Bull. Soc. math. France
134 (4), 2006, p. 487–508

H^∞ CALCULUS AND DILATIONS

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ABSTRACT. — We characterise the boundedness of the H^∞ calculus of a sectorial operator in terms of dilation theorems. We show e. g. that if $-A$ generates a bounded analytic C_0 semigroup (T_t) on a UMD space, then the H^∞ calculus of A is bounded if and only if (T_t) has a dilation to a bounded group on $L^2([0,1],X)$. This generalises a Hilbert space result of C. Le Merdy. If X is an L^p space we can choose another L^p space in place of $L^2([0,1],X)$.

RÉSUMÉ (*Calcul H^∞ et dilatations*). — Nous donnons une condition nécessaire et suffisante en termes de théorèmes de dilatation pour que le calcul H^∞ d'un opérateur sectoriel soit borné. Nous montrons par exemple que, si A engendre un semigroupe C_0 analytique borné (T_t) sur un espace UMD, alors le calcul H^∞ de A est borné si et seulement si (T_t) admet une dilatation en un groupe borné sur $L_2([0,1],X)$. Ceci généralise un résultat de C. Le Merdy sur les espaces de Hilbert. Si X est un espace L_p, on peut choisir un autre espace L_p à la place de $L_2([0,1],X)$.

1. Introduction

In recent years, the holomorphic functional calculus for sectorial operators as introduced in [19] and [4] has received a lot of attention because of its

Texte reçu le 15 avril 2005, accepté le 25 octobre 2005.

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2000 Mathematics Subject Classification. — 47A60, 47A20, 47D06.

Key words and phrases. — H^∞ functional calculus, dilation theorems, spectral operators, square functions, C_0 groups, UMD spaces.

applications to evolution equations (e. g., interpolation of domains and maximal regularity [14], [17]) and to Kato's square root problem [1], [2]. In particular, the boundedness of the H^∞ functional calculus was shown for large classes of elliptic differential operators – see [5], [22], and the literature cited there.

One of the first results in this direction was the observation that an accretive operator A on a Hilbert space H has a bounded H^∞ calculus. This follows from the Sz.-Nagy dilation theorem for contractions. More recently, C. Le Merdy [18] has shown that there is a converse to this statement: a sectorial operator A of type $< \frac{1}{2}\pi$ on H has a bounded H^∞ calculus if and only if it is accretive in an equivalent Hilbert space norm, and therefore, by the dilation theorem, A has a bounded H^∞ calculus if and only if there is a second Hilbert space G, an isomorphic embedding $J : H \hookrightarrow G$ and a C_0 group of isometries (U_t) on G such that

$$JT_t = PU_tJ \quad \text{for all } t > 0, \tag{1.1}$$

where (T_t) is the analytic semigroup generated by $-A$ and $P : G \to J(H)$ is the orthogonal projection onto $J(H)$.

In this paper, we show that this characterisation of the bounded H^∞ calculus can be extended to the class of Banach spaces of finite cotype. These are Banach spaces that do not contain ℓ_n^∞ uniformly for all dimensions n. If X is a UMD space and A is R-sectorial (or almost R-sectorial) of type $< \frac{1}{2}\pi$, then our result takes a particular simple form (Corollary 5.4): A has a bounded H^∞ calculus if and only if there is an isomorphic embedding $J : X \to L^2([0,1], X)$, a bounded projection $P : L^2([0,1], X) \to J(X)$ and a group of isometries (U_t) on $L^2([0,1], X)$ such that (1.1) holds. If X is an $L^p(\Omega, \mu)$ space we can even replace $L^2([0,1], X)$ in this statement by another $L^p(\Omega_0, \mu_0)$ space.

Furthermore, our construction shows that the generator of U_t does not just have an H^∞ calculus but can be chosen to be a spectral operator of scalar type in the sense of Dunford and Schwartz [6], and in this form our characterisation also holds in Banach spaces of finite cotype (see Theorem 5.1 and Corollary 5.3). Spectral operators of scalar type are quite rare on Banach spaces that are not Hilbert spaces. Therefore it seems remarkable that the rather large class of operators with a bounded H^∞ calculus can be characterised by dilations to operators in this small class of Banach space operators which have a spectral theory as rich as the spectral theory of normal operators on a Hilbert space.

We also get a dilation theorem for general sectorial operators whose type is not smaller than $\frac{1}{2}\pi$ (Theorem 5.5 and Corollary 5.6). In this case we obtain a result that may be new even in Hilbert space: a sectorial operator on a Hilbert space has a bounded H^∞ calculus if and only if it has a dilation to a normal operator (Corollary 5.7).

Our proofs are based on the square function characterisation of the boundedness of the H^∞ calculus. This technique was introduced for Hilbert spaces by McIntosh [19] and extended to L^p spaces in [4]. (For subspaces of L^p spaces, see also [16].) We use square functions in a general Banach space setting as introduced in [13] and [12]. These definitions and further preliminary information on sectorial operators and spectral theory will be given in Sections 2, 3 and 4. In Section 5 we describe our main results and Section 6 contains the construction of the dilation.

We would like to thank the referee for suggesting several improvements of our presentation.

2. H^∞ calculus and spectral operators

We start with some notation. Let $X \neq \{0\}$ be a complex Banach space;

▷ $\mathcal{B}(X)$ will denote the space of all bounded linear operators on X with the operator norm, and

▷ $\mathcal{C}(X)$ is the set of closed linear operators on X; we write

▷ $\mathcal{D}(A)$ for the domain of an operator A and $\mathcal{R}(A)$ for its range;

▷ $\mathcal{N}(A)$ is the kernel.

For $\theta \in (0,\pi)$ we define the sector $S_\theta := \{\, z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta \,\}$. Let

▷ $H(S_\theta)$ be the set of all functions holomorphic on S_θ and let

▷ $H^\infty(S_\theta)$ be the set of all functions in $H(S_\theta)$ that are bounded.

Furthermore, we define

$$\Psi(S_\theta) := \bigl\{\psi \in H(S_\theta) \mid \exists\, c, s > 0,\ \forall z \in S_\theta : |\psi(z)| \leqslant c \min\{|z|^s, |z|^{-s}\}\bigr\}.$$

Definition 2.1. — An operator $A \in \mathcal{C}(X)$ is *of type* μ, where $\mu \in (0,\pi)$, if

1) $\sigma(A) \subset \overline{S}_\mu$ and

2) for all $\theta \in (\mu,\pi)$, there exists a constant C_θ such that

$$\bigl\|R(z,A)\bigr\| \leqslant C_\theta |z|^{-1} \quad \text{for all } z \notin \overline{S}_\theta.$$

If A is of type μ for some $\mu \in (0,\pi)$, we say that A is a *sectorial operator*, and by $\omega(A)$ we denote the infimum over all such μ.

Remark 2.2. — An operator A is densely defined and sectorial of type $< \frac{1}{2}\pi$ if and only if $-A$ generates a bounded analytic semigroup [7, Thm 4.6].

Cowling, Doust, McIntosh, and Yagi [4] have introduced a functional calculus for sectorial operators, based on earlier work by McIntosh [19]: if A is of type μ, we can define $\psi(A) \in \mathcal{B}(X)$ for $\psi \in \Psi(S_\theta)$ with $\theta > \mu$ by the contour integral

$$\psi(A) = \frac{1}{2\pi i} \int_{\gamma_\alpha} \psi(z) R(z,A) \mathrm{d}z,$$

where γ_α is the edge of the sector S_α (with $\mu < \alpha < \theta$), oriented in the positive sense. (Compare this with $\psi(\lambda) = \frac{1}{2\pi i}\int_{\gamma_\alpha}\psi(z)(z-\lambda)^{-1}\mathrm{d}z$ for all $\lambda \in S_\alpha$, which follows from Cauchy's formula.)

