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SEMISTABILITY OF FROBENIUS DIRECT IMAGES OVER CURVES

BY VIKRAM B. MEHTA & CHRISTIAN PAULY

ABSTRACT. — Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$. Given a semistable vector bundle E over X , we show that its direct image F_*E under the Frobenius map F of X is again semistable. We deduce a numerical characterization of the stable rank- p vector bundles F_*L , where L is a line bundle over X .

RÉSUMÉ (*Semi-stabilité des images directes par Frobenius sur les courbes*)

Soit X une courbe projective lisse de genre ≥ 2 définie sur un corps k algébriquement clos de caractéristique $p > 0$. Étant donné un fibré vectoriel semi-stable E sur X , nous montrons que l'image directe F_*E par le morphisme de Frobenius F de X est aussi semi-stable. Nous déduisons une caractérisation numérique du fibré vectoriel stable F_*L de rang p , où L est un fibré en droites sur X .

1. Introduction

Let X be a smooth projective curve of genus $g \geq 2$ defined over an algebraically closed field k of characteristic $p > 0$ and let $F : X \rightarrow X_1$ be the relative k -linear Frobenius map. It is by now a well-established fact that on

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any curve X there exist semistable vector bundles E such that their pull-back under the Frobenius map F^*E is not semistable [4, 5]. In order to control the degree of instability of the bundle F^*E , one is naturally lead (through adjunction $\text{Hom}_{\mathcal{O}_X}(F^*E, E') = \text{Hom}_{\mathcal{O}_{X_1}}(E, F_*E')$) to ask whether semistability is preserved by direct image under the Frobenius map. The answer is (somewhat surprisingly) yes. In this note we show the following result.

THEOREM 1.1. — *Assume that $g \geq 2$. If E is a semistable vector bundle over X (of any degree), then F_*E is also semistable.*

Unfortunately we do not know whether also stability is preserved by direct image under Frobenius. It has been shown that F_*L is stable for a line bundle L (see [4, Proposition 1.2]) and that in small characteristics the bundle F_*E is stable for any stable bundle E of small rank (see [3]). The main ingredient of the proof is Faltings’ cohomological criterion of semistability. We also need the fact that the generalized Verschiebung V , defined as the rational map from the moduli space $\mathcal{M}_{X_1}(r)$ of semistable rank- r vector bundles over X_1 with fixed trivial determinant to the moduli space $\mathcal{M}_X(r)$ induced by pull-back under the relative Frobenius map F ,

$$V_r : \mathcal{M}_{X_1}(r) \dashrightarrow \mathcal{M}_X(r), \quad E \longmapsto F^*E$$

is dominant for large r . We actually show a stronger statement for large r .

PROPOSITION 1.2. — *If $\ell \geq g(p-1)+1$ and ℓ prime, then the generalized Verschiebung V_ℓ is generically étale for any curve X . In particular V_ℓ is separable and dominant.*

As an application of Theorem 1.1 we obtain an upper bound of the rational invariant ν of a vector bundle E , defined as

$$\nu(E) := \mu_{\max}(F^*E) - \mu_{\min}(F^*E),$$

where μ_{\max} (resp. μ_{\min}) denotes the slope of the first (resp. last) piece in the Harder-Narasimhan filtration of F^*E .

PROPOSITION 1.3. — *For any semistable rank- r vector bundle E*

$$\nu(E) \leq \min((r-1)(2g-2), (p-1)(2g-2)).$$

We note that the inequality $\nu(E) \leq (r-1)(2g-2)$ was proved in [10, Corollary 2], and in [11, Theorem 3.1]. We suspect that the relationship between both inequalities comes from the conjectural fact that the length (= number of pieces) of the Harder-Narasimhan filtration of F^*E is at most p for semistable E .

Finally we show that direct images of line bundles under Frobenius are characterized by maximality of the invariant ν .

PROPOSITION 1.4. — *Let E be a stable rank- p vector bundle over X . Then the following statements are equivalent.*

- 1) *There exists a line bundle L such that $E = F_*L$.*
- 2) $\nu(E) = (p - 1)(2g - 2)$.

We do not know whether the analogue of this proposition remains true for higher rank.

2. Reduction to the case $\mu(E) = g - 1$

In this section we show that it is enough to prove Theorem 1.1 for semistable vector bundles E with slope $\mu(E) = g - 1$.

