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THE STACK OF MICROLOCAL PERVERSE SHEAVES

BY INGO WASCHKIES

ABSTRACT. — In this paper we construct the abelian stack of microlocal perverse sheaves on the projective cotangent bundle of a complex manifold. Following ideas of Andronikof we first consider microlocal perverse sheaves at a point using classical tools from microlocal sheaf theory. Then we will use Kashiwara-Schapira's theory of analytic ind-sheaves to globalize our construction. This presentation allows us to formulate explicitly a global microlocal Riemann-Hilbert correspondence.

RÉSUMÉ (*Le champ des faisceaux pervers microlocaux*). — Nous construisons le champ abélien des faisceaux pervers microlocaux sur le fibré cotangent projectif d'une variété analytique complexe. Suivant des idées d'Andronikof, nous considérons d'abord les germes de faisceaux pervers microlocaux en un point en utilisant les outils classiques de la théorie microlocale des faisceaux. Ensuite nous utilisons la théorie des ind-faisceaux analytiques de Kashiwara-Schapira pour globaliser notre construction. Cette présentation nous permettra de formuler explicitement une version globale de la correspondance de Riemann-Hilbert microlocale.

Contents

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1. Introduction

In [8] Kashiwara constructed the stack of microdifferential modules on a complex contact manifold, generalizing the stack of modules over the ring of microdifferential operators $\mathcal{E}_X$ on the projective cotangent bundle P^*X of a complex manifold X. In his discussion Kashiwara asked for the construction of a stack of microlocal perverse sheaves that should be equivalent to the stack of regular holonomic modules by a microlocal Riemann-Hilbert correspondence. Such a stack should be defined over any field k, but the Riemann-Hilbert morphism only makes sense over $\mathbb{C}$.

There have been several attempts to construct a local version of such a stack. In [1] Andronikof defined a prestack on P^*X and announced the microlocal Riemann-Hilbert correspondence on the stalks. However, at that time there did not exist tools to define a global microlocal Riemann-Hilbert morphism. Another topological construction was proposed in [6], but to our knowledge this project has neither been completed nor published.

Our approach makes use of the theory of analytic ind-sheaves, recently introduced in [12] by Kashiwara and Schapira. Hence, microlocal perverse sheaves on a $\mathbb{C}^{\times}$-conic open subset $U \subset T^*X$ will be ind-sheaves (or more precisely objects of the derived category of ind-sheaves) on U contrary to the construction of [1], in which microlocal perverse sheaves on $U \subset T^*X$ were represented by complexes of sheaves on the base space X. The theory of ind-sheaves provides us with a nice representative of the stack associated to the prestack of [1] and allows us to use the machinery developed in [12]. The essential tool in this description is Kashiwara's functor of ind-microlocalization $\mu : \mathrm{D}^{\mathrm{b}}(k_X) \to \mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{T^*X}))$ of [9]. This functor enables us to define explicitly a global Riemann-Hilbert morphism when $k = \mathbb{C}$.

In the future, we will hopefully show that we can actually patch (a twisted version of) this stack on a complex contact manifold and prove the Riemann-Hilbert theorem in the complex case.

In more detail, the contents of this paper are as follows.

In Section 2 we recall first the theory of microlocalization of [11] on a real manifold X. We do not review in detail the theory of the micro-support of sheaves but concentrate on the definition of the microlocal category $\mathrm{D}^{\mathrm{b}}(k_X, S)$ where $S \subset T^*X$ is an arbitrary subset. It is defined as the localization of the category $\mathrm{D}^{\mathrm{b}}(k_X)$ by the objects $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X)$ whose micro-support does not intersect S. For any $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X)$ we get a natural morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,S)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big).$$

In the case where $S = \{p\}$, $p \in T^*X$ the category $\mathrm{D}^{\mathrm{b}}(k_X, p)$ has been intensively studied in [11], and in particular it is proved that the morphism above is an isomorphism. We will show that this result is still valid in the category $\mathrm{D}^{\mathrm{b}}(k_X, \{x\} \times \dot{\delta})$ where x is a point of X, $\delta \subset T^*_x X$ a closed cone and $\dot{\delta} = \delta \setminus \{0\}$. Later we will be mainly interested in the case where δ is a complex line. The main tool is the refined microlocal cut-off lemma for non-convex sets, which we recall adding a few comments. We will also need the cut-off functor in Section 5.

Section 3 extends the definitions and results of Section 3 first to $\mathbb{R}$-constructible then to $\mathbb{C}$-constructible sheaves. There are two natural ways to define the microlocalization of the derived category of $\mathbb{R}$-constructible sheaves. We either localize the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ by sheaves whose micro-support does not intersect S or we take the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X, S)$ whose objects are represented by $\mathbb{R}$-constructible sheaves. Following [2] we will use the first definition. One important question is whether or not the two definitions coincide. The main result of this section is that this is the case when $S = \{x\} \times \dot{\delta}$.

In Section 4 we show that the constructions of Section 3 are locally "invariant under quantized contact transformations".

Section 5 is devoted to the study of microlocally $\mathbb{C}$-constructible sheaves in the category $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times p)$. In Section 4 we have shown that the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ is invariant by quantized contact transformation. Hence we are reduced to study microlocally $\mathbb{C}$-constructible sheaves in generic position, *i.e.*, complexes of sheaves whose micro-support is contained in $T^*_Z X$ for a complex (not necessarily smooth) hypersurface Z in a neighborhood of p. We give a complete proof that microlocally $\mathbb{C}$-constructible sheaves in generic position may be represented by $\mathbb{C}$-constructible sheaves (as announced in [1]).

Following [1], we define in Section 6 the category of microlocal perverse sheaves as a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$. An object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ is perverse if for any non-singular point $q \in \mathrm{SS}(\mathcal{F})$ in a neighborhood of $\mathbb{C}^\times p$ the complex $\mathcal{F}$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times q)$ to a constant sheaf $M_Y[d_Y]$ supported on a closed submanifold $Y \subset X$. This definition is natural in view of the microlocal characterization of perverse sheaves of [11] and also leads to the definition of a prestack of microlocal perverse sheaves on P^*X. Then we prove that the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ is abelian as has been announced in [1]. Our proof gives a refined result which allows us to conclude that the stack associated to this prestack is abelian. This stack is the stack of microlocal perverse sheaves on P^*X.

In Section 7 we finally define microlocal perverse sheaves as particular objects of the derived category of ind-sheaves on conic open subsets of T^*X. In Section 6 we have constructed the category of microlocal perverse sheaves at any $p \in P^*X$ (or on $\mathbb{C}^\times p \subset \dot{T}^*X$) which will be equivalent to the stalk of the stack $\mu\mathrm{Perv}$ of microlocal perverse sheaves. The idea of the construction of $\mu\mathrm{Perv}$ is to use the fact Kashiwara's functor μ of ind-microlocalization induces a fully faithful functor from $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ into the stalk of the prestack of bounded derived categories of ind-sheaves on $\mathbb{C}^\times$-conic subsets of T^*X. Then we can define a microlocal perverse sheaf on a conic open subset $U \subset T^*X$ as an object of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_U))$ that is isomorphic to a microlocal perverse sheaf of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ at any point of $p \in U$. We show that the stack of microlocal perverse sheaves is canonically equivalent to the stack associated to the prestack of the last section. Finally, we state without proof the microlocal Riemann-Hilbert theorem which will be the subject of a forthcoming paper.

Appendix A recalls the concepts of 2-limits and 2-colimits in the category of all small categories.

Appendix B gives a short introduction to stacks with emphasis on the special properties resulting from the fact that we work on a topological space. Then we give a criterion for subprestacks of the prestack of derived categories of ind-sheaves on a manifold to be stacks. It is a generalization of a proof of [11] showing that the prestack of perverse sheaves is a substack of the prestack of derived categories of sheaves with $\mathbb{C}$-constructible cohomology. Then we investigate abelian stacks on a topological space. Roughly speaking, an additive stack on a topological space is abelian if and only if its stalks are abelian categories and we have a “lifting property” for kernels and cokernels.

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2. Microlocalization of sheaves

2.1. Notations. — Let $\mathbb{R}^+$ denote the group of strictly positive real numbers and $\mathbb{C}^\times$ the group of non-zero complex numbers. We will mainly work on a fixed complex manifold[(1)] X of complex dimension $\dim_{\mathbb{C}} X = d_X$. Let T^*X be

(1) All manifolds (complex or real) in this paper are supposed to be finite dimensional with a countable base of open sets.

its cotangent bundle and $T^*_X X$ the zero section. Set $\dot{T}^*X = T^*X \setminus T^*_X X$ and let $P^*X = \dot{T}^*X/\mathbb{C}^\times$ be the projective cotangent bundle. We denote the natural map by

$$\gamma : \dot{T}^*X \longrightarrow P^*X.$$

If $\Lambda \subset \dot{T}^*X$ is a subset, we define the antipodal set Λ^a as

$$\Lambda^a = \big\{(x;\xi) \mid (x;-\xi) \in \Lambda\big\},$$

and we set

$$\mathbb{R}^+\Lambda = \big\{(x;\xi) \in \dot{T}^*X \mid \exists\, \alpha \in \mathbb{R}^+,\ (x;\alpha\xi) \in \Lambda\big\}.$$

We define similarly $\mathbb{C}^\times\Lambda$. Hence $\mathbb{C}^\times\Lambda = \gamma^{-1}\gamma(\Lambda)$. If $\Lambda = \{p\}$ is a point, we will write $\mathbb{C}^\times p$ instead of $\mathbb{C}^\times\{p\}$.

We say that a subset $\Lambda \subset \dot{T}^*X$ is $\mathbb{R}^+$-conic (resp. $\mathbb{C}^\times$-conic) if it is stable under the action of $\mathbb{R}^+$ (resp. $\mathbb{C}^\times$), *i.e.* if $\mathbb{R}^+\Lambda = \Lambda$ (resp. $\mathbb{C}^\times\Lambda = \Lambda$).

In the sequel, we will often deal with $\mathbb{R}^+$-conic subsets that are only locally $\mathbb{C}^\times$-conic. More precisely, a subset $\Lambda \subset \dot{T}^*X$ is called $\mathbb{C}^\times$-conic at $p \in \dot{T}^*X$ if there exists an open neighborhood U of p such that $U \cap \mathbb{C}^\times\Lambda = U \cap \Lambda$. Note that this definition still makes sense if Λ is a germ of a subset at p. An open subset is always $\mathbb{C}^\times$-conic at each $p \in U$.

Let $S \subset \dot{T}^*X$ be another subset, and suppose that Λ is defined on a germ of a neighborhood of S. Then we say that Λ is $\mathbb{C}^\times$-conic on S if it is $\mathbb{C}^\times$-conic at every point of S. Clearly this is equivalent to the statement that there exists an open neighborhood U of S such that $U \cap \mathbb{C}^\times\Lambda = U \cap \Lambda$. In particular, Λ is $\mathbb{C}^\times$-conic on $\dot{T}^*X$ if and only if it is $\mathbb{C}^\times$-conic.

Finally we call the following easy topological lemma to the reader's attention.

Lemma 2.1.1. — *Let $S \subset \dot{T}^*X$ be a $\mathbb{C}^\times$-conic set and $U \supset S$ an $\mathbb{R}^+$-conic open neighborhood. Then there exists a $\mathbb{C}^\times$-conic open set V such that $S \subset V \subset U$.*

Now let us fix the conventions for sheaves. All sheaves considered here are sheaves of vector spaces over a given field k. We will consider the following categories:

- $\mathrm{D^b}(k_X)$ is the derived category of bounded complexes of sheaves of k vector spaces;
- $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ is the full subcategory of $\mathrm{D^b}(k_X)$ whose objects have $\mathbb{R}$-constructible cohomology;
- $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ whose objects have $\mathbb{C}$-constructible cohomology;
- $\mathrm{Perv}(k_X)$ is the full abelian subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ whose objects are perverse sheaves. We will follow the conventions for the shift of [11], which imply that a perverse sheaf is concentrated in degrees $-d_X$ to 0.

We will not recall the construction of these categories here, for more details see for instance [11].

2.2. Microlocalization of sheaves. — In this section we recall the construction and some properties of the microlocalization of $\mathrm{D}^{\mathrm{b}}(k_X)$ on a subset $S \subset T^*X$ (see [11, Section VI]) which we will then discuss from the (pre)stack-theoretical point of view. Note that all definitions and statements below which do not involve $\mathbb{C}^{\times}$-conic subsets of T^*X are valid on a real manifold.

Recall that if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X)$, then one can associate to $\mathcal{F}$ a closed $\mathbb{R}^+$-conic involutive subset $\mathrm{SS}(\mathcal{F})$ of T^*X called the micro-support of $\mathcal{F}$. The theory of the micro-support can be found in [11]. It is the set of codirections in which $\mathcal{F}$ "does not propagate". More precisely, a point $p \in T^*X$ is not a point of the micro-support if and only if there exists an open neighborhood U of p such that for any $x \in X$ and any real map ψ of class C^1 with $\psi(x) = 0$ and $(\mathrm{d}\psi)_x \in U$ we have

$$\big(\mathrm{R}\Gamma_{\{x \mid \psi(x) \geqslant 0\}}(\mathcal{F})\big)_x \simeq 0.$$

Let $S \subset T^*X$ be an arbitrary subset. Set

$$\mathcal{N}_S = \big\{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X) \mid \mathrm{SS}(\mathcal{F}) \cap S = \varnothing\big\}.$$

It is easily verified that $\mathcal{N}_S$ defines a full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(k_X)$. Note that if $x \in S \cap T^*_X X$ then $\mathcal{F} \in \mathcal{N}_S$ implies $\mathcal{F} \simeq 0$ in a neighborhood of x.

Definition 2.2.1. — The microlocalization of $\mathrm{D}^{\mathrm{b}}(k_X)$ on S is the localization of the triangulated category $\mathrm{D}^{\mathrm{b}}(k_X)$ by the full triangulated subcategory $\mathcal{N}_S$, which we denote by

$$\mathrm{D}^{\mathrm{b}}(k_X, S) = \mathrm{D}^{\mathrm{b}}(k_X)/\mathcal{N}_S.$$

If $S = \{p\}$, we will write $\mathrm{D}^{\mathrm{b}}(k_X, p)$ for $\mathrm{D}^{\mathrm{b}}(k_X, \{p\})$.

Note that an object $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(k_X, S)$ is isomorphic to zero if and only if $\mathcal{F} \oplus \mathcal{F}[1] \in \mathcal{N}_S$ and since $\mathrm{SS}(\mathcal{F} \oplus \mathcal{F}[1]) = \mathrm{SS}(\mathcal{F})$ this is equivalent to $\mathrm{SS}(\mathcal{F}) \cap S = \varnothing$.

A morphism $\mathcal{F} \to \mathcal{G}$ of $\mathrm{D}^{\mathrm{b}}(k_X)$ is called an isomorphism on S if it is an isomorphism in $\mathrm{D}^{\mathrm{b}}(k_X, S)$. This is equivalent to the existence of a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(k_X)$

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+}$$

with $\mathrm{SS}(\mathcal{H}) \cap S = \varnothing$. From this it follows easily that if $\mathcal{F} \xrightarrow{\sim} \mathcal{G}$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(k_X, S)$, then $\mathrm{SS}(\mathcal{F}) \cap S = \mathrm{SS}(\mathcal{G}) \cap S$. Hence the micro-support of $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X, S)$ is well-defined in a germ of a neighborhood of S.

Let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X, S)$. By definition we have

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,S)}(\mathcal{F}, \mathcal{G}) \simeq \varinjlim_{\substack{\mathcal{F}' \xrightarrow{\sim} \mathcal{F} \\ \text{on } S}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}', \mathcal{G}).$$

Consider $x \in X$. For any open neighborhood $U \ni x$ one gets a functor

$$\mathrm{D}^{\mathrm{b}}(k_U) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, x) \simeq \mathrm{D}^{\mathrm{b}}\big(k_X, \pi^{-1}(x)\big) \longrightarrow \mathrm{D}^{\mathrm{b}}\big(k_X, \dot{\pi}^{-1}(x)\big).$$

These functors define

$$\mathrm{D}^{\mathrm{b}}(k_X)_x \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, x) \longrightarrow \mathrm{D}^{\mathrm{b}}\big(k_X, \dot{\pi}^{-1}(x)\big)$$

where $\mathrm{D}^{\mathrm{b}}(k_X)_x$ denotes the stalk at x of the prestack $U \mapsto \mathrm{D}^{\mathrm{b}}(k_U)$. Set

$$\mathcal{LC}_x = \big\{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X)_x \mid \mathrm{SS}(\mathcal{F}) \subset T^*_X X \text{ in a neighborhood of } x\big\}.$$

Clearly, $\mathcal{LC}_x$ defines a full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(k_X)_x$ and we easily get

Lemma 2.2.2. — *The diagram*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}(k_X)_x & \xrightarrow{\sim} & \mathrm{D}^{\mathrm{b}}(k_X, x) \\ \downarrow & & \downarrow \\ \mathrm{D}^{\mathrm{b}}(k_X)_x/\mathcal{LC}_x & \xrightarrow[\sim]{} & \mathrm{D}^{\mathrm{b}}(k_X, \dot{\pi}^{-1}(x)) \end{array}$$

commutes up to isomorphism and the horizontal functors are equivalences of categories.

Remark 2.2.3. — Note that by Proposition 6.6.1 of [11] the objects of $\mathcal{LC}_x$ are precisely the germs of local systems at x, hence

$$\mathcal{LC}_x = \big\{\mathcal{F} \in \mathrm{D}^{\mathrm{b}}(k_X)_x \mid \exists\, U \ni x,\ \exists\, M \in \mathrm{D}^{\mathrm{b}}(\mathrm{Vect}(k)) : \mathcal{F} \simeq M_X \text{ in } \mathrm{D}^{\mathrm{b}}(k_X)_x\big\}.$$

We will have constant recourse to the following easy lemma.

Lemma 2.2.4. — *Let $S \subset \dot{T}^*X$ be any subset. Consider a morphism $\mathcal{F} \to \mathcal{G}$ of $\mathrm{D}^{\mathrm{b}}(k_X)$ that is an isomorphism on S. Then there exists an $\mathbb{R}^+$-conic open neighborhood U of S such that $\mathcal{F} \to \mathcal{G}$ is an isomorphism on U. In particular $\mathrm{SS}(\mathcal{F}) \cap U = \mathrm{SS}(\mathcal{G}) \cap U$. If moreover S is $\mathbb{C}^\times$-conic, then we can choose U to be $\mathbb{C}^\times$-conic.*

Proof. — By hypothesis there exists a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(k_X)$

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+}$$

such that $\mathrm{SS}(\mathcal{H}) \cap S = \varnothing$. Since $\dot{\mathrm{SS}}(\mathcal{H})$ is a closed $\mathbb{R}^+$-conic subset of $\dot{T}^*X$, the set $U = \complement\dot{\mathrm{SS}}(\mathcal{H})$ is an open and $\mathbb{R}^+$-conic neighborhood of S such that $\mathrm{SS}(\mathcal{H}) \cap U = \varnothing$. Now suppose that S is $\mathbb{C}^\times$-conic. To prove the last statement we use the fact that every $\mathbb{R}^+$-conic open neighborhood V of S contains a $\mathbb{C}^\times$-conic open neighborhood of S. ☐

Recall that to any $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X)$ we can associate the object $\mu\mathrm{hom}(\mathcal{F}, \mathcal{G}) \in \mathrm{D}^{\mathrm{b}}(k_{T^*X})$ (see [11], Section IV). This complex satisfies

$$\mathrm{supp}\big(\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big) \subset \mathrm{SS}(\mathcal{F}) \cap \mathrm{SS}(\mathcal{G}).$$

Therefore $\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})_S$ is well-defined for $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X, S)$.

For an arbitrary subset S there is a natural morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,S)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big). \tag{2.2.1}$$

Let us recall its construction. For any two objects $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{D}^{\mathrm{b}}(k_X)$ we have a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}_1, \mathcal{F}_2) \simeq \mathrm{H}^0\big(T^*X, \mu\mathrm{hom}(\mathcal{F}_1, \mathcal{F}_2)\big)$$

which defines a morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}_1, \mathcal{F}_2) \longrightarrow \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}_1, \mathcal{F}_2)\big).$$

Now if $\mathcal{F}' \to \mathcal{F}$ is an isomorphism on S we get an induced isomorphism

$$\mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big) \xrightarrow{\sim} \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}', \mathcal{G})\big).$$

Thus we get morphisms

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}', \mathcal{G}) \longrightarrow \mathrm{H}^0\big(S, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big)$$

which induce the morphism (2.2.1).

There is a well-known situation in which this morphism is an isomorphism (see [11], Theorem 6.1.2).

Proposition 2.2.5. — *Let $p \in T^*X$ and $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(X, p)$. Then the morphism* (2.2.1)

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(X,p)}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{H}^0\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})_p$$

is an isomorphism.

The idea is to calculate both sides by using microlocal cut-off functors. We will show that a similar strategy works in the case of a closed cone in $\dot{T}^*_x X$, $x \in X$.

However, the morphism (2.2.1) is not an isomorphism in general (*cf.* [11, Exercise VI.6], which gives a counter-example on an open subset).

Remark 2.2.6. — The correspondence

$$T^*X \supset U \longmapsto \mathrm{D}^{\mathrm{b}}(k_X, U)$$

defines a prestack on T^*X, which we will usually denote by $\mathrm{D}^{\mathrm{b}}(k_X, *)$. Further, we denote by $\mathrm{D}^{\mathrm{b}}(k_X, *)_p$ its stalk at $p \in T^*X$. Note that $\gamma_* \mathrm{D}^{\mathrm{b}}(k_X, *)_{|\dot{T}^*X}$ defines a prestack on P^*X.

Proposition 2.2.7. — *Let S be a subset of $\dot{T}^*X$. Then the natural functor*

$$2\varinjlim_{\substack{S \subset U \subset \dot{T}^*X \\ U\ \mathbb{R}^+\text{-conic}}} \mathrm{D}^{\mathrm{b}}(k_X, U) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, S)$$

is an equivalence.

Proof. — The functor is obviously essentially surjective. Let us show that it is fully faithful. Let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X)$. By Lemma 2.2.4 we get

$$\begin{aligned}
\mathrm{Hom}_{2\varinjlim_{\substack{S\subset U\subset \dot{T}^*X\\ U\ \mathbb{R}^+\text{-conic}}} \mathrm{D}^{\mathrm{b}}(k_X,U)}(\mathcal{F},\mathcal{G}) &\simeq \varinjlim_{\substack{S\subset U\subset \dot{T}^*X\\ U\ \mathbb{R}^+\text{-conic}}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,U)}(\mathcal{F},\mathcal{G})\\
&\simeq \varinjlim_{\substack{S\subset U\subset \dot{T}^*X\\ U\ \mathbb{R}^+\text{-conic}}} \varinjlim_{\substack{\mathcal{F}'\xrightarrow{\sim}\mathcal{F}\\ \text{on } U}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}',\mathcal{G})\\
&\simeq \varinjlim_{S\subset U\subset \dot{T}^*X} \varinjlim_{\substack{\mathcal{F}'\xrightarrow{\sim}\mathcal{F}\\ \text{on } U}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{F}',\mathcal{G})\\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,S)}(\mathcal{F},\mathcal{G}).
\end{aligned}$$

□

Corollary 2.2.8. — *Let S be a $\mathbb{C}^\times$-conic subset of $\dot{T}^*X$ and $p \in T^*X$.*

(i) *The natural functor*

$$2\varinjlim_{\substack{S\subset U\subset \dot{T}^*X\\ U\ \mathbb{C}^\times\text{-conic}}} \mathrm{D}^{\mathrm{b}}(k_X,U) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X,S)$$

is an equivalence.

(ii) *The natural functor*

$$\mathrm{D}^{\mathrm{b}}(k_X,*)_p \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X,p)$$

is an equivalence. If moreover $p \in \dot{T}^*X$ then

$$\gamma_*\big(\mathrm{D}^{\mathrm{b}}(k_X,*)_{|\dot{T}^*X}\big)_{\gamma(p)} \simeq \mathrm{D}^{\mathrm{b}}\big(k_X,\gamma^{-1}\gamma(p)\big) = \mathrm{D}^{\mathrm{b}}(k_X,\mathbb{C}^\times p).$$

Proof. — Part (i) follows from the proposition by Lemma 2.2.7 and (ii) follows from Lemma 2.2.7 and (i). □

2.3. Refined microlocal cut-off.

— Let us recall the basic idea of a microlocal cut-off functor. Let X be a finite dimensional real vector space, $U \ni 0$ a relatively compact open neighborhood of 0 and consider an open cone $\gamma \subset X^*$.

Definition 2.3.1. — A microlocal cut-off functor on $U \times \gamma$ is a functor

$$\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}(k_X) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X)$$

such that

(i) $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \subset X \times \overline{\gamma}$;

(ii) $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \cap U \times \gamma = \mathrm{SS}(\mathcal{F}) \cap U \times \gamma$;

(iii) $\Phi_{U,\gamma}$ is equipped with a morphism of functors $\alpha : \Phi_{U,\gamma} \to \mathrm{Id}$ such that α induces an isomorphism in $\mathrm{D^b}(k_X, U \times \gamma)$ which can be visualized by

$$\begin{array}{ccc} \mathrm{D^b}(k_X) & \xrightarrow{\quad\Phi_{U,\gamma}\quad} & \mathrm{D^b}(k_X) \\ \downarrow & & \downarrow \\ \mathrm{D^b}(k_X, U \times \gamma) & \xrightarrow[\Phi_{U,\gamma} \simeq \mathrm{Id}]{} & \mathrm{D^b}(k_X, U \times \gamma). \end{array}$$

Note that condition (iii) implies (ii).

If the cut-off functor $\Phi_{U,\gamma}$ allows us to estimate the micro-support of $\Phi_{U,\gamma}(\mathcal{F})$ in the fiber $\{0\} \times X^*$, we usually call it a refined microlocal cut-off.