If, moreover, A has dense domain and dense range (which implies that A is one-to-one, too), this calculus $\Psi(S_\theta) \to \mathcal{B}(X)$ can be extended to the class of functions $f \in H(S_\theta)$ with $\psi_0^n f \in \Psi(S_\theta)$ for some $n \in \mathbb{N}$ and $\psi_0(z) := z/(1+z)^2$ using the definition

$$f(A) := \psi_0(A)^{-n}(\psi_0^n f)(A),$$
$$\mathcal{D}\big(f(A)\big) := \big\{x \in X : (\psi_0 f)(A)y \in \mathcal{R}(\psi_0^n(A))\big\}.$$

Note that $f(A)$ is a densely defined closed operator but not necessarily a bounded operator for all $f \in H^\infty(S_\theta)$.

However, we always have $f_\lambda(A) = R(\lambda, A)$ for $f_\lambda(z) := (\lambda - z)^{-1}$ and $|\arg\lambda| > \mu$. Furthermore,

$$f(A)g(A) = (fg)(A) \qquad \text{on} \quad \mathcal{D}((fg)(A)) \cap \mathcal{D}(g(A)) \text{ and}$$
$$f(A) + g(A) = (f+g)(A) \quad \text{on} \quad \mathcal{D}(f(A)) \cap \mathcal{D}(g(A)).$$

Note that $g(A) = A$ for $g(z) := z$ and $h(tA) = T_t$ for $h(z) := \mathrm{e}^{-z}$ if $-A$ generates a C_0 semigroup (T_t) and $\omega(A) < \frac{1}{2}\pi$. We can also define A^z for all $z \in \mathbb{C}$.

Remark 2.3. — This functional calculus has the following convergence property which is an immediate consequence of Lebesgue's convergence theorem: *if $f_n, f \in H^\infty(S_\theta)$ are uniformly bounded and $f_n(z) \to f(z)$ for every $z \in S_\theta$, we have*

$$(f_n\psi)(A) \xrightarrow{n\to\infty} (f\psi)(A) \quad \textit{for every } \psi \in \Psi(S_\theta).$$

This is often used in connection with an "approximate identity" such as

$$\psi_n(z) := \frac{nz - \frac{1}{n}z}{(n+z)(\frac{1}{n}+z)} = -\frac{n}{-n-z} + \frac{\frac{1}{n}}{-\frac{1}{n}-z}$$

which satisfies $\psi_n(A)x \xrightarrow{n\to\infty} x$ for all $x \in \overline{\mathcal{D}(A)} \cap \overline{\mathcal{R}(A)}$.

Definition 2.4. — Let A be of type μ, with dense domain and dense range. We say that A has a *bounded $H^\infty(S_\theta)$ functional calculus*, where $\theta \in (\mu, \pi)$, if $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(S_\theta)$. By $\omega_{H^\infty}(A)$ we denote the infimum over all such θ.

In this case, there exists a constant C such that $\|f(A)\| \leqslant C\|f\|_\infty$ for all $f \in H^\infty(S_\theta)$. To check that A has a bounded $H^\infty(S_\theta)$ functional calculus it suffices [4, Cor. 2.2] to show $\|\psi(A)\| \leqslant C\|\psi\|_\infty$ for all $\psi \in \Psi(S_\theta)$.

If a sectorial operator has a bounded H^∞ functional calculus (short: a bounded H^∞ calculus), we always have weak estimates of the following form:

Proposition 2.5 (see [4, Thm 4.2]). — *Let A be a sectorial operator of type μ with dense domain and dense range. If A has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta \in (\mu, \pi)$, then for every $\psi \in \Psi(S_\theta)$, there exists a constant $C > 0$ satisfying*

$$\int_0^\infty \big|\langle \psi(tA)x, x'\rangle\big| \frac{\mathrm{d}t}{t} \leqslant C\|x\| \cdot \|x'\| \quad \textit{for all } x \in X \textit{ and } x' \in X'.$$

For some sectorial operators we can define $f(A) \in \mathcal{B}(X)$ for every bounded Borel function $f \in B_b(\sigma(A))$ on $\sigma(A)$ and get a functional calculus which has the same properties as the functional calculus of normal operators on Hilbert spaces.

Definition 2.6. — A sectorial operator A is said to be a *spectral operator of scalar type* if there exists a functional calculus $\Phi : B_b(\sigma(A)) \to \mathcal{B}(X)$ with the following properties:

1) The operator Φ is bounded, linear and multiplicative.
2) For $\lambda \notin \sigma(A)$ we have $\Phi((\lambda - \cdot)^{-1}) = R(\lambda, A)$, and $\Phi(1) = \mathrm{id}_X$.
3) Let $f_n, f \in B_b(\sigma(A))$ and $f_n(z) \to f(z)$ for almost all $z \in \sigma(A)$.
   If $\|f_n\|_\infty \leqslant C$ for all $n \in \mathbb{N}$, then $\Phi(f_n)x \to \Phi(f)x$ for all $x \in X$.

Such operators are studied in detail in [6, Section XVIII.2.8], where a definition in terms of spectral representations and spectral measures is given. However, Theorem 11 in [6, XVIII.2.8], shows that our definition is equivalent. Of course, the strong functional calculus of Definition 2.6 implies the boundedness of the H^∞ calculus.

Proposition 2.7. — *If A is a spectral operator of scalar type with $\sigma(A) \subset \overline{S}_\mu$, then A has a bounded $H^\infty(S_\theta)$ functional calculus for every $\theta > \mu$.*

Proof. — Obviously, A is sectorial of type μ. We will show that $\Phi(\psi) = \psi(A)$ for all $\psi \in \Psi(S_\theta)$, where $\psi(A)$ is the operator defined via the functional calculus for sectorial operators. (The boundedness of Φ gives the desired result then.) By property 2) we have

$$\psi(A) = \frac{1}{2\pi i}\int_\gamma \psi(z)R(z, A)\mathrm{d}z = \frac{1}{2\pi i}\int_\gamma \psi(z)\Phi\big((z - \cdot)^{-1}\big)\mathrm{d}z,$$

and using linearity of Φ and the convergence property 3) the claim follows. (The latter can be applied because the existence of the integrals is clear.) □

3. The moon dual

Let $A \in \mathcal{C}(X)$ be densely defined. Thus, if A is sectorial, the dual operator $A' \in \mathcal{C}(X')$ is sectorial, too, but in general, it will not be densely defined. Consequently, we can not plug A' into the functional calculus.

Therefore, we will consider the moon dual space $X^{\#}$, a space that is even smaller than the sun dual $X^{\odot} = \overline{\mathcal{D}(A')}$ considered in [21], but still large enough to norm X. (Another notation for the moon dual space is $X^{☾}$.)

Definition 3.1. — Let $A \in \mathcal{C}(X)$ be a densely defined sectorial operator with dense range. The *moon dual space* $X^{\#}$ is defined by

$$X^{\#} := \overline{\mathcal{D}(A')} \cap \overline{\mathcal{R}(A')} \quad \text{with } \|\,.\,\|_{X'} \text{ as its norm.}$$

The *moon dual operator* $A^{\#}$ is the part of A' in $X^{\#}$, i.e.,

$$A^{\#}x^{\#} = A'x^{\#}$$

for all $x^{\#} \in \mathcal{D}(A^{\#}) = \{x' \in X^{\#} : x' \in \mathcal{D}(A') \text{ and } A'x' \in \overline{\mathcal{D}(A')}\}$.

Lemma 3.2. — *Let $A \in \mathcal{C}(X)$ be a densely defined sectorial operator with dense range. The embedding $\iota : X \to (X^{\#})'$ defined by $(\iota x)(x^{\#}) := x^{\#}(x)$ and its inverse are bounded, i.e.,*

$$C^{-1}\|x\| \leqslant \|x\|^{\#} := \sup\big\{|\langle x, x^{\#}\rangle| : x^{\#} \in X^{\#},\ \|x^{\#}\| \leqslant 1\big\} \leqslant \|x\|,$$

where $C = 2\sup\{\|tR(-t, A)\| : t > 0\}$.