Let E be a semistable vector bundle over X of rank r and let s be the integer defined by the equality

$$\mu(E) = g - 1 + \frac{s}{r}.$$

Applying the Grothendieck-Riemann-Roch theorem to the Frobenius map $F : X \rightarrow X_1$, we obtain

$$\mu(F_*E) = g - 1 + \frac{s}{pr}.$$

Let $\pi : \widetilde{X} \rightarrow X$ be a connected étale covering of degree n and let $\pi_1 : \widetilde{X}_1 \rightarrow X_1$ denote its twist by the Frobenius of k (see [9, Section 4]). The diagram

$$(2.1) \quad \begin{array}{ccc} \widetilde{X} & \xrightarrow{F} & \widetilde{X}_1 \\ \pi \downarrow & & \downarrow \pi_1 \\ X & \xrightarrow{F} & X_1 \end{array}$$

is Cartesian and we have an isomorphism

$$\pi_1^*(F_*E) \cong F_*(\pi^*E).$$

Since semistability is preserved under pull-back by a separable morphism of curves, we see that π^*E is semistable. Moreover if $F_*(\pi^*E)$ is semistable, then F_*E is also semistable.

Let L be a degree d line bundle over \widetilde{X}_1 . The projection formula

$$F_*(\pi^*E \otimes F^*L) = F_*(\pi^*E) \otimes L$$

shows that semistability of $F_*(\pi^*E)$ is equivalent to semistability of

$$F_*(\pi^*E \otimes F^*L).$$

Let \tilde{g} denote the genus of \tilde{X} . By the Riemann-Hurwitz formula, one has $\tilde{g} - 1 = n(g - 1)$. We compute

$$\mu(\pi^*E \otimes F^*L) = n(g - 1) + n\frac{s}{r} + pd = \tilde{g} - 1 + n\frac{s}{r} + pd,$$

which gives

$$\mu(F_*(\pi^*E \otimes F^*L)) = \tilde{g} - 1 + n\frac{s}{pr} + d.$$

LEMMA 2.1. — *For any integer m there exists a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$ for some k .*

Proof. — If the p -rank of X is nonzero, the statement is clear. If the p -rank is zero, we know by [9, Corollaire 4.3.4], that there exist connected étale coverings $Y \rightarrow X$ of degree p^t for infinitely many integers t (more precisely for all t of the form $(\ell - 1)(g - 1)$ where ℓ is a large prime). Now we decompose $m = p^s u$ with p not dividing u . We then take a covering $Y \rightarrow X$ of degree p^t with $t \geq s$ and a covering $\tilde{X} \rightarrow Y$ of degree u . □

Now the lemma applied to the integer $m = pr$ shows existence of a connected étale covering $\pi : \tilde{X} \rightarrow X$ of degree $n = p^k m$. Hence $n\frac{s}{pr}$ is an integer and we can take d such that $n\frac{s}{pr} + d = 0$.

To summarize, we have shown that for any semistable E over X there exists a covering $\pi : \tilde{X} \rightarrow X$ and a line bundle L over \tilde{X}_1 such that the vector bundle $\tilde{E} := \pi^*E \otimes F^*L$ is semistable with $\mu(\tilde{E}) = \tilde{g} - 1$ and such that semistability of $F_*\tilde{E}$ implies semistability of F_*E .

3. Proof of Theorem 1.1

In order to prove semistability of F_*E we shall use the cohomological criterion of semistability due to Faltings [2].

PROPOSITION 3.1 (see [6, Théorème 2.4]). — *Let E be a rank- r vector bundle over X with $\mu(E) = g - 1$ and ℓ an integer $> \frac{1}{4}r^2(g - 1)$. Then E is semistable if and only if there exists a rank- ℓ vector bundle G with trivial determinant such that*

$$h^0(X, E \otimes G) = h^1(X, E \otimes G) = 0.$$

Moreover if the previous condition holds for one bundle G , it holds for a general bundle by upper semicontinuity of the function $G \mapsto h^0(X, E \otimes G)$.

REMARK. — The proof of this proposition (see [6, Section 2.4]) works over any algebraically closed field k .