A cut-off functor is easily constructed in the case of a convex open cone (see Proposition 5.2.3 of [11]). A generalization to non-convex cones is stated in Exercise V.8 of [11] (for a proof see [4]). These tools will allow us in Section 2.4 to calculate sections of μhom along a complex line (or more generally along closed cones of $T^*_x X$ where $x \in X$). The result will imply that the morphism (2.2.1) is an isomorphism in this case.

Later we will need to construct a functor $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, \mathbb{C}^\times p) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \pi(p))$ if Λ is in generic position at p (Section 5.1). For this purpose we will need the refined microlocal cut-off of [4]. It is an extension of the classical "refined microlocal cut-off lemma" (Proposition 6.1.4 of [11]) to non-convex cones with a good estimate for the micro-support.

Let us recall the cut-off functor of [11], Exercise V.8. Let X be a real, finite dimensional vector space, $\dot{X} = X \setminus \{0\}$, $U \subset X$ an open subset and $\gamma \subset X^*$ an open cone.

We have the following natural morphisms:

$$\begin{array}{ccccc} & & X \times X & & \\ & \swarrow^{q_1} & \widetilde{s}\downarrow\downarrow s & {}^{q_2}\searrow & \\ X & & X & & X \end{array}$$

where q_1 and q_2 are the natural projections and

$$s : X \times X \longrightarrow X, \quad (x, y) \longmapsto x + y;$$

$$\widetilde{s} : X \times X \longrightarrow X, \quad (x, y) \longmapsto x - y.$$

We define the functor $\Phi_{U,\gamma}$ by setting for any $\mathcal{F} \in \mathrm{D^b}(k_X)$

$$\Phi_{U,\gamma}(\mathcal{F}) = k^{\wedge}_{\gamma^a} * \mathcal{F}_U = \mathrm{R}s_!(q_1^{-1} k^{\wedge}_{\gamma^a} \otimes q_2^{-1} \mathcal{F}_U),$$

where $(.)^\wedge$ denotes the Fourier-Sato Transformation (see [11]). It can be shown that $\Phi_{U,\gamma}$ is a microlocal cut-off functor in the sense of Definition 2.3.1. Let us add two easy lemmas which will be useful in the next section.

LEMMA 2.3.2. — *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}(k_X)$*. Then we have a canonical isomorphism*

$$\mathrm{R}s_!(q_1^{-1}\mathcal{F} \otimes q_2^{-1}\mathcal{G}) \simeq \mathrm{R}q_{1!}(\widetilde{s}^{-1}\mathcal{F} \otimes q_2^{-1}\mathcal{G})$$

LEMMA 2.3.3. — *Let* $\gamma_1, \gamma_2 \subset X^*$ *be two open cones and* $U \subset X$ *open. Then there is a natural distinguished triangle*

$$\Phi_{U,\gamma_1\cap\gamma_2}(\mathcal{F}) \longrightarrow \Phi_{U,\gamma_1}(\mathcal{F}) \oplus \Phi_{U,\gamma_2}(\mathcal{F}) \longrightarrow \Phi_{U,\gamma_1\cup\gamma_2}(\mathcal{F}) \xrightarrow{+} .$$

Next, we recall D'Agnolo's condition under which the cut-off functor $\Phi_{U,\gamma}$ is refined. These results will not be needed until Section 5.1.

DEFINITION 2.3.4. — If $\gamma \subset X^*$ is an open cone, set

$$\partial^\circ\gamma = \pi\chi\big(\mathrm{SS}(\mathbb{C}_\gamma) \setminus \{0;0\}\big)$$

where $\chi : T^*X^* \to T^*X$ is defined by $\chi(\xi; x) = (x; -\xi)$.

One says that (U, γ) is a *refined cutting pair* at 0, if $U \subset X$ is a relatively compact open neighborhood of 0 and for any $x \in \partial U \cap \partial^\circ\gamma$ there exists $\xi \in \dot{X}$ such that $\mathrm{N}^*_x(U) = \mathbb{R}_{\geqslant 0}\xi$ and $\chi(\mathrm{SS}(k_\gamma)) \cap \pi^{-1}(x) = \mathbb{R}_{\leqslant 0}\xi$. Here $N^*_x(U)$ denotes the conormal cone to U at x.

This allows one to give a good estimate of the micro-support of $\Phi_{U,\gamma}(\mathcal{F})$ at the origin.

PROPOSITION 2.3.5. — *Let* (U, γ) *be a refined cutting pair at* 0*. Then*

$$\begin{aligned}&\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{F})\big) \cap \dot{\pi}^{-1}(0)\\ &\qquad \subset \big\{\xi \in \gamma \mid (0;\xi) \in \mathrm{SS}(\mathcal{F})\big\} \cup \big\{\xi \in \partial\gamma \mid \exists\, x \in \overline{U} : (x,\xi) \in \mathrm{SS}(\mathcal{F})\big\}.\end{aligned}$$

Let us add a useful corollary:

COROLLARY 2.3.6. — *Let* (U, γ) *be a refined cutting pair at* 0 *and suppose that* $\mathrm{SS}(\mathcal{F}) \cap (\overline{U} \times \partial\gamma) = \varnothing$*. Then there exists an open neighborhood* V *of* 0 *such that*

$$\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{F})\big) \cap \dot{\pi}^{-1}(V) = \mathrm{SS}(\mathcal{F}) \cap (V \times \gamma).$$

Proof. — Since $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \subset X \times \overline{\gamma}$ and

$$\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{F})\big) \cap (U \times \gamma) = \mathrm{SS}(\mathcal{F}) \cap (U \times \gamma),$$

it is enough to show that $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \cap V \times \partial\gamma = \varnothing$ for some neighborhood V of 0. D'Agnolo's estimate of the micro-support implies that this is at least true at 0.

Now suppose that such a neighborhood V does not exist. Then we can construct a sequence (x_n, ξ_n) such that $x_n \to 0$ and $\xi_n \in \mathrm{SS}_{x_n}(\Phi_{U,\gamma}(\mathcal{F})) \cap \partial\gamma$. Since both sets are invariant by $\mathbb{R}_{\geqslant 0}$ we can assume that $|\xi_n| = 1$, hence by extracting a subsequence we can suppose that $\xi_n \to \xi$. Since $\partial\gamma$ and $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F}))$ are closed we get by the estimate of the micro-support that there exists $x \in \overline{U}$ such that $(x, \xi) \in \mathrm{SS}(\mathcal{F})$ which is impossible by hypothesis. ☐

Finally, let us state an existence lemma for refined cutting pairs.

LEMMA 2.3.7. — *Let X be a real vector space and $L \subset X$ a subspace of X. Then there exists a fundamental system of open conic neighborhoods γ of $(\dot{T}^*_L X)_0$ such that for each γ there exists a fundamental system of open neighborhoods U of 0 in X such that (U, γ) is a refined cutting pair.*

Proof. — This lemma is shown in [4] during the proof of Corollary 3.4 using Lemma 3.3. □

2.4. Morphisms in $\mathrm{D}^{\mathrm{b}}(k_X, \{x\} \times \dot{\delta})$. — In this section we will show that the natural morphism

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X, \{x\}\times\dot{\delta})}(\mathcal{F}, \mathcal{G}) \longrightarrow \mathrm{H}^0\big(\{x\} \times \dot{\delta}, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big)$$

is an isomorphism, where $x \in X$ and $\delta \subset T^*_x X$ is a closed cone. In order to prove this, we will first consider the composition

$$(*)_\delta \quad \varinjlim_{U,\gamma} \mathrm{H}^0\, \mathrm{RHom}\big(\Phi_{U,\gamma}(\mathcal{F})_U, \mathcal{G}\big) \longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X, \{x\}\times\dot{\delta})}(\mathcal{F}, \mathcal{G}) \\ \longrightarrow \mathrm{H}^0\big(\{x\} \times \dot{\delta}, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big)$$

and show that it is an isomorphism. Here U runs through the family of relatively compact open neighborhoods of 0 and γ through the set of open cones containing $\dot{\delta}$.

LEMMA 2.4.1. — *Let X be a real vector space and consider a closed convex proper cone $\delta \subset X^*$. Then the natural morphism*

$$(*)_\delta \qquad \varinjlim_{U,\gamma} \mathrm{H}^n \mathrm{RHom}\left(\Phi_{U,\gamma}(\mathcal{F})_U, \mathcal{G}\right) \longrightarrow \mathrm{H}^n\big(\{0\} \times \dot{\delta}, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big)$$

is an isomorphism. Here U runs through the family of relatively compact open neighborhoods of 0 and γ through the set of open cones containing $\dot{\delta}$.

Proof. — First note that since $\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})$ is conic, we have

$$\mathrm{H}^n\big(\{0\} \times \dot{\delta}, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big) \simeq \varinjlim_{U,\gamma} \mathrm{H}^n\big(U \times \gamma, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big).$$

Since γ is open, convex and proper, we have by [11], Theorem 4.3.2

$$\mathrm{H}^n\big(U \times \gamma, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big) \simeq \varinjlim_{V,Z} \mathrm{H}^n_{Z\cap V}\big(V, \mathrm{R}\mathcal{H}om(q_2^{-1}\mathcal{F}, q_1^!\mathcal{G})\big)$$

where V runs through the family of open subsets of $T_{\Delta_X}(X\times X)\simeq X\times X$ such that $V\cap\Delta_X=U$ (*i.e.* $q_1(V)=U$) and Z through the family of closed subsets such that the inclusion $\mathrm{C}_{\Delta_X}Z\subset U\times\gamma^\circ$ holds[(2)] .

We have the following chain of isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_{Z\cap V}\big(V,\mathrm{R}\mathcal{H}om(q_2^{-1}\mathcal{F},q_1^!\mathcal{G})\big)&\simeq\mathrm{R}\Gamma\big(V,\mathrm{R}\Gamma_{Z\cap V}\mathrm{R}\mathcal{H}om(q_2^{-1}\mathcal{F},q_1^!\mathcal{G})\big)\\ &\simeq\mathrm{R}\Gamma\big(V,\mathrm{R}\mathcal{H}om((q_2^{-1}\mathcal{F})_{Z\cap V},q_1^!\mathcal{G})\big)\simeq\mathrm{R}\Gamma\big(U\times X,\mathrm{R}\mathcal{H}om((q_2^{-1}\mathcal{F})_{Z\cap V},q_1^!\mathcal{G})\big)\\ &\simeq\mathrm{R}\Gamma\big(U,\mathrm{R}\mathcal{H}om(\mathrm{R}q_{1!}(q_2^{-1}\mathcal{F})_{Z\cap V},\mathcal{G}\big)\simeq\mathrm{RHom}\,\big((\mathrm{R}q_{1!}(k_{Z\cap V}\otimes q_2^{-1}\mathcal{F})_U,\mathcal{G}\big).\end{aligned}$$

Hence we have

$$\mathrm{H}^n\big(\{0\}\times\dot\delta,\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)\simeq\varinjlim_{U,\gamma}\varinjlim_{V,Z}\mathrm{H}^n\mathrm{RHom}\,\big((\mathrm{R}q_{1!}(k_{Z\cap V}\otimes q_2^{-1}\mathcal{F}))_U,\mathcal{G}\big).$$

Now fix U,γ and V,Z. Then V contains a small relatively compact open neighborhood of 0 of type $U'\times U'$. Moreover we may assume by cofinality that Z is of the form $\widetilde{s}^{-1}\gamma^\circ$ in a neighborhood of 0. Hence $Z\cap V$ contains $\widetilde{s}^{-1}\gamma^\circ\cap U'\times U'$. We can therefore remove the second limit by replacing $V\cap Z$ with $\widetilde{s}^{-1}\gamma^\circ\cap U\times U$. Then we get

$$\begin{aligned}\mathrm{RHom}((\mathrm{R}q_{1!}(k_{Z\cap V}\otimes q_2^{-1}\mathcal{F}))_U,\mathcal{G})&\simeq\mathrm{RHom}((\mathrm{R}q_{1!}(\widetilde{s}^{-1}k_{\gamma^\circ}\otimes q_2^{-1}\mathcal{F}_U))_U,\mathcal{G})\\ &\simeq\mathrm{RHom}((\mathrm{R}s_!(q_1^{-1}k^\wedge_{\gamma^a}\otimes q_2^{-1}\mathcal{F}_U))_U,\mathcal{G})\simeq\mathrm{RHom}(\Phi_{U,\gamma}(\mathcal{F})_U,\mathcal{G}).\end{aligned}$$

Therefore

$$\mathrm{H}^n\big(\{0\}\times\dot\delta,\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)\simeq\varinjlim_{U,\gamma}\mathrm{H}^n\,\mathrm{RHom}\,\big(\Phi_{U,\gamma}(\mathcal{F})_U,\mathcal{G}\big).\qquad\square$$

Lemma 2.4.2. — *For any closed cone $\delta\subset X^*$ consider the morphism $(*)_\delta$. Let $\delta_1,\delta_2\subset X^*$ be two closed cones such that the morphisms $(*)_{\delta_1},(*)_{\delta_2}$ and $(*)_{\delta_1\cap\delta_2}$ are isomorphisms. Then $(*)_{\delta_1\cup\delta_2}$ is an isomorphism.*

Proof. — By Lemma 2.3.3 we get a morphism of distinguished triangles such that the vertical morphisms are given by $(*)_{\delta_1\cap\delta_2}$, $(*)_{\delta_1}\oplus(*)_{\delta_2}$ and $(*)_{\delta_1\cup\delta_2}$. Then the lemma follows from the Five Lemma. □

Proposition 2.4.3. — *Let X be a real vector space and consider a closed cone $\delta\subset X^*$. Then the natural morphism*

$$(*)_\delta\qquad\varinjlim_{U,\gamma}\mathrm{H}^n\mathrm{RHom}\,\big(\Phi_{U,\gamma}(\mathcal{F})_U,\mathcal{G}\big)\longrightarrow\mathrm{H}^n\big(\{0\}\times\dot\delta,\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)$$

---

(2) Here $\mathrm{C}_{\Delta_X}Z$ denotes the normal cone to Z along the diagonal Δ_X (see [11], Def. 4.1.1) and γ° denotes the polar cone of γ, *i.e.*,

$$\gamma^\circ=\{x\in X^*\mid\langle y,x\rangle\geqslant0\text{ for all }y\in\gamma\}.$$

is an isomorphism. Here U runs through the family of relatively compact open subsets of 0 *and* γ *through the set of open cones containing* $\dot{\delta}$.

Proof. — First suppose that δ can be written as a finite union of closed convex proper cones. Note that the intersection of two proper, closed, convex cones is again proper, closed and convex. Therefore, if δ' is the union of n closed, convex, proper cones then the intersection of a closed, convex, proper cone with δ' can be written as a union of n closed convex proper cones. Using Lemma 2.4.1 and the previous lemma we can then easily show the proposition by induction on the number of closed, convex, proper cones that cover δ.

Now let us consider the general case. Every closed cone is a decreasing intersection of closed cones δ_i that can be covered by a finite number of closed convex proper cones(3). Then $(*)_\delta = \varinjlim (*)_{\delta_i}$ is an isomorphism. ☐

THEOREM 2.4.4. — *Let X be a real manifold, $x \in X$ and $\delta \subset T^*_x X$ a closed cone. Then the natural morphism*

$$\mathrm{Hom}_{\mathrm{D^b}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{H}^0\big(\{x\}\times\dot{\delta}, \mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)$$

is an isomorphism.

Proof. — By Proposition 2.4.3 we know that the morphism $(*)_\delta$ is an isomorphism. Therefore the morphism of the theorem is surjective.

Let us prove that it is injective. Let $\mathcal{F} \to \mathcal{G}$ be a morphism of $\mathrm{D^b}(k_X,\{x\}\times\dot{\delta})$ that is zero in $\mathrm{H}^0(\{x\}\times\dot{\delta}, \mu\mathrm{hom}(\mathcal{F},\mathcal{G}))$. Then we may represent this morphism by a morphism $\mathcal{F}' \to \mathcal{G}$ in $\mathrm{D^b}(k_X)$ and a morphism $\mathcal{F}' \to \mathcal{F}$ that is an isomorphism on $\{x\}\times\dot{\delta}$. We get a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathrm{D^b}(k_X)}(\mathcal{F}',\mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathrm{D^b}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) & \longrightarrow & \varinjlim_{U,\gamma} \mathrm{R}^0\,\mathrm{Hom}\big(\Phi_{U,\gamma}(\mathcal{F})_U,\mathcal{G}\big) \\
\Big\downarrow \mathrm{Id} & & \Big\downarrow \wr & & \Big\downarrow \wr \\
\mathrm{Hom}_{\mathrm{D^b}(k_X)}(\mathcal{F}',\mathcal{G}) & \longrightarrow & \mathrm{Hom}_{\mathrm{D^b}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F}',\mathcal{G}) & \longrightarrow & \varinjlim_{U,\gamma} \mathrm{R}^0\,\mathrm{Hom}\big(\Phi_{U,\gamma}(\mathcal{F}')_U,\mathcal{G}\big).
\end{array}$$

Using the diagram, we see that there exists (U,γ) such that $\Phi_{U,\gamma}(\mathcal{F}')_U \to \mathcal{F}' \to \mathcal{G}$ is the zero map in $\mathrm{D^b}(k_X)$. But $\Phi_{U,\gamma}(\mathcal{F}')_U \to \mathcal{F}'$ is an isomorphism on $\{x\}\times\dot{\delta}$, and therefore $\mathcal{F}' \to \mathcal{G}$ represents the zero morphism in $\mathrm{D^b}(k_X,\{x\}\times\dot{\delta})$. ☐

(3) The proof is done by a simple compacity argument in $\dot{X}/\mathbb{R}^+$. For every $p \in \delta$ choose a closed convex proper cone with angle ε that contains p. Then a finite number of these cones, say $\gamma_1, \gamma_2, \dots, \gamma_n$, will cover δ and we set $\delta_1 = \bigcup_{i=1,\dots,n} \gamma_i$. The next cone δ_2 is constructed by choosing for each point a closed convex proper cone with angle $\frac{1}{2}\varepsilon$. Again a finite number will cover δ, say $\gamma'_1, \dots, \gamma'_m$. Then we define δ_2 as the union of all intersections $\gamma_i \cap \gamma'_j$ and we proceed by induction. It is clear by construction that the intersection of the δ_i is the cone δ.

3. Microlocalization of constructible sheaves

3.1. Microlocalization of $\mathbb{R}$-constructible sheaves. — Consider the full triangulated subcategory $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{D}^{\mathrm{b}}(k_X)$ and a subset $S \subset T^*X$. There are two obvious ways to define the microlocalization of the derived category of $\mathbb{R}$-constructible sheaves on S that we recall now. Set

$$\mathcal{N}_{\mathbb{R}\text{-c},S} = \mathcal{N}_S \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X).$$

Then the inclusion $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X) \subset \mathrm{D}^{\mathrm{b}}(k_X)$ induces a triangulated functor

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, S). \tag{3.1.1}$$

Clearly the objects of the image of this functors are complexes in $\mathrm{D}^{\mathrm{b}}(k_X, S)$ with $\mathbb{R}$-constructible cohomology. But the functor is not fully faithful in general, and therefore another possible definition of microlocal $\mathbb{R}$-constructible sheaves would be the full subcategory of $\mathrm{D}^{\mathrm{b}}(k_X, S)$ defined by its image. For our purpose it will be convenient to work with the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)/\mathcal{N}_{\mathbb{R}\text{-c},S}$ as does Andronikof in [1], [2].

Definition 3.1.1. — We set

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S) = \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)/\mathcal{N}_{\mathbb{R}\text{-c},S}.$$

Remark 3.1.2. — Note that although the natural functor $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S) \to \mathrm{D}^{\mathrm{b}}(k_X, S)$ is not fully faithful we have nevertheless by definition that $\mathcal{F} \simeq 0$ if and only if $\mathrm{SS}(\mathcal{F}) \cap S = \varnothing$. Hence if $\mathcal{F} \to \mathcal{G}$ is a morphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ we get that $\mathcal{F} \to \mathcal{G}$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ if and only if it is an isomorphism in $\mathrm{D}^{\mathrm{b}}(k_X, S)$, hence if and only if there is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+}$$

such that $\mathrm{SS}(\mathcal{H}) \cap S = \varnothing$. More generally we get

Proposition 3.1.3. — *The natural functor*

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X, S) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, S)$$

is conservative, i.e., a morphism $\mathcal{F} \to \mathcal{G}$ of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X, S)$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X, S)$ if and only if it is an isomorphism in $\mathrm{D}^{\mathrm{b}}(k_X, S)$.

Proof. — We embed $\mathcal{F} \to \mathcal{G}$ in a distinguished triangle

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+}$$

in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$. If $\mathcal{F} \to \mathcal{G}$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}(k_X, S)$ then $\mathcal{H} \simeq 0$ in $\mathrm{D}^{\mathrm{b}}(k_X, S)$. Hence $\mathrm{SS}(\mathcal{H}) \cap S = \varnothing$. Therefore $\mathcal{H} \simeq 0$ in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ and $\mathcal{F} \to \mathcal{G}$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$. □

However there are some situations when the functor (3.1.1) is fully faithful. For instance, Andronikof remarked that the proof of Proposition 2.2.5 holds in the constructible case, hence

Proposition 3.1.4. — *Let* $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,p)$. *Then there is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,p)}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^0\mu\mathrm{hom}(\mathcal{F},\mathcal{G})_p$$

and the natural functor $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,p) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X,p)$ *is fully faithful.*

3.2. The category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})$.

— We will see that the natural morphism (3.1.1) is a an isomorphism for orbits of $\mathbb{C}^\times$ and more generally for closed cones $\dot{\delta}$ in $\dot{T}^*_xX$ for some $x \in X$.

Proposition 3.2.1. — *Let* $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X)$. *Then the natural morphism*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G})$$

is an isomorphism.

Proof. — We may assume that X is a vector space. Note that if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X)$, then $\Phi_{U,\gamma}(\mathcal{F})_U$ is $\mathbb{R}$-constructible for any relatively compact subanalytic open subset $U \ni 0$ and any subanalytic open cone $\gamma \subset X$. Then the results of Section 2.4 hold in the $\mathbb{R}$-constructible case. More precisely, we see first (as in Proposition 2.4.3) that the composition

$$\begin{aligned}\varinjlim_{U,\gamma} \mathrm{H}^0\,\mathrm{RHom}\big(\Phi_{U,\gamma}(\mathcal{F})_U,\mathcal{G}\big) &\longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G})\\ &\longrightarrow \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \longrightarrow \mathrm{H}^0\big(\{x\}\times\dot{\delta},\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)\end{aligned}$$

is an isomorphism and then (as in Theorem 2.4.4) that the composition

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \to \mathrm{H}^0\big(\{x\}\times\dot{\delta},\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)$$

is an isomorphism. Since the second morphism of the last composition is an isomorphism, we get the result. □

Combining Proposition 3.2.1 and Theorem 2.4.4 we get the following theorem.

Theorem 3.2.2. — *Let* $\delta \subset T^*_xX$ *be a closed cone. Then the natural functor*

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}\big(k_X,\{x\}\times\dot{\delta}\big) \longrightarrow \mathrm{D}^{\mathrm{b}}\big(k_X,\{x\}\times\dot{\delta}\big)$$

is fully faithful. Moreover for every $\mathcal{F},\mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})$ *we have*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X,\{x\}\times\dot{\delta})}(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \mathrm{H}^0\big(\{x\}\times\dot{\delta},\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big).$$

3.3. Microlocally $\mathbb{C}$-constructible sheaves. — Recall the microlocal characterization of complexes with $\mathbb{C}$-constructible cohomology sheaves given in [11, Theorem 8.5.5].

PROPOSITION 3.3.1. — *A complex $\mathcal{F}$ in $\mathrm{D}^{\mathrm{b}}(k_X)$ has $\mathbb{C}$-constructible cohomology if and only if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X)$ and $\mathrm{SS}(\mathcal{F})$ is a $\mathbb{C}^\times$-conic subset of T^*X.*

If $S \subset T^*X$ is a not necessarily $\mathbb{C}^\times$-conic subset, then this suggests the definition of a microlocally $\mathbb{C}$-constructible sheaf on S as follows:

DEFINITION 3.3.2. — (i) An object $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ (or $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S')$ for $S' \supset S$) is called *microlocally $\mathbb{C}$-constructible* on S if $\mathrm{SS}(\mathcal{F})$ is $\mathbb{C}^\times$-conic on S.

(ii) We denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$ be the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ consisting of microlocally $\mathbb{C}$-constructible sheaves on S.

REMARK 3.3.3. — Note that the category of microlocally $\mathbb{C}$-constructible sheaves (on S) is different from the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)/(\mathcal{N}_S \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X))$ (*i.e.*, the microlocalization of $\mathbb{C}$-constructible sheaves). There is a natural functor

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)/(\mathcal{N}_S \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S),$$

but in general, an object in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$ cannot be represented by a complex with $\mathbb{C}$-constructible cohomology sheaves. One shall keep in mind that by definition an object in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ (the microlocalization of $\mathbb{R}$-constructible sheaves) is represented by an $\mathbb{R}$-constructible sheaf on X.

REMARK 3.3.4. — Of course Definition 3.3.2, (i) is equivalent to the statement

(i) A sheaf $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ (resp. $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S')$ for $S' \supset S$) is microlocally $\mathbb{C}$-constructible on S if for every point p of S there exists an open neighborhood U of p such that $U \cap \mathrm{SS}(\mathcal{F}) = U \cap \mathbb{C}^\times \mathrm{SS}(\mathcal{F})$.

Obviously $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, T^*X) = \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ and if $x \in X$, then $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ defines an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, x)$ if and only if $\mathcal{F}_{|V}$ is $\mathbb{C}$-constructible for some neighborhood V of x. However the category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$ is not very easy to understand in general, especially if S is not $\mathbb{C}^\times$-conic.