Proof. — Because of $X^{\#} \subset X'$ we have $\|x\|^{\#} \leqslant \|x\|$ for all $x \in X$. For $x' \in X'$ with $\|x'\| \leqslant 1$ let $x_n^{\#} := \psi_n(A')x'$ with ψ_n from Remark 2.3. Then we have $x_n^{\#} \in X^{\#}$ and

$$\langle x, x_n^{\#}\rangle = \big\langle x, \psi_n(A')x'\big\rangle = \big\langle \psi_n(A)x, x'\big\rangle \xrightarrow{n\to\infty} \langle x, x'\rangle$$

for every $x \in X$. Since $\|x_n^{\#}\| = \|\psi_n(A')x'\| \leqslant C\|x'\| \leqslant C$ by Remark 2.3,

$$\|x\|^{\#} \geqslant \lim_{n\to\infty} \big|\langle x, x_n^{\#}/C\rangle\big| = \frac{|\langle x, x'\rangle|}{C} \quad \text{for all } x \in X.$$

Taking the supremum for $x' \in X'$ with $\|x'\| \leqslant 1$ yields $C\|x\|^{\#} \geqslant \|x\|$. □

Next, we take a look at the moon dual operator and show that it has the properties that are necessary for the functional calculus.

Proposition 3.3. — *If $A \in \mathcal{C}(X)$ is a densely defined sectorial operator of type μ with dense range, the operator $A^{\#}$ in $X^{\#}$ has the following properties:*

1) *$\varrho(A^{\#}) = \varrho(A)$ and $R(z, A^{\#}) = R(z, A)'|_{X^{\#}}$ for $z \in \varrho(A)$.*
2) *$A^{\#}$ is densely defined, sectorial of type μ and has dense range.*
3) *If A has a bounded $H^{\infty}(S_\theta)$ functional calculus, $A^{\#}$ has one, too.*

Proof

1) For $z \in \varrho(A)$ and $x^{\#} \in \mathcal{D}(A^{\#})$, we have

$$\big\langle x, R(z, A)'(z - A^{\#})x^{\#}\big\rangle = \big\langle (z - A)R(z, A)x, x^{\#}\big\rangle = \langle x, x^{\#}\rangle$$

for all $x \in X$, i.e., $R(z, A)'(z - A^{\#}) = \mathrm{id}$ on $\mathcal{D}(A^{\#})$. If, on the other hand, we choose an $x^{\#} \in X^{\#}$ and let $y^{\#} := R(z, A)'x^{\#}$, we can show $y^{\#} \in \mathcal{D}(A^{\#})$

as follows: obviously, $y^\# \in \mathcal{D}(A')$. Because $R(z,A)'$ is bounded and maps $\mathcal{R}(A')$ into $\mathcal{R}(A')$, it maps $\overline{\mathcal{R}(A')}$ into $\overline{\mathcal{R}(A')}$, i.e., we have $y^\# \in \overline{\mathcal{R}(A')}$. Since $A'R(z,A)'$ is bounded and maps $\mathcal{D}(A')$ into $\mathcal{D}(A')$, we have $A'y^\# \in \overline{\mathcal{D}(A')}$. Thus, we know $y^\# \in \mathcal{D}(A^\#)$ and get

$$\langle x, (z-A^\#)R(z,A)'x^\#\rangle = \big\langle R(z,A)(z-A)x, x^\#\big\rangle = \langle x, x^\#\rangle$$

for every $x \in \mathcal{D}(A)$. Since this is true for every $x^\# \in \mathcal{D}(A^\#)$, it follows that $(z-A^\#)R(z,A)' = \mathrm{id}$ on $X^\#$. We have proven that $z \in \varrho(A^\#)$ and $R(z,A^\#) = R(z,A)'|_{X^\#}$.

It remains to show that $\varrho(A^\#) \subset \varrho(A)$. Lemma 3.2 tells us that there exists a constant $\alpha > 0$ such that for every $x \in X$, there exists an $x^\# \in X^\#$ with $\|x^\#\| \leqslant 1$ and $|\langle x, x^\#\rangle| \geqslant \alpha\|x\|$. For $z \in \varrho(A^\#)$ and $\beta := \|R(z,A^\#)\|^{-1}$ we get for every $x \in X$ and the corresponding $x^\# \in X^\#$

$$\begin{aligned}\big\|(z-A)x\big\| &\geqslant \beta\big|\langle (z-A)x, R(z,A^\#)x^\#\rangle\big| \\ &= \beta\big|\langle x, (z-A^\#)R(z,A^\#)x^\#\rangle\big| = \beta\big|\langle x, x^\#\rangle\big| \geqslant \alpha\beta\|x\|.\end{aligned}$$

This implies that the operator $z-A$ is one-to-one and has closed range. Furthermore, the operator has dense range: for $z=0$, this is clear, so we can assume $z \neq 0$. If $z-A$ *doesn't* have dense range, there exists $x' \in X'$ with $x' \neq 0$ and $\langle (z-A)x, x'\rangle = 0$ for all $x \in \mathcal{D}(A)$. This implies $x' \in \mathcal{D}(A')$ and $(z-A')x' = 0$, i.e., $A'x' = zx' \in \mathcal{D}(A')$ and $x' = A'x'/z \in \mathcal{R}(A')$. The result is $x' \in \mathcal{D}(A^\#)$ and $(z-A^\#)x' = 0$. Because of $z \in \varrho(A^\#)$ this gives $x' = 0$, a contradiction.

2) This follows from [4, Thm 3.8] because of $X^\# = (X')_\infty \cap (X')_{00}$ in the notation of the latter paper.

3) The definition of the functional calculus and $R(z,A^\#) = R(z,A')|_{X^\#}$ imply $\psi(A^\#) = \psi(A')|_{X^\#}$ for all $\psi \in \Psi(S_\theta)$. Because A' has a bounded $H^\infty(S_\theta)$ functional calculus, this gives the desired result. □

4. Square functions

Kalton and Weis [13] have shown how the notion of square function estimates of the form

$$\Big\|\Big(\int_0^\infty \big|\psi(tA)x(\,.\,)\big|^2\,\frac{\mathrm{d}t}{t}\Big)^{\frac12}\Big\|_{L^p} \leqslant C\|x\|_{L^p} \qquad (x \in L^p)$$

can be generalised to a Banach space X. In this section we summarise some of their definitions and results. For this purpose, let $g_1, g_2, \ldots$ be independent, $N(0,1)$ distributed random variables.

Definition 4.1. — If H is a Hilbert space and X is a Banach space, $\gamma_+(H,X)$ denotes the space of all linear operators $u : H \to X$ satisfying

$$\|u\|_\gamma := \sup_{(e_n)} \Big(\mathbb{E} \big\| \sum g_n u(e_n) \big\|^2 \Big)^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all finite orthonormal systems (e_n) in H. This norm is certainly finite for finite rank operators and we let $\gamma(H,X)$ be the closure of all finite rank operators in $\gamma_+(H,X)$.

With the norm $\|.\|_\gamma$, the two linear spaces $\gamma_+(H,X)$ and $\gamma(H,X)$ are Banach spaces.

Definition 4.2. — If H is a Hilbert space and X is a Banach space, $\gamma'_+(H,X')$ denotes the space of all linear operators $v : H \to X'$ satisfying

$$\|v\|_{\gamma'} := \sup\big\{|\mathrm{trace}(v'u)| : u \in \gamma(H,X),\ \dim u(H) < \infty,\ \|u\|_\gamma \leqslant 1\big\} < \infty.$$

Furthermore, we define $\gamma'(H,X')$ to be the closure of all finite rank operators in $\gamma'_+(H,X')$.

The spaces $\gamma'_+(H,X')$ and $\gamma'(H,X')$ are Banach spaces, too. The dual space of $\gamma(H,X)$ can be written in the following form:

Proposition 4.3 (see [13, Prop. 5.1 and 5.2]). — *The space $\gamma'_+(H,X')$ is the dual of $\gamma(H,X)$ with respect to trace duality $\langle u\,,v\rangle_\gamma :=$ trace$(v'u)$. Furthermore, $\gamma_+(H,X') \subset \gamma'_+(H,X')$ and $\|v\|_{\gamma'} \leqslant \|v\|_\gamma$ for $v \in \gamma_+(H,X')$.*

Now we will consider the special case $H = L^2(\Omega)$, where $(\Omega, \mathfrak{A}, \nu)$ is a σ-finite measure space.

Definition 4.4. — Let $P_2(\Omega,X)$ denote the linear space of Bochner measurable functions $f : \Omega \to X$ such that $x' \circ f \in L^2(\Omega)$ for all $x' \in X'$.