By Proposition 1.2 (proved in Section 4) we know that V_ℓ is dominant when ℓ is a large prime number. Hence a general vector bundle $G \in \mathcal{M}_X(\ell)$ is of the form F^*G' for some $G' \in \mathcal{M}_{X_1}(\ell)$. Consider a semistable E with $\mu(E) = g - 1$. Then by Proposition 3.1 $h^0(X, E \otimes G) = 0$ for general $G \in \mathcal{M}_X(\ell)$. Assuming G general, we can write $G = F^*G'$ and we obtain by adjunction

$$h^0(X, E \otimes F^*G') = h^0(X_1, F_*E \otimes G') = 0.$$

This shows that F_*E is semistable by Proposition 3.1.

4. Proof of Proposition 1.2

According to [7, Section 2], it will be enough to prove the existence of a stable vector bundle $E \in \mathcal{M}_{X_1}(\ell)$ satisfying F^*E stable and

$$h^0(X_1, B \otimes \text{End}_0(E)) = 0,$$

because the vector space $H^0(X_1, B \otimes \text{End}_0(E))$ can be identified with the kernel of the differential of V_ℓ at the point $E \in \mathcal{M}_{X_1}(\ell)$. Here B denotes the sheaf of locally exact differentials over X_1 (see [9, Section 4]).

Let $\ell \neq p$ be a prime number and let $\alpha \in JX_1[\ell] \cong JX[\ell]$ be a nonzero ℓ -torsion point. We denote by

$$\pi : \widetilde{X} \longrightarrow X \quad \text{and} \quad \pi_1 : \widetilde{X}_1 \longrightarrow X_1$$

the associated cyclic étale cover of X and X_1 and by σ a generator of the Galois group $\text{Gal}(\widetilde{X}/X) = \mathbb{Z}/\ell\mathbb{Z}$. We recall that the kernel of the Norm map

$$\text{Nm} : J\widetilde{X} \longrightarrow JX$$

has ℓ connected components and we denote by

$$i : P := \text{Prym}(\widetilde{X}/X) \subset J\widetilde{X}$$

the associated Prym variety, *i.e.*, the connected component containing the origin. Then we have an isogeny

$$\pi^* \times i : JX \times P \longrightarrow J\widetilde{X}$$

and taking direct image under π induces a morphism

$$P \longrightarrow \mathcal{M}_X(\ell), \quad L \longmapsto \pi_*L.$$

Similarly we define the Prym variety $P_1 \subset JX_1$ and the morphism $P_1 \rightarrow \mathcal{M}_{X_1}(\ell)$ (obtained by twisting with the Frobenius of k). Note that $\pi_{1*}L$ is semistable for any $L \in P_1$ and stable for general $L \in P_1$ (see *e.g.*, [1]). Since $F^*(\pi_{1*}L) \cong \pi_*(F^*L)$ — see diagram (2.1) — and since F^* induces the Verschiebung $V_P : P_1 \rightarrow P$, which is surjective, we obtain that $\pi_{1*}L$ and $F^*(\pi_{1*}L)$ are stable for general $L \in P_1$.

Therefore Proposition 1.2 will immediately follow from the next proposition.

PROPOSITION 4.1. — *If $\ell \geq g(p - 1) + 1$ then there exists a cyclic degree ℓ étale cover $\pi_1 : \widetilde{X}_1 \rightarrow X_1$ with the property that*

$$h^0(X_1, B \otimes \text{End}_0(\pi_{1*}L)) = 0$$

for general $L \in P_1$.

Proof. — By relative duality for the étale map π_1 we have $(\pi_{1*}L)^* \cong \pi_{1*}L^{-1}$. Therefore

$$\text{End}(\pi_{1*}L) \cong \pi_{1*}L \otimes \pi_{1*}L^{-1} \cong \pi_{1*}(L^{-1} \otimes \pi_1^*\pi_{1*}L)$$

by the projection formula. Moreover since π_1 is Galois étale we have a direct sum decomposition

$$\pi_1^*\pi_{1*}L \cong \bigoplus_{i=0}^{\ell-1} (\sigma^i)^*L.$$

Putting these isomorphisms together we find that

$$\begin{aligned} H^0(X_1, B \otimes \text{End}(\pi_{1*}L)) &= H^0(X_1, B \otimes \pi_{1*} \left(\bigoplus_{i=0}^{\ell-1} L^{-1} \otimes (\sigma^i)^*L \right)) \\ &= \bigoplus_{i=0}^{\ell-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^*L)) \\ &= H^0(X_1, B \otimes \pi_{1*}\mathcal{O}_{\widetilde{X}_1}) \\ &\quad \oplus \bigoplus_{i=1}^{\ell-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^*L)). \end{aligned}$$

Moreover $\pi_*\mathcal{O}_{\widetilde{X}_1} = \bigoplus_{i=0}^{\ell-1} \alpha^i$, which implies that

$$(4.1) \quad H^0(X_1, B \otimes \text{End}_0(\pi_{1*}L)) = \bigoplus_{i=1}^{\ell-1} H^0(X_1, B \otimes \alpha^i) \oplus \bigoplus_{i=1}^{\ell-1} H^0(X_1, B \otimes \pi_{1*}(L^{-1} \otimes (\sigma^i)^*L)).$$

Let us denote for $i = 1, \dots, \ell - 1$ by ϕ_i the isogeny

$$\phi_i : P_1 \longrightarrow P_1, \quad L \longmapsto L^{-1} \otimes (\sigma^i)^*L.$$

Since the function $L \mapsto h^0(X_1, B \otimes \text{End}_0(\pi_{1*}L))$ is upper semicontinuous, it will be enough to show the existence of a cover $\pi_1 : \widetilde{X}_1 \rightarrow X_1$ satisfying

- 1) for $i = 1, \dots, \ell - 1$, $h^0(X_1, B \otimes \alpha^i) = 0$ (or equivalently, P is an ordinary abelian variety);
- 2) for M general in P , $h^0(X_1, B \otimes \pi_{1*}M) = 0$.

Note that these two conditions imply that the vector space (4.1) equals $\{0\}$ for general $L \in P_1$, because the ϕ_i 's are surjective.

We recall that $\ker(\pi_1^* : JX_1 \rightarrow \widetilde{JX_1}) = \langle \alpha \rangle \cong \mathbb{Z}/\ell\mathbb{Z}$ and that

$$P_1[\ell] = P_1 \cap \pi_1^*(JX_1) \cong \alpha^\perp / \langle \alpha \rangle$$

where $\alpha^\perp = \{\beta \in JX_1[\ell] \text{ with } \omega(\alpha, \beta) = 1\}$ and $\omega : JX_1[\ell] \times JX_1[\ell] \rightarrow \mu_\ell$ denotes the symplectic Weil form. Consider a $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. Then $\pi_{1*}\beta \in P_1[\ell]$ and

$$\pi_{1*}\pi_1^*\beta = \bigoplus_{i=0}^{\ell-1} \beta \otimes \alpha^i.$$

Again by upper semicontinuity of the function $M \mapsto h^0(X_1, B \otimes \pi_{1*}M)$ one observes that conditions 1) and 2) are satisfied because of the following lemma (take $M = \pi_1^*\beta$).

LEMMA 4.2. — *If $\ell \geq g(p-1) + 1$ then there exists $(\alpha, \beta) \in JX_1[\ell] \times JX_1[\ell]$ satisfying*

- 1) $\alpha \neq 0$ and $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$;
- 2) for $i = 1, \dots, \ell - 1$, $h^0(X_1, B \otimes \alpha^i) = 0$;
- 3) for $i = 0, \dots, \ell - 1$, $h^0(X_1, B \otimes \beta \otimes \alpha^i) = 0$.

Proof. — We adapt the proof of [9, Lemme 4.3.5]. We denote by \mathbb{F}_ℓ the finite field $\mathbb{Z}/\ell\mathbb{Z}$. Then there exists a symplectic isomorphism $JX_1[\ell] \cong \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$, where the latter space is endowed with the standard symplectic form. Note that composition is written multiplicatively in $JX_1[\ell]$ and additively in \mathbb{F}_ℓ^{2g} . A quick computation shows that the number of isotropic 2-planes in $\mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ equals

$$N(\ell) = \frac{(\ell^{2g} - 1)(\ell^{2g-2} - 1)}{(\ell^2 - 1)(\ell - 1)}.$$

Let $\Theta_B \subset JX_1$ denote the theta divisor associated to B . Then by [9, Lemma 4.3.5], the cardinality $A(\ell)$ of the finite set

$$\Sigma(\ell) := JX_1[\ell] \cap \Theta_B$$

satisfies

$$A(\ell) \leq \ell^{2g-2}g(p-1).$$

Suppose that there exists an isotropic 2-plane $\Pi \subset \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ which contains $\leq \ell - 2$ points of $\Sigma(\ell)$. Then we can find a pair (α, β) satisfying the three properties of the lemma as follows: any nonzero point $x \in \Pi$ determines a line (= \mathbb{F}_ℓ -vector space of dimension 1). Since a line contains $\ell - 1$ nonzero points, we obtain at most $(\ell - 1)(\ell - 2)$ nonzero points lying on lines generated by $\Sigma(\ell) \cap \Pi$. Since $(\ell - 1)(\ell - 2) < \ell^2 - 1$ there exists a nonzero α in the complement of these lines. Now we note that there are $\ell - 1$ affine lines parallel

to the line generated by α and the ℓ points on any of these affine lines are of the form $\beta\alpha^i$ for $i = 0, \dots, \ell - 1$ for some $\beta \in \alpha^\perp \setminus \langle \alpha \rangle$. The points $\Sigma(\ell) \cap \Pi$ lie on at most $\ell - 2$ such affine lines, hence there exists at least one affine line parallel to $\langle \alpha \rangle$ avoiding $\Sigma(\ell)$. This gives β .

Finally let us suppose that any isotropic 2-plane contains $\geq \ell - 1$ points of $\Sigma(\ell)$. Then we will arrive at a contradiction as follows: we introduce the set

$$S = \{(x, \Pi) \mid x \in \Pi \cap \Sigma(\ell) \text{ and } \Pi \text{ isotropic 2-plane}\}.$$

with cardinality $|S|$. Then by our assumption we have

$$(4.2) \quad |S| \geq (\ell - 1)N(\ell).$$

On the other hand, since any nonzero $x \in \mathbb{F}_\ell^g \times \mathbb{F}_\ell^g$ is contained in $(\ell^{2g-2} - 1)/(\ell - 1)$ isotropic 2-planes, we obtain

$$(4.3) \quad |S| \leq \frac{\ell^{2g-2} - 1}{\ell - 1} A(\ell).$$

Putting (4.2) and (4.3) together, we obtain

$$A(\ell) \geq \frac{\ell^{2g} - 1}{\ell + 1}.$$

But this contradicts the inequality $A(\ell) \leq \ell^{2g-2}g(p - 1)$ if $\ell \geq g(p - 1) + 1$. \square

This completes the proof of Proposition 4.1. \square

REMARK. — It has been shown [8, Theorem A.6], that V_r is dominant for any rank r and any curve X , by using a versal deformation of a direct sum a r line bundles.

REMARK. — We note that V_r is not separable when p divides the rank r and X is non-ordinary. In that case the Zariski tangent space at a stable bundle $E \in \mathcal{M}_{X_1}(r)$ identifies with the quotient $H^1(X_1, \text{End}_0(E))/\langle e \rangle$ where e denotes the nonzero extension class of $\text{End}_0(E)$ by \mathcal{O}_{X_1} given by $\text{End}(E)$. Then the inclusion of homotheties $\mathcal{O}_{X_1} \hookrightarrow \text{End}_0(E)$ induces an inclusion $H^1(X_1, \mathcal{O}_{X_1}) \subset H^1(X_1, \text{End}_0(E))/\langle e \rangle$ and the restriction of the differential of V_r at the point E to $H^1(X_1, \mathcal{O}_{X_1})$ coincides with the non-injective Hasse-Witt map.

5. Proof of Proposition 1.3

Since we already know that $\nu(E) \leq (r-1)(2g-2)$ (see [10, 11]) it suffices to show that $\nu(E) \leq (p-1)(2g-2)$.

We consider the quotient $F^*E \rightarrow Q$ with minimal slope, *i.e.*, $\mu(Q) = \mu_{\min}(F^*E)$ and Q semistable. By adjunction we obtain a nonzero morphism $E \rightarrow F_*(Q)$, from which we deduce (using Theorem 1.1) that

$$\mu(E) \leq \mu(F_*Q) = \frac{1}{p}(\mu_{\min}(F^*E) + (p-1)(g-1))$$

hence

$$\mu(F^*E) \leq \mu_{\min}(F^*E) + (p-1)(g-1).$$

Similarly we consider the subbundle $S \hookrightarrow F^*E$ with maximal slope, *i.e.*, $\mu(S) = \mu_{\max}(F^*E)$ and S semistable. Taking the dual and proceeding as above, we obtain that

$$\mu(F^*E) \geq \mu_{\max}(F^*E) - (p-1)(g-1).$$

Now we combine both inequalities and we are done.

REMARK. — We note that the inequality of Proposition 1.3 is sharp. The maximum $(p-1)(2g-2)$ is obtained for the bundles $E = F_*E'$ (see [3, Theorem 5.3]).

6. Characterization of direct images

Consider a line bundle L over X . Then the direct image F_*L is stable (see [4], Proposition 1.2) and the Harder-Narasimhan filtration of F^*F_*L is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = F^*F_*L, \quad \text{with } V_i/V_{i-1} \cong L \otimes \omega_X^{p-i}.$$

In particular $\nu(F_*L) = (p-1)(2g-2)$. In this section we will show a converse statement.

More generally let E be a stable rank- rp vector bundle with $\mu(E) = g-1 + \frac{d}{rp}$ for some integer d and satisfying

- 1) the Harder-Narasimhan filtration of F^*E has ℓ terms.
- 2) $\nu(E) = (p-1)(2g-2)$.

QUESTIONS. — Do we have $\ell \leq p$? Is E of the form $E = F_*G$ for some rank- r vector bundle G ? We will give a positive answer in the case $r = 1$ (Proposition 6.1).

Let us denote the Harder-Narasimhan filtration by

$$0 = V_0 \subset V_1 \subset \dots \subset V_{\ell-1} \subset V_\ell = F^*E, \quad V_i/V_{i-1} = M_i$$

satisfying the inequalities

$$\mu_{\max}(F^*E) = \mu(M_1) > \mu(M_2) > \dots > \mu(M_\ell) = \mu_{\min}(F^*E).$$

The quotient $F^*E \rightarrow M_\ell$ gives via adjunction a nonzero map $E \rightarrow F_*M_\ell$. Since F_*M_ℓ is semistable, we obtain that $\mu(E) \leq \mu(F_*M_\ell)$. This implies that $\mu(M_\ell) \geq g - 1 + d/r$. Similarly taking the dual of the inclusion $M_1 \subset F^*E$ gives a map $F^*(E^*) \rightarrow M_1^*$ and by adjunction $E^* \rightarrow F_*(M_1^*)$. Let us denote

$$\mu(M_1^*) = g - 1 + \delta,$$

so that $\mu(F_*(M_1^*)) = g - 1 + \delta/p$. Because of semistability of $F_*(M_1^*)$, we obtain

$$-\left(g - 1 + \frac{d}{rp}\right) = \mu(E^*) \leq \mu(F^*(M_1^*)),$$

hence $\delta \geq -2p(g - 1) - d/r$. This implies $\mu(M_1) \leq (2p - 1)(g - 1) + d/r$. Combining this inequality with $\mu(M_\ell) \geq g - 1 + d/r$ and the assumption $\mu(M_1) - \mu(M_\ell) = (p - 1)(2g - 2)$, we obtain that

$$\mu(M_1) = (2p - 1)(g - 1) + \frac{d}{r}, \quad \mu(M_\ell) = g - 1 + \frac{d}{r}.$$

Let us denote by r_i the rank of the semistable bundle M_i . We have the equality

$$(6.1) \quad \sum_{i=1}^{\ell} r_i = rp.$$

Since E is stable and $F_*(M_\ell)$ is semistable and since these bundles have the same slope, we deduce that $r_\ell \geq r$. Similarly we obtain that $r_1 \geq r$.

Note that it is enough to show that $r_\ell = r$. Since E is stable and F_*M_ℓ semistable and since the two bundles have the same slope and rank, they will be isomorphic.

We introduce for $i = 1, \dots, \ell - 1$ the integers

$$\delta_i = \mu(M_{i+1}) - \mu(M_i) + 2(g - 1) = \mu(M_{i+1} \otimes \omega) - \mu(M_i).$$

Then we have the equality

$$(6.2) \quad \sum_{i=1}^{\ell-1} \delta_i = \mu(M_\ell) - \mu(M_1) + 2(\ell - 1)(g - 1) = 2(\ell - p)(g - 1).$$

We note that if $\delta_i < 0$, then $\text{Hom}(M_i, M_{i+1} \otimes \omega) = 0$.

PROPOSITION 6.1. — *Let E be stable rank- p vector bundle with*

$$\mu(E) = g - 1 + d/p \quad \text{and} \quad \nu(E) = (p - 1)(2g - 2).$$

*Then $E = F_*L$ for some line bundle L of degree $g - 1 + d$.*

Proof. — Let us first show that $\ell = p$. We suppose that $\ell < p$. Then

$$\sum_{i=1}^{\ell-1} \delta_i = 2(\ell - p)(g - 1) < 0$$

so that there exists a $k \leq \ell - 1$ such that $\delta_k < 0$. We may choose k minimal, *i.e.*, $\delta_i \geq 0$ for $i < k$. Then we have

$$(6.3) \quad \mu(M_k) > \mu(M_i) + 2(g - 1) \quad \text{for } i > k.$$

We recall that $\mu(M_i) \leq \mu(M_{k+1})$ for $i > k$. The Harder-Narasimhan filtration of V_k is given by the first k terms of the Harder-Narasimhan filtration of F^*E . Hence $\mu_{\min}(V_k) = \mu(M_k)$.

Consider now the canonical connection ∇ on F^*E and its first fundamental form

$$\phi_k : V_k \hookrightarrow F^*E \xrightarrow{\nabla} F^*E \otimes \omega_X \longrightarrow (F^*E/V_k) \otimes \omega_X.$$

Since $\mu_{\min}(V_k) > \mu(M_i \otimes \omega)$ for $i > k$ we obtain $\phi_k = 0$. Hence ∇ preserves V_k and since ∇ has zero p -curvature, there exists a subbundle $E_k \subset E$ such that $F^*E_k = V_k$.

We now evaluate $\mu(E_k)$. By assumption $\delta_i \geq 0$ for $i < k$. Hence

$$\mu(M_i) \geq \mu(M_1) - 2(i - 1)(g - 1) \quad \text{for } i \leq k,$$

which implies that

$$\deg(V_k) = \sum_{i=1}^k r_i \mu(M_i) \geq \text{rk}(V_k) \mu(M_1) - 2(g - 1) \sum_{i=1}^k r_i (i - 1).$$

Hence we obtain

$$p\mu(E_k) = \mu(V_k) \geq \mu(M_1) - 2(g - 1)C,$$

where C denotes the fraction $(\sum_{i=1}^k r_i (i - 1)) / \text{rk}(V_k)$. We will prove in a moment that $C \leq \frac{1}{2}(p - 1)$, so that we obtain by substitution

$$p\mu(E_k) \geq (2p - 1)(g - 1) + d - (g - 1)(p - 1) = p(g - 1) + d = p\mu(E),$$

contradicting stability of E . Now let us show that $C \leq \frac{1}{2}(p - 1)$ or equivalently

$$\sum_{i=1}^k i r_i \leq \frac{1}{2}(p + 1) \sum_{i=1}^k r_i.$$

But that is obvious if $k \leq \frac{1}{2}(p - 1)$. Now if $k > \frac{1}{2}(p - 1)$ we note that passing from E to E^* reverses the order of the δ_i 's, so that the index k^* for E^* satisfies $k^* \leq \frac{1}{2}(p - 1)$. This proves that $\ell = p$.

Because of (6.1) we obtain $r_i = 1$ for all i and therefore $E = F_*M_p$. □

7. Stability of F_*E ?

Is stability also preserved by F_* ?

We show the following result in that direction.

PROPOSITION 7.1. — *Let E be a stable vector bundle over X . Then F_*E is simple.*

Proof. — Using relative duality $(F_*E)^* \cong F_*(E^* \otimes \omega_X^{1-p})$ we obtain

$$H^0(X_1, \text{End}(F_*E)) = H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}).$$

Moreover the Harder-Narasimhan filtration of F^*F_*E is of the form (see [3])

$$0 = V_0 \subset V_1 \subset \dots \subset V_{p-1} \subset V_p = F^*F_*E, \quad \text{with} \quad V_i/V_{i-1} \cong E \otimes \omega_X^{p-i}.$$

We deduce that

$$H^0(X, F^*F_*E \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, V_1 \otimes E^* \otimes \omega_X^{1-p}) = H^0(X, \text{End}(E)),$$

and we are done. □

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