LEMMA 3.3.5. — *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X)$ and $S \subset \dot{T}^*X$.*

(i) *The object $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible on S if and only if $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible on $\mathbb{R}^+S$.*

(ii) *Suppose that $S \subset \dot{T}^*X$ is $\mathbb{C}^\times$-conic. Then $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible on S if and only if $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible in $\gamma^{-1}(U)$ where U is a germ of a neighborhood of $\gamma(S)$ in P^*X.*

Proof. — Statement (i) is a consequence of Lemma 2.2.4 and (ii) follows from (i) and the fact that any $\mathbb{R}^+$-conic neighborhood of S contains a $\mathbb{C}^\times$-conic neighborhood. □

Remark 3.3.6. — There is an obvious functor $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^{\times} S) \to \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$. One might ask the question whether or not a sheaf $\mathcal{F}$ of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$ can be lifted to $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^{\times} S)$ and if there is a tool to produce an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^{\times} S)$ that is isomorphic to $\mathcal{F}$ on S.

There does not seem to be an obvious answer as the following example shows: Consider $X = \mathbb{C}^2$ and the sheaf $\mathcal{F} = \mathbb{C}_{\mathbb{C}\times\{(0,0)\}} \oplus \mathbb{C}_{\{0\}\times\mathbb{R}\times\mathbb{R}\times\{0\}}$. Then

$$\mathrm{SS}(\mathcal{F}) = T^*_{\mathbb{C}\times\{(0,0)\}} X \cup T^*_{\{0\}\times\mathbb{R}\times\mathbb{R}\times\{0\}} X.$$

Take $p = ((0,0,0,0);(0,0,1,1)) \in \dot{T}^*X$. Then $\mathrm{SS}(\mathcal{F}) = \mathbb{C}^{\times} p$ in a neighborhood of p. But if $U \supset \mathbb{C}^{\times} p$ is an arbitrary neighborhood of $\mathbb{C}^{\times} p$, $\mathrm{SS}(\mathcal{F})$ is not $\mathbb{C}^{\times}$-conic on U, hence $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible at p but not on $\mathbb{C}^{\times} p$. However $\mathcal{F}$ is isomorphic in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, p)$ to the sheaf $\mathbb{C}_{\mathbb{C}\times\{(0,0)\}}$ which is globally $\mathbb{C}$-constructible. The problem is how to construct $\mathbb{C}_{\mathbb{C}\times\{(0,0)\}}$ functorially from $\mathcal{F}$. It cannot be done by a cut-off functor which will always preserve the micro-support in a neighborhood of $\mathbb{C}^{\times} p$.

Hence microlocally $\mathbb{C}$-constructible sheaves should be defined on P^*X rather then T^*X and Definition 3.3.2 will mostly be used for a $\mathbb{C}^{\times}$-conic subset X.

3.4. The category $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \{x\} \times \dot{\delta})$.

— The results from Section 2.4 hold in the $\mathbb{C}$-constructible case. We get:

Proposition 3.4.1. — *The natural functors*

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}\big(k_X, \{x\} \times \dot{\delta}\big) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}\big(k_X, \{x\} \times \dot{\delta}\big) \longrightarrow \mathrm{D}^{\mathrm{b}}\big(k_X, \{x\} \times \dot{\delta}\big)$$

are fully faithful. Moreover for every $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X, \{x\} \times \dot{\delta})$ *we have*

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, \{x\}\times\dot{\delta})}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{H}^0\big(\{x\} \times \dot{\delta}, \mu\mathrm{hom}(\mathcal{F}, \mathcal{G})\big).$$

Proof. — The first functor is fully faithful by definition, the second by Theorem 3.2.2. The second part follows again from Theorem 3.2.2. □

4. Invariance by quantized contact transformations

Let $\Omega_X \subset T^*X$ be an open subset of a real manifold X. In [11], Kashiwara-Schapira showed that the category $\mathrm{D}^{\mathrm{b}}(k_X, \Omega_X)$ (or more generally the prestack $\mathrm{D}^{\mathrm{b}}(k_X, *)_{|\Omega_X}$) is invariant under "quantized contact transformations".

Let us briefly explain this statement. Consider real manifolds X, Y of the same dimension and open subsets $\Omega_X \subset \dot{T}^*X$, $\Omega_Y \subset \dot{T}^*Y$. An $\mathbb{R}^+$-homogeneous symplectic isomorphism

$$\chi : \Omega_X \xrightarrow{\sim} \Omega_Y$$

is often called a contact transformation (although strictly speaking, the contact structures are defined on the projective bundles). Invariance under "quantized contact transformations" means that locally we can construct from χ an equivalence of categories

$$\Phi_{\mathcal{K}} : \mathrm{D}^{\mathrm{b}}(k_X, \Omega_X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}(k_Y, \Omega_Y).$$

The equivalence $\Phi_{\mathcal{K}}$ is explicitly given by an integral transform and depends on the choice of a kernel $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(k_{Y\times X})$. The main result [11, Cor. 7.2.2] is:

THEOREM 4.0.2. — *Let X, Y be two real manifolds, $\Omega_X \subset T^*X$, $\Omega_Y \subset T^*Y$ open subsets and $\chi : \Omega_X \longrightarrow \Omega_Y$ a real contact transformation. Set*

$$\Lambda = \big\{((y;\eta),(x;\xi)) \in \Omega_Y \times \Omega_X^a \mid (y,\eta) = \chi(x,-\xi)\big\}.$$

Let $p_X \in \Omega_X$ and $p_Y = \chi(p_X)$.

There exist open neighborhoods X' of $\pi(p_X)$, Y' of $\pi(p_Y)$, Ω'_X of p_X, Ω'_Y of p_Y with $\Omega'_X \subset T^*X' \cap \Omega_X$, $\Omega'_Y \subset T^*Y' \cap \Omega_Y$ and a kernel $\mathcal{K} \in \mathrm{D}^{\mathrm{b}}(k_{Y'\times X'})$ such that:

1) *χ induces a contact transformation $\Omega'_X \xrightarrow{\sim} \Omega'_Y$;*
2) *for every open subsets $\Omega''_X \subset \Omega'_X$ and $\Omega''_Y = \chi(\Omega''_X)$,*

$$\big((\Omega''_Y \times T^*X') \cup (T^*Y' \times \Omega''^{\,a}_X)\big) \cap \mathrm{SS}(\mathcal{K}) \subset \Lambda \cap (\Omega''_Y \times \Omega''^{\,a}_X);$$

3) *composition with $\mathcal{K}$ induces an equivalence of prestacks*

$$\Phi_{\mathcal{K}} = \mathcal{K}\circ : \chi_*\mathrm{D}^{\mathrm{b}}(k_{X'}, *)_{|\Omega'_X} \longrightarrow \mathrm{D}^{\mathrm{b}}(k_{Y'}, *)_{|\Omega'_Y},$$

a quasi-inverse being given by $\Phi_{\mathcal{K}^*}$ with $\mathcal{K}^* = r_*\,\mathrm{R}\mathcal{H}om(\mathcal{K}, \omega_{Y\times X|X})$ where $r : Y\times X \to X\times Y$ switches the factors;

4) $\mathrm{SS}(\Phi_{\mathcal{K}}(\mathcal{F})) \cap \Omega''_Y = \chi(\mathrm{SS}(\mathcal{F}) \cap \Omega''_X)$;
5) $\chi_*\mu\hom(\mathcal{F},\mathcal{G})_{|\Omega_{X'}} \simeq \mu\hom(\Phi_{\mathcal{K}}(\mathcal{F}), \Phi_{\mathcal{K}}(\mathcal{G}))_{|\Omega_{Y'}}$.

Now let us consider constructible sheaves. It is not immediately obvious that the equivalence 3) of Theorem 4.2.1 should induce an equivalence on the microlocalization of $\mathbb{R}$-constructible (resp. on microlocally $\mathbb{C}$-constructible or later on microlocally perverse) sheaves

$$\chi_*\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, *)_{|\Omega_X} \xrightarrow{?} \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y, *)_{|\Omega_Y}$$

since the functor $\Phi_{\mathcal{K}}$ is not well defined on $\mathbb{R}$-constructible sheaves. This problem can be solved at a point $p \in T^*X$ by using the microlocal composition of [11]. In [2], Andronikof uses this tool to construct the functor $\Phi^{\mu}_{\mathcal{K}}$. Then one can treat a variety of kernels $\mathcal{K}$ but *a priori*, one can no longer work in an open neighborhood.

However, under the hypothesis of Theorem 4.0.2, we do not need to use the microlocal composition of kernels. We can always define the functor

$$\chi_*\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, *)_{|\Omega_X} \longrightarrow \chi_*\mathrm{D}^{\mathrm{b}}(k_X, *)_{|\Omega_X} \xrightarrow{\Phi_{\mathcal{K}}} \mathrm{D}^{\mathrm{b}}(k_Y, *)_{|\Omega_Y}$$

and hope that it factors through $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y,*)_{|\Omega_Y}$. However, one has to beware that $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y,*)_{|\Omega_Y}$ is not a full subprestack of $\mathrm{D}^{\mathrm{b}}(k_Y,*)_{|\Omega_Y}$, hence it is not sufficient to show that $\Phi_{\mathcal{K}}(\mathcal{F})$ is isomorphic to an $\mathbb{R}$-constructible sheaf in $\mathrm{D}^{\mathrm{b}}(k_Y,*)_{|\Omega_Y}$. We encounter this problem in Section 4.1 (see Theorem 4.1.2).

4.1. Quantized contact transformations and $\mathbb{R}$-constructible sheaves

Recall the definition of the full subcategory $\mathrm{N}(Y,X,\Omega_Y,\Omega_X)$ of the category $\mathrm{D}^{\mathrm{b}}(k_{Y\times X},\Omega_Y\times T^*X)$. Its objects are kernels $\mathcal{K}$ on $Y\times X$ such that

(i) $\mathrm{SS}(\mathcal{K})\cap(\Omega_Y\times T^*X)\subset\Omega_Y\times\Omega_X^a$,

(ii) the projection $p_1:\mathrm{SS}(\mathcal{K})\cap(\Omega_Y\times T^*X)\to\Omega_Y$ is proper.

If $V=\pi_X(\Omega_X)$ is a subanalytic relatively compact open subset of X we set

$$\mathrm{N}_{\mathbb{R}\text{-c}}(Y,X,\Omega_Y,\Omega_X)=\mathrm{N}(Y,X,\Omega_Y,\Omega_X)\cap\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_{Y\times X},\Omega_Y\times T^*X).$$

Definition 4.1.1. — Let $\mathcal{K}\in\mathrm{N}_{\mathbb{R}\text{-c}}(Y,X,\Omega_Y,\Omega_X)$. We define the functor

$$\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}:\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X,\Omega_X)\longrightarrow\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y,\Omega_Y),\quad \mathcal{F}\longmapsto\mathcal{K}\circ\mathcal{F}_{\overline{V}}.$$

Note that $\mathcal{K}\circ\mathcal{F}_{\overline{V}}$ is $\mathbb{R}$-constructible since V is subanalytic and relatively compact. We may visualize the situation by the following diagram

$$\begin{array}{ccc}\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X,\Omega_X) & \xrightarrow{\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}} & \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y,\Omega_Y)\\ \downarrow & & \downarrow\\ \mathrm{D}^{\mathrm{b}}(k_X,\Omega_X) & \xrightarrow{\mathcal{K}\circ} & \mathrm{D}^{\mathrm{b}}(k_Y,\Omega_Y).\end{array}$$

The square is commutative up to natural isomorphism since the natural morphism $\mathcal{F}\to\mathcal{F}_{\overline{V}}$ induces an isomorphism $\mathcal{K}\circ\mathcal{F}\xrightarrow{\sim}\mathcal{K}\circ\mathcal{F}_{\overline{V}}$ in $\mathrm{D}^{\mathrm{b}}(k_Y,\Omega_Y)$.

Now suppose that $\mathcal{K}\in\mathrm{N}_{\mathbb{R}\text{-c}}(Z,Y,\Omega_Z,\Omega_Y)$ and $\mathcal{L}\in\mathrm{N}_{\mathbb{R}\text{-c}}(Y,X,\Omega_Y,\Omega_X)$. Then their composition $\mathcal{K}\circ\mathcal{L}$ is well defined in $\mathrm{N}(Z,X,\Omega_Z,\Omega_X)$ but not necessarily in $\mathrm{N}_{\mathbb{R}\text{-c}}(Z,X,\Omega_Z,\Omega_X)$. Note however that by definition $\mathcal{L}\in\mathrm{D}^{\mathrm{b}}(k_{Y\times X},\Omega_Y\times T^*X)$. Hence, if we set $W=\pi_Y(\Omega_Y)$, we do not distinguish between $\mathcal{L}$ and $\mathcal{L}_{\overline{W}\times X}$ in $\mathcal{L}\in\mathrm{N}(Y,X,\Omega_Y,\Omega_X)$. In other words we get a natural isomorphism

$$\mathcal{K}\circ\mathcal{L}\xrightarrow{\sim}\mathcal{K}\circ\mathcal{L}_{\overline{W}\times X}$$

in $\mathrm{D}^{\mathrm{b}}(k_{Z\times X},\Omega_Z\times T^*X)$. Then we get natural isomorphisms

$$\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}\Phi^{\mathbb{R}\text{-c}}_{\mathcal{L}}(\mathcal{F})\simeq\mathcal{K}\circ(\mathcal{L}\circ\mathcal{F}_{\overline{V}})_{\overline{W}}\simeq(\mathcal{K}\circ\mathcal{L}_{\overline{W}\times X})\circ\mathcal{F}_{\overline{V}}\simeq\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}\circ\mathcal{L}_{\overline{W}\times X}}(\mathcal{F}).$$

Hence, the theory of microlocal kernels (Section 7.1. of [11]) works well in the $\mathbb{R}$-constructible case if we restrict ourselves to relatively compact subanalytic open sets.

Finally suppose that $\Omega'_X \subset \Omega_X$, $\Omega'_Y \subset \Omega_Y$ and that $\mathcal{K} \in \mathrm{N}_{\mathbb{R}\text{-c}}(Y, X, \Omega_Y, \Omega_X) \cap \mathrm{N}_{\mathbb{R}\text{-c}}(Y, X, \Omega'_Y, \Omega'_X)$. Then we get a diagram

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, \Omega_X) & \xrightarrow{\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}} & \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y, \Omega_Y) \\ \downarrow & & \downarrow \\ \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, \Omega'_X) & \xrightarrow{\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}} & \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y, \Omega'_Y) \end{array}$$

that is commutative up to natural isomorphism induced by $(\mathcal{F}_{\overline{V}})_{\overline{V}'} \simeq \mathcal{F}_{\overline{V}'}$ where $V' = \pi(\Omega'_Y)$.

Now suppose that we are given a contact transformation

$$\chi : \Omega_X \xrightarrow{\sim} \Omega_Y$$

where we assume that $\pi(\Omega_X), \pi(\Omega_Y)$ are relatively compact subanalytic open sets. If there exists an object $\mathcal{K}$ such that $\mathcal{K} \in \mathrm{N}_{\mathbb{R}\text{-c}}(Y, X, \Omega'_Y, \Omega'_X)$ for all open subsets $\Omega'_X \subset \Omega_X$ and $\Omega'_Y = \chi(\Omega'_X)$ then we get a commutative diagram of functors of prestacks

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, *)|_{\Omega_X} & \xrightarrow{\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}} & \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_Y, *)|_{\Omega_Y} \\ \downarrow & & \downarrow \\ \mathrm{D}^{\mathrm{b}}(k_X, *)|_{\Omega_X} & \xrightarrow{\Phi_{\mathcal{K}}} & \mathrm{D}^{\mathrm{b}}(k_Y, *)|_{\Omega_Y} \end{array}$$

All compatibility conditions are easily verified by diagram chases.

We are now ready to quantize contact transformations for $\mathbb{R}$-constructible sheaves. All we need to know is that the kernel produced in Theorem 4.0.2 can be taken $\mathbb{R}$-constructible.

THEOREM 4.1.2. — *Theorem* 4.0.2 *holds when replacing* $\mathrm{D}^{\mathrm{b}}(k_X, S)$ *by* $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X, S)$ *and* $\Phi_{\mathcal{K}}$ *by* $\Phi^{\mathbb{R}\text{-}c}_{\mathcal{K}}$.

Proof. — Let us show that in the situation of Theorem 4.0.2 we can choose the kernel $\mathcal{K}$ to be $\mathbb{R}$-constructible.

First assume that Λ is the conormal bundle of a smooth hypersurface $S \subset Y \times X$. Then one can take the kernel $\mathcal{K} = k_S$ (*cf.* [11], Corollary 7.2.2). Recall that locally χ may be decomposed as

$$\chi = \chi_1 \circ \chi_2 : \Omega_X \longrightarrow \Omega_Z \longrightarrow \Omega_Y$$

where the Lagrangian manifold Λ_i associated to the contact transformation χ_i is the conormal bundle to a smooth hypersurface S_i. By shrinking Ω_X and Ω_Y we may assume that $\pi_Z(\Omega_Z)$ is subanalytic and relatively compact. Then $k_{S_1} \circ k_{S_2 \cap (\pi_Z(\Omega_Z) \times X)}$ is $\mathbb{R}$-constructible and satisfies 1), 2), 4) and 5) of the theorem (*cf.* [11, Cor. 7.2.2]).

Moreover we know that $\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}$ and $\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}^*}$ are well-defined and that they are quasi-inverse functors in the non-constructible case. By definition, we have

$$\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}} \circ \Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}^*} \simeq \Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}\circ\mathcal{K}^*_{\overline{V}\times Y}}.$$

Recall that there is a natural isomorphism

$$k_{\Delta_X} \longrightarrow \mathcal{K}\circ\mathcal{K}^*$$

in $\mathrm{D^b}(k_X,\Omega_X)$. Hence we get an isomorphism

$$k_{\Delta_X} \longrightarrow \mathcal{K}\circ\mathcal{K}^* \longrightarrow \mathcal{K}\circ\mathcal{K}^*_{\overline{V}\times Y} \tag{4.1.1}$$

in $\mathrm{D^b}(k_X,\Omega_X)$. It is sufficient to prove that this morphism is well-defined in $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X,\Omega_X)$.

Denote by q_{12}, q_{13}, q_{23} the obvious projections from $X\times X\times Y$. We get a commutative diagram

$$\begin{array}{ccccc}
k_{\Delta_X} & \longrightarrow & \mathrm{R}q_{12*}\,\mathrm{R}\mathcal{H}om(q_{13}^{-1}\mathcal{K}, q_{23}^{!}\mathcal{K}) & \longleftarrow & \mathcal{K}\circ\mathcal{K}^* \\
& \searrow & \downarrow & & \downarrow \\
& & \mathrm{R}q_{12*}\,\mathrm{R}\mathcal{H}om(q_{13}^{-1}\mathcal{K}, q_{23}^{!}\mathcal{K}_{\overline{V}\times Y}) & \longleftarrow & \mathcal{K}\circ\mathcal{K}^*_{\overline{V}\times Y}
\end{array}$$

that is defined in $\mathrm{D^b}(k_X)$. Note that the lower part is well-defined in $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X)$. All morphisms become isomorphisms in $\mathrm{D^b}(k_X,\Omega_X)$ (*cf.* [11], Theorem 7.2.1). Since the natural functor $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X,\Omega_X)\to\mathrm{D^b}(k_X,\Omega_X)$ is conservative, this shows that (4.1.1) is a well-defined isomorphism in $\mathrm{D^b_{\mathbb{R}\text{-c}}}(k_X,\Omega_X)$.

Similarly one shows that the kernel $r^{-1}\,\mathrm{R}\mathcal{H}om(\mathcal{K},\omega_{X\times Y|X})$ defines a right inverse of $\mathcal{K}$ which proves that $\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}$ is an equivalence. Then $\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}^*}$ is actually a quasi-inverse since it is a left inverse of an equivalence. □

4.2. Quantized contact transformations and $\mathbb{C}$-constructible sheaves. — It is now easy to transfer the results of the last section to microlocally $\mathbb{C}$-constructible sheaves. Consider complex manifolds X, Y of the same dimension and open $\mathbb{C}^\times$-conic subsets $\Omega_X\subset\dot{T}^*X$, $\Omega_Y\subset\dot{T}^*Y$. We will call a $\mathbb{C}^\times$-homogeneous symplectic isomorphism

$$\chi:\Omega_X \xrightarrow{\ \sim\ } \Omega_Y$$

a contact transformation (omitting "complex" since we will never consider real contact transformations when dealing with microlocally $\mathbb{C}$-constructible sheaves). Then we get the analogous statements of Theorems 4.0.2 and 4.1.2 by replacing open sets with $\mathbb{C}^\times$-conic open sets.

THEOREM 4.2.1. — *Let X and Y be two complex manifolds, $\Omega_X\subset\dot{T}^*X$, $\Omega_Y\subset\dot{T}^*Y$ open subsets and*

$$\chi:\Omega_X\longrightarrow\Omega_Y$$

a homogeneous complex contact transformation. Then Theorem 4.0.2 *holds when replacing* $\mathrm{D}^{\mathrm{b}}(k_X, S)$ *by* $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, S)$.

Proof. — Since by definition $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \Omega''_X)$ is a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, \Omega''_X)$, it is enough to show that for any $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \Omega_X)$ the object $\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}(\mathcal{F})$ is an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_Y, \Omega_Y)$. Hence we have to show that $\mathrm{SS}(\Phi(\mathcal{F}))$ is $\mathbb{C}^\times$-conic on Ω_Y. Since χ is $\mathbb{C}^\times$-homogeneous this is easily verified by the formula

$$\mathrm{SS}\big(\Phi^{\mathbb{R}\text{-c}}_{\mathcal{K}}(\mathcal{F})\big) \cap \Omega''_X = \chi(\mathrm{SS}(\mathcal{F}) \cap \Omega''_Y). \qquad \square$$

5. Microlocally complex constructible sheaves on $\mathbb{C}^\times p$

Let $p \in \dot{T}^*X$. As a special case of Proposition 3.4.1 we get that the natural functor

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times p)$$

is fully faithful. Moreover, morphisms in $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times p)$ between microlocally $\mathbb{C}$-constructible sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ are given by sections of $\mu\mathrm{hom}(\mathcal{F}, \mathcal{G})$ on $\mathbb{C}^\times p$.

In this section we will give a description of the objects of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ using quantized contact transformation and the generic position theorem. More precisely, we will show in Section 6.1 that if $\mathcal{F}$ is microlocally $\mathbb{C}$-constructible on $\mathbb{C}^\times p$ and $\mathrm{SS}(\mathcal{F})$ is in generic position (*i.e.*, $\mathrm{SS}(\mathcal{F}) \cap \mathbb{C}^\times p$ is isolated in $\pi^{-1}\pi(p)$) then $\mathcal{F}$ is isomorphic in $\mathrm{D}^{\mathrm{b}}(k_X, \mathbb{C}^\times p)$ to an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$.

It will often be convenient to fix an $\mathbb{R}^+$-conic Lagrangian variety Λ in T^*X and to consider only sheaves whose micro-support is contained in Λ. We introduce the following categories:

Definition 5.0.2. — Let $\Lambda \subset T^*X$ be an $\mathbb{R}^+$-conic Lagrangian variety defined in a neighborhood of a subset $S \subset T^*X$. Then we define the following two categories:

1) $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda}(k_X, S) \subset \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ is the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$ whose objects $\mathcal{F}$ satisfy $\mathrm{SS}(\mathcal{F}) \subset \Lambda$ in a neighborhood of S.

2) $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, S) = \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S) \cap \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda}(k_X, S)$.

Of course, the second definition is mainly of interest when S is $\mathbb{C}^\times$-conic and Λ is $\mathbb{C}^\times$-conic on the subset S.

5.1. Microlocally complex constructible sheaves in generic position

In this section we will show that at a generic point of its micro-support a microlocally complex constructible sheaf is naturally isomorphic to a complex constructible sheaf.

Definition 5.1.1. — Let $p \in \dot{T}^*X$ and $\Lambda \subset T^*X$ be an $\mathbb{R}^+$-conic Lagrangian subset such that Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$. We say that Λ is in *generic position* at p if

$$\Lambda \cap \pi^{-1}\big(\pi(p)\big) \subset \mathbb{C}^\times p$$

in a neighborhood of $\mathbb{C}^\times p$.

Remark 5.1.2. — If Λ is in generic position at p, then Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$ by definition. Hence either $\Lambda \cap \pi^{-1}\big(\pi(p)\big) = \mathbb{C}^\times p$ or $\Lambda \cap \pi^{-1}\big(\pi(p)\big) = \varnothing$ in a neighborhood of $\mathbb{C}^\times p$. Moreover, being in generic position is an open property. Also note that if Λ is in generic position then it is in generic position in the sense of [10, Section I.6], where Λ is supposed to be only locally $\mathbb{C}^\times$-conic in a neighborhood of p.

Proposition 5.1.3. — *Let $\Lambda \subset T^*X$ be an $\mathbb{R}^+$-conic Lagrangian variety that is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$. Suppose that Λ is in generic position at p. Then there exists a fundamental system of conic open subanalytic neighborhoods γ of $\mathbb{C}^\times p$ in $\dot{T}^*_{\pi(p)}X$ and for each γ a fundamental system of open relatively compact subanalytic neighborhoods U of $\pi(p)$ such that*

1) *the microlocal cut-off functor $\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}(k_X) \to \mathrm{D}^{\mathrm{b}}(k_X)$ induces a functor*

$$\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,\Lambda}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, \mathbb{C}^\times p),$$

and this functor factors as

$$\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,\Lambda}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)} \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, \mathbb{C}^\times p);$$

2) *there is an isomorphism of functors $\Phi_{U,\gamma} \xrightarrow{\sim} \iota$ where*

$$\iota : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,\Lambda}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, \mathbb{C}^\times p)$$

is the natural functor;

3) $\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{F})\big) \cap \dot{\pi}^{-1}(V) = \mathrm{SS}(\mathcal{F}) \cap (V \times \gamma)$ *for sufficiently small open neighborhoods V of $\pi(p)$;*

4) $\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{F})\big) \cap \dot{\pi}^{-1}\pi(p) \subset \mathbb{C}^\times p$.