For $f \in P_2(\Omega,X)$, a linear operator $u_f : L^2(\Omega) \to X$ is defined [13] by

$$\big\langle u_f(\varphi), x' \big\rangle = \int_\Omega \big\langle f(\omega), x' \big\rangle \varphi(\omega) \mathrm{d}\nu(\omega), \quad x' \in X',\ \varphi \in L^2(\Omega).$$

We write $\|f\|_\gamma := \|u_f\|_\gamma$ and we say that the operator u_f is represented by the function f. Similarly, we write $\|g\|_{\gamma'} := \|v_g\|_{\gamma'}$ for $g \in P_2(\Omega,X')$.

Note that $\|f\|_\gamma = \infty$ is possible, even if $f \in L^2(\Omega,X)$. Nevertheless, for certain functions in $L^2(\Omega,X)$ we always get finite norm.

Remark 4.5. — We use the notation

$$L^2 \otimes X := \Big\{ \sum_{n=1}^N f_n(\,.\,) x_n \;\Big|\; N \in \mathbb{N},\ f_n \in L^2(\mathbb{R}),\ x_n \in X \Big\}.$$

Then we have $u_f \in \gamma(L^2(\mathbb{R}), X)$ for every $f \in L^2 \otimes X$, and $\{u_f : f \in L^2 \otimes X\}$ is dense in $\gamma(L^2(\mathbb{R}), X)$. Note that for $h \in L^2(\Omega)$ and $x \in X$, we have

$$\big\|h(\,.\,)x\big\|_\gamma = \|h\|_{L^2} \cdot \|x\|.$$

For operators represented by functions, the duality can be computed using these functions [13, Cor. 5.5].

Proposition 4.6. — *For functions $f \in P_2(\Omega, X)$ and $g \in P_2(\Omega, X')$ with $u_f \in \gamma(L^2(\Omega), X)$ and $v_g \in \gamma'_+(L^2(\Omega), X')$, we have*

$$\langle u_f\,, v_g\rangle_\gamma = \int_\Omega \big\langle f(\omega), g(\omega)\big\rangle \mathrm{d}\nu(\omega).$$

Bounded operators can be extended from $L^2(\Omega)$ to $\gamma(L^2(\Omega), X)$ using the following result [13, Cor. 4.8].

Theorem 4.7. — *Let $M \in \mathcal{B}(L^2(\Omega_1), L^2(\Omega_2))$. If $M' \in \mathcal{B}(L^2(\Omega_2), L^2(\Omega_1))$ is the dual operator with respect to $\langle f, g\rangle = \int f(\omega)g(\omega)\mathrm{d}\nu(\omega)$, then*

$$\mathcal{M}u := u \circ M'$$

defines an operator

$$\mathcal{M} : \gamma\big(L^2(\Omega_1), X\big) \longrightarrow \gamma(L^2(\Omega_2), X)$$

with $\|\mathcal{M}\| \leqslant \|M\|$, such that for $f \in P_2(\Omega_1, X)$ and $g \in P_2(\Omega_2, X)$ with $\mathcal{M}u_f = u_g$, we have

$$\big\langle g(\,.\,), x'\big\rangle = M\big(\big\langle f(\,.\,), x'\big\rangle\big)$$

for all $x' \in X'$.

Remark 4.8. — For $f = h(\,.\,)x$ with $h \in L^2(\Omega)$ and $x \in X$ this means

$$\big\langle(\mathcal{M}u_f)\varphi, x'\big\rangle = \langle u_f M'\varphi, x'\rangle = \int_\Omega \big\langle f(\omega), x'\big\rangle (M'\varphi)(\omega)\mathrm{d}\nu(\omega),$$

i.e., $\mathcal{M}u_{h(\,.\,)x}$ is represented by the function $(Mh)(\,.\,)x$, since

$$M\big(\big\langle f(\,.\,), x'\big\rangle\big) = M\big(\langle x, x'\rangle h(\,.\,)\big) = \big\langle xMh(\,.\,), x'\big\rangle.$$

Corollary 4.9. — *Every isometric isomorphism $M : L^2(\Omega_1) \to L^2(\Omega_2)$ can be extended to an isometric isomorphism*

$$\mathcal{M} : \gamma\big(L^2(\Omega_1), X\big) \longrightarrow \gamma\big(L^2(\Omega_2), X\big)$$

such that for $h \in L^2(\Omega_1)$ and $x \in X$ the operator $\mathcal{M}u_{h(\,.\,)x}$ is represented by the function $(Mh)(\,.\,)x$.

Lemma 4.10 (see [13, Lemma 4.10 b) and Rem. 5.4]). — *Let $f_n, f \in P_2(\Omega, X)$. If $f_n(\omega) \to f(\omega)$ almost everywhere, then*

$$\|f\|_\gamma \leqslant \liminf \|f_n\|_\gamma.$$

The same is true for the $\|\,.\,\|_{\gamma'}$ norm.

5. The main results

If X is not a Hilbert space there are only few spectral operators of scalar type on X, i.e., operators with an $B_b(\sigma(A))$ calculus. For example, $A = \mathrm{d}/\mathrm{d}x$ is not a spectral operator on $L^p(\mathbb{R})$ for $p \neq 2$. The reason for this is a general lack of spectral projections outside the Hilbert space setting. Nevertheless, we can characterise the H^∞ calculus by dilations to these very special operators.

Recall that a Banach space X has *cotype* $q \in [2, \infty)$ if there exists a constant $C > 0$ with

$$\Big(\sum_{n=1}^{m} \|x_n\|^q\Big)^{1/q} \leqslant \Big(\int_0^1 \|\sum_{n=1}^{m} r_n(t) x_n\|^q \,\mathrm{d}t\Big)^{1/q}$$

for all $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$. Here, (r_n) is the sequence of Rademacher functions. If X has some cotype $q \in [2, \infty)$ we say that X has *finite cotype.*

First, we consider a sectorial operator A with $\omega(A) < \frac{1}{2}\pi$.

THEOREM 5.1. — *Let $-A$ be the generator of a bounded analytic semigroup in a Banach space X with finite cotype. Let A have dense range, and assume that X has finite cotype. Then we get* 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) *with*

1) *A has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$.*
2) *There exist a Banach space Y, an isomorphic embedding $J : X \to Y$, a bounded projection P of Y onto $J(X)$, and a spectral operator of scalar type M on Y with $\sigma(M) = i\mathbb{R}$ such that*

$$JR(z, A) = PR(z, M)J \quad \textit{for } \operatorname{Re} z < 0. \tag{5.1}$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{R(z,M)} & Y \\ \cup & & \big\downarrow P \\ J(X) & & J(X) \\ J\big\uparrow & & \big\uparrow J \\ X & \xrightarrow{R(z,A)} & X \end{array}$$

3) *There exist Y, J and M as in* 2) *such that for the semigroups (T_t) and (N_t) generated by $-A$ and $-M$ respectively, we have*

$$JT_t = PN_tJ \quad \textit{for } t > 0. \tag{5.2}$$

4) *A has a bounded $H^\infty(S_\theta)$ functional calculus for all $\theta > \frac{1}{2}\pi$.*

REMARK 5.2. — The proof of this theorem will be given in Section 6, where we will see that we can choose $Y = \gamma(L^2(\mathbb{R}), X)$. Note that $\gamma(L^2(\mathbb{R}), X)$ is isomorphic to the subspace $\mathrm{Rad}(X)$ of $L^2([0,1],X)$ with

$$\mathrm{Rad}(X) := \overline{\mathrm{span}}\Big\{\sum_n x_n r_n : x_n \in X\Big\},$$

where (r_n) is the sequence of the Rademacher functions. If X is B-convex (cf. [20]) then $\mathrm{Rad}(X)$ is complemented in $L^2([0,1],X)$ and it is easy to see (as in the proof of Corollary 5.4) that we can choose $Y = L^2([0,1],X)$ in 2) and 3) above.

For Hilbert spaces, a result of McIntosh [19] states that $\omega_{H^\infty}(A) = \omega(A)$, provided that A has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta \in (0,\pi)$, so that in this case 4) ⇒ 1) and we get a full characterisation. This Hilbert space result is due to Le Merdy [18].

In general Banach spaces one may have $\omega_{H^\infty}(A) > \omega(A)$ (see [10]) and we have to make an additional assumption to obtain a full characterisation.

A sectorial operator is called *almost R-sectorial* if for some $\omega \geqslant \omega(A)$ the set $\{zAR(z,A)^2 : z \notin \overline{S}_\omega\}$ is R-bounded. By $\omega_r(A)$ we denote the infimum over all such ω. It is shown in [11] that every operator A with a bounded H^∞ calculus is almost R-bounded and, furthermore, we have $\omega_r(A) = \omega_{H^\infty}(A)$. Hence we obtain a characterisation for almost R-bounded operators.

Corollary 5.3. — *If the assumptions of Theorem* 5.1 *hold and A is almost R-bounded with $\omega_r(A) < \frac{1}{2}\pi$, then* 1), 2), 3), *and* 4) *in Theorem* 5.1 *are equivalent.*

If X is a UMD space (see [3] for information on this class), we can restate our characterisation in a manner more closely related to the Sz.-Nagy dilation theorem.

Corollary 5.4. — *Let X be a UMD space and A an almost R-sectorial operator with $\omega_r(A) < \frac{1}{2}\pi$ that has dense domain and dense range. Then A has a bounded H^∞ calculus with*

$$\omega_{H^\infty}(A) = \omega_r(A) < \tfrac{1}{2}\pi$$

if and only if there are an isomorphic embedding $J : X \to L^2([0,1],X)$, a bounded projection $P : L^2([0,1],X) \to J(X)$ and a bounded group (N_t) on $L^2([0,1],X)$, such that for the semigroup (T_t) generated by $-A$ we have

$$JT_t = PN_tJ \quad \text{for } t > 0. \tag{5.3}$$

If $X = L^p(\Omega,\mu)$ with $1 < p < \infty$, then $L^2([0,1],X)$ can be replaced in this statement by a space $L^p(\Omega_0,\mu_0)$.

Proof. — Assume that we have a bounded group (N_t) on $L^2([0,1],X)$ with generator $-M$ satisfying (5.3). Since $L^2([0,1],X)$ is also a UMD space it follows from [9] that the operator M has a bounded $H^\infty(S_\Theta)$ calculus for all $\Theta > \frac{1}{2}\Pi$. As in the proof of Theorem 5.1 we see that $\omega_{H^\infty}(A) \leqslant \omega_{H^\infty}(M)$ and by assumption $\omega_{H^\infty}(A) = \omega_r(A) < \frac{1}{2}\pi$.

Conversely, if A has a bounded H^∞ calculus, we construct the group (N_t) on $Y = \operatorname{Rad}(X)$ as in the proof of Theorem 5.1. Since a UMD space has non-trivial type [3], the space $\operatorname{Rad}(X)$ is a complemented subspace of $L^2([0,1],X)$ (see [20]), i.e., $L^2([0,1],X) \cong \operatorname{Rad} X \oplus Z$ with a closed subspace Z of $L^2([0,1],X)$. Now the group $\tilde{N}_t := N_t \oplus \mathrm{e}^{it} \operatorname{id}_Z$ on $L^2([0,1],X)$ has the required properties.

If $X = L^p(\Omega,\mu)$, then X is a UMD space if $1 < p < \infty$. The argument given above also works for $L^p([0,1],X)$ in place of $L^2([0,1],X)$. But $L^p([0,1],L^p(\Omega,\mu))$ is of the form $L^p(\Omega_0,\mu_0)$. □

Now we consider sectorial operators without the restriction $\omega(A) < \frac{1}{2}\pi$.

Theorem 5.5. — *Let $-A$ be a sectorial operator of type μ in a Banach space X with finite cotype. Let A have dense domain and dense range, and assume that X has finite cotype. For $\mu < \omega < \pi$ we get* 1) $\Rightarrow$ 2) $\Rightarrow$ 3) *with*

1) *A has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta < \omega$.*
2) *There exist a Banach space Y, an isomorphic embedding $J : X \to Y$, a bounded projection P of Y onto $J(X)$ and a spectral operator of scalar type M on Y with $\sigma(M) = \partial S_\omega$ such that*
$$JR(z,A) = PR(z,M)J \quad \textit{for } |\arg z| > \omega.$$
3) *A has a bounded $H^\infty(S_\theta)$ functional calculus for all $\theta > \omega$.*

Again the additional assumption of almost R-boundedness allows us to obtain a full characterisation. Since $\omega_{H^\infty}(A) = \omega_r(A)$ (see [11]) we get from Theorem 5.5:

Corollary 5.6. — *If the assumptions of Theorem* 5.5 *hold and A is almost R-sectorial, we have the following equivalence:* 2) *from Theorem* 5.5 *is true for some $\omega > \omega_r(A)$ if and only if A has a bounded $H^\infty(S_\theta)$ functional calculus for all $\theta > \omega_r(A)$.*

For $\omega_{H^\infty}(A) > \frac{1}{2}\pi$, Corollary 5.6 gives a new result even in Hilbert spaces since every sectorial operator is almost R-sectorial in this setting.

Corollary 5.7. — *Let X be a Hilbert space. A sectorial operator A on X has a bounded H^∞ calculus if and only if A has a dilation to a normal operator, i.e., there is a Hilbert space Y, an isomorphic embedding $J : X \to Y$ and a normal operator M on Y such that*
$$JR(z,A) = PR(z,M)J \quad \textit{for all } z \in \mathbb{R}^-$$
with a bounded projection $P : Y \to J(X)$.

6. The construction

In this section, we will prove Theorem 5.1 and Theorem 5.5. Until stated otherwise assume that $-A \in \mathcal{C}(X)$ generates a bounded analytic semigroup (T_t) and has dense range; then A has type $\mu < \frac{1}{2}\pi$ by Remark 2.2. Furthermore, we assume that A has a bounded $H^\infty(S_\theta)$ functional calculus for some $\theta < \frac{1}{2}\pi$.

We choose $Y = \gamma(L^2(\mathbb{R}), X)$, which was defined in Section 4.

Next, we will define the dilation (N_t), the embedding J and the projection P satisfying the dilation equation (5.2).

The dilation (N_t). — We define

$$N_t : \gamma\big(L^2(\mathbb{R}), X\big) \longrightarrow \gamma\big(L^2(\mathbb{R}), X\big)$$

to be the extension (see Theorem 4.7) of the bounded multiplication operator in $L^2(\mathbb{R})$ that maps $h \in L^2(\mathbb{R})$ to the function $s \mapsto \mathrm{e}^{-ist}h(s)$. This means

$$(N_t u)(\varphi) = u\big(s \mapsto \mathrm{e}^{-ist}\varphi(s)\big) \quad \text{for } u \in \gamma(L^2(\mathbb{R}), X) \text{ and } \varphi \in L^2(\mathbb{R}).$$

Proposition 6.1. — *With this definition, (N_t) is a C_0 group of isometries. If we call its generator $-M$, then M is a spectral operator of scalar type.*

Proof. — By Corollary 4.9, N_t is an isometry and for $h \in L^2(\mathbb{R})$ and $x \in X$, the operator $N_t u_{h(\,.\,)x}$ is represented by the function $s \mapsto \mathrm{e}^{-ist}h(s)x$. Therefore, we have

$$\|N_t u_{h(\,.\,)x} - u_{h(\,.\,)x}\|_\gamma = \big\|s \mapsto (\mathrm{e}^{-ist} - 1)h(s)\big\|_{L^2} \cdot \|x\| \xrightarrow{t\to 0} 0.$$

This implies $N_t u_f \to u_f$ for $t \to 0$ and all $f \in L^2 \otimes X$. Since the operators corresponding to this set are dense in $\gamma(L^2(\mathbb{R}), X)$, we have proven that (N_t) is a C_0 group (the group properties are obvious).

Now let's take at look at M. If we choose $x \in X$ and $h \in L^2(\mathbb{R})$ such that the function $s \mapsto sh(s)$ is in $L^2(\mathbb{R})$,

$$\frac{N_t u_{h(\,.\,)x} - u_{h(\,.\,)x}}{t} \quad \text{is represented by} \quad s \mapsto \frac{\mathrm{e}^{-ist} - 1}{t} h(s)x.$$

For $t \to 0$ this function converges to $s \mapsto -ish(s)x$ with respect to $\|\,.\,\|_\gamma$, so $Mu_{h(\,.\,)x}$ is represented by $s \mapsto ish(s)x$. For $z \notin i\mathbb{R}$, we consider the operator $\mathcal{M}_z$ that is the extension of the multiplication operator in $L^2(\mathbb{R})$ mapping h to $s \mapsto h(s)/(z - is)$. Then we have

$$\mathcal{M}_z(z - M)u_f = (z - M)\mathcal{M}_z u_f = u_f$$

for all $f \in L^2 \otimes X$ by Remark 4.8 and a density argument, and since the set $K := \{u_f : f \in L^2 \otimes X\}$ is dense in $\gamma(L^2(\mathbb{R}), X)$ by Remark 4.5 and $N_t(K) \subset K$, it is a core for M and we have $(z - M)\mathcal{M}_z = \mathrm{id}$ on $\gamma(L^2(\mathbb{R}), X)$ and $\mathcal{M}_z(z - M) = \mathrm{id}$ on $\mathcal{D}(M)$, i.e., $z \in \varrho(M)$.

Now we will show that M is a spectral operator of scalar type. From the considerations above it follows that $\sigma(M) = i\mathbb{R}$. We define $\Phi(f)$ to be the extension of the multiplication operator $(B_f h)(t) := f(it)h(t)$. This functional calculus Φ is bounded, linear and multiplicative. Property 2) of sectorial operators follows from $\Phi((z-\cdot)^{-1}) = \mathcal{M}_z$.

This leaves us with 3): Since $\|\Phi(f_n)\| \leqslant \|B_{f_n}\| \leqslant \|f_n\|_\infty$, it suffices (Banach-Steinhaus) to show $\Phi(f_n)u \to \Phi(f)u$ for every u represented by a function in $L^2 \otimes X$, and for this it is enough to consider u represented by $h(.)x$ with $h \in L^2(\mathbb{R})$ and $x \in X$. In this case $\Phi(f_n)u$ is represented by the function $t \mapsto f_n(it)h(t)x$ and $\Phi(f)u$ by $t \mapsto f(it)h(t)x$. Because of $f_n(it)h(t) \to f(it)h(t)$ in $L^2(\mathbb{R})$ the claim follows from Remark 4.5. □

The embedding *J*. — Now we assume that the Banach space X has finite cotype. Under this assumption, we want to show that one can define an embedding $J : X \to \gamma(L^2(\mathbb{R}), X)$ by

$$\langle (Jx)(\varphi), x' \rangle := \int_{-\infty}^{\infty} \langle A^{\frac12} R(it, A)x, x' \rangle \varphi(t)\,\mathrm{d}t$$

for all $x \in X$, $x' \in X'$ and $\varphi \in L^2(\mathbb{R})$. In other words, we want to show that Jx belongs to $\in \gamma(L^2(\mathbb{R}), X)$ and is represented by the function

$$t \longmapsto A^{\frac12} R(it, A)x.$$

To ease matters later on, we consider a second embedding $J_\#$. For $x^\# \in X^\#$, the operator $J_\# x^\#$ will be represented by the function

$$t \longmapsto (A^\#)^{\frac12} R(-it, A^\#)x^\#.$$

Note that $A^\#$ has the same properties as A by Proposition 3.3.

Proposition 6.2. — *The operator $J : X \to \gamma(L^2(\mathbb{R}), X)$ is well defined and bounded, and the same is true for the operator $J_\# : X^\# \to \gamma'_+(L^2(\mathbb{R}), X')$.*

Proof. — Since X has finite cotype and A has a bounded $H^\infty(S_\theta)$ functional calculus with $\theta < \frac12\pi$, we know from [13, Thm 7.2] that $Jx \in \gamma(L^2(\mathbb{R}), X)$ and that $\|Jx\|_\gamma \leqslant C\|x\|$ for all $x \in \mathcal{D}(A) \cap \mathcal{R}(A)$. Since this set is dense in X, the claim follows with Lemma 4.10 if we can prove that for every $x \in X$ the function $t \mapsto A^{\frac12} R(it, A)x$ is in $P_2(\mathbb{R}, X)$, i.e.,

$$\int_{-\infty}^{\infty} \left|\langle A^{\frac12} R(it, A)x, x' \rangle\right|^2 \mathrm{d}t < \infty$$

for all $x \in X$ and $x' \in X'$. This may be seen as follows:

$$\left|\langle A^{\frac12} R(it, A)x, x' \rangle\right|^2 \leqslant \left\|A^{\frac12} R(it, A)x\right\| \cdot \|x'\| \cdot \left|\langle A^{\frac12} R(it, A)x, x' \rangle\right|.$$

The first factor is $\leqslant D/\sqrt{t}$ with some constant D since $\|\psi(A/t)\| \leqslant D$ for $\psi(z) := z^{\frac12}/(i-z)$. Now the claim follows from Proposition 2.5.

The proof for $J_\#$ is the same since [13, Thm 7.2] gives $J_\# x^\# \in \gamma'_+(L^2(\mathbb{R}), X')$ and $\|J_\# x^\#\|_{\gamma'} \leqslant C\|x^\#\|$ for $x^\# \in \mathcal{D}(A^\#) \cap \mathcal{R}(A^\#)$. □

Now we will prove that the embedding J gives rise to an equivalent norm on X. To this end, we will use the following result.

LEMMA 6.3. — *For all $x \in X$ and $x^\# \in X^\#$ we have*

$$\langle Jx\,, J_\# x^\# \rangle_\gamma = \pi \langle x, x^\# \rangle.$$

Proof. — From Proposition 4.6 it follows that

$$\begin{aligned}\langle Jx\,, J_\# x^\# \rangle_\gamma &= \int_{-\infty}^{\infty} \big\langle A^{\frac12} R(it, A)x, (A^\#)^{\frac12} R(-it, A^\#)x^\# \big\rangle \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \big\langle AR(it, A)R(-it, A)x, x^\# \big\rangle \mathrm{d}t \\ &= 2\int_0^{\infty} \big\langle (A/t)R(i, A/t)R(-i, A/t)x, x^\# \big\rangle \frac{\mathrm{d}t}{t} \\ &= 2\int_0^{\infty} \big\langle sAR(i, sA)R(-i, sA)x, x^\# \big\rangle \frac{\mathrm{d}s}{s}\cdot\end{aligned}$$

According to Proposition 2.5 this is continuous in x. Thus, the claim follows from Lemma 6.4 and

$$\int_0^{\infty} \frac{s}{(i-s)(-i-s)} \frac{\mathrm{d}s}{s} = \int_0^{\infty} \frac{1}{1+s^2} \mathrm{d}s = \frac{1}{2}\pi$$

because $\mathcal{D}(A) \cap \mathcal{R}(A)$ is dense in X (see Remark 2.3). □

LEMMA 6.4 (see [19]). — *Under the assumptions of this section we have*

$$\int_0^{\infty} \psi(tA)x \frac{\mathrm{d}t}{t} = x \quad \textit{for all } x \in \mathcal{D}(A) \cap \mathcal{R}(A)$$

if $\psi \in \Psi(S_\theta)$ has the property $\int_0^\infty \psi(t)\,\frac{\mathrm{d}t}{t} = 1$.

PROPOSITION 6.5. — *There exist constants $\alpha, \beta > 0$ such that for all $x \in X$*

$$\alpha\|x\| \leqslant \|Jx\|_\gamma \leqslant \beta\|x\|.$$

Proof. — The existence of β follows from Proposition 6.2. Furthermore, this proposition tells us that $J_\#$ is bounded, too. Lemma 6.3 implies

$$\big|\langle x, x^\# \rangle\big| \leqslant \pi^{-1} \cdot \|Jx\|_\gamma \cdot \|J_\# x^\#\|_{\gamma'} \leqslant \pi^{-1} \cdot \|Jx\|_\gamma \cdot \|J_\#\| \cdot \|x^\#\|,$$

and taking the supremum for all $x^\#$ with $\|x^\#\| \leqslant 1$ gives the desired result (using Lemma 3.2). □

The projection P. — Our idea is to define $P = \pi^{-1} J (J_\#)'$, but since $(J_\#)'$ is an operator from $\gamma'_+(L^2(\mathbb{R}), X')'$ to $(X^\#)'$, we have to deal with certain embedding operators.

DEFINITION 6.6. — The embedding

$$\iota_\gamma : \gamma\big(L^2(\mathbb{R}), X\big) \longrightarrow \gamma'_+\big(L^2(\mathbb{R}), X'\big)'$$

is defined, for $u \in \gamma(L^2(\mathbb{R}), X)$ and $v \in \gamma'_+(L^2(\mathbb{R}), X')$, by

$$(\iota_\gamma u)(v) := \langle u\,, v\rangle_\gamma.$$

REMARK 6.7. — The embedding ι_γ is well defined and bounded by Proposition 4.3.

Now we can consider the operator $(J_\#)' \circ \iota_\gamma : \gamma(L^2(\mathbb{R}), X) \to (X^\#)'$, and we can show that it is, in a certain sense, X-valued.

LEMMA 6.8. — *For u_f represented by a function $f \in L^2 \otimes X$ we have*

$$\big\langle (J_\#)'(\iota_\gamma u_f), x^\# \big\rangle_{((X^\#)', X^\#)} = \Big\langle \int_0^\infty A^{\frac12} R(-it, A) f(t)\,\mathrm{d}t, x^\# \Big\rangle_{(X, X')}$$

for all $x^\# \in X^\#$.

Proof. — We have

$$\begin{aligned}\big\langle (J_\#)'(\iota_\gamma u_f), x^\# \big\rangle_{((X^\#)', X^\#)} &= \langle \iota_\gamma u_f, J_\# x^\# \rangle_{(\gamma'_+(L^2(\mathbb{R}), X')', \gamma'_+(L^2(\mathbb{R}), X'))} \\ &= \langle u_f\,, J_\# x^\# \rangle_\gamma = \int_0^\infty \big\langle f(t), (A^\#)^{\frac12} R(-it, A^\#) x^\# \big\rangle \mathrm{d}t,\end{aligned}$$

and the claim follows. (Note that the integral in the stated equality exists according to the inequality shown in the proof of Proposition 6.2) ☐

From Lemma 3.2 we know that the bounded embedding $\iota : X \to (X^\#)'$ defined by $(\iota x)(x^\#) := x^\#(x)$ has a bounded inverse on $\iota(X)$. Lemma 6.8 tells us that

$$(J_\#)'(\iota_\gamma u_f) \in \iota(X) \quad \text{for } f \in L^2 \otimes X.$$

COROLLARY 6.9. — *For every $u \in \gamma(L^2(\mathbb{R}), X)$, we have*

$$(J_\#)'(\iota_\gamma u) \in \iota(X).$$

Proof. — Since $\{u_f : f \in L^2 \otimes X\}$ is dense in $\gamma(L^2(\mathbb{R}), X)$ with respect to $\|\,.\,\|_\gamma$ according to Remark 4.5 and $\iota(X)$ is a closed subspace of $(X^\#)'$, this follows from Lemma 6.8. ☐

Now we are ready to define the projection P. Let

$$P : \gamma\big(L^2(\mathbb{R}), X\big) \longrightarrow \gamma\big(L^2(\mathbb{R}), X\big)$$

be defined by

$$P := \pi^{-1} J \circ \iota^{-1} \circ (J_\#)' \circ \iota_\gamma .$$

PROPOSITION 6.10. — *The operator P is a bounded projection onto $\mathcal{R}(J)$.*

Proof. — The boundedness of P is clear. Since $\mathcal{R}(P) \subset \mathcal{R}(J)$, we just have to show that $Pu = u$ for all $u \in \mathcal{R}(J)$, i.e., $\pi^{-1}\iota^{-1}(J_\#)'\iota_\gamma J = \mathrm{id}$. This follows using Lemma 6.3: for every $x^\# \in X^\#$ we get

$$\begin{aligned}\big\langle \iota^{-1}(J_\#)'\iota_\gamma Jx, x^\#\big\rangle_{(X,X')} &= \big\langle (J_\#)'\iota_\gamma Jx, x^\#\big\rangle_{((X^\#)',X^\#)}\\ &= \langle \iota_\gamma Jx, J_\# x^\#\rangle_{(\gamma'_+(L^2(\mathbb{R}),X')',\gamma'_+(L^2(\mathbb{R}),X'))}\\ &= \langle Jx\,, J_\# x^\#\rangle_\gamma = \pi\langle x, x^\#\rangle .\end{aligned}$$

□

Proof of Theorem 5.1

1) ⇒ 2). Let J, P and M be defined as above. It remains to check the dilation equation $JR(z,A) = PR(z,M)J$. For $\operatorname{Re} z < 0$ we have (using the resolvent equation)

$$A^{\frac12}R(it,A)R(z,A)x = \frac{A^{\frac12}R(it,A)x}{z-it} - \frac{A^{\frac12}R(z,A)x}{z-it},$$

and this is the function representing $JR(z,A)x$. Thus,

$$\begin{aligned}&\big\langle JR(z,A)x\,, J_\# x^\#\big\rangle_\gamma\\ &\qquad= \int_{-\infty}^{\infty} \big\langle A^{\frac12}R(it,A)R(z,A)x, (A^\#)^{\frac12}R(-it,A^\#)x^\#\big\rangle \,\mathrm{d}t\\ &\qquad= \int_{-\infty}^{\infty} \frac{\langle A^{\frac12}R(it,A)x, (A^\#)^{\frac12}R(-it,A^\#)x^\#\rangle}{z-it}\,\mathrm{d}t\\ &\qquad\qquad - \int_{-\infty}^{\infty} \frac{\langle A^{\frac12}R(z,A)x, (A^\#)^{\frac12}R(-it,A^\#)x^\#\rangle}{z-it}\,\mathrm{d}t .\end{aligned}$$

On operators represented by functions the operator $R(z,M)$ acts as multiplication by $(z-it)^{-1}$. Therefore, the first integral equals

$$\big\langle R(z,M)Jx\,, J_\# x^\#\big\rangle_\gamma .$$

The second integral equals 0. This can be seen as follows: on every sector S_ν with $\frac12\pi < \nu < \min\{|\arg z|, \pi-\theta\}$ the function

$$\lambda \longmapsto \frac{\langle A^{\frac12}R(z,A)x, (A^\#)^{\frac12}R(-\lambda,A^\#)x^\#\rangle}{z-\lambda}$$

is analytic. Furthermore, $\|\psi(A^{\#}/\lambda)\| \leqslant C$ for $\psi(z) := z^{\frac{1}{2}}(-1-z)^{-1}$, i.e.,

$$\big\|(A^{\#})^{\frac{1}{2}} R(-\lambda, A^{\#})x^{\#}\big\| \leqslant \frac{C}{|\lambda|^{\frac{1}{2}}},$$

and the claim follows using Cauchy's theorem. (Consider a semi-circle in the right half plane with radius $r \to \infty$.) Our result is

$$\big\langle JR(z,A)x\,, J_{\#}x^{\#}\big\rangle_{\gamma} = \big\langle R(z,M)Jx\,, J_{\#}x^{\#}\big\rangle_{\gamma},$$

and Lemma 6.3 implies

$$\begin{aligned}\pi\,\big\langle R(z,A)x, x^{\#}\big\rangle &= \big\langle JR(z,A)x\,, J_{\#}x^{\#}\big\rangle_{\gamma} = \big\langle R(z,M)Jx\,, J_{\#}x^{\#}\big\rangle_{\gamma}\\ &= \big\langle \iota_{\gamma} R(z,M)Jx, J_{\#}x^{\#}\big\rangle_{(\gamma'_{+}(L^2(\mathbb{R}),X')', \gamma'_{+}(L^2(\mathbb{R}),X'))}\\ &= \big\langle (J_{\#})'\iota_{\gamma} R(z,M)Jx, x^{\#}\big\rangle_{((X^{\#})', X^{\#})}.\end{aligned}$$

We have proven

$$\pi R(z,A) = \iota^{-1}(J_{\#})'\iota_{\gamma} R(z,M)J,$$

and the definition of P gives us $JR(z,A) = PR(z,M)J$.

2) ⇒ 3). Apply the exponential formula

$$S_t x = \lim_{n\to\infty}\Big[\frac{n}{t}R\Big(\frac{n}{t}, B\Big)\Big]^n x$$

for the generator B of a C_0 semigroup (S_t) to $B = -A$ and $B = -M$. Then (5.2) follows from (5.1).

3) ⇒ 4). From Proposition 2.7 and Proposition 6.1 it follows that M has an $H^{\infty}(S_{\theta})$ functional calculus for all $\theta > \frac{1}{2}\pi$. The Laplace formula $R(\lambda, B) = \int_0^{\infty} \mathrm{e}^{-\lambda t} S_t \,\mathrm{d}t$ for a semigroup generator B applied to $B = -A$ and $B = -M$ implies the dilation equation (5.1). Now the definition of the functional calculus implies $J\psi(A) = P\psi(M)J$ for all $\psi \in \Psi(S_{\theta})$, and with the properties of J and P this gives us the estimate $\|\psi(A)\| \leqslant C\|\psi\|_{\infty}$. □

Remark 6.11. — There is an alternative construction of the dilation in (5.2); more details can be found in [8]. Let (U_t) be defined by

$$(U_t u)(\varphi) := u\big(\varphi(\cdot - t)\big) \quad \text{for } u \in \gamma\big(L^2(\mathbb{R}), X\big) \text{ and } \varphi \in L^2(\mathbb{R}).$$

Then one can show that $(U_t)_{t\in\mathbb{R}}$ is a C_0 group of isometries. If we call its generator $-B$, then B is a spectral operator of scalar type with $\sigma(B) = i\mathbb{R}$. The embedding J is defined as follows: for $x \in X$ the operator Jx is represented by the function

$$t \longmapsto \begin{cases} A^{\frac{1}{2}} T_t x & t > 0,\\ 0 & t \leqslant 0.\end{cases}$$

The projection P can be defined as

$$P := 2J \circ \iota^{-1} \circ (J_{\#})' \circ \iota_{\gamma},$$

where $J_\#$ is defined with $(A^\#)^{\frac{1}{2}}T_t^\#$. The properties of J and P follow from the identity

$$\int_0^\infty \langle AT_t x, x'\rangle \mathrm{d}t = \langle x, x'\rangle \quad \text{for all } x \in X \text{ and } x' \in X'.$$

In this case the dilation equation can be proven somewhat simpler: we use that $U_t J x$ is represented by

$$s \longmapsto \begin{cases} A^{\frac{1}{2}} T_{s+t} x & s > -t, \\ 0 & s \leqslant -t \end{cases}$$

and the fact that $Pu_f = Pu_{f\cdot\chi_{\mathbb{R}+}}$ by the definition of J and P. Essentially, the first construction is the "Fourier image" of the present one.

We still have to prove Theorem 5.5, and for this we need the following known result which we state for convenience. (Note that a stronger result holds true: A^β is sectorial of type $\beta\mu$ and has a bounded $H^\infty(S_{\beta\theta})$ functional calculus. However, only the weaker result is needed here.)

LEMMA 6.12. — *Let A be a sectorial operator of type μ with dense domain and dense range. Assume that A has a bounded $H^\infty(S_\theta)$ functional calculus and let $\beta \in (0, \pi/\theta)$. Then the operator A^β, defined by the functional calculus, is sectorial of type $\beta\theta$, has dense domain and dense range and a bounded $H^\infty(S_\tau)$ functional calculus for every $\tau > \beta\theta$. For $f \in H^\infty(S_\tau)$, we have $f(A^\beta) = f_\beta(A)$, where $f_\beta(z) := f(z^\beta)$.*

Proof. — From the properties of the functional calculus it follows that A^β is closed and densely defined. It has dense range because $A^{-\beta} = (A^\beta)^{-1}$ and $A^{-\beta}$ is densely defined.

Now we will show that A^β is sectorial of type $\beta\theta$. Obviously, we have

$$zR(z, A^\beta) = f_z(A) \quad \text{with} \quad f_z(\lambda) := \frac{z}{z - \lambda^\beta}$$

for $|\arg z| > \beta\theta$. For $|\arg z| > \beta\theta + \varepsilon$ the functions f_z are uniformly bounded on S_θ and the resolvent estimate follows from the fact that A has a bounded $H^\infty(S_\theta)$ functional calculus.

We still have to show the last statement: for $x \in X$ we have

$$\begin{aligned} \psi(A^\beta)AR(-1, A)x &= \frac{1}{2\pi i}\int_\gamma \psi(z)R(z, A^\beta)AR(-1, A)x\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_\gamma \psi(z)\frac{1}{2\pi i}\int_{\gamma_1} \frac{\lambda}{(z-\lambda^\beta)(-1-\lambda)}R(\lambda, A)x\,\mathrm{d}\lambda\,\mathrm{d}z \\ &= \frac{1}{2\pi i}\int_{\gamma_1} \psi(\lambda^\beta)\frac{\lambda}{-1-\lambda}R(\lambda, A)x\,\mathrm{d}\lambda \\ &= \psi_\beta(A)AR(-1, A)x, \end{aligned}$$

where γ is the edge of a sector between $S_{\beta\theta}$ and S_τ and γ_1 is the edge of a sector between S_μ and S_θ. Since $\mathcal{R}(AR(-1,A)) = \mathcal{R}(A)$ is dense in X, this implies the desired identity for $\psi \in \Psi(S_\tau)$. For $f \in H^\infty(S_\tau)$, we can use an approximation argument. □

Proof of Theorem 5.5. — We define $\alpha := \frac{2}{\pi}\omega$, which gives us $\omega = \frac{1}{2}\alpha\pi$.

1) $\Rightarrow$ 2). We consider the operator $A_0 := A^{1/\alpha}$. It is sectorial and has a bounded $H^\infty(S_{\theta_0})$ functional calculus for every $\theta_0 > \theta/\alpha$. Because of $\theta/\alpha < \omega/\alpha = \frac{1}{2}\pi$ we can choose $\theta_0 < \frac{1}{2}\pi$. Now, Theorem 5.1 gives us an embedding J_0, a projection P_0 and a spectral operator M_0 (with $\sigma(M_0) = i\mathbb{R}$) such that $J_0R(z,A_0) = P_0R(z,M_0)J_0$ for $\operatorname{Re} z < 0$. This implies $J_0\psi_0(A_0) = P_0\psi_0(M_0)J_0$ for every $\psi_0 \in \Psi(S_\tau)$ with $\tau > \frac{1}{2}\pi$. For $f_0 \in H^\infty(S_\tau)$ we consider $f_n := f_0 \cdot \psi_n \in \Psi(S_\tau)$ with ψ_n from Remark 2.3. For $n \to \infty$ we get $J_0f_0(A_0)x = P_0f_0(M_0)J_0x$ for every $f_0 \in H^\infty(S_\tau)$ and every $x \in X$.

Let $J := J_0$ and $P := P_0$. Let $M := M_0^\alpha$. Similarly to the proof of Lemma 6.12 one can show that M is a spectral operator of scalar type with $\sigma(M) = \partial S_{\frac{1}{2}\alpha\pi}$, and for $f_0 \in H^\infty(S_\tau)$ and $f(\lambda) := f_0(\lambda^{1/\alpha})$ we have $f_0(M_0) = f(M)$.

For z with $|\arg z| > \omega$ we choose $\tau \in (\frac{1}{2}\pi, |\arg z|/\alpha)$. Then the function

$$f_0(\lambda) := \frac{1}{z - \lambda^\alpha}$$

lies in $H^\infty(S_\tau)$, and with $f(\lambda) := (z-\lambda)^{-1}$ we get

$$JR(z,A) = Jf(A) = J_0f_0(A_0) = P_0f_0(M_0)J_0 = Pf(M)J = PR(z,M)J.$$

2) $\Rightarrow$ 3). Since M has a bounded $H^\infty(S_\theta)$ functional calculus for every $\theta > \omega$, this implication can be shown as in the proof of Theorem 5.1. □

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