a is a common divisor of $u - 1, v - 1$ and a has no zeros in S, hence an application of Theorem CZ yields

$$\delta \leq 3\sqrt[3]{2}(\deg u \deg v(\#S - 2))^{1/3} \leq 3\sqrt[3]{2}(4\delta^2(\frac{\delta}{216} - 1))^{1/3} < 3\sqrt[3]{2}(4\delta^2\frac{\delta}{216})^{1/3} = \delta.$$

This contradiction proves the theorem. □

Appendix

Our aim is to complete the proof of Theorem 1.2 bis, by proving that the surface $\mathcal{S}$ appearing in the statement of Theorem 1.2 is of general type, which in turn implies via Theorem 1.2 the finiteness of curves of genus one on $\mathcal{S}$.

We recall some facts from classical theory of algebraic surfaces.

Let $\tilde{\mathcal{S}}$ be a smooth projective surface. A divisor D on $\tilde{\mathcal{S}}$ is said to be big if

$$\lim\sup_{n\to\infty} (h^0(nD)/n^2) > 0.$$

Equivalently (see [7], chap I, Corollary 1.30 and ex. 8), a divisor is big if it admits a positive multiple $nD \sim A + B$, which is linearly equivalent to the sum of an ample divisor A and an effective divisor B. Another equivalent condition is that the linear system $|nD|$, for (all) sufficiently large integers n, provides a dominant rational map to a surface. Using this third condition for bigness, we obtain the following:

FACT 1. — *Let $\pi : \tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ be a dominant morphism of smooth projective surfaces. For every big divisor D on $\tilde{\mathcal{S}}$, the pull-back $\pi^*(D)$ is a big divisor on $\tilde{\mathcal{X}}$.*

A smooth projective surface $\tilde{\mathcal{S}}$ is said to be of *general type* if its Kodaira dimension is maximal, i.e. equal to 2. Equivalently, it is of general type if one (hence every) canonical divisor $K_{\tilde{\mathcal{S}}}$ is big.

If $\pi : \tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ is a dominant morphism of smooth projective surfaces, and $K_{\tilde{\mathcal{S}}}$ denotes a canonical divisor for $\tilde{\mathcal{S}}$, then a canonical divisor for $\tilde{\mathcal{X}}$ is

$$K_{\tilde{\mathcal{X}}} = \pi^*(K_{\tilde{\mathcal{S}}}) + \mathrm{Ram}(\pi),$$

where $\mathrm{Ram}(\pi)$ denotes the ramification divisor (see [7], formula (1.11)). The latter is an effective divisor and its support contains in particular all the curves which are contracted by π to a point (note that we do not suppose that π is a finite morphism).

From the above relation, Fact 1 and the fact that the sum of a big and an effective divisor is big, we obtain the well-known

FACT 2. — *If $\pi : \tilde{\mathcal{X}} \to \tilde{\mathcal{S}}$ is a dominant morphism of smooth projective surfaces, and $\tilde{\mathcal{S}}$ is of general type, then $\tilde{\mathcal{X}}$ is also of general type.*

We say that a smooth affine surface is of *general type* if it admits a smooth compactification which is of general type. Then it is known that every compactification will be a surface of general type.

PROPOSITION A. — *Let a, c be positive integers with $c \geq 2$ and*

$$a > \frac{3c}{c-1}. \tag{A1}$$

Then the surface $\mathcal{S} \subset \mathbf{A}^3$ of equation

$$\mathcal{S}: \quad x^a + y^a + z^c = 1$$

is of general type.

REMARK. — Whenever c divides a, it can be proved that the converse implication also holds, so $\mathcal{S}$ is of general type if and only if (A1) holds. Take e.g. $a = 6, c = 2$, where we would have an equality in (A1): it is known (see [2, remarque VIII.16]) that a degree two cover ramified over a smooth sextic is a K-3 surface, so its canonical divisor vanishes.