Proof. — Since the functor $\Phi_{U,\gamma}$ sends $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ to $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ (*cf.* Prop. 3.2.1), it induces

$$\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, U \times \gamma).$$

Since $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \cap U \times \gamma = \mathrm{SS}(\mathcal{F}) \cap U \times \gamma$ the functor $\Phi_{U,\gamma}$ preserves microlocally $\mathbb{C}$-constructible sheaves and we get the functor of (1) as

$$\Phi_{U,\gamma} : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, U \times \gamma) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p).$$

Recall that there exist a fundamental system of conic subanalytic open neighborhoods γ of $\mathbb{C}^\times p$ and for each γ a fundamental system of relatively compact subanalytic open neighborhoods U of $\pi(p)$ such that (U, γ) is a refined cutting pair. If γ is sufficiently small then $\gamma \cap \pi^{-1}\pi(p) \cap \Lambda = \mathbb{C}^\times p$. Next we choose U

sufficiently small such that $U \cap \partial\gamma \cap \Lambda = \varnothing$. Then 2) follows from the refined microlocal cut-off lemma, 3) from Corollary 2.3.6 and 4) from 3).

Finally let us prove the factorization of 1). For this purpose let us first write $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, U\times\gamma)$ as a localization of a full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$.

Denote by $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ such that

$$\mathrm{SS}(\mathcal{F}) \cap U\times\gamma \subset \Lambda.$$

The category $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)$ is obviously a full triangulated subcategory. Now we localize $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)$ by complexes whose micro-support is disjoint from $U\times\gamma$ (hence by $\mathcal{N}_{\mathbb{R}\text{-c},U\times\gamma}$). Then we get a natural functor

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)/\mathcal{N}_{\mathbb{R}\text{-c},U\times\gamma} \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda}(k_X, U\times\gamma). \tag{5.1.1}$$

If $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)$ and $(\mathrm{SS}(\mathcal{F}) \cap U\times\gamma) \subset (\Lambda\cap U\times\gamma)$, then any object $\mathcal{F}'$ that is isomorphic to $\mathcal{F}$ on $U\times\gamma$ is also an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)$. Hence we get that (5.1.1) is an equivalence.

By assumption Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$. Hence we may assume that Λ is $\mathbb{C}^\times$-conic on $U\times\gamma$. One can show that if $\mathrm{SS}(\mathcal{F}) \subset \Lambda$ then $\mathrm{SS}(\mathcal{F})$ is $\mathbb{C}^\times$-conic on $U\times\gamma$ (*cf.* Theorem 8.5.5 of [11]). Let us recall the idea of the proof. First one shows that $\mathrm{SS}(\mathcal{F})$ is open in Λ (on $U\times\gamma$) and therefore locally $\mathbb{C}^\times$-conic, *i.e.*, for every $\mathbb{C}^\times$-orbit S the set $\mathrm{SS}(\mathcal{F})\cap S\cap U\times\gamma$ is open in $S\cap U\times\gamma$ (Lemma 8.3.14 of [11]). Then, by Proposition 8.5.2 of [11], one gets that Λ is $\mathbb{C}$-analytic on $U\times\gamma$. Hence Λ is $\mathbb{C}$-analytic and $\mathbb{R}^+$-conic and therefore $\mathbb{C}^\times$-conic. Thus, we get the equivalence

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X)/\mathcal{N}_{\mathbb{R}\text{-c},U\times\gamma} \simeq \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda}(k_X, U\times\gamma) \simeq \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, U\times\gamma).$$

By 3) and the assumption that Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$ we get the functor

$$\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X) \xrightarrow{\Phi_{U,\gamma}} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(p)}$$

for sufficiently small (U,γ). If $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c},\Lambda,U\times\gamma}(k_X) \cap \mathcal{N}_{\mathbb{R}\text{-c},U\times\gamma}$ then again by 3) we have $\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \subset T^*_X X$ in a neighborhood of $\pi(p)$, hence $\Phi_{U,\gamma}(\mathcal{F})$ is constant in a neighborhood of $\pi(p)$ which implies the factorisation 1). □

Lemma 5.1.4. — *In the situation of Proposition* 5.1.3, *suppose that we have two pairs* $(V,\delta) \subset (U,\gamma)$ *such that* 1), 2), 3), 4) *are satisfied. Then the natural morphism*

$$\Phi_{V,\delta}(\mathcal{F}) \longrightarrow \Phi_{U,\gamma}(\mathcal{F})$$

is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$.

Proof. — Embed the morphism in a distinguished triangle

$$\Phi_{V,\delta}(\mathcal{F}) \longrightarrow \Phi_{U,\gamma}(\mathcal{F}) \longrightarrow \mathcal{H} \xrightarrow{+}$$

in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$. Note that we have $\mathrm{SS}(\mathcal{H}) \cap V \times \delta = \varnothing$ because $\Phi_{V,\delta}(\mathcal{F}) \to \Phi_{U,\gamma}(\mathcal{F})$ is an isomorphism on $V \times \delta$. By 4), we get

$$\mathrm{SS}(\mathcal{H}) \cap \dot{\pi}^{-1}\pi(p) \subset \mathbb{C}^\times p \subset V \times \delta.$$

Hence $\mathrm{SS}(\mathcal{H}) \cap \pi^{-1}\pi(p) \subset T^*_X X$ in a neighborhood of $\pi(p)$ and $\mathcal{H}$ is constant in a neighborhood of $\pi(p)$. □

THEOREM 5.1.5. — *Suppose that Λ is in generic position at p. Then the microlocal cut-off functors induce a fully faithful functor*

$$\Phi : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,\Lambda}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$$

Proof. — We obviously have the equivalence

$$2\varinjlim_{\mathbb{C}^\times p \in U \times \gamma} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, U \times \gamma) \xrightarrow{\ \sim\ } \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, \mathbb{C}^\times p).$$

We may assume by cofinality that (U, γ) is a sufficiently small refined cutting pair such that Proposition 5.1.3 holds. By Lemma 5.1.4 the functors $\Phi_{U,\gamma}$ of Proposition 5.1.3 induce a functor

$$\Phi : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}.$$

Let us show that Φ is fully faithful.

Let $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, \mathbb{C}^\times p)$. Note that by Proposition 5.1.3, 3) if $\mathcal{H} \to \mathcal{F}$ is an isomorphism on $\mathbb{C}^\times p$ then $\Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{F})$ is an isomorphism on $\dot{\pi}^{-1}(V)$ for sufficiently small (U, γ) and $V \ni \pi(p)$. In particular, $\Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{F})$ is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$. Consider the following chain of morphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)}(\mathcal{F}, \mathcal{G}) &= \varinjlim_{\substack{\mathcal{H} \to \mathcal{F} \\ \text{iso on } \mathbb{C}^\times p}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}(\mathcal{H}, \mathcal{G}) \\
&\longrightarrow \varinjlim_{\substack{V, \Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{F}) \\ \text{iso on } \dot{\pi}^{-1}(V)}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}\big(\Phi_{U,\gamma}(\mathcal{H})_{|V}, \Phi_{U,\gamma}(\mathcal{G})_{|V}\big) \\
&\longrightarrow \varinjlim_{\substack{V, \mathcal{H}' \to \Phi_{U,\gamma}(\mathcal{F}) \\ \text{iso on } \dot{\pi}^{-1}(V)}} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(k_X)}\big(\mathcal{H}'_{|V}, \Phi_{U,\gamma}(\mathcal{G})_{|V}\big) \\
&= \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}}\big(\Phi_{U,\gamma}(\mathcal{F}), \Phi_{U,\gamma}(\mathcal{G})\big) \\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}}\big(\Phi(\mathcal{F}), \Phi(\mathcal{G})\big).
\end{aligned}$$

We have to check that the composition is an isomorphism.

To prove that this map is surjective it is sufficent to note that if we consider a morphism $\mathcal{H}' \to \Phi_{U,\gamma}(\mathcal{G})$ and an isomorphism $\mathcal{H}' \to \Phi_{U,\gamma}(\mathcal{F})$ on $\dot{\pi}^{-1}\pi(p)$, then $\mathcal{H}' \to \mathcal{F}$ is an isomorphism on $\mathbb{C}^\times p$ and the morphism $\mathcal{H}' \to \mathcal{G}$ defines a morphism in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ which is sent to the same morphism $\Phi(F) \to \Phi(G)$ as $\mathcal{H}' \to \Phi_{U,\gamma}(\mathcal{G})$.

Let us show that the map is injective. If $\Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{G})$ is zero in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \dot{\pi}^{-1}\pi(p))$ then there exist $\mathcal{K}$ and an isomorphism $\mathcal{K} \to \Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{F})$ on $\dot{\pi}^{-1}\pi(p)$ such that $\mathcal{K} \to \Phi_{U,\gamma}(\mathcal{H}) \to \Phi_{U,\gamma}(\mathcal{G})$ is the zero morphism. Then $\mathcal{K} \to \Phi_{U,\gamma}(\mathcal{H}) \to \mathcal{H} \to \mathcal{F}$ is an isomorphism on $\mathbb{C}^\times p$ and $\mathcal{K} \to \Phi_{U,\gamma}(\mathcal{H}) \to \mathcal{H} \to \mathcal{G}$ is zero hence $\mathcal{H} \to \mathcal{G}$ is zero in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$. □

5.2. The generic position theorem. — Let us recall Kashiwara-Kawai's generic position theorem (*cf.* [10, Section I.6]).

Proposition 5.2.1. — *Let $p \in \dot{T}^*X$ and $\Lambda \subset T^*X$ be an $\mathbb{R}^+$-conic Lagrangian subset such that Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$. Then there exists a complex contact transformation $\chi : T^*X \longrightarrow T^*X$, defined in a neighborhood of $\mathbb{C}^\times p$, such that $\chi(\Lambda)$ is in generic position at $q = \chi(p)$.*

Proof. — Kashiwara-Kawai show this result on a neighborhood of p for locally $\mathbb{C}^\times$-conic Lagrangian varieties assuming only that $\Lambda \cap \pi^{-1}\pi(p) = \mathbb{C}^\times p$ in a neighborhood of p. Hence if we suppose that Λ is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$, we get that $\chi(\Lambda)$ is in generic position at q since χ is $\mathbb{C}^\times$-homogeneous. □

Remark 5.2.2. — Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$. By the generic position theorem and invariance under quantized contact transformations we can find a contact transformation $\chi : T^*X \to T^*X$ defined in a neighborhood of $\mathbb{C}^\times p$ and an equivalence of categories

$$\Phi^{\mathbb{C}\text{-c}}_{\mathcal{K}} : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times q)$$

such that $\mathrm{SS}(\Phi^{\mathbb{C}\text{-c}}_{\mathcal{K}}(\mathcal{F})) = \chi(\mathrm{SS}(\mathcal{F}))$ is in generic position at q. Then $\Phi^{\mathbb{C}\text{-c}}_{\mathcal{K}}(\mathcal{F})$ is functorially isomorphic in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times q)$ to an object of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)_{\pi(q)}$ (modulo constant sheaves). Hence many problems in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, \mathbb{C}^\times p)$ can be reduced to the study of germs of complex constructible sheaves in generic position.

6. Microlocal perverse sheaves in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X, S)$

6.1. Andronikof's prestack of microlocally perverse sheaves

In this section we will first recall Andronikof's definition of the prestack of microlocal perverse sheaves. In [2] he suggests a definition of a microlocal perverse sheaf on an arbitrary subset S of T^*X which is based on the microlocal characterization of perverse sheaves (see Proposition 6.1.1 below). However, he is not very precise concerning $\mathbb{C}^\times$-conicity. In particular the proof of his main tool (Proposition 3.2) is incomplete under the given assumptions. Hence we will recall a slightly more precise version of his prestack and we will restrict ourselves from the beginning to $\mathbb{C}^\times$-conic sets. Moreover we will add complete proofs to the statements of [2] which hold in this case. Also, while Andronikof restricts his studies to points and $\mathbb{C}^\times$-orbits we will work from the beginning with the entire prestack.

In [12] we find the following microlocal characterization of perverse sheaves.

PROPOSITION 6.1.1. — *Let $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X)$. Then we have equivalence between*

(P1) *$\mathcal{F}$ is a perverse sheaf.*

(P2) *For every non-singular point p of* $\mathrm{SS}(\mathcal{F})$ *such that the projection $\pi : \mathrm{SS}(\mathcal{F}) \to X$ has constant rank on a neighborhood of p, there exist a submanifold $Y \subset X$ and an object $M \in \mathrm{Mod}(k)$ such that $\mathcal{F} \simeq M_Y[\dim Y]$ in $\mathrm{D}^{\mathrm{b}}(k_X, p)$.*

(P3) *The assertion* (P2) *is true for some point p of any irreducible component of* $\mathrm{SS}(\mathcal{F})$.

This naturally leads to the following

DEFINITION 6.1.2. — Let $S \subset \dot{T}^*X$ be a $\mathbb{C}^\times$-conic subset.

(i) A sheaf $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ is called *microlocally perverse* on S, if it is microlocally $\mathbb{C}$-constructible on S and there exists an open neighborhood U of S such that (P2) is satisfied on U, *i.e.*, for every non-singular point p of $\mathrm{SS}(\mathcal{F}) \cap U$ such that the projection $\pi : \mathrm{SS}(\mathcal{F}) \to X$ has constant rank on a neighborhood of p, there exists a submanifold $Y \subset X$ and an object $M \in \mathrm{Mod}(k)$ such that $\mathcal{F} \simeq M_Y[\dim Y]$ in $\mathrm{D}^{\mathrm{b}}(k_X, p)$.

(ii) We will denote by $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, S)$ the full subcategory of $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X, S)$ whose objects are perverse on S (*i.e.*, which may be represented by a microlocally perverse sheaf on S).

(iii) In particular if $\Omega \subset \dot{T}^*X$ is an open $\mathbb{C}^\times$-conic subset, we get the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \Omega)$. Clearly this defines a prestack of categories on P^*X, denoted by $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *)$. This is *Andronikof's prestack* of microlocal perverse sheaves.

(iv) Let $\Lambda \subset T^*X$ be an $\mathbb{R}^+$-conic Lagrangian variety that is $\mathbb{C}^\times$-conic in a neighborhood of $\mathbb{C}^\times p$. Then we set

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) = \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c},\Lambda}(k_X, \mathbb{C}^\times p) \cap \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p).$$

PROPOSITION 6.1.3. — *Let $p \in \dot{T}^*X$. The stalk of Andronikof's prestack at $\gamma(p)$ is naturally equivalent to the category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$.

Proof. — Consider the functor

$$2\varinjlim_{\substack{\mathbb{C}^\times p \subset U \subset \dot{T}^*X \\ U\ \mathbb{C}^\times\text{-conic}}} \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, U) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p).$$

It is essentially surjective by definition of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$. The proof that it is fully faithful is analogous to Proposition 2.2.7 and Corollary 2.2.8. ☐

Proposition 6.1.4. — *The duality functor*

$$\mathrm{D} : \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-}c}(k_X), \quad \mathcal{F} \longmapsto \mathrm{R}\mathcal{H}om(\mathcal{F}, \omega_X)$$

induces contravariant equivalences of prestacks

$$\mathrm{D} : \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, *) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X, *), \quad \mathrm{D} : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *).$$

Proof. — The first functor is well-defined because $\mathrm{SS}(\mathrm{D}\mathcal{F}) = \mathrm{SS}(\mathcal{F})$. Let p in $\mathrm{SS}(\mathcal{F})$ such that $\mathcal{F} \simeq M_Y[\dim Y]$ in $\mathrm{D}^{\mathrm{b}}(k_X, p)$. Then $\mathrm{D}\mathcal{F} \simeq \mathrm{D}M_Y[\dim Y]$ in $\mathrm{D}^{\mathrm{b}}(k_X, p)$ since D is an anti-equivalence in $\mathrm{D}^{\mathrm{b}}(k_X, p)$. But $\mathrm{D}M_Y[\dim Y] \simeq M_Y[\dim Y]$. Hence the second functor is well-defined.

The two functors are equivalences because $\mathrm{D}^2 \simeq \mathrm{Id}$ in $\mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$. □

Andronikof's prestack is invariant by quantized contact transformations:

Theorem 6.1.5. — *Let X and Y be two complex manifolds, $\Omega_X \subset \dot{T}^*X$, $\Omega_Y \subset \dot{T}^*Y$ open $\mathbb{C}^{\times}$-conic subsets and $\chi : \Omega_X \xrightarrow{\sim} \Omega_Y$ a contact transformation. Then Theorem* 4.0.2 *holds when replacing* $\mathrm{D}^{\mathrm{b}}(k_X, S)$ *by* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, S)$.

Proof. — It is enough to show the proposition in the case in which Λ is the conormal bundle to a smooth hypersurface S and we can choose $\mathcal{K} \simeq k_S$. Then the fact that $\Phi^{\mathbb{C}\text{-c}}_{\mathcal{K}}$ preserves microlocal perverse sheaves follows from Proposition 7.4.6 of [11]. □

6.2. The abelian category $\mathbf{D}^{\mathbf{b}}_{\mathbf{perv}}(\boldsymbol{k_X}, \mathbb{C}^{\times}\boldsymbol{p})$. — In general, one does not know much about the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, S)$ even if S is $\mathbb{C}^{\times}$-conic. In [2], it is announced that if $p \in \dot{T}^*X$, then $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(X, p)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(X, \mathbb{C}^{\times}p)$ are abelian. While we do not know if this is true for $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(X, p)$, we will give a proof here for the category $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(X, \mathbb{C}^{\times}p)$.

Let $\mathcal{D}$ be a triangulated category with a t-structure and heart $\mathcal{C}$. Recall that if $\mathcal{N} \subset \mathcal{D}$ is a full triangulated category that is stable under truncation functors, then $\mathcal{N} \cap \mathcal{C}$ is a thick subcategory of $\mathcal{C}$ and we get a fully faithful functor

$$\mathcal{C}/\mathcal{N} \cap \mathcal{C} \longrightarrow \mathcal{D}/\mathcal{N}$$

where $\mathcal{C}/\mathcal{N} \cap \mathcal{C}$ the localisation of $\mathcal{C}$ by morphisms whose kernel and cokernel are objects of $\mathcal{N}$ (in particular $\mathcal{C}/\mathcal{N} \cap \mathcal{C}$ is abelian (see [5])).

We will fix a point $p \in \dot{T}^*X$ and an $\mathbb{R}^+$-conic Lagrangian subvariety Λ which is $\mathbb{C}^{\times}$-conic in a neighborhood of $\mathbb{C}^{\times}p$.

Proposition 6.2.1. — *Suppose that Λ is in generic position at p. Then the functor Φ of Theorem* 5.1.5 *induces a commutative diagram of fully faithful* *functors*

$$\begin{array}{ccc} \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c,\Lambda}(k_X,\mathbb{C}^\times p) & \xrightarrow{\Phi} & \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-}c}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)} \\ \uparrow & & \uparrow \\ \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X,\mathbb{C}^\times p) & \longrightarrow & \mathrm{Perv}(X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}. \end{array}$$

Proof. — It is sufficient to prove that if $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,\mathbb{C}^\times p)$ then $\Phi(\mathcal{F})$ is perverse in a neighborhood of $\pi(p)$. Since we have

$$\mathrm{SS}\big(\Phi(\mathcal{F})\big) \cap \dot\pi^{-1}(V) = \mathrm{SS}(\mathcal{F}) \cap (V \times \gamma)$$

for some small neighborhood $V \times \gamma$ of $\mathbb{C}^\times p$, we get the result by the characterization of perverse sheaves (*cf.* Proposition 6.1.1). □

Also note that $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X,\mathbb{C}^\times p)$ is additive. Hence $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X,\mathbb{C}^\times p)$ is a full additive subcategory of $\mathrm{Perv}(X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$. In order to prove that it is actually a full abelian subcategory, we will need two lemmas:

LEMMA 6.2.2. — *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of perverse sheaves such that* $\mathrm{SS}(\mathcal{F}) \cup \mathrm{SS}(\mathcal{G}) \subset \Lambda$ *in a neighborhood of p. Then* $\mathrm{SS}(\ker(\varphi)) \cup \mathrm{SS}(\mathrm{coker}(\varphi)) \subset \Lambda$ *in a neighborhood at p.*

Proof. — Recall (see [11, Exercise X.6]) that the micro-support of a $\mathbb{C}$-constructible sheaf $\mathcal{F} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_X)$ can be calculated as

$$\mathrm{SS}(\mathcal{F}) = \bigcup_{i\in\mathbb{Z}} {}^{\mathrm{p}}\mathrm{SS}\big(\mathrm{H}^i(\mathcal{F})\big)$$

where ${}^{\mathrm{p}}\mathrm{H}^i(\mathcal{F})$ denotes the i-th perverse cohomology sheaf of $\mathcal{F}$. Now let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of perverse sheaves. We embed it into a distinguished triangle

$$\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \xrightarrow{+} .$$

Then we consider the canonical distinguished triangle

$${}^{\mathrm{p}}\tau^{\leqslant -1}(\mathcal{H}) \longrightarrow \mathcal{H} \longrightarrow {}^{\mathrm{p}}\tau^{\geqslant 0}(\mathcal{H}) \xrightarrow{+}$$

where ${}^{\mathrm{p}}\tau^{\leqslant -1}, {}^{\mathrm{p}}\tau^{\geqslant 0}$ denote the perverse truncation functors. Then ${}^{\mathrm{p}}\tau^{\leqslant -1}(\mathcal{H})[-1]$ is the kernel and ${}^{\mathrm{p}}\tau^{\geqslant 0}(\mathcal{H})$ is the cokernel of φ. Therefore

$$\mathrm{SS}(\ker\varphi) \cup \mathrm{SS}(\mathrm{coker}\,\varphi) = \mathrm{SS}(\mathcal{H}) \subset \mathrm{SS}(\mathcal{F}) \cup \mathrm{SS}(\mathcal{G}). \qquad □$$

LEMMA 6.2.3. — *Let Λ be in generic position at p. Consider a morphism $\mathcal{F} \to \mathcal{G}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X,\mathbb{C}^\times p)$. Let $\Phi(\mathcal{F}) \to \Phi(\mathcal{G})$ be the corresponding morphism in $\mathrm{Perv}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$ and $\mathcal{L} \to \Phi(\mathcal{F})$ a kernel (resp. $\Phi(\mathcal{G}) \to \mathcal{L}'$ a cokernel) in $\mathrm{Perv}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$. Then $\Phi(\mathcal{L}) \to \mathcal{L}$ (resp. $\Phi(\mathcal{L}') \to \mathcal{L}'$) is an isomorphism in $\mathrm{Perv}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$.*

Proof. — Fix U, γ such that $\Phi(\mathcal{F}) \simeq \Phi_{U,\gamma}(\mathcal{F})$, $\Phi(\mathcal{G}) \simeq \Phi_{U,\gamma}(\mathcal{G})$ and $\Phi(\mathcal{L}) \simeq \Phi_{U,\gamma}(\mathcal{L})$. Embed $\Phi_{U,\gamma}(\mathcal{L}) \to \mathcal{L}$ in a distinguished triangle

$$\Phi_{U,\gamma}(\mathcal{L}) \longrightarrow \mathcal{L} \longrightarrow \mathcal{H} \xrightarrow{+} .$$

Then there is an open neighborhood $V \subset U$ of $\pi(p)$ such that

$$\mathrm{SS}\big(\Phi_{U,\gamma}(\mathcal{L})\big) \cap \dot{\pi}^{-1}(V) = \mathrm{SS}(\mathcal{L}) \cap V \times \gamma.$$

Moreover, by Lemma 6.2.2, we have

$$\mathrm{SS}(\mathcal{L}) \cap \dot{\pi}^{-1}(V) \subset \big(\mathrm{SS}(\Phi_{U,\gamma}(\mathcal{F})) \cup \mathrm{SS}(\Phi_{U,\gamma}(\mathcal{G})\big) \cap \dot{\pi}^{-1}(V) \subset \Lambda \cap V \times \gamma.$$

Hence if V is sufficiently small we get

$$\mathrm{SS}(\mathcal{L}) \cap \dot{\pi}^{-1}(V) \subset \Lambda \cap V \times \gamma.$$

Since $\Phi_{U,\gamma}(\mathcal{L}) \to \mathcal{L}$ is an isomorphism on $U \times \gamma$ we get that $\mathrm{SS}(\mathcal{H}) \cap V \times \gamma = \varnothing$ and therefore $\mathrm{SS}(\mathcal{H}) \subset T^*_X X$ on V and $\mathcal{H}$ is constant in a neighborhood of $\pi(p)$. The proof for the cokernel is similar. □

PROPOSITION 6.2.4. — *The additive category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p)$ *is equivalent to an abelian subcategory of* $\mathrm{Perv}(k_X)_{\pi(p)}/\mathcal{LC}_{\pi(p)}$.

Proof. — By Lemma 6.2.3 the full additive subcategory $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p)$ is stable by kernels and cokernels. Since it is a full subcategory, it is abelian. □

LEMMA 6.2.5. — *Let* Λ *be a* $\mathbb{C}^\times$*-conic Lagrangian variety* (*we do not ask* Λ *to be in generic position at* p)*. Then* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p)$ *is abelian. Moreover if* Λ' *is another* $\mathbb{C}^\times$*-conic Lagrangian variety with* $\Lambda \subset \Lambda'$*, then the natural functor*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda'}(k_X, \mathbb{C}^\times p)$$

is exact.

Proof. — Consider $\Lambda \subset \Lambda'$. Let χ be a canonical transformation such that $\chi(\Lambda')$ is in generic position at $q = \chi(p)$. Then $\chi(\Lambda)$ is also in generic position at q and we get a diagram

$$\begin{array}{ccccc}
\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) & \xrightarrow{\sim} & \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\chi(\Lambda)}(k_X, \mathbb{C}^\times q) & \longrightarrow & \mathrm{Perv}(k_X)_{\pi(q)}/\mathcal{LC}_{\pi(q)} \\
\downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda'}(k_X, \mathbb{C}^\times p) & \xrightarrow{\sim} & \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\chi(\Lambda')}(k_X, \mathbb{C}^\times q) & \longrightarrow & \mathrm{Perv}(k_X)_{\pi(q)}/\mathcal{LC}_{\pi(q)}.
\end{array}$$

This diagram is commutative up to isomorphism. The horizontal functors are exact and fully faithful. By Proposition 6.2.4, the categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\chi(\Lambda)}(k_X, \mathbb{C}^\times q)$ and $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\chi(\Lambda')}(k_X, \mathbb{C}^\times q)$ are abelian subcategories of $\mathrm{Perv}(X)_{\pi(q)}/\mathcal{LC}_{\pi(q)}$. Hence β (and therefore α) is exact. □

PROPOSITION 6.2.6. — *The category* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ *is abelian. Moreover, for every germ of a* $\mathbb{C}^\times$*-conic Lagrangian variety* $\Lambda \subset T^*X$ *defined in a neighborhood of* $\mathbb{C}^\times p$*, the inclusion functor*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$$

is exact.

Proof. — We have the equivalence

$$2 \varinjlim_{\Lambda \supset \mathbb{C}^\times p} \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p).$$

Filtered 2-colimits of abelian categories with exact restriction functors are abelian. □

Now let us prove the following "lifting property" for kernels and cokernels of microlocal perverse sheaves:

PROPOSITION 6.2.7. — *Let* $\varphi : \mathcal{F} \to \mathcal{G}$ *be a morphism in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$*. Then there exists a neighborhood* V *of* $\mathbb{C}^\times p$*, objects* $\mathcal{K}, \mathcal{K}' \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, V)$ *and morphisms* $\mathcal{K} \to \mathcal{F}$*,* $\mathcal{G} \to \mathcal{K}'$ *in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, V)$ *such that these morphisms induce kernel and cokernel of* φ *in* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times q)$ *for all* $q \in V$*.*

Proof. — Choose a Lagrangian variety Λ such that $\mathrm{SS}(\mathcal{F}) \cup \mathrm{SS}(\mathcal{G}) \subset \Lambda$ in a neighborhood of $\mathbb{C}^\times p$. By the generic position theorem, we can find a contact transformation defined in a $\mathbb{C}^\times$-conic open neighborhood of $\mathbb{C}^\times p$

$$\chi : (T^*X, \mathbb{C}^\times p) \longrightarrow (T^*X, \mathbb{C}^\times p')$$

such that $\chi(\Lambda)$ is in generic position at p'. Then $\chi(\Lambda)$ is isomorphic to the conormal bundle to a closed hypersurface in a neighborhood of $\mathbb{C}^\times p'$. In particular, there exists a $\mathbb{C}^\times$-conic open neighborhood V' of $\chi(\Lambda)$ such that $\chi(\Lambda)$ is in generic position at any point of $\chi(\Lambda) \cap V'$.

Hence, by Theorem 6.1.5, we can assume that Λ is in generic position at any point in a $\mathbb{C}^\times$-conic neighborhood Ω of $\mathbb{C}^\times p$. Consider the functor

$$\Phi : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times p) \longrightarrow \mathrm{Perv}_{\pi(p)} / \mathcal{LC}_{\pi(p)}.$$

Note that $\mathrm{Perv}_{\pi(p)} / \mathcal{LC}_{\pi(p)} \simeq 2 \varinjlim_{U \ni x} \mathrm{Perv}(k_U)/\mathcal{LC}(U)$ (where $\mathcal{LC}(U)$ is the thick subcategory of local systems) and that we can find (U, γ) such that the morphism $\Phi(\mathcal{F}) \to \Phi(\mathcal{G})$ is defined as $\Phi_{U,\gamma}(\mathcal{F}) \to \Phi_{U,\gamma}(\mathcal{G})$ in $\mathrm{Perv}(V)/\mathcal{LC}(V)$ for some small neighborhood V of $\pi(p)$. Let $\mathcal{L} \to \Phi_{U,\gamma}(\mathcal{F})$ (resp. $\Phi_{U,\gamma}(\mathcal{G}) \to \mathcal{L}'$) be a kernel (resp. a cokernel) in $\mathrm{Perv}(V)/\mathcal{LC}(V)$. Choose a $\mathbb{C}^\times$-conic open neighborhood of $\mathbb{C}^\times p$ such that $\Omega' \subset \Omega \cap V \times \gamma$. We will show that $\mathcal{L} \to \Phi_{U,\gamma}(\mathcal{F})$ (resp. $\Phi_{U,\gamma}(\mathcal{G}) \to \mathcal{L}'$) is a kernel (resp. a cokernel) in $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times q)$ for all q in Ω'. We have distinguished triangles in $\mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(k_V)/\mathcal{LC}(V)$

$$\Phi_{U,\gamma}(\mathcal{F}) \longrightarrow \Phi_{U,\gamma}(\mathcal{G}) \longrightarrow \mathcal{H} \xrightarrow{+} \quad \text{and} \quad \mathcal{L}[1] \longrightarrow \mathcal{H} \longrightarrow \mathcal{L}' \xrightarrow{+}.$$

Now let q be a point of $\Lambda \cap \Omega'$. Since Λ is in generic position at q we get the functor

$$\Phi^q : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times q) \longrightarrow \mathrm{Perv}_{\pi(q)} / \mathcal{LC}_{\pi(q)}.$$

Applying Φ^q to the two triangles, we get distinguished triangles

$$\Phi^q \Phi_{U,\gamma}(\mathcal{F}) \to \Phi^q \Phi_{U,\gamma}(\mathcal{G}) \to \Phi^q(\mathcal{H}) \xrightarrow{+} \text{ and } \Phi^q \mathcal{L}[1] \to \Phi^q \mathcal{H} \to \Phi^q \mathcal{L}' \xrightarrow{+}.$$

Since $\Phi^q(\mathcal{L})$ and $\Phi^q(\mathcal{L}')$ are perverse in a neighborhood of $\pi(q)$ we get (by construction of kernels (resp. cokernels) in $\mathrm{Perv}_{\pi(q)} / \mathcal{LC}_{\pi(q)}$) that $\Phi^q(\mathcal{L}) \to \Phi^q \Phi_{U,\gamma}(\mathcal{F})$ (resp. $\Phi^q \Phi_{U,\gamma}(\mathcal{G}) \to \Phi^q(\mathcal{L}')$) is a kernel (resp. cokernel) in $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times q)$. Finally since Φ^q is a cut-off functor we have $\Phi^q(\mathcal{L}) \simeq \mathcal{L}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv},\Lambda}(k_X, \mathbb{C}^\times q)$. □

Theorem 6.2.8. — *The stack associated to Andronikof's prestack of microlocal perverse sheaves is abelian.*

Proof. — We have shown that the stalks of this additive prestack are abelian categories (Proposition 6.2.6). Further, we have shown that kernels and cokernels in the stalks may be lifted to small open neighborhoods (Proposition 6.2.7). Therefore the conditions of Proposition B.7.1 are satisfied, and the stack of microlocal perverse sheaves is abelian. □

7. Microlocal perverse sheaves on P^*X

7.1. Microlocal perverse sheaves. — We are now ready to give a first definition of the stack of microlocal perverse sheaves.

Definition 7.1.1. — The *stack of microlocal perverse sheaves* on P^*X is the stack associated to Andronikof's prestack.

By Theorem 6.2.8 we know that the stack of microlocal perverse sheaves is abelian. Furthermore, since the underlying prestack is invariant by quantized contact transformations (by Theorem 6.1.5) we easily get that the stack of microlocal perverse sheaves is invariant by quantized contact transformations.

7.2. Kashiwara's functor of ind-microlocalization. — We will give an explicit description of microlocal perverse sheaves in terms of ind-sheaves. For this purpose we will construct a subprestack

$$\mu\mathrm{Perv} \subset \gamma_*\big(\mathrm{D}^{\mathrm{b}}\big(\mathrm{I}(k_*)\big)_{|\dot{T}^*X}\big).$$

Here, $\gamma_*(\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_*))_{|\dot{T}^*X})$ is the prestack of bounded derived categories of ind-sheaves on $\mathbb{C}^\times$-conic open subsets of $\dot{T}^*X$. Then we will show that $\mu\mathrm{Perv}$ is actually a stack and construct a morphism $\mu : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *) \to \mu\mathrm{Perv}$ that induces equivalences in the stalks. Hence $\mu\mathrm{Perv}$ can be identified to the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *)$. In particular it is an abelian stack.

Recall that one denotes by $\operatorname{Ind}\mathcal{C}$ the full subcategory of $\widehat{\mathcal{C}}$ formed by small filtered colimits of representable objects and calls it the category of ind-objects of $\mathcal{C}$. Then $\operatorname{Ind}\mathcal{C}$ admits all small filtered colimits. If $\mathcal{C}$ is abelian then $\operatorname{Ind}\mathcal{C}$ is abelian and the Yoneda-functor induces an exact fully faithful functor $\mathcal{C} \to \operatorname{Ind}\mathcal{C}$.

Now let X be a locally compact topological space with a countable base of open sets and fix a field k. One sets (*cf.* [12])

$$\mathrm{I}(k_X) = \operatorname{Ind}\mathcal{M}od^c(k_X)$$

where $\mathcal{M}od^c(k_X)$ denotes the full subcategory of $\mathcal{M}od(k_X)$ formed by sheaves with compact support. We call $\mathrm{I}(k_X)$ the category of ind-sheaves (of k-vector spaces). One can show that the prestack $X \supset U \mapsto \mathrm{I}(k_U)$ is a proper stack, in particular it is an abelian stack. One identifies $\mathcal{M}od(k_X)$ with a full subcategory of $\mathrm{I}(k_X)$ by the fully faithful exact functor

$$\iota : \mathcal{M}od(k_X) \longrightarrow \mathrm{I}(k_X), \quad \mathcal{F} \longmapsto \text{“}\varinjlim_{U \subset\subset X}\text{”}\, \mathcal{F}_U.$$

In [9], Kashiwara establishes the following theorem

Theorem 7.2.1. — *There exists a functor*

$$\mu : \mathrm{D^b}\big(\mathrm{I}(k_X)\big) \longrightarrow \mathrm{D^b}\big(\mathrm{I}(k_{T^*X})\big)$$

such that for any $\mathcal{F}, \mathcal{G} \in \mathrm{D^b}(k_X)$ *we have a natural isomorphism*

$$\mathrm{R}\mathcal{H}om(\mu\mathcal{F}, \mu\mathcal{G}) \simeq \mathrm{R}\mathcal{H}om(\pi^{-1}\mathcal{F}, \mu\mathcal{G}) \simeq \mu\hom(\mathcal{F}, \mathcal{G}).$$

Remark 7.2.2. — Note that if $\mathcal{F} \in \mathrm{D^b}(k_X)$, then

$$\operatorname{supp}(\mu\mathcal{F}) = \operatorname{supp}\big(\mathrm{R}\mathcal{H}om(\mu\mathcal{F}, \mu\mathcal{F})\big) = \operatorname{supp}\big(\mu\hom(\mathcal{F}, \mathcal{F})\big) = \mathrm{SS}(\mathcal{F}).$$

The construction of μ and the proof of the theorem is rather straightforward using the machinery developed in [12] which we will not recall here. Let us state the definition of μ with the notations of *loc. cit.*

The normal deformation of the diagonal in $T^*X \times T^*X$ can be visualized by the following diagram

$$\begin{array}{ccccccc}
TT^*X & \xrightarrow{\ \sim\ } & T_{\Delta_{T^*X}}(T^*X \times T^*X) & \xhookrightarrow{\ s\ } & \widetilde{T^*X \times T^*X} & \xhookleftarrow{\ j\ } & \Omega \\
 & & \downarrow{\scriptstyle \tau_{T^*X}} & & {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \widetilde{p}} & \\
 & & T^*X & \xhookrightarrow{\ \Delta_{T^*X}\ } & T^*X \times T^*X & &
\end{array}$$

Note that $\widetilde{p}$ is smooth but p is not. Also, the square is not Cartesian. Set

$$\mathrm{K}_X = \mathrm{R}p_{!!}\big(k_{\overline{\Omega}} \otimes \beta(k_P)\big) \otimes \beta(\omega^{\otimes -1}_{\Delta_{T^*X}|T^*X \times T^*X})$$

where the set $P \subset TT^*X$ is defined by

$$P = \big\{(x, \xi; v_x, v_\xi) \mid \langle v_x, \xi\rangle \geqslant 0\big\}.$$

Definition 7.2.3. — Kashiwara's functor of ind-microlocalization is defined as

$$\mu : \mathrm{D^b}\big(\mathrm{I}(k_X)\big) \longrightarrow \mathrm{D^b}\big(\mathrm{I}(k_{T^*X})\big), \quad \mathcal{F} \longmapsto \mu\mathcal{F} = \mathrm{K}_X \circ \pi^{-1}\mathcal{F}.$$

Lemma 7.2.4. — *Let $S \subset T^*X$ be an arbitrary subset. Then μ defines functors*

$$\mu : \mathrm{D^b}(k_X, S) \longrightarrow \mathrm{D^b}\big(\mathrm{I}(k_S)\big).$$

If one considers these functors for open subsets $U \subset T^*X$, they define functors of prestacks.

Proof. — It is enough to show the existence of the first functor. If $\mu(\mathcal{F})_{|S} \simeq 0$, then $\mathrm{supp}(\mu(\mathcal{F})) \cap S = \varnothing$, hence $\mathrm{SS}(\mathcal{F}) \cap S = \varnothing$ and $\mu(.)_{|S}$ factors through $\mathrm{D^b}(k_X, S)$. □

7.3. The stack of microlocal perverse sheaves. — Kashiwara's functor of ind-microlocalization naturally leads to the following definition:

Definition 7.3.1. — Let $\Omega \subset P^*X$ be an open subset.

1) An object $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(k_{\gamma^{-1}\Omega}))$ is *microlocally perverse* (on Ω) if for all $p \in \gamma^{-1}(\Omega)$ there exist a $\mathbb{C}^\times$-conic neighborhood $V \supset \mathbb{C}^\times p$ and an object $\mathcal{G} \in \mathrm{D^b_{perv}}(k_X, V)$ such that $\mu\mathcal{G}_{|V} \simeq \mathcal{F}_{|V}$.

2) We denote by $\mu\mathrm{Perv}(\Omega)$ the full subcategory of $\mathrm{D^b}(\mathrm{I}(k_{\gamma^{-1}\Omega}))$ whose objects are microlocally perverse.

Remark 7.3.2. — The functor μ induces a functor in the stalks

$$\mu : \mathrm{D^b_{perv}}(k_X, \mathbb{C}^\times p) \longrightarrow \gamma_*\mathrm{D^b}\big(\mathrm{I}(k_*)\big)_{\gamma(p)}$$

and the definition of a microlocal perverse sheaf is clearly equivalent to

1′) An object $\mathcal{F} \in \mathrm{D^b}(\mathrm{I}(k_{\gamma^{-1}\Omega}))$ is microlocally perverse (on Ω) if for all $p \in \gamma^{-1}(\Omega)$ there exists an object $\mathcal{G} \in \mathrm{D^b_{perv}}(k_X, \mathbb{C}^\times p)$ such that $\mu\mathcal{G} \simeq \mathcal{F}$ in $\gamma_*\mathrm{D^b}(\mathrm{I}(k_*))_{\gamma(p)}$.

Here, one shall keep in mind that the natural functor

$$\gamma_*\mathrm{D^b}\big(\mathrm{I}(k_*)\big)_{\gamma(p)} \longrightarrow \mathrm{D^b}\big(\mathrm{I}(k_{\mathbb{C}^\times p})\big)$$

is not fully faithful. Therefore germs of microlocal perverse sheaves should not be interpreted as complexes of ind-sheaves on $\mathbb{C}^\times p$.

Remark 7.3.3. — Note that in order to formulate Definition 7.3.1 it is not necessary to construct the categories $\mathrm{D^b_{perv}}(k_X, S)$ for any $\mathbb{C}^\times$-conic subset $S \subset T^*X$ but only the categories $\mathrm{D^b_{perv}}(k_X, \mathbb{C}^\times p)$. However, the prestack $\mathrm{D^b_{perv}}(k_X, *)$ can sometimes be useful to define functors on microlocal perverse sheaves and therefore we decided to include it in our presentation.

Clearly $\mu\,\mathrm{Perv}(\Omega)$ is an additive subcategory of $\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{\gamma^{-1}(\Omega)}))$, and the correspondence

$$P^*X \supset \Omega \longmapsto \mu\,\mathrm{Perv}(\Omega)$$

defines an additive prestack on P^*X. Hence $\mu\,\mathrm{Perv}$ is a full additive subprestack of $\gamma_*\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_*))$, and we can see directly from the construction that it is defined by a local property. Moreover, by definition, for any open subset $\Omega \subset P^*X$ the functor μ induces a natural functor

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}\big(k_X, \gamma^{-1}(\Omega)\big) \longrightarrow \mu\,\mathrm{Perv}(\Omega).$$

These functors define a functor of prestacks

$$\mu : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *) \longrightarrow \mu\,\mathrm{Perv}\,. \tag{7.3.1}$$

We want to show that the functor (7.3.1) induces equivalences of categories in the stalks. Let $p \in \dot{T}^*X$. The definition of $\mu\,\mathrm{Perv}$ immediately implies that

$$\mu : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p) \longrightarrow \mu\,\mathrm{Perv}_{\gamma(p)}$$

is essentially surjective.

PROPOSITION 7.3.4. — (i) *Let* $\mathcal{F}, \mathcal{G} \in \mu\,\mathrm{Perv}(\Omega)$ *and* $p \in \gamma^{-1}(\Omega)$. *Let* $\widetilde{\mathcal{F}}$ *and* $\widetilde{\mathcal{G}}$ *be two objects of* $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$ *such that* $\mu\widetilde{\mathcal{F}} \simeq \mathcal{F}$ *and* $\mu\widetilde{\mathcal{G}} \simeq \mathcal{G}$ *in* $\mu\,\mathrm{Perv}_{\gamma(p)}$. *Then we have*

$$\mathrm{Hom}_{\mu\,\mathrm{Perv}_{\gamma(p)}}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{H}^0\big(\mathbb{C}^\times p, \mu\hom\big(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}\big)\big).$$

(ii) *Let* $\mathcal{F}, \mathcal{G} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)$. *Then*

$$\mathrm{Hom}_{\mu\,\mathrm{Perv}_{\gamma(p)}}(\mu\mathcal{F}, \mu\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p)}(\mathcal{F}, \mathcal{G}).$$

(iii) *The functor* μ *induces a canonical equivalence of categories*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \mathbb{C}^\times p) \xrightarrow{\ \sim\ } \mu\,\mathrm{Perv}_{\gamma(p)}\,.$$

Proof. — Note that the functor of (iii) is obviously essentially surjective. Hence (iii) follows from (ii). Moreover (ii) follows from (i) and Proposition 2.4.4. Let us prove (i). We have

$$\begin{aligned}
\mathrm{Hom}_{\mu\,\mathrm{Perv}_{\gamma(p)}}(\mathcal{F}, \mathcal{G}) &\simeq \varinjlim_{\gamma(p)\in V\subset P^*X} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{\gamma^{-1}(V)}))}\big(\mathcal{F}_{|\gamma^{-1}(V)}, \mathcal{G}_{|\gamma^{-1}(V)}\big)\\
&\simeq \varinjlim_{\gamma(p)\in V\subset P^*X} \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{\gamma^{-1}(V)}))}\big(\mu\widetilde{\mathcal{F}}_{|\gamma^{-1}(V)}, \mu\widetilde{\mathcal{G}}_{|\gamma^{-1}(V)}\big)\\
&\simeq \varinjlim_{\gamma(p)\in V\subset P^*X} \mathrm{H}^0\big(V, \mathrm{RHom}(\mu\widetilde{\mathcal{F}}, \mu\widetilde{\mathcal{G}})\big)\\
&\simeq \varinjlim_{\gamma(p)\in V\subset P^*X} \mathrm{H}^0\big(V, \mu\hom(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}})\big) \simeq \mathrm{H}^0\,\big(\mathbb{C}^\times p, \mu\hom(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}})\big).
\end{aligned}$$

□

Proposition 7.3.4 implies that if $\mu\mathrm{Perv}$ is a stack then it is equivalent to the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,*)$. In order to prove that $\mu\mathrm{Perv}$ is a stack we will apply the general results of Section 2.

Let us recall the following well-known proposition with a partial proof.

Proposition 7.3.5. — *Let $\mathcal{F},\mathcal{G}$ be two perverse sheaves on X. Then the sheaf $\mu\mathrm{hom}(\mathcal{F},\mathcal{G})[d_X]$ is perverse and $\mu\mathrm{hom}(\mathcal{F},\mathcal{G})$ is concentrated in positive degrees.*

Proof. — According to [11, Cor. 10.3.20], $\mu\mathrm{hom}(\mathcal{F},\mathcal{G})[d_X]$ is a perverse sheaf on T^*X. Hence for any complex analytic subset S of T^*X we have

$$\mathrm{H}^{j+d_X}_S\big(\mu\mathrm{hom}(\mathcal{F},\mathcal{G})\big)_{|S}\simeq 0$$

if $j<-\dim S$. Now recall that $\mathrm{supp}(\mu\mathrm{hom}(\mathcal{F},\mathcal{G}))\subset \mathrm{SS}(\mathcal{F})\cap\mathrm{SS}(\mathcal{G})$. Since $\mathcal{F},\mathcal{G}$ are perverse sheaves their micro-supports are Lagrangian subsets of T^*X, hence are of dimension d_X. Therefore $\mathrm{H}^j(\mu\mathrm{hom}(\mathcal{F},\mathcal{G}))\simeq 0$ for $j<0$. □

Proposition 7.3.6. — *The prestack $\mu\mathrm{Perv}$ of microlocal perverse sheaves is separated.*

Proof. — Let $\mathcal{F},\mathcal{G}$ be two microlocal perverse sheaves of $\mu\mathrm{Perv}(\Omega)$. Recall that $\mathcal{F},\mathcal{G}\in\mathrm{D}^{\mathrm{b}}(\mathrm{I}(k_{\gamma^{-1}(\Omega)}))$ and that we have by definition

$$\mathcal{H}om_{\mu\mathrm{Perv}}(\mathcal{F},\mathcal{G})\simeq\gamma_*\mathcal{H}om_{\mathrm{D^b(I}(k_*))}(\mathcal{F},\mathcal{G}).$$

Hence it is sufficient to prove that $\mathcal{H}om_{\mathrm{D^b(I}(k_*))}(\mathcal{F},\mathcal{G})$ is a sheaf.

We will first show that the complex $\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})$ is concentrated in positive degrees. This is a local question, hence we may assume that $\mathcal{F}\simeq\mu\widetilde{\mathcal{F}}$, $\mathcal{G}\simeq\mu\widetilde{\mathcal{G}}$ for two objects $\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}$ of $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,\gamma^{-1}(\Omega))$. Since

$$\mathrm{R}\mathcal{H}om(\mu(\widetilde{\mathcal{F}}),\mu(\widetilde{\mathcal{G}}))\simeq\mu\mathrm{hom}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})$$

we are reduced to study $\mu\mathrm{hom}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})_p$ for any $p\in\gamma^{-1}(\Omega)$. By invariance of quantized contact transformation (*cf.* Theorem 6.1.5) we may assume that $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ are perverse sheaves on a neighborhood of $\pi(p)$. Hence $\mu\mathrm{hom}(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}})_p$ is concentrated in positive degrees by the last proposition.

Therefore $\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})$ is concentrated in positive degrees. Then the presheaf $\mathcal{H}om_{\mathrm{D^b(I}(k_*))}(\mathcal{F},\mathcal{G})$ is a sheaf since

$$\mathcal{H}om_{\mathrm{D^b(I}(k_*))}(\mathcal{F},\mathcal{G})(V)\simeq\mathrm{H}^0\big(V,\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})\big)\simeq\Gamma\big(V,\mathrm{H}^0\,\mathrm{R}\mathcal{H}om(\mathcal{F},\mathcal{G})\big).\quad\square$$

Theorem 7.3.7. — *The prestack $\mu\mathrm{Perv}$ on P^*X is an abelian stack. Moreover the functor μ induces an equivalence of abelian stacks*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,*)^{\ddagger}\xrightarrow{\sim}\mu\mathrm{Perv},$$

where $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,*)^{\ddagger}$ denotes the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X,*)$.

Proof. — Since microlocal perverse sheaves form a separated subprestack of the prestack of ind-sheaves and are obviously defined by a local property, they form a stack. Moreover, Proposition 7.3.4 states that the functor of prestacks

$$\mu : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *) \longrightarrow \mu\,\mathrm{Perv}$$

induces equivalences of categories in the stalks. Hence μ identifies $\mu\,\mathrm{Perv}$ with the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *)$. □

7.4. Autoduality. — The stack of microlocal perverse sheaves is autodual, *i.e.*, it is equivalent to its opposite stack.

Proposition 7.4.1. — *Let* $\mathcal{F}, \mathcal{G} \in \mu\,\mathrm{Perv}(\Omega)$. *Then* $\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G})[d_X]$ *is a perverse sheaf on* $\gamma^{-1}(\Omega)$.

Proof. — Locally we can find $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}} \in \mathrm{D}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(k_X)$ such that

$$\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mu\hom(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}).$$

By invariance of quantized contact transformations we may assume that $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}$ are perverse sheaves. Then the result follows from the fact that $\mu\hom(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}})[d_X]$ is a perverse sheaf (see [11, Cor. 10.3.20]). □

Proposition 7.4.2. — *The stack* $\mu\,\mathrm{Perv}$ *is autodual, i.e., it is equivalent to its opposite stack. More precisely, there exists a contravariant functor of stacks*

$$D : \mu\,\mathrm{Perv} \longrightarrow \mu\,\mathrm{Perv}$$

such that

(i) $D \circ D \simeq \mathrm{Id}$;

(ii) $\mathrm{R}\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq \mathrm{R}\mathcal{H}om(D\mathcal{G}, D\mathcal{F})$;

(iii) *if* $\mathcal{F} \in \mu\,\mathrm{Perv}(\Omega)$ *is isomorphic to* $\mu\widetilde{\mathcal{F}}$ *and* $\widetilde{\mathcal{F}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \gamma^{-1}(\Omega))$ *then we have a natural isomorphism* $D\mathcal{F} \simeq \mu\,\mathrm{R}\mathcal{H}om(\widetilde{\mathcal{F}}, \omega_X)$.

Proof. — Recall that the functor $\mathrm{D} = \mathrm{R}\mathcal{H}om(\,.\,, \omega_X)$ induces a contravariant equivalence of prestacks

$$\mathrm{D} : \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *).$$

Hence we get a contravariant equivalence D on the stack associated to $\mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, *)$ which satisfies by definition (i) and (iii).

Let $\mathcal{F}, \mathcal{G} \in \mu\,\mathrm{Perv}(\Omega)$. It is enough to prove (ii) locally. Hence we may assume that there are objects $\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{perv}}(k_X, \gamma^{-1}\Omega)$ such that $\mu\widetilde{\mathcal{F}} \simeq \mathcal{F}$ and $\mu\widetilde{\mathcal{G}} \simeq \mathcal{G}$. Recall (*cf.* [11, Exercise IV.4]) that on $\gamma^{-1}\Omega$ we have

$$\mu\hom(\widetilde{\mathcal{F}}, \widetilde{\mathcal{G}}) \simeq \mu\hom(\mathrm{D}\widetilde{\mathcal{G}}, \mathrm{D}\widetilde{\mathcal{F}}).$$

Then on $\gamma^{-1}\Omega$ we get

$$\begin{aligned}\mathrm{R}\mathcal{H}om(\mu\widetilde{\mathcal{F}},\mu\widetilde{\mathcal{G}}) &\simeq \mu\hom(\widetilde{\mathcal{F}},\widetilde{\mathcal{G}}) \simeq \mu\hom(\mathrm{D}\widetilde{\mathcal{G}},\mathrm{D}\widetilde{\mathcal{F}})\\ &\simeq \mathrm{R}\mathcal{H}om(\mu\mathrm{D}\widetilde{\mathcal{G}},\mu\mathrm{D}\widetilde{\mathcal{F}}) \simeq \mathrm{R}\mathcal{H}om(\mathrm{D}\mathcal{G},\mathrm{D}\mathcal{F}).\end{aligned}$$ □

7.5. Microlocal Riemann-Hilbert correspondence. — In this section we formulate the microlocal Riemann-Hilbert correspondance as a pair of qusi-inverse functors between the stack of microlocal perverse sheaves and the stack of regular holonomic microdifferential operators. The proof involves deeper results on Kashiwara's functor μ and is postponed to a forthcoming paper.

Let $\mathcal{D}_X$ be the ring of holomorphic differential operators, $\mathcal{E}_X$ be the ring of micro-differential operators of [14] and $\mathcal{O}^t \in \mathrm{D^b}(\mathrm{I}(k_X))$ the "ring" of tempered holomorphic functions. Recall that following Kashiwara [7] the classical Riemann-Hilbert correspondence can be translated (using [12]) as

Theorem 7.5.1. — *The functors*

$$\mathrm{Perv}(\mathbb{C}_X) \underset{\mathrm{R}\,\mathcal{H}om_{\mathcal{D}_X}(\,\cdot\,,\mathcal{O}_X)[d_X]}{\overset{\mathrm{R}\,\mathcal{H}om(\,\cdot\,,\mathcal{O}^t)[-d_X]}{\rightleftarrows}} \mathcal{H}ol\mathcal{R}eg(\mathcal{D}_X).$$

define quasi-inverse equivalences of stacks.

Recall from [12] that $\mathrm{R}\mathcal{I}\mathcal{H}om$ denotes the derived functor of the internal Hom-functor of ind-sheaves and $\beta : \mathrm{D^b}(k_X) \to \mathrm{D^b}(\mathrm{I}(k_X)$ the fully faithful left adjoint functor of $\alpha : \mathrm{D^b}(\mathrm{I}(k_X)) \to \mathrm{D^b}(k_X)$, where $\alpha(\text{“}\varinjlim\text{”}F_i) = \varinjlim F_i$. The microlocal Riemann-Hilbert theorem can then be formulated as

Theorem 7.5.2. — *The functors*

$$\mu\,\mathrm{Perv} \underset{\mathrm{R}\,\mathcal{I}\mathcal{H}om_{\beta\mathcal{E}_X}(\beta(\,\cdot\,),\mu\mathcal{O}_X)\,[d_X]}{\overset{\gamma^{-1}\,\mathrm{R}\,\gamma_*\,\mathrm{R}\,\mathcal{H}om(\,\cdot\,,\mu\mathcal{O}^t)[-d_X]}{\rightleftarrows}} \mathcal{H}ol\mathcal{R}eg(\mathcal{E}_X).$$

define quasi-inverse equivalences of stacks.

Appendix A
2-colimits and 2-limits in $\mathcal{CAT}$

In this appendix we work in $\mathcal{CAT}$, the 2-category of all small categories. However this is just a precaution to avoid set-theoretical problems and we will actually apply our results to big categories.

We do not recall the concept of a (strict) 2-category here (see [13] for example), but all definitions given below generalize easily to the context of arbitrary

2-categories. Since we are only interested in 2-colimits and 2-limits in the category of all (small) categories we restrict ourselves to this case which sometimes simplifies the notations.

We call a morphism of functors a natural transformation and an isomorphism of functors a natural equivalence. Note that there are two compositions of natural transformations. Following [13] we will denote vertical composition by '$\circ$' and horizontal composition by the symbol '$\bullet$'.

If $\mathcal{I}$ is a (small) category for indexing direct or inverse systems, we will use the notation $I = \operatorname{Ob}\mathcal{I}$.

A.1. 2-functors, 2-natural transformations and modifications

We recall some definitions from the theory of 2-categories applied to $\mathcal{CAT}$ mainly to fix our notations. The concepts are classical, however the notations and terminology vary considerably in the literature (for instance, a 2-functor is often called a pseudo-functor). A standard reference is [15].

Definition A.1.1. — Let $\mathcal{I}$ be a category. A *2-functor* $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ is given by the following data:

1) a category $\mathfrak{a}(i)$ of $\operatorname{Ob}\mathcal{CAT}$ for any $i \in I$;

2) a functor $\mathfrak{a}(s) : \mathfrak{a}(i) \to \mathfrak{a}(j)$ of $\operatorname{Mor}\mathcal{CAT}$ for any morphism $s : i \to j$ of $\operatorname{Mor}\mathcal{I}$;

3) a natural equivalence $\Phi(i) : \mathfrak{a}(\mathrm{id}_i) \xrightarrow{\sim} \mathrm{Id}_{\mathfrak{a}(i)}$ for any $i \in I$;

4) and a natural equivalence $\Phi(s,t) : \mathfrak{a}(t \circ s) \xrightarrow{\sim} \mathfrak{a}(t) \circ \mathfrak{a}(s)$ for any two composable morphisms $s, t \in \operatorname{Mor}\mathcal{I}$;

these data satisfying the following axioms:

(2F1) $(\mathfrak{a}(s) \bullet \Phi(i)) \circ \Phi(\mathrm{id}_i, s) = \mathrm{Id}_{\mathfrak{a}(s)}$ and $(\Phi(j) \bullet \mathfrak{a}(s)) \circ \Phi(s, \mathrm{id}_j) = \mathrm{Id}_{\mathfrak{a}(s)}$ for all morphisms $s : i \to j$ of $\operatorname{Mor}\mathcal{I}$ as visualized by

$$\begin{array}{ccc}
\mathfrak{a}(s) = \mathfrak{a}(\mathrm{id}_j \circ s) = \mathfrak{a}(s \circ \mathrm{id}_i) & \xrightarrow[\sim]{\Phi(\mathrm{id}_i, s)} & \mathfrak{a}(s)\mathfrak{a}(\mathrm{id}_i) \\
\Phi(s, \mathrm{id}_j) \big\downarrow \wr & \searrow^{\mathrm{id}_{\mathfrak{a}(s)}} & \wr \big\downarrow \mathfrak{a}(s) \bullet \Phi(i) \\
\mathfrak{a}(\mathrm{id}_j)\mathfrak{a}(s) & \xrightarrow[\Phi(j) \bullet \mathfrak{a}(s)]{\sim} & \mathfrak{a}(s).
\end{array}$$

(2F2) For any three composable morphisms $s : i \to j$, $t : j \to k$, $u : k \to \ell$ of $\operatorname{Mor}\mathcal{I}$ we have

$$\big(\mathfrak{a}(u) \bullet \Phi(s,t)\big) \circ \Phi(t \circ s, u) = \big(\Phi(t,u) \bullet \mathfrak{a}(s)\big) \circ \Phi(s, u \circ t),$$

as visualized by the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{a}(u\circ t\circ s) & \xrightarrow[\sim]{\Phi(t\circ s,u)} & \mathfrak{a}(u)\mathfrak{a}(t\circ s) \\
{\scriptstyle \Phi(s,u\circ t)}\Big\downarrow\wr & & \wr\Big\downarrow{\scriptstyle \mathfrak{a}(u)\bullet\Phi(s,t)} \\
\mathfrak{a}(u\circ t)\mathfrak{a}(s) & \xrightarrow[\sim]{\Phi(t,u)\bullet\mathfrak{a}(s)} & \mathfrak{a}(u)\mathfrak{a}(t)\mathfrak{a}(s).
\end{array}$$

A 2-*functor with strict identity* is a 2-functor such that $\Phi(i)$ is the identity for all $i\in I$.

A *strict* 2-*functor* is a 2-functor such that $\Phi(i)$ and $\Phi(s,t)$ are identities for all $i\in I$ and all composable $s,t\in\operatorname{Mor}\mathcal{I}$. Hence it is just a functor from $\mathcal{I}$ to the underlying 1-category of $\mathcal{CAT}$.

Next we need the notion of a morphism of 2-functors (which we will call a 2-natural transformation) and that of morphisms of such morphisms (which are called modifications).

Definition A.1.2. — Let $\mathfrak{a},\mathfrak{b}:\mathcal{I}\to\mathcal{CAT}$ be two 2-functors. A 2-*natural transformation* of 2-functors $\mathfrak{f}:\mathfrak{a}\to\mathfrak{b}$ consists of the following data:

1) a functor $\mathfrak{f}_i:\mathfrak{a}(i)\to\mathfrak{b}(i)$ of $\operatorname{Mor}\mathcal{CAT}$ for all $i\in I$;

2) a natural equivalence $\Theta^{\mathfrak{f}}_s:\mathfrak{b}(s)\mathfrak{f}_i\xrightarrow{\sim}\mathfrak{f}_j\mathfrak{a}(s)$ for any morphism $s:i\to j$ of $\operatorname{Mor}\mathcal{I}$

such that

(2NT1) for any $i\in I$ we have the equation

$$\big(\mathfrak{f}_i\bullet\Phi^{\mathfrak{a}}(i)\big)\circ\Theta^{\mathfrak{f}}_{\mathrm{id}_i}=\Phi^{\mathfrak{b}}(i)\bullet\mathfrak{f}_i,$$

visualized by

$$\begin{array}{ccc}
\mathfrak{b}(id_i)\mathfrak{f}_i & \xrightarrow[\sim]{\Theta^{\mathfrak{f}}_{id_i}} & \mathfrak{f}_i\mathfrak{a}(id_i) \\
{\scriptstyle \Phi^{\mathfrak{b}}(i)\bullet\mathfrak{f}_i}\ \searrow\sim & & \sim\swarrow\ {\scriptstyle \mathfrak{f}_i\bullet\Phi^{\mathfrak{a}}(i)} \\
 & \mathfrak{f}_i &
\end{array}$$

(2NT2) for any two composable morphisms $s:i\to j$, $t:j\to k$ of $\operatorname{Mor}\mathcal{I}$ we have

$$\big(\Theta^{\mathfrak{f}}_t\bullet\mathfrak{a}(s)\big)\circ\big(\mathfrak{b}(t)\bullet\Theta^{\mathfrak{f}}_s\big)\circ\big(\Phi^{\mathfrak{b}}(s,t)\bullet\mathfrak{f}_i\big)=\big(\mathfrak{f}_k\bullet\Phi^{\mathfrak{a}}(s,t)\big)\circ\Theta^{\mathfrak{f}}_{t\circ s},$$

as visualized by

$$
\begin{array}{ccc}
\mathfrak{b}(t\circ s)\mathfrak{f}_i & \xrightarrow[\sim]{\Phi^{\mathfrak{b}}(s,t)\bullet\mathfrak{f}_i} & \mathfrak{b}(t)\mathfrak{b}(s)\mathfrak{f}_i \\
 & & \wr\big\downarrow \mathfrak{b}(t)\bullet\Theta^{\mathfrak{f}}_s \\
\Theta^{\mathfrak{f}}_{t\circ s}\big\downarrow\wr & & \mathfrak{b}(t)\mathfrak{f}_j\mathfrak{a}(s) \\
 & & \wr\big\downarrow \Theta^{\mathfrak{f}}_t\bullet\mathfrak{a}(s) \\
\mathfrak{f}_k\mathfrak{a}(t\circ s) & \xrightarrow[\mathfrak{f}_k\bullet\Phi^{\mathfrak{a}}(s,t)]{\sim} & \mathfrak{f}_k\mathfrak{a}(t)\mathfrak{a}(s).
\end{array}
$$

A strict 2-natural transformation is a 2-natural transformation such that all $\Theta^{\mathfrak{f}}_s$ are identity transformations.

NOTATION A.1.3. — If $\mathfrak{f}:\mathfrak{a}\to\mathfrak{a}'$ and $\mathfrak{g}:\mathfrak{a}'\to\mathfrak{a}''$ are 2-natural transformations, there is an obvious way to define their composition $\mathfrak{g}\mathfrak{f}:\mathfrak{a}\to\mathfrak{a}''$.

This composition is clearly associative, admits an identity and preserves strict 2-natural transformations, hence gives rise to two categories:

(i) $2\mathcal{F}(\mathcal{I},\mathcal{CAT})$, the category of 2-functors with 2-natural transformations as morphisms;

(ii) $\mathcal{S}2\mathcal{F}(\mathcal{I},\mathcal{CAT})$, the category of strict 2-functors with strict 2-natural transformations as morphisms.

We will be primarily interested in the category $2\mathcal{F}(\mathcal{I},\mathcal{CAT})$ which we will denote simply by $2\mathcal{F}$ if the index-category $\mathcal{I}$ is fixed.

An isomorphism in $2\mathcal{F}$ will sometimes be called a 2-natural equivalence, an isomorphism of $\mathcal{S}2\mathcal{F}$ a strict 2-natural equivalence. Although this terminology would be parallel to the notations of $\mathcal{CAT}$ it might lead to confusion with the definition of equivalent 2-functors later. Hence note that two 2-functors $\mathfrak{a},\mathfrak{a}':\mathcal{I}\to\mathcal{CAT}$ are called isomorphic if there exists a 2-natural equivalence between them and not equivalent. Similarly to the situation in category theory that we have few isomorphisms of categories there are not many 2-natural equivalences as shown by the following lemma whose proof is straightforward.

LEMMA A.1.4. — *Let $\mathfrak{f}:\mathfrak{a}\to\mathfrak{b}$ be a 2-natural transformation. Then $\mathfrak{f}$ is a 2-natural equivalence if and only if $\mathfrak{f}_i:\mathfrak{a}(i)\to\mathfrak{b}(i)$ is an isomorphism of categories for all $i\in I$.*

DEFINITION A.1.5. — Let $\mathfrak{a},\mathfrak{a}':\mathcal{I}\to\mathcal{C}$ be two 2-functors and $\mathfrak{f},\mathfrak{g}:\mathfrak{a}\to\mathfrak{a}'$ be two 2-natural transformations. A *modification* (of 2-natural transformations of 2-functors) $\Lambda:\mathfrak{f}\to\mathfrak{g}$ consists of

1) a natural transformation $\Lambda_i:\mathfrak{f}_i\longrightarrow\mathfrak{g}_i$ for any object $i\in I$ such that

(M) for any morphism $s : i \to j$ in $\mathcal{I}$ we have the following equation

$$\big(\Lambda_j \bullet \mathfrak{a}(s)\big) \circ \Theta^{\mathfrak{f}}_s = \Theta^{\mathfrak{g}}_s \circ \big(\mathfrak{a}'(s) \bullet \Lambda_i\big),$$

visualized by the diagram

$$\begin{array}{ccc} \mathfrak{a}'(s)\mathfrak{f}_i & \xrightarrow[\sim]{\Theta^{\mathfrak{f}}_s} & \mathfrak{f}_j\mathfrak{a}(s) \\ {\scriptstyle \mathfrak{a}'(s)\bullet\Lambda_i}\downarrow & & \downarrow{\scriptstyle \Lambda_j\bullet\mathfrak{a}(s)} \\ \mathfrak{a}'(s)\mathfrak{g}_i & \xrightarrow[\sim]{\Theta^{\mathfrak{g}}_s} & \mathfrak{g}_j\mathfrak{a}(s). \end{array}$$

Remark A.1.6. — One can easily define vertical and horizontal composition of modifications and a straightforward verification shows that 2-functors, 2-natural transformations and modifications define a strict 2-category. In particular we get the category of 2-natural transformations between two 2-functors and therefore the notion of an isomorphism of 2-natural transformations.

Definition A.1.7. — Two 2-functors $\mathfrak{a}, \mathfrak{b} : \mathcal{I} \to \mathcal{CAT}$ are called *equivalent* if there are 2-natural transformations $\mathfrak{f} : \mathfrak{a} \to \mathfrak{b}$ and $\mathfrak{g} : \mathfrak{b} \to \mathfrak{a}$ such that the compositions $\mathfrak{g}\mathfrak{f}$ and $\mathfrak{f}\mathfrak{g}$ are isomorphic to the identity.

Equivalent 2-functors are essentially the same in the framework of 2-categories. Hence the following lemma shows that we can always work with 2-functors with strict identity.

Lemma A.1.8. — *Every 2-functor is equivalent to a 2-functor with strict identity.*

A.2. 2-colimits. — It is well known that the category of all small categories admits all small limits and colimits, essentially because this is true for the category of sets. However these objects are not of much practical use since we rarely encounter direct or inverse systems in $\mathcal{CAT}$. The reason is that we mostly work up to equivalence of categories and not up to isomorphism and therefore we will get direct (resp. inverse) systems up to equivalence. The universal objects associated to such direct (resp. inverse) systems up to equivalence are called 2-colimits (resp. 2-limits) which we now introduce in detail. Although the (usually implicit) use of 2-limits and 2-colimits is widespread, there does not seem to be a standard reference as [13] for category theory.

In the sequel $\mathcal{I}$ will denote a small category and $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ a 2-functor which is nothing but a direct system up to equivalence. A 2-colimit of $\mathfrak{a}$ should be an object of $\mathcal{CAT}$ unique up to essentially unique equivalence of categories that factors (up to natural equivalence) the following type of data:

(i) for any $i \in I$ a functor $\rho_i : \mathfrak{a}(i) \to \mathcal{C}$ to some fixed category $\mathcal{C}$ and

(ii) for any morphism $s : i \to j$ of $\operatorname{Mor}\mathcal{I}$ a natural equivalence $\Theta^\rho_s : \rho_i \xrightarrow{\sim} \rho_j\mathfrak{a}(s)$, visualized by

$$\begin{array}{ccc} \mathfrak{a}(i) & & \\ & \searrow^{\rho_i} & \\ \mathfrak{a}(s)\downarrow & \sim\Swarrow \Theta^\rho_s & \mathcal{C} \\ & \nearrow_{\rho_j} & \\ \mathfrak{a}(j) & & \end{array}$$

Of course this data should satisfy a certain compatibility condition with the 2-functor $\mathfrak{a}$:

(A) $(\rho_i \bullet \Phi^{\mathfrak{a}}_{\mathrm{id}_i}) \circ \Theta^\rho_{\mathrm{id}_i} = \mathrm{Id}_{\rho_i}$ visualized by

$$\begin{array}{ccc} \rho_i & \xrightarrow[\sim]{\Theta^\rho_{\mathrm{id}_i}} & \rho_i\mathfrak{a}(\mathrm{id}_i) \\ & {}_{\mathrm{Id}_{\rho_i}}\searrow \quad \swarrow^{\sim}_{\rho_i \bullet \Phi^{\mathfrak{a}}_{\mathrm{id}_i}} & \\ & \rho_i & \end{array}$$

(B) for any two composable morphisms $s : i \to j$ and $t : j \to k$ of $\operatorname{Mor}\mathcal{I}$ the equation

$$\big(\rho_k \bullet \Phi(s,t)\big) \circ \Theta^\rho_{t\circ s} = \big(\Theta^\rho_t \bullet \alpha(s)\big) \circ \Theta^\rho_s$$

should hold. This may be visualized by the following commutative diagram:

$$\begin{array}{ccc} \rho_i & \xrightarrow[\sim]{\Theta^\rho_s} & \rho_j\mathfrak{a}(s) \\ {\scriptstyle\Theta^\rho_{t\circ s}}\downarrow\wr & & \wr\downarrow{\scriptstyle\Theta^\rho_t \bullet \mathfrak{a}(s)} \\ \rho_k\mathfrak{a}(t\circ s) & \xrightarrow[\sim]{\rho_k \bullet \Phi(s,t)} & \rho_k\mathfrak{a}(t)\mathfrak{a}(s). \end{array}$$

Note that for any category $\mathcal{C}$ we have a constant (strict) 2-functor, denoted by $\mathcal{C} : \mathcal{I} \to \mathcal{CAT}$ that sends every $i \in I$ to $\mathcal{C}$ and every morphism of $\mathcal{I}$ to the identity of $\mathcal{C}$. The key remark to the definition of direct 2-colimits then is that the data above just describes a 2-natural transformation $\mathfrak{a} \to \mathcal{C}$.

Definition A.2.1. — Let $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ be a 2-functor. The system $\mathfrak{a}$ admits a 2-colimit if and only if there exists

1) a category $2\varinjlim_{i\in I}\mathfrak{a}(i)$ and

2) a 2-natural transformation $\sigma : \mathfrak{a} \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$,

such that for any category $\mathcal{C}$ the functor

$$(\circ\,\sigma) : \underline{\mathrm{Hom}}_{\mathcal{CAT}}\big(2\varinjlim_{i\in I}\mathfrak{a}(i), \mathcal{C}\big) \longrightarrow \underline{\mathrm{Hom}}_{2\mathcal{F}}(\mathfrak{a}, \mathcal{C})$$

is an equivalence of categories.

We say that a 2-colimit has the *strong factorisation property* if $(\circ\,\sigma)$ is an isomorphism of categories.

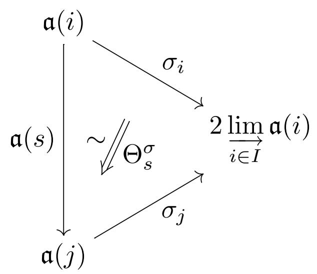
Let us look more in detail at the definition of 2-colimits.

First consider a 2-functor $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$. Then $\mathfrak{a}$ has a 2-colimit in $\mathcal{CAT}$ if and only if there exists

1) a category $2\varinjlim_{i\in I}\mathfrak{a}(i)$;

2) functors $\sigma_i : \mathfrak{a}(i) \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ for any $i \in I$;

3) and a natural equivalence $\Theta^\sigma_s : \sigma_i \xrightarrow{\sim} \sigma_j\mathfrak{a}(s)$ for any morphism $s : i \to j$ of $\mathrm{Mor}\,\mathcal{I}$ visualized by

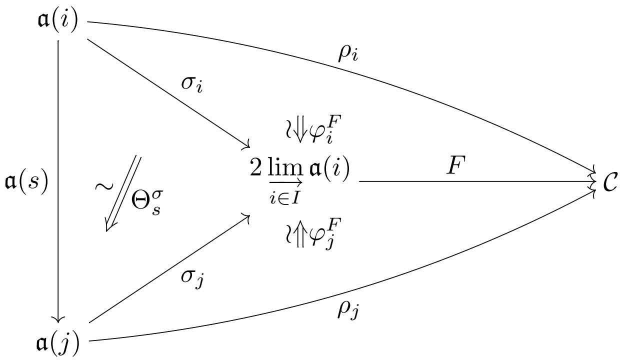


such that $\sigma : \mathfrak{a} \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ is a 2-natural transformation (*i.e.*, satisfies the axioms (2NT1) and (2NT2) of Definition A.1.2). This data should satisfy the following 2-universal property (here (2CL1) translates the fact that $(\circ\sigma)$ is essentially surjective and (2CL2) is just the condition that $(\circ\sigma)$ is fully faithful):

(2CL1) For any category $\mathcal{C}$, any 2-natural transformation $\rho : \mathfrak{a} \to \mathcal{C}$ (*i.e.*, the type of data described in the beginning of this section) there exists a functor

$$F : 2\varinjlim_{i\in I}\mathfrak{a}(i) \longrightarrow \mathcal{C}$$

and an isomorphism $\varphi^F : \rho \to F\sigma$, which is a modification given by a natural equivalence $\varphi_i^F : \rho_i \xrightarrow{\sim} F\sigma_i$ for every $i \in I$. This may be visualized by



The compatibility condition is given by

$$\big(F \bullet \Theta_s^\sigma\big) \circ \varphi_i^F = \big(\varphi_j^F \bullet \mathfrak{a}(s)\big) \circ \Theta_s^\rho,$$

which means that the following diagram commutes

$$\begin{array}{ccc} \rho_i & \xrightarrow[\sim]{\varphi_i^F} & F\sigma_i \\ \Theta_s^\rho \big\downarrow \wr & & \wr \big\downarrow F \bullet \Theta_s^\sigma \\ \rho_j \mathfrak{a}(s) & \xrightarrow[\sim]{\varphi_j^F \bullet \mathfrak{a}(s)} & F\sigma_j \mathfrak{a}(s). \end{array}$$

The pair (F, φ^F) is called a lax factorization (or factorization up to natural equivalence) of the system ρ.

If the 2-colimit has the strong factorization property then there exists a unique factorization such that φ^F is the identity.

(2CL2) Let $\rho, \rho' : \mathfrak{a} \to \mathcal{C}$ be two 2-natural transformations and $\lambda : \rho \to \rho'$ a modification. Then for any lax factorization $F : 2\varinjlim_{i\in I} \mathfrak{a}(i) \to \mathcal{C}$ of ρ and $G : 2\varinjlim_{i\in I} \mathfrak{a}(i) \to \mathcal{C}$ of ρ' there exists a unique natural transformation $\Lambda : F \to G$ such that

$$\varphi_i^G \circ \lambda_i = \big(\Lambda_i \bullet \sigma_i\big) \circ \varphi_i^F,$$

visualized by

$$\begin{array}{ccc} \rho_i & \xrightarrow{\lambda_i} & \rho_i' \\ \varphi_i^F \big\downarrow \wr & & \wr \big\downarrow \varphi_i^G \\ F\sigma_i & \xrightarrow{\Lambda_i \bullet \sigma_i} & G\sigma_i. \end{array}$$

Equivalently we can state that every modification $\Xi : F\sigma \to G\sigma$ can be written as $\lambda \bullet \sigma$ for a uniquely determined natural transformation $\lambda : F \to G$. In particular this gives a precise meaning to the terminology unique up to essentially unique equivalence.

A.3. Existence of 2-colimits in $\mathcal{CAT}$. — It is well known that all small 2-colimits exist in $\mathcal{CAT}$. We will recall the construction here and look more closely to the case of filtered 2-colimits.

We will start with Grothendieck's construction of the category $\int_{\mathcal{I}}\mathfrak{a}$ for any 2-functor $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ which are used to construct homotopy colimits in $\mathcal{CAT}$ *cf.* [16]). In our terminology $\int_{\mathcal{I}}\mathfrak{a}$ would be a "lax" 2-colimit of $\mathfrak{a}$ (satisfying the strong factorization property), that factors families of functors described above as ρ which are compatible up to natural transformation (and not necessarily up to natural equivalence).

Then the 2-colimit is obtained by a localization of the category $\int_{\mathcal{I}}\mathfrak{a}$. In the case that $\mathcal{I}$ is filtered this localization is given by a calculus of fractions and the description of the 2-colimit is particularly simple.

Notation A.3.1. — Let us recall the explicit construction of the small category $\int_{\mathcal{I}}\mathfrak{a}$. We set

$$\operatorname{Ob}\Big(\int_{\mathcal{I}}\mathfrak{a}\Big) = \big\{(i,X) \mid i \in I \text{ and } X \in \operatorname{Ob}(\mathfrak{a}(i))\big\}.$$

In other words the set of objects is just the disjoint union of the sets of objects of the direct system. For any two objects $(i,X),(j,Y) \in \operatorname{Ob}\big(\int_{\mathcal{I}}\mathfrak{a}\big)$ we set

$$\begin{aligned}&\operatorname{Hom}_{\int_{\mathcal{I}}\mathfrak{a}}\big((i,X),(j,Y)\big)\\ &\qquad = \big\{(s,f) \mid (s : i \to j) \in \operatorname{Mor}\mathcal{I},\ (f : \mathfrak{a}(s)(X) \to Y) \in \operatorname{Mor}\mathfrak{a}(j)\big\}.\end{aligned}$$

Composition of two morphisms $(s,f) : (i,X) \to (j,Y)$, $(t,g) : (j,Y) \to (k,Z)$ is defined such that the following diagram is commutative

$$\begin{array}{ccc} \mathfrak{a}(t\circ s)(X) & \longrightarrow & Z \\ {\scriptstyle \Phi(s,t)_X}\big\downarrow\wr & & \big\uparrow{\scriptstyle g} \\ \mathfrak{a}(t)\mathfrak{a}(s)(X) & \xrightarrow[\mathfrak{a}(t)(f)]{} & \mathfrak{a}(t)(Y) \end{array}$$

hence we have to set

$$(t,g)\circ(s,f) = \big(t\circ s, g\circ\mathfrak{a}(t)(f)\circ\Phi(s,t)_X\big).$$

One checks easily that the composition of morphisms is associative (using the compatibility condition (2F2)). Furthermore for an object $(i,X) \in \operatorname{Ob}\big(\int_{\mathcal{I}}\mathfrak{a}\big)$ the morphism $(\mathrm{id}_i, \Phi^{\mathfrak{a}}(i))$ has the properties of an identity morphism which is a direct consequence of axiom (2F1). Thus we defined a category $\int_{\mathcal{I}}\mathfrak{a}$.

Note that any morphism $(s,f):(i,X)\to(j,Y)$ may be factored as $(s,f)=(\mathrm{id}_j, f\circ\Phi^{\mathfrak{a}}(j))\circ(s,\mathrm{id}_{\mathfrak{a}(s)(X)})$.

For each $i\in I$ we have a natural functor

$$\sigma_i:\mathfrak{a}(i)\longrightarrow\int_{\mathcal{I}}\mathfrak{a}$$

that maps an object $X\in\mathrm{Ob}\,(\mathfrak{a}(i))$ to (i,X) and a morphism f of $\mathfrak{a}(i)$ to $(\mathrm{id}_i, f\circ\Phi^{\mathfrak{a}}(i))$. Note that for any morphism $s:i\to j$ we have $\sigma_j\mathfrak{a}(s)(X)=(j,\mathfrak{a}(s)(X))$, thus we get a morphism

$$\big(s,\mathrm{id}_{\mathfrak{a}(s)(X)}\big):\sigma_i(X)=(i,X)\longrightarrow\big(j,\mathfrak{a}(s)(X)\big)=\sigma_j\mathfrak{a}(s)(X).$$

Set $(\Phi_s)_X=(s,\mathrm{id}_{\mathfrak{a}(s)(X)})$ for $X\in\mathrm{Ob}\,(\mathfrak{a}(i))$. One checks easily, that these morphisms define a natural transformation $\Phi_s:\sigma_i\to\sigma_j\mathfrak{a}(s)$ and that (σ,Φ) defines a 2-natural transformation $\mathfrak{a}\to\int_{\mathcal{I}}\mathfrak{a}$.

PROPOSITION A.3.2. — *Let $\mathfrak{a}\to\mathcal{C}$ be a 2-natural transformation. Then it factors uniquely through $\int_{\mathcal{I}}\mathfrak{a}$.*

REMARK A.3.3. — Note that $\int_{\mathcal{I}}\mathfrak{a}$ is not a 2-colimit of $\mathfrak{a}$ in general since the natural transformations $\Phi_s:\sigma_i\to\sigma_j\mathfrak{a}(s)$ do not need to be natural equivalences.

THEOREM A.3.4. — *Let $\mathcal{I}$ be a small category and $\mathfrak{a}:\mathcal{I}\to\mathcal{CAT}$ a 2-functor. Then $\mathfrak{a}$ admits a 2-colimit. Moreover one may find a class $\mathcal{S}$ of morphisms in $\int_{\mathcal{I}}\mathfrak{a}$ such that the localization $\int_{\mathcal{I}}\mathfrak{a}[\mathcal{S}^{-1}]$ is a 2-colimit of $\mathfrak{a}$ satisfying the strong factorization property.*

If $\mathcal{I}$ is filtered (resp. cofiltered) then $\mathcal{S}$ can be naturally chosen to be right multiplicative (resp. left multiplicative) and left saturated (in both cases) and $\int_{\mathcal{I}}\mathfrak{a}[\mathcal{S}^{-1}]$ admits a calculus of fractions.

Proof. — We localize $\int_{\mathcal{I}}\mathfrak{a}$ with respect to the following class (actually this is a set) of morphisms

$$\mathcal{S}=\big\{(s,f):(i,X)\to(j,Y)\mid f:\mathfrak{a}(s)(X)\to Y\text{ is an isomorphism}\big\}.$$

In any case ($\mathcal{I}$ filtered or not) we may define a small category

$$2\varinjlim_{i\in I}\mathfrak{a}(i)=\Big(\int_{\mathcal{I}}\mathfrak{a}\Big)[\mathcal{S}^{-1}].$$

Here we may choose the unique (up to isomorphism of categories) localization that has the strong factorization property.

Denote by $Q_{\mathfrak{a}}:\int_{\mathcal{I}}\mathfrak{a}\to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ the canonical functor and set

$$\widetilde{\sigma}_i=Q_{\mathfrak{a}}\sigma_i:\mathfrak{a}(i)\longrightarrow\int_{\mathcal{I}}\mathfrak{a}\longrightarrow 2\varinjlim_{i\in I}\mathfrak{a}(i).$$

For any morphism $s : i \to j$ of $\operatorname{Mor}\mathcal{I}$ we have a natural transformation

$$\widetilde{\Phi}_s = Q_{\mathfrak{a}} \bullet \Phi_s : \widetilde{\sigma}_i \longrightarrow \widetilde{\sigma}_j \mathfrak{a}(s)$$

defined by $(\widetilde{\Phi}_s)_X = Q_{\mathfrak{a}}((\Phi_s)_X) = Q_{\mathfrak{a}}(s, \mathrm{id}_{\mathfrak{a}(s)(X)})$ for $X \in \operatorname{Ob}\big(\mathfrak{a}(i)\big)$. These morphisms are isomorphisms in the localization, hence $\widetilde{\Phi}_s$ is a natural equivalence. It can easily be shown that $2\varinjlim_{i\in I}\mathfrak{a}(i)$ satisfies the strong factorization property and also (2CL2), using the proposition above and the (2-) universal property of the localization of categories. If we suppose $\mathcal{I}$ to be filtered (resp. cofiltered), it is not hard to show that the class $\mathcal{S}$ is left (resp. right) multiplicative and moreover that

$$\mathcal{S}' = \big\{(s, \mathrm{id}_{\mathfrak{a}(s)(X)}) : (i, X) \to (j, \mathfrak{a}(s)(X))\big\}$$

is cofinal. □

Remark A.3.5. — Let us describe morphisms in the 2-colimit. Let $\mathcal{I}$ be a filtered category. We obviously have

$$\operatorname{Hom}_{\int_{\mathcal{I}}\mathfrak{a}}\big((i,X),(j,Y)\big) = \bigsqcup_{s:i\to j} \operatorname{Hom}_{\mathfrak{a}(j)}\big(\mathfrak{a}(s)(X), Y\big).$$

Then we get

$$\operatorname{Hom}\big((i,X),(j,Y)\big) = \varinjlim_{(j,Y)\to(k,\mathfrak{a}(s)(Y))} \bigsqcup_{t:i\to k} \operatorname{Hom}_{\mathfrak{a}(k)}\big(\mathfrak{a}(t)(X), \mathfrak{a}(t)\mathfrak{a}(s)(Y)\big).$$

If we apply this formula to the special case in which $\mathcal{I}$ is a quasi-ordered set, we get the following proposition:

Proposition A.3.6. — *Let $\mathcal{I}$ be a filtered category such that between two given objects there is at most one morphism. Let $\mathfrak{a}$ be a 2-functor. Let $X \in \mathfrak{a}(i)$, $Y \in \mathfrak{a}(j)$. Then*

$$\operatorname{Hom}_{2\varinjlim_{i\in I}\mathfrak{a}(i)}(X,Y) = \varinjlim_{\substack{i\to k\\ j\to k}} \operatorname{Hom}_{\mathfrak{a}(k)}\big(\mathfrak{a}(k)(X), \mathfrak{a}(k)(Y)\big).$$

A.4. 2-limits. — The 2-limit is the dual construction of the 2-colimit. However there are three notions of an opposite 2-category. Here we will use the convention that in the (2-)opposite 2-category of $\mathcal{CAT}$ that we replace the categories $Hom(C,D)$ by $Hom(D,C)^{\circ}$.

Definition A.4.1. — Let $\mathcal{I}$ be a category. A contravariant 2-functor $\mathfrak{b} : \mathcal{I} \to \mathcal{C}$ is a 2-functor $\mathfrak{b} : \mathcal{I}^{\circ} \to \mathcal{CAT}$. Hence we also get the notion of a 2-natural transformation of contravariant 2-functors and that of modifications of such transformations.

Remark A.4.2. — The 2-limit of a contravariant 2-functor is defined by duality, hence it is the 2-colimit in the opposite 2-category of $\mathcal{CAT}$. We will not state the detailed description dual to Section 2.2.

As in the case of ordinary limits and colimits, 2-colimits can be defined by using only 2-limits (see Proposition A.4.9). However we will mostly work with 2-colimits and therefore chose to present them in detail.

Lemma A.4.3. — *Let $\mathfrak{f}, \mathfrak{g} : \mathfrak{a} \to \mathfrak{a}'$ be two 2-natural transformations of 2-functors and $\Lambda : \mathfrak{f} \to \mathfrak{g}$ be a modification. Then we get 2-natural transformations of contravariant 2-functors*

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{f}, X), \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{g}, X) : \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{a}', X) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{a}, X)$$

and a modification

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(\Lambda, X) : \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{f}, X) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathfrak{g}, X).$$

We will now give a very explicit description of the 2-limit in $\mathcal{CAT}$. Since it is a subcategory of the "lax" 2-limit, we will start with the lax version although we will not use it:

Notation A.4.4. — Consider a contravariant 2-functor $\mathfrak{b} : \mathcal{I} \to \mathcal{CAT}$. Let us call a $\mathfrak{b}$-*admissible pair* (X, ϑ^X) the following data:

1) an object $X_i \in \mathrm{Ob}\, \mathfrak{b}(i)$ for any $i \in I$;

2) a morphism $\vartheta^X_s : X_i \to \mathfrak{b}(s)(X_j)$ for any $s \in \mathrm{Mor}\, \mathcal{I}$,

such that the two following conditions hold

(A) for any $i \in I$ we have $\Phi(i)_{X_i} \circ \vartheta^X_{\mathrm{id}_i} = \mathrm{id}_{X_i}$ as visualized by

$$\begin{array}{ccc} X_i & \xrightarrow{\vartheta^X_{\mathrm{id}_i}} & \mathfrak{b}(\mathrm{id}_i)(X_i) \\ & {\scriptstyle \mathrm{id}_{X_i}} \searrow \quad \swarrow {\scriptstyle \Phi(i)_{X_i}}\ \sim & \\ & X_i & \end{array}$$

(B) and for any two composable morphisms $s : i \to j$, $t : j \to k$ the equation

$$\mathfrak{b}(s)(\vartheta^X_t) \circ \vartheta^X_s = \Phi(s,t)_{X_k} \circ \vartheta^X_{t \circ s}$$

holds as visualized by the following diagram

$$\begin{array}{ccc} X_i & \xrightarrow{\quad \vartheta^X_s \quad} & \mathfrak{b}(s)(X_j) \\ {\scriptstyle \vartheta^X_{t\circ s}} \downarrow & & \downarrow {\scriptstyle \mathfrak{b}(s)(\vartheta^X_t)} \\ \mathfrak{b}(t \circ s)(X_k) & \xrightarrow[\sim]{\Phi(s,t)_{X_k}} & \mathfrak{b}(s)\mathfrak{b}(t)(X_k) \end{array}$$

A strictly $\mathfrak{b}$-admissible pair is a $\mathfrak{b}$-admissible pair (X, ϑ^X) such that all morphisms of (B) are isomorphisms. The notion of a $\mathfrak{b}$-admissible pair is not of much interest and will only be used in this section. The only aim of the description of this type of data is to give details to the formula of Proposition A.4.8.

Let $(X, \vartheta^X), (Y, \vartheta^Y)$ be two $\mathfrak{b}$-admissible pairs. A morphism of $\mathfrak{b}$-admissible pairs $\varphi : (X, \vartheta^X) \to (Y, \vartheta^Y)$ is given by a family of morphisms $\varphi_i : X_i \to Y_i$ of $\operatorname{Mor} \mathfrak{b}(i)$ (indexed by $i \in I$) satisfying for any $s : i \to j$ of $\operatorname{Mor} \mathcal{I}$ the equation

$$\mathfrak{b}(s)(\varphi_j) \circ \vartheta_s^X = \vartheta_s^Y \circ \varphi_i$$

as visualized by

$$\begin{array}{ccc} X_i & \xrightarrow{\vartheta_s^X} & \mathfrak{b}(s)(X_j) \\ {\scriptstyle \varphi_i}\downarrow & & \downarrow{\scriptstyle \mathfrak{b}(s)(\varphi_j)} \\ Y_i & \xrightarrow{\vartheta_s^Y} & \mathfrak{b}(s)(Y_j) \end{array}$$

A morphism of strictly $\mathfrak{b}$-admissible pairs is a morphism of the underlying of $\mathfrak{b}$-admissible pairs. Configure Proposition A.4.8 for an interpretation of the data of a $\mathfrak{b}$-admissible pair.

Notation A.4.5. — Denote by $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ the category of $\mathfrak{b}$-admissible pairs as defined in the notation above. Hence the objects of $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ are the $\mathfrak{b}$-admissible pairs and morphisms in $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ are morphisms of $\mathfrak{b}$-admissible pairs.

Composition of morphisms is defined in an obvious way, and it is immediately verified that the composition of a two morphisms of $\mathfrak{b}$-admissible pairs is in fact a morphism of $\mathfrak{b}$-admissible pairs, that it is associative and finally that for any object (X, ϑ^X) of $\operatorname{Ob}(\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I})$ the morphisms id_{X_i} (for $i \in I$) define the identity morphism of (X, ϑ^X). Hence $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ is a well-defined category.

For each $i \in I$ there is a natural functor

$$\pi_i : \frac{\mathrm{d}\mathfrak{b}}{\mathrm{d}\mathcal{I}} \longrightarrow \mathfrak{b}(i)$$

that projects an object (X, ϑ^X) to X_i and a morphism $\varphi : (X, \vartheta^X) \to (Y, \vartheta^Y)$ to $\varphi_i : X_i \to Y_i$.

Let $s : i \to j$ be a morphism of $\operatorname{Mor} \mathcal{I}$ and (X, ϑ^X) an object of $\operatorname{Ob}(\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I})$. Then we have a morphism

$$\vartheta_s^X : \pi_i(X, \vartheta^X) = X_i \longrightarrow \mathfrak{b}(s)(X_j) = \mathfrak{b}(s)\pi_j(X, \vartheta^X).$$

Put $(\Phi_s)_{(X,\vartheta^X)} = \vartheta_s^X$. The definition of morphisms of $\mathfrak{b}$-admissible pairs immediately implies that Φ_s defines actually a natural transformation $\Phi_s : \pi_i \to \mathfrak{b}(s)\pi_j$ and one checks that (π, Φ) is a 2-natural transformation.

Proposition A.4.6. — *Let $\mathcal{C} \to \mathfrak{b}$ be a 2-natural transformation. Then it factors uniquely through* $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$.

Theorem A.4.7. — *Let $\mathcal{I}$ be a small category and $\mathfrak{b} : \mathcal{I} \to \mathcal{CAT}$ a contravariant 2-functor. Then $\mathfrak{b}$ admits a 2-limit. Moreover one may choose the 2-limit to be a suitable full subcategory of $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$. In that case it will satisfy the strong factorization property.*

Proof. — Consider the category $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$. Define $\mathrm{d^s}\mathfrak{b}/\mathrm{d}\mathcal{I}$ to be the full subcategory of $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ whose objects are the strictly $\mathfrak{b}$-admissible pairs.

The projections $\pi_i : \mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I} \to \mathfrak{b}(i)$ induce functors $\pi_i^s : \mathrm{d^s}\mathfrak{b}/\mathrm{d}\mathcal{I} \to \mathfrak{b}(i)$ by restriction. It follows immediately from the proposition above that any strictly $\mathfrak{b}$-admissible family factors uniquely through $\mathrm{d^s}\mathfrak{b}/\mathrm{d}\mathcal{I} \to \mathfrak{b}(i)$. Moreover since the inclusion $\iota : \mathrm{d^s}\mathfrak{b}/\mathrm{d}\mathcal{I} \to \mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$ is fully faithful we get the 2-property from the 2-property of the 2-limit $\mathrm{d}\mathfrak{b}/\mathrm{d}\mathcal{I}$. □

Proposition A.4.8. — *Let $\mathfrak{b} : \mathcal{I} \to \mathcal{CAT}$ be a contravariant 2-functor. Then we have a natural isomorphism of categories*

$$\frac{\mathrm{d^s}\mathfrak{b}}{\mathrm{d}\mathcal{I}} \simeq \underline{\mathrm{Hom}}_{2\mathcal{F}}(e_{\mathcal{I}}, \mathfrak{b}).$$

(*Here $e_{\mathcal{I}} : \mathcal{I} \to \mathcal{CAT}$ is the functor that associates to every $i \in I$ the point category e that has only one morphism.*) *Hence there is a natural equivalence:*

$$2 \varprojlim_{i\in I} \mathfrak{b}(i) \simeq \underline{\mathrm{Hom}}_{2\mathcal{F}}(e_{\mathcal{I}}, \mathfrak{b})$$

that is an isomorphism if and only if the chosen representative of $2\varprojlim_{i\in I}\mathfrak{b}(i)$ satisfies to the strong factorization property.

Proposition A.4.9. — *Let $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ be a 2-functor and $\mathfrak{b} : \mathcal{I} \to \mathcal{CAT}$ a contravariant 2-functor. Then we have canonical natural equivalences*

(i) $$\underline{\mathrm{Hom}}_{\mathcal{CAT}}\big(2\varinjlim_{i\in I} \mathfrak{a}(i), \mathcal{C}\big) \simeq 2\varprojlim_{i\in I} \underline{\mathrm{Hom}}_{\mathcal{CAT}}\big(\mathfrak{a}(i), \mathcal{C}\big),$$

(ii) $$\underline{\mathrm{Hom}}_{\mathcal{CAT}}\big(\mathcal{C}, 2\varprojlim_{i\in I} \mathfrak{b}(i)\big) \simeq 2\varprojlim_{i\in I} \underline{\mathrm{Hom}}_{\mathcal{CAT}}\big(\mathcal{C}, \mathfrak{b}(i)\big).$$

Remark A.4.10. — Note that these formulas can be used to define 2-limits and 2-colimits in any 2-categories by only using 2-limits in $\mathcal{CAT}$ which are rather simple objects.

A.5. Properties of 2-colimits and 2-limits. — There are a lot of formal consequences of the 2-universal properties of 2-limits and 2-colimits.

Proposition A.5.1. — *2-limits and 2-colimits are functorial.*

REMARK A.5.2. — Actually the result can be made more precise. The category $2\mathcal{F}(\mathcal{I})$ naturally carries the structure of a 2-category and the correspondence

$$2\mathcal{F}(\mathcal{I}) \longrightarrow \mathcal{CAT}, \quad \mathfrak{a} \mapsto 2\varinjlim_{i\in I} \mathfrak{a}(i)$$

can be extended to a 2-functor between 2-categories (which we did not define here). A similar statement holds for 2-limits.

First consider a filtered category $\mathcal{I}$.

PROPOSITION A.5.3. — *Let $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ be a 2-functor. Suppose that $\mathfrak{a}(i)$ is an additive category for any $i \in I$ and that $\mathfrak{a}(s)$ is an additive functor for every morphism $s \in \operatorname{Mor}\mathcal{I}$. Then $2\varinjlim_{i\in I}\mathfrak{a}(i)$ is an additive category and the natural functors $\sigma_i : \mathfrak{a}(i) \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ are additive.*

Proof. — Finite products (resp. coproducts) can be constructed in $\mathfrak{a}(i)$ for some $i \in I$. Then their image will define products (resp. coproducts) in the 2-colimit. □

PROPOSITION A.5.4. — *Let $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ be a 2-functor. Suppose that $\mathfrak{a}(i)$ is a triangulated category for any $i \in I$ and that $\mathfrak{a}(s)$ is an exact functor (i.e., it maps distinguished triangles to distinguished triangles) for every morphism $s \in \operatorname{Mor}\mathcal{I}$. Then $2\varinjlim_{i\in I}\mathfrak{a}(i)$ is a triangulated category and the natural functors $\sigma_i : \mathfrak{a}(i) \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ are exact.*

Proof. — A triangle in $2\varinjlim_{i\in I}\mathfrak{a}(i)$ is distinguished if it is the image of a distinguished triangle in $\mathfrak{a}(i)$ for some $i \in I$. □

PROPOSITION A.5.5. — *Let $\mathfrak{a} : \mathcal{I} \to \mathcal{CAT}$ be a 2-functor. Suppose that $\mathfrak{a}(i)$ is an abelian category for any $i \in I$ and that $\mathfrak{a}(s)$ is an exact functor for every morphism $s \in \operatorname{Mor}\mathcal{I}$. Then $2\varinjlim_{i\in I}\mathfrak{a}(i)$ is an abelian category and the natural functors $\sigma_i : \mathfrak{a}(i) \to 2\varinjlim_{i\in I}\mathfrak{a}(i)$ are exact.*

Proof. — Every morphism can be lifted to $\mathfrak{a}(i)$ for some $i \in I$ where we can construct its kernel and its cokernel. □

Appendix B
Abelian substacks of a prestack on a topological space

Perverse sheaves on a complex manifold X are local objects — they form an abelian stack which is a subprestack of the prestack of (derived) sheaves on X (see [3] for the general theory of perverse sheaves, see also [11], Section X, for a microlocal approach to perverse sheaves). In Section B.5 we will generalize the method used in [11] in order to prove that this subprestack is actually a stack. In particular we will show that a similar method can be applied to find substacks of the prestack of (derived) ind-sheaves. The abelian structure of the stack of perverse sheaves is defined by a t-structure on the triangulated prestack of derived categories of sheaves with $\mathbb{C}$-constructible cohomology. However, the category of microlocal perverse sheaves will not be defined as the heart of a t-structure. Our strategy is based on the idea that a stack is "almost" abelian, if its stalks are abelian categories. Roughly speaking, an additive stack is abelian if and only if its stalks are abelian and kernels and cokernels can be lifted to small neighborhoods. We will investigate this statement more precisely in Sections B.6 and B.7.

B.1. Prestacks. — A prestack is a "presheaf of categories up to equivalence". More precisely, let X be a topological space and denote by $\mathcal{T}(X)$ the category of open sets of X. A prestack on X is just a 2-functor $\mathcal{C} : \mathcal{T}(X)^\circ \to \mathcal{CAT}$. We get immediately the notion of a functor of prestacks (being a 2-natural transformation of the underlying 2-functors) and the notion of a natural transformation of functors of prestacks (being a modification of the underlying 2-natural transformations). In particular we get the concept of an equivalence of prestacks and we may define the (2-)category $\mathcal{PST}(X)$ of prestacks on X.

Remark B.1.1. — Let $\mathcal{C}$ be a prestack on X, $U \subset X$ an open subset and $A, B \in \operatorname{Ob}\mathcal{C}(U)$. For $V \subset U$, we set

$$\mathcal{H}om_{\mathcal{C}_{|U}}(A,B)(V) = \operatorname{Hom}_{\mathcal{C}(V)}(A_{|V}, B_{|V})$$

If $W \subset V \subset U$ the restriction functor ρ_{WV} and the natural equivalence Φ_{WVU} define a restriction map and one easily verifies that $\mathcal{H}om_{\mathcal{C}_{|U}}(A,B)$ is a presheaf.

Let us add some notations.

Definition B.1.2. — 1) A prestack $\mathcal{C}$ is called *additive* if for any $U \subset X$ the category $\mathcal{C}(U)$ is additive and the restriction functors are additive.

2) An additive prestack $\mathcal{C}$ is called *triangulated* if for any $U \subset X$ the category $\mathcal{C}(U)$ is triangulated and the restriction functors are exact.

3) An additive prestack $\mathcal{C}$ is called *abelian* if for any $U \subset X$ the category $\mathcal{C}(U)$ is abelian and the restriction functors are exact.

We then get the obvious concept of an additive (resp. exact) functor between additive (resp. triangulated or abelian) prestacks.

Let $f : X \to Y$ be a continuous map and $\mathcal{C}$ a prestack on X. Then there is a natural prestack $f_*\mathcal{C}$ on Y defined by $f_*\mathcal{C}(V) \simeq \mathcal{C}(f^{-1}(V))$. Let $\mathcal{D}$ be a prestack on Y. Then one defines a prestack $f_p^{-1}\mathcal{D}$ on X by

$$f_p^{-1}\mathcal{D}(U) = 2 \varinjlim_{f(U)\subset V} \mathcal{D}(V).$$

These operations are adjoint:

PROPOSITION B.1.3. — *The operations* f_* *and* f_p^{-1} *are* (2-)*adjoint to each other, i.e., there is a* (2-)*natural equivalence of categories*

$$\underline{\mathrm{Hom}}_{\mathcal{PST}(X)}(f_p^{-1}\mathcal{D}, \mathcal{C}) \simeq \underline{\mathrm{Hom}}_{\mathcal{PST}(Y)}(\mathcal{D}, f_*\mathcal{C}).$$

B.2. Stalks. — Since prestacks (and stacks) are often treated in the more general framework of sites, we will describe here in detail the notion of a stalk of a prestack on a topological space. Of course if $p \in X$ and $i : \{p\} \hookrightarrow X$ is the inclusion, then the stalk $\mathcal{C}_p$ of a prestack $\mathcal{C}$ at p is nothing but $i_p^{-1}\mathcal{C}$.

DEFINITION B.2.1. — Let $\mathcal{C}$ be a prestack on X and $p \in X$ a point. Consider the category $\mathcal{T}_p(X)$ of open sets that contain the point p. Note that since the set of open sets containing p is stable by union and intersection the category $\mathcal{T}_p(X)$ is filtered and cofiltered.

The prestack $\mathcal{C}$ induces a 2-functor $\alpha_p : \mathcal{T}_p(X)^\circ \to \mathcal{CAT}$. We set

$$\mathcal{C}_p = 2\varinjlim_{U \ni p} \mathcal{C}(U) = 2\varinjlim_{U \in \mathcal{T}_p(X)} \alpha_p(U)$$

and call $\mathcal{C}_p$ the stalk of $\mathcal{C}$ at p or the category of germs of $\mathcal{C}$ at p.

Hence the stalk of a prestack is defined up to canonical equivalence of categories. Moreover by the preceding paragraph we have a canonical construction of the stalk at any point p that gives us even a strict 2-colimit with strong factorization property. We will call this stalk the canonical stalk of $\mathcal{C}$ at p (which is unique up to canonical isomorphism of categories) and is of some theoretical use. The canonical stalk can easily be described using the explicit construction of Theorem A.3.4 and Proposition A.3.6. We get

$$\mathrm{Ob}\,\mathcal{C}_p = \big\{(U, A) \mid p \in U \subset X \text{ open and } A \in \mathrm{Ob}\,\mathcal{C}(U)\big\} = \bigsqcup_{p \in U \subset X} \mathrm{Ob}\,\mathcal{C}(U).$$

Let (U, A),(V, B) be two objects of $\mathcal{C}_p$. Then

$$\mathrm{Hom}_{\mathcal{C}_p}\big((U, A), (V, B)\big) = \varinjlim_{p \in W \subset U \cap V} \mathrm{Hom}_{\mathcal{C}(W)}(A_{|W}, B_{|W}).$$

Hence a morphism $f : (U, A) \to (V, B)$ is defined on a small neighborhood $W \subset U \cap V$ of p. In particular we get

PROPOSITION B.2.2. — *Let* $\mathcal{C}$ *be a prestack on* X, $p \in X$ *a point,* $U \subset X$ *an open set containing* p *and* $A, B \in \mathrm{Ob}\,\mathcal{C}(U)$ *two objects. Then we have a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{C}_{|U}}(A, B)_p \xrightarrow{\ \sim\ } \mathrm{Hom}_{\mathcal{C}_p}(A, B).$$

This isomorphism is compatible with the composition maps in $\mathcal{C}$ *in the following way: Let* $U \subset X$ *be an open set containing* p *and* $A, B, C \in \mathrm{Ob}\,\mathcal{C}(U)$. *Then the morphism of sheaves*

$$\mathcal{H}om_{\mathcal{C}_{|U}}(A, B) \times \mathcal{H}om_{\mathcal{C}_{|U}}(B, C) \longrightarrow \mathcal{H}om_{\mathcal{C}_{|U}}(A, C)$$

induces in its stalks the composition in the stalk:

$$\mathrm{Hom}_{\mathcal{C}_p}(A,B)\times \mathrm{Hom}_{\mathcal{C}_p}(B,C)\longrightarrow \mathrm{Hom}_{\mathcal{C}_p}(A,C).$$

Hence germs of morphisms may be seen as morphisms in the category of germs. The compatibility with the composition map has an obvious corollary:

Corollary B.2.3. — *Let $\mathcal{C}$ be a prestack on X, $p\in X$ a point, $U,V\subset X$ two open sets containing p and $A\in \mathrm{Ob}\,\mathcal{C}(U)$, $B\in \mathrm{Ob}\,\mathcal{C}(V)$ two objects. Then A and B are isomorphic in $\mathcal{C}_p$ if and only if they are isomorphic on an open neighborhood of p.*

Let us observe that an object $A\in \mathrm{Ob}\,\mathcal{C}(U)$ (with $p\in U$) is isomorphic to all its restrictions to sets $V\in p$ but there is no equivalence relation imposed on the objects.

If $A\in \mathrm{Ob}\,\mathcal{C}(U)$ we will still denote by A its image in $\mathcal{C}_p$. If $f:A\to B$ is a morphism in $\mathcal{C}(U)$ then we note $f_p:A\to B$ its image in $\mathcal{C}_p$. The reason why we do not write A_p is given by the following remark.

Remark B.2.4. — Consider a sheaf of rings $\mathcal{A}$ and the stack $\mathcal{MOD}(\mathcal{A})$. One shall beware that the natural functor

$$\mathcal{MOD}(\mathcal{A})_p\longrightarrow \mathrm{Mod}(\mathcal{A}_p)$$

is not an equivalence of categories because the morphism

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{F},\mathcal{G})_p\longrightarrow \mathrm{Hom}_{\mathcal{A}_p}(\mathcal{F}_p,\mathcal{G}_p)$$

is not an isomorphism in general.

Proposition B.2.5. — 1) *If $\mathcal{C}$ is additive then its stalks are additive categories and the natural functors into the stalks are additive.*

2) *If $\mathcal{C}$ is triangulated then its stalks are triangulated categories and the natural functors into the stalks are exact.*

3) *If $\mathcal{C}$ is abelian then its stalks are abelian categories and the natural functors into the stalks are exact.*

Proof. — It's a direct application of the propositions (A.5.3), (A.5.4) and (A.5.5). □

B.3. Stacks. — A stack is a "lax" version of a sheaf of categories, hence objects and morphisms in a stack are determined up to unique isomorphism by their local data.

Definition B.3.1. — A prestack $\mathcal{C}$ on X is *separated* if for all open subsets $U\subset X$ and all objects $A,B\in \mathrm{Ob}\,\mathcal{C}(U)$ the presheaf $\mathcal{H}om_{\mathcal{C}_{|U}}(A,B)$ is a sheaf.

Definition B.3.2. — A prestack $\mathcal{C}$ on X is a stack if the following two conditions are satisfied:

(i) the prestack $\mathcal{C}$ is separated;

(ii) let $U = \bigcup_{i\in I} U_i$ be an open covering of an open subset $U \subset X$ and suppose that we are given the following data

(a) for every $i \in I$ an object $A_i \in \mathcal{C}(U_i)$,

(b) for every $i, j \in I$ an isomorphism $\sigma_{ij} : A_{j|U_{ij}} \xrightarrow{\sim} A_{i|U_{ij}}$ such that for any $i, j, k \in I$ the equation $\sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$ holds on U_{ijk}.

Then there exist an object $A \in \mathcal{C}(U)$ and isomorphisms $\rho_i : A_{|U_i} \to A_i$ such that $\sigma_{ij} \circ \rho_j = \rho_i$.

Proposition B.3.3. — *Let $\mathcal{C}, \mathcal{C}'$ be two stacks on X. Consider a functor $F : \mathcal{C} \to \mathcal{C}'$. Then we have*

1) *F is faithful if and only if F_p is faithful for all $p \in X$;*

2) *F is fully faithful if and only if F_p is fully faithful for all $p \in X$;*

3) *F is an equivalence of stacks if and only if F_p is an equivalence of categories for all $p \in X$.*

Proof. — We know that F is faithful (resp. fully faithful) if and only if the morphisms of sheaves

$$\mathcal{H}om_{\mathcal{C}_{|U}}(A, B) \longrightarrow \mathcal{H}om_{\mathcal{C}'_{|U}}\big(F(A), F(B)\big)$$

are monomorphisms (resp. isomorphisms). Since these two properties are verified in the stalks, we immediately get 1) and 2).

Let us prove 3). Note that the condition is clearly necessary.

Now suppose that F_p is an equivalence of categories for all $p \in X$. By 2) we know that F is fully faithful. Hence it is sufficient to show that F is essentially surjective for any open set $U \subset X$. Let $A' \in \operatorname{Ob} \mathcal{C}'(U)$. For any point $p \in U$ there is an object $A_p \in \mathcal{C}_p$ such that $F_p(A_p) \simeq A'_p$. Hence by corollary (B.2.3) there is an open neighborhood $V(p)$ of p, an object $A(p) \in \operatorname{Ob}\big(\mathcal{C}(V(p))\big)$ and an isomorphism $\vartheta(p) : F(A(p)) \simeq A'_{|V(p)}$. These isomorphisms define a cocycle that patches together the objects $F(A(p))$ to an object isomorphic to A'. Since F is fully faithful this cocycle can be lifted to a cocycle in $\mathcal{C}'$ where the $A(p)$ patch together to an object A such that $F(A)$ is isomorphic to A'. □

B.4. The stack associated to a prestack. — In this paragraph we will describe the stack associated to a prestack on a topological space. As is the case of the sheaf associated to a presheaf this can be done explicitly and is less complicated than on an arbitrary site.

Proposition B.4.1. — *Let $\mathcal{C}$ be a prestack on X. Then there exists a separated prestack $\mathcal{C}^\dagger$ on X and a canonical functor $\eta^\dagger_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\dagger$ that induces an equivalence of categories on the stalks, such that any morphism $\mathcal{C} \to \mathcal{D}$ into a separated prestack $\mathcal{D}$ factors uniquely through $\mathcal{C}^\dagger$ (up to unique equivalence).*

Moreover for any functor $F : \mathcal{C} \to \mathcal{D}$ there exists a functor $F^\dagger : \mathcal{C}^\dagger \to \mathcal{D}^\dagger$ such that the diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathcal{C}^\dagger & \xrightarrow{F^\dagger} & \mathcal{D}^\dagger. \end{array}$$

Proof. — Let $U \subset X$ be an open subset. Let us define a category $\mathcal{C}^\dagger(U)$. Set

$$\operatorname{Ob} \mathcal{C}^\dagger(U) = \operatorname{Ob} \mathcal{C}(U)$$

now let $A, B \in \operatorname{Ob} \mathcal{C}(U)$ be two objects. Put

$$\operatorname{Hom}_{\mathcal{C}^\dagger(U)}(A, B) = \Gamma\big(U, \mathcal{H}om_{\mathcal{C}_{|U}}(A, B)^\dagger\big),$$

where $\mathcal{H}om_{\mathcal{C}_{|U}}(A, B)^\dagger$ is the sheaf associated to $\mathcal{H}om_{\mathcal{C}_{|U}}(A, B)$. Note that we have a canonical map $\operatorname{Hom}_{\mathcal{C}(U)}(A, B) \to \operatorname{Hom}_{\mathcal{C}^\dagger(U)}(A, B)$ which is just the natural morphism from the presheaf $\operatorname{Hom}_{\mathcal{C}(U)}(A, B)$ into its associated sheaf. The map

$$\mathcal{H}om_{\mathcal{C}_{|U}}(A, B) \times \mathcal{H}om_{\mathcal{C}_{|U}}(B, C) \longrightarrow \mathcal{H}om_{\mathcal{C}_{|U}}(A, C)$$

induces the composition in $\mathcal{C}^\dagger$:

$$\mathcal{H}om_{\mathcal{C}^\dagger_{|U}}(A, B) \times \mathcal{H}om_{\mathcal{C}^\dagger_{|U}}(B, C) \longrightarrow \mathcal{H}om_{\mathcal{C}^\dagger_{|U}}(A, C).$$

The restriction functors of $\mathcal{C}^\dagger$ and the equivalences can easily be constructed by the universal property of the sheaf associated to a presheaf which also implies that all the axioms are verified. The universal property of the separated prestack associated to a prestack also follows from the universal property of the sheaf associated to a presheaf. □

THEOREM B.4.2. — *Let $\mathcal{C}$ be a prestack on X. Then there exists a stack $\mathcal{C}^\ddagger$ on X together with a canonical functor of prestacks $\eta^\ddagger_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\ddagger$ such that any morphism $\mathcal{C} \to \mathcal{D}$ into some stack $\mathcal{D}$ factors uniquely through $\mathcal{C}^\ddagger$. Moreover for any stack $\mathcal{D}$ and any morphism of prestacks $F : \mathcal{C} \to \mathcal{D}$ there exists a canonical functor of stacks $F^\ddagger$ such that the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ {\scriptstyle \eta^\ddagger_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \eta^\ddagger_{\mathcal{C}}} \\ \mathcal{C}^\ddagger & \xrightarrow{F^\ddagger} & \mathcal{D}^\ddagger. \end{array}$$

Finally the functor $\eta^\ddagger_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\ddagger$ induces equivalences of categories on the stalks at every point of X.

Proof. — By the previous proposition we may assume that $\mathcal{C}$ is a separated prestack, *i.e.*, all associated presheaves are actually sheaves.

Let $U \subset X$ be an open subset. We have to define a category $\mathcal{C}^{\ddagger}(U)$.

Consider families $A = \{(A_p, U_p^A)\}_{p\in U}$ where U_p^A is an open neighborhood of p with $A_p \in \mathcal{C}(U_p^A)$ and families of morphisms $\theta^A = \{\theta_{pq}^A\}_{p,q\in U}$ where $\theta_{pq}^A : A_{q|U_{pq}^A} \xrightarrow{\sim} A_{p|U_{pq}^A}$ is an isomorphism for all $p, q \in U$ (here $U_{pq}^A = U_p^A \cap U_q^A$) satisfying the cocycle condition. During the proof let us call a pair (A, θ^A) a cocycle on U.

We shall now define morphisms of cocycles. A morphism $f : (A, \theta^A) \to (B, \theta^B)$ consists of a family of germs morphisms $f_p : (A_p, U_p^A) \to (B_p, U_p^B)$ of $\operatorname{Mor}\mathcal{C}_p$ such that for any point $p \in U$ there is an open set U_p^f on which f_p is represented as a morphism $f_p : A_{p|U_p^f} \to B_{p|U_p^f}$ (where $U_p^f \subset U_p^{AB} = U_p^A \cap U_p^B$ is an open neighborhood of p) satisfying the following compatibility condition: the diagram

$$\begin{array}{ccc} A_{q|U_{pq}^f} & \xrightarrow{\theta^A_{pq|U_{pq}^f}} & A_{p|U_{pq}^f} \\ {\scriptstyle f_{q|U_{pq}^f}}\downarrow & & \downarrow{\scriptstyle f_{p|U_{pq}^f}} \\ B_{q|U_{pq}^f} & \xrightarrow{\theta^B_{pq|U_{pq}^f}} & B_{p|U_{pq}^f} \end{array}$$

should be commutative for all $p, q \in U$.

Now define $\mathcal{C}^{\ddagger}(U)$ to be the category of cocycles. The obvious restriction maps define a prestack $\mathcal{C}^{\ddagger}$ on U (which is actually a presheaf). It is now tedious but straightforward, that $\mathcal{C}^{\ddagger}$ is a stack that satisfies the universal property. Note that we need the assumption that $\mathcal{C}$ is separated when proving the patching condition. □

COROLLARY B.4.3. — *Let $\mathcal{C}$ be a stack. Then there exists a stack $\mathcal{C}'$, canonically isomorphic to $\mathcal{C}$, which is also a presheaf of categories.*

COROLLARY B.4.4. — *Let $\mathcal{C}$ be a prestack and $F : \mathcal{C} \to \mathcal{D}$ be a morphism into a stack $\mathcal{D}$. Suppose that F induces equivalences of categories in the stalks. Then $\mathcal{D}$ is equivalent to the stack associated to $\mathcal{C}$.*

B.5. A criterion for substacks. — The basic definitions from the theory of stacks (on a topological space) are recalled in Appendix B. The results on proper stacks and ind-sheaves that we will use can be found in [12].

DEFINITION B.5.1. — Consider a prestack $\mathcal{C}$ on a topological space X. We say that a full subprestack $\mathcal{C}' \subset \mathcal{C}$ is defined by a local property (with respect to $\mathcal{C}$) if the following conditions are satisfied:

(i) the prestack $\mathcal{C}'$ is stable by isomorphisms, *i.e.*, if $U \subset X$ is open, $A \in \operatorname{Ob}\mathcal{C}'(U)$ then any object $B \in \operatorname{Ob}\mathcal{C}(U)$ isomorphic to A is also an object of $\operatorname{Ob}\mathcal{C}'(U)$,

(ii) if $U \subset X$ is open and $A \in \operatorname{Ob}\mathcal{C}(U)$ then $A \in \operatorname{Ob}\mathcal{C}'(U)$ if and only if there is an open covering $U = \bigcup_{i\in I} U_i$ such that $A_{|U_i} \in \operatorname{Ob}\mathcal{C}'(U_i)$ for all $i \in I$.

Remark B.5.2. — Consider a full subprestack $\mathcal{C}' \subset \mathcal{C}$ and a point $p \in X$. Then the natural functor $\mathcal{C}'_p \to \mathcal{C}_p$ is fully faithful. Therefore the subprestack $\mathcal{C}'$ is defined by a local property if and only if for any object $A \in \operatorname{Ob}\mathcal{C}(U)$ the statements below are equivalent:

(a) $A \in \operatorname{Ob}\mathcal{C}'(U)$;

(b) for every $p \in X$ the object A is in the essential image of the functor $\mathcal{C}'_p \to \mathcal{C}_p$, *i.e.*, there exists an object $B \in \operatorname{Ob}\mathcal{C}'_p$ such that A is isomorphic to B in $\mathcal{C}_p$.

Lemma B.5.3. — *Let $\mathcal{C}$ be a triangulated prestack. Assume moreover that*

1) *for any $V \subset U$ the restriction functor i_{VU}^{-1} has a fully faithful left adjoint $i_{VU!}$*(4)*;*

2) *these functors satisfy the base change theorem, i.e., for any Cartesian square of open subsets*

$$\begin{array}{ccc} U_{12} & \longrightarrow & U_1 \\ \downarrow & \square & \downarrow \\ U_2 & \longrightarrow & V \end{array}$$

we have $i_{U_{12}U_2!}i_{U_{12}U_1}^{-1} \simeq i_{U_2V}^{-1} i_{U_1V!}$, where $U_{12} = U_1 \cap U_2$.

Consider the union of two open sets $U = U_1 \cup U_2$ and suppose that we are given

(i) *objects $A_1 \in \operatorname{Ob}\mathcal{C}(U_1)$ and $A_2 \in \operatorname{Ob}\mathcal{C}(U_2)$;*

(ii) *an isomorphism $f_{21} : A_{1|U_{12}} \xrightarrow{\sim} A_{2|U_{12}}$ in $\mathcal{C}(U_{12})$.*

Then there exist an object $A \in \operatorname{Ob}\mathcal{C}(U)$ and isomorphisms $f_1 : A_{|U_1} \xrightarrow{\sim} A_1$, $f_2 : A_{|U_2} \xrightarrow{\sim} A_2$ that are compatible with f_{21} on U_{12}, i.e., the following diagram commutes:

$$\begin{array}{ccccc} A_{1|U_{12}} & & \xrightarrow[\sim]{f_{21}} & & A_{2|U_{12}} \\ & {\scriptstyle f_{1|U_{12}}}\nwarrow\sim & & \sim\nearrow{\scriptstyle f_{2|U_{12}}} & \\ & & A_{|U_{12}} & & \end{array}$$

(4) Recall that $i_{VU!}$ is fully faithful if and only if the adjunction morphism $\mathrm{Id} \to i_{VU}^{-1} i_{VU!}$ is an isomorphism. Also note that for any three open subsets $W \subset V \subset U$ the isomorphism $i_{WV}^{-1} i_{VU}^{-1} \simeq i_{WU}^{-1}$ induces an isomorphism $i_{WV!}i_{VU!} \simeq i_{WU!}$.

Proof. — The object A is obtained by choosing a distinguished triangle

$$i_{12!}(A_{1|U_{12}}) \xrightarrow{\binom{g_1}{g_2}} i_{1!}A_1 \oplus i_{2!}A_2 \xrightarrow{(h_1,h_2)} A \xrightarrow{+}$$

where the first morphism defining the triangle is given by

$$g_1 : i_{12!}(A_{1|U_{12}}) \xrightarrow{\sim} i_{1!}i_{12,1!}(A_{1|U_{12}}) \longrightarrow i_{1!}(A_1),$$

$$g_2 : i_{12!}(A_{1|U_{12}}) \xrightarrow[\sim]{-f_{21}} i_{12!}(A_{2|U_{12}}) \xrightarrow{\sim} i_{2!}i_{12,2!}(A_{2|U_{12}}) \longrightarrow i_{2!}(A_2).$$

□

Remark B.5.4. — In the situation of the preceding Lemma B.5.3, suppose that we are given a full (but not necessarily triangulated) subprestack $\mathcal{C}' \subset \mathcal{C}$ that is defined by a local property. Then the lemma holds in $\mathcal{C}'$, *i.e.*, if the objects A_1,A_2 are in $\mathcal{C}'$ then the object A lies also in $\mathcal{C}'$. Indeed, we may patch the given objects A_1, A_2 of $\mathcal{C}'$ to an object A in the prestack $\mathcal{C}$ using Lemma B.5.3. Then the axioms (*cf.* Definition B.5.1) immediately imply that A is an object of $\mathcal{C}'$.

Note that if moreover $\mathcal{C}'$ is separated, then the object A is unique up to unique isomorphism(5).

Let us note that we can apply Lemma B.5.3 to the prestack of bounded derived categories of ind-sheaves (*cf.* [12]). Denote by $\mathrm{D^b}(\mathrm{I}(k_*))$ the prestack $U \mapsto \mathrm{D^b}(\mathrm{I}(k_U))$ on a locally compact space X with a countable base of open sets.

We can now state Proposition 10.2.9 of [11] in a slightly more general context and change the proof so that we may adapt it later to the case of ind-sheaves.

Proposition B.5.5. — *Let X be a locally compact paracompact space with a countable base of open sets. Consider a proper stack*(6) *$\mathcal{A}$ such that for every open subset $U \subset X$ the category $\mathcal{A}(U)$ has enough injective objects. Denote by $\mathrm{D^b}(\mathcal{A})$ the associated prestack of bounded derived categories.*

Let $\mathcal{C} \subset \mathrm{D^b}(\mathcal{A})$ be a separated full subprestack that is defined by a local property. Then $\mathcal{C}$ is a stack.

Proof. — We have to show that $\mathcal{C}$ satisfies the patching condition (*cf.* Definition B.3.2 in Appendix B). Since X is paracompact we have to verify the patching condition only for countable coverings. Since the conditions of Lemma B.5.3 are satisfied for the prestack $\mathrm{D^b}(\mathcal{A})$ and $\mathcal{C}$ is defined by a local property, we know

(5) More precisely, if A' is another object with isomorphisms $f'_i : A'_{|U_i} \xrightarrow{\sim} A_i$ for $i \in \{1,2\}$ such that $f_{21} \circ f'_{1|U_{12}} = f'_{2|U_{12}}$ then there exists a unique isomorphism $\varphi : A \xrightarrow{