COROLLARY. — *Let $\mathcal{E}$ be either an elliptic or a ruled surface. Then no rational map $\mathcal{E} \to \mathcal{S}$ is dominant.*

Proof of the corollary. — Applying Fact 1, we can replace the surface $\mathscr{E}$ by a smooth projective surface $\tilde{\mathscr{E}}$ birational to $\mathscr{E}$, and reduce to the case of a morphism $\pi : \tilde{\mathscr{E}} \to \tilde{\mathscr{S}}$, for some smooth compactification $\tilde{\mathscr{S}}$ of $\mathscr{S}$. By Fact 2, if π were dominant, $\tilde{\mathscr{E}}$ would also be of general type. Now, if $\mathscr{E}$ is ruled, its Kodaira dimension is $-\infty$ (see [2, Chap. VII]), so it is not of general type. If it is elliptic (in the sense of [2, IX.2 b]), and not ruled, then its Kodaira dimension is zero or one [2, IX.3]. □

Before proving the Proposition, let us recall some facts about surface singularities. We are interested in surfaces locally (i.e. analytically) isomorphic to a surface of equation

$$z^c = f(x, y),$$

for a polynomial $f(x, y) \in \mathbf{C}[x, y]$, without multiple factors, and an integer $c \geq 2$. More generally, x, y can be taken to be local parameters at a given point on an affine smooth surface U. We shall identify the relevant points with their x, y coordinates (remembering that every smooth surface is analytically locally isomorphic to the plane).

It is easy to check that such a surface is smooth over the points $(x_0, y_0) \in U$ on which f does not vanish, or (x_0, y_0) is a simple point of the plane curve of equation $f(x, y) = 0$. If, on the contrary, (x_0, y_0) is a singular point of the curve $f(x, y) = 0$, then the equation $z^c = f(x, y)$ defines a surface singular at $(x_0, y_0, 0)$. In this case a minimal desingularisation is obtained by repeatedly blowing up a point over the singular point (x_0, y_0) of the curve $f(x, y) = 0$; a single blow-up suffices if $c = 2$ and (x_0, y_0) is a simple node for the curve $f(x, y) = 0$. In our case, all we need to keep in mind is that the smooth surface $\mathscr{S}$ obtained in this way is endowed with a projection $\mathscr{S} \to U$ such that: 1) each point of U outside the curve $f(x, y) = 0$ has c distinct pre-images; 2) each smooth point of the curve $f(x, y) = 0$ has a single pre-image; 3) the pre-image of the singular points of the curve $f(x, y) = 0$ form a finite union of divisors with negative self-intersection (a single divisor E with $E^2 = -2$ if $c = 2$). In the proof of Proposition A, the regular function f will be either the polynomial $x^a + y^a - 1$ or a local equation for the union of the line at infinity and the curve $x^a + y^a = 1$. In any case, the singularities of the curve $f(x, y) = 0$, if any, are of normal crossing type.

Proof of Proposition A. — Consider the projection $\pi : \mathscr{S} \to \mathbf{A}^2$ sending $(x, y, z) \to (x, y)$: this is a cyclic cover of the affine plane, ramified over the (Fermat) curve $\mathscr{C} \subset \mathbf{A}^2$ given by the equation

$$\mathscr{C} : \quad x^a + y^a = 1.$$

We want to construct a surface $\tilde{\mathscr{S}}$ and a degree c map $\pi : \tilde{\mathscr{S}} \to \mathbf{P}_2$, where $\tilde{\mathscr{S}}$ is projective, smooth and birational to $\mathscr{S}$. Denote by $\bar{\mathscr{C}}$ the projective closure of

$\mathscr{C}$ in $\mathbf{P}_2$ and let $H \subset \mathbf{P}_2$ be the line at infinity. It will turn out that the cover $\pi : \tilde{\mathscr{S}} \to \mathbf{P}_2$ will be ramified over the curve $\bar{\mathscr{C}}$ and, in case the integer c does not divide a, also over the line H.

Let us first consider the case $c|a$, so the only branched curve will be $\bar{\mathscr{C}}$. The surface will be constructed by covering $\mathbf{P}_2$ by open sets U_i, taking local equations $f_i = 0$ for $\bar{\mathscr{C}} \cap U_i$, taking $\mathscr{S}_i$ to be the surface $z^c = f_i$ and gluing them together. We obtain a smooth surface, as remarked, endowed with a morphism $\pi : \tilde{\mathscr{S}} \to \mathbf{P}_2$ which is totally ramified over the curve $\bar{\mathscr{C}}$. Let us denote by L any line on $\mathbf{P}_2$, so that a canonical divisor on $\mathbf{P}_2$ is $-3L$. Then the mentioned formula for the canonical divisor on $\tilde{\mathscr{S}}$ gives

$$K_{\tilde{\mathscr{S}}} = -3\pi^*(L) + (c-1)\bar{\mathscr{C}}^*,$$

where $\bar{\mathscr{C}}^* = \pi^{-1}(\bar{\mathscr{C}})$ is the set theoretic pre-image of $\bar{\mathscr{C}}$. Since $aL \sim \bar{\mathscr{C}}$ and the morphism π is totally ramified on $\bar{\mathscr{C}}^*$, we have $c \cdot \bar{\mathscr{C}}^* = \pi^*(\bar{\mathscr{C}}) \sim a\pi^*(L)$; then the canonical divisor satisfies

$$cK_{\tilde{\mathscr{S}}} \sim (-3c + a(c-1))\pi^*(L),$$

so is big (actually ample, but we don't need it) as soon as $a > 3c/(c-1)$, which is our hypothesis.

Let us now consider the case when c does not divide a, so the branched locus will be the reducible curve $\bar{\mathscr{C}} \cup H$, which has normal crossing singularities (at the a distinct points at infinity of $\mathscr{C}$).

The smooth surface $\tilde{\mathscr{S}}$, which is birationally defined as the normalization of $\mathbf{P}_2$ in the field $\kappa(\mathbf{P}_2)(\sqrt[c]{x^a + y^a - 1})$ (where $x = X/Z, y = Y/Z$ and $(X : Y : Z)$ are homogeneous coordinates in $\mathbf{P}_2$), can be constructed via the desingularization procedure described above. Locally the normalization has an equation of the form $z^c = f(x,y)$, where f is an equation for $\bar{\mathscr{C}} \cup H$, so it is a curve with normal-crossing singularities; on the open set $\mathbf{A}^2$ (which contains no singular point of $\bar{\mathscr{C}} \cup H$) one can take for f the regular function $f(x,y) = 1 - x^a - y^a$; on any neighborhood of a singular point p of $\bar{\mathscr{C}} \cup H$ (i.e. a point at infinity of $\bar{\mathscr{C}}$), the regular function f will be of the form $f = \xi\eta$, where ξ, η are local parameters at p. Then the morphism $\pi : \tilde{\mathscr{S}} \to \mathbf{P}_2$ will have the following property: 1) each point of $\mathbf{A}^2 \setminus \mathscr{C}$ has exactly c pre-images; 2) each point of $\mathscr{C}$ has one pre-image; 3) the pre-image of each singular point p_i of $\bar{\mathscr{C}} \cup H$ is a curve E_i (for $i = 1, \dots, a$), which has $c-1$ components, forming a tree of type A_{c-1} as described in [1, Ch. III, Sect. 5 & Sect. 7].

The ramification divisor of π contains the closure of the pre-image of the curve $\mathscr{C} \subset \mathbf{A}^2$, i.e. the strict transform $\bar{\mathscr{C}}^*$ of $\bar{\mathscr{C}}$ and of the strict transform H^* of the line at infinity, counted with the appropriate multiplicity. Such multiplicity turns out to be $c-1$ for the component $\bar{\mathscr{C}}^*$, since the cover is totally ramified over $\mathscr{C}$. Over the line at infinity H, it is $d-1$, where $d = c/\gcd(a,c)$

is the denominator in the reduced fraction of a/c (so there is no ramification if c divides a). So, it is of the form

$$\operatorname{Ram}(\pi) = (c-1)\bar{\mathscr{C}}^* + (d-1)H^* + E_{\operatorname{Ram}},$$

for a suitable effective divisor E_{Ram}, whose support is contained in $E_1 \cup \cdots \cup E_a$.

Let L denote any line on $\mathbf{P}_2$ not passing through any singular point of $\bar{\mathscr{C}} \cup H$. Let $L^* = \pi^*(L)$ be its pull-back, which coincides with its strict transform.

Let us now compute the equivalence class of a canonical divisor $K_{\tilde{\mathscr{S}}}$ on $\tilde{\mathscr{S}}$. We have

$$K_{\tilde{\mathscr{S}}} = -3L^* + (c-1)\bar{\mathscr{C}}^* + (d-1)H^* + E_{\operatorname{Ram}} \tag{A2}$$

Since $\bar{\mathscr{C}} \sim aL$, we have $\pi^*(\bar{\mathscr{C}}) \sim aL^*$; taking into account that the map π is totally ramified over $\bar{\mathscr{C}}$, we have $\pi^*(\bar{\mathscr{C}}) = c\,\bar{\mathscr{C}}^* + E_{\bar{\mathscr{C}}}$, for an effective divisor $E_{\bar{\mathscr{C}}}$ whose support is contained in $E_1 \cup \cdots \cup E_a$. Then

$$c\,\bar{\mathscr{C}}^* = \pi^*(\bar{\mathscr{C}}) - E_{\bar{\mathscr{C}}} \sim aL^* - E_{\bar{\mathscr{C}}}.$$

Multiplying by c in ($A2$) and using the above relation, we obtain

$$\begin{aligned} cK_{\tilde{\mathscr{S}}} &\sim -3cL^* + c(c-1)\bar{\mathscr{C}}^* + cE_{\operatorname{Ram}} + c(d-1)H^* \\ &\sim -3cL^* + (c-1)(aL^* - E_{\bar{\mathscr{C}}}) + cE_{\operatorname{Ram}} + c(d-1)H^* \\ &= ((c-1)a - 3c)\,L^* + c(E_{\operatorname{Ram}} - E_{\bar{\mathscr{C}}}) + E_{\bar{\mathscr{C}}} + c(d-1)H^* \\ &= ((c-1)a - 3c)\,L^* + E_K + c(d-1)H^*. \end{aligned}$$

Here $E_K := c(E_{\operatorname{Ram}} - E_{\bar{\mathscr{C}}}) + E_{\bar{\mathscr{C}}}$ has its support contained in $E_1 \cup \cdots \cup E_a$. Clearly, the first term is a big divisor, since it is a positive multiple of the pull-back of an ample divisor. The last term $c(d-1)H^*$ is effective. We claim (next lemma) that the term E_K is also effective, after which the proof will be complete.

Lemma. — *Each component of E_K in the above decomposition appears with non-negative weight.*

Proof. — Let us recall that the curve E_i (for $i = 1, \ldots, a$) is the fiber of the i-th singular point of the curve $\bar{\mathscr{C}} \cup H$. Also, each E_i is of the form $E_i = E_{i,1} \cup \cdots \cup E_{i,c-1}$, where each $E_{i,j}$ is smooth of genus zero, and the corresponding intersection matrix is of type A_{c-1} (see [1, Chap. III, Sect. 7]). This means that for $j = 1, \ldots, c-1$, $E_{i,j}^2 = -2$; for $|j-l| = 1$, $E_{i,j} \cdot E_{i,l} = 1$; for $|j-l| \geq 2$, $E_{i,j} \cdot E_{i,l} = 0$.

From the fact that $E_{i,j}^2 = -2$ and $E_{i,j} \simeq \mathbf{P}_1$ it follows, using the adjunction formula [2, I.15], that $K_{\tilde{\mathscr{S}}} \cdot E_{i,j} = 0$.

Let us fix an index $i \in \{1, \ldots, a\}$ and let $l_1 E_{i,1} + \cdots + l_{c-1} E_{i,c-1}$ be the decomposition of the part of E_K having its support in E_i. We want to prove that $l_j \geq 0$ for all $j = 1, \ldots, c-1$. For simplicity let us put $l_j = 0$ for all $j \geq c$

and for $j=-1$. Let us suppose by contradiction that some l_j is negative and let $\bar{j}$ the minimum $j\in\{1,\dots,c-1\}$ with this property. We shall prove by induction that the sequence $\{0,1,\dots\}\ni h\mapsto l_{\bar{j}+h}$ is a decreasing sequence of negative numbers, contradicting the fact that it is eventually zero. Let then $h\geq 0$ be a non-negative integer, and suppose $l_{\bar{j}+h-1}>l_{\bar{j}+h}$ (note that this is true for $h=0$, even when $\bar{j}=0$, by our convention that $l_{-1}=0$); we show that $l_{\bar{j}+h+1}<l_{\bar{j}+h}$. We have

$$\begin{aligned}0=c\,K_{\tilde{\mathcal{S}}}\cdot E_{i,\bar{j}+h}&\geq\Big(\sum_{j=1}^{c-1}l_jE_{i,j}\Big)\cdot E_{i,\bar{j}+h}\\&=(l_{\bar{j}+h-1}E_{i,\bar{j}+h-1}+l_{\bar{j}+h}E_{i,\bar{j}+h}+l_{\bar{j}+h+1}E_{i,\bar{j}+h+1})\cdot E_{i,\bar{j}+h}\\&=l_{\bar{j}+h-1}-2l_{\bar{j}+h}+l_{\bar{j}+h+1}.\end{aligned}$$

By the inductive assumption, $l_{\bar{j}+h}<l_{\bar{j}+h-1}$, so $0=c\,K_{\tilde{\mathcal{S}}}\cdot E_{i,\bar{j}+h}>-l_{\bar{j}+h}+l_{\bar{j}+h+1}$, completing the proof. □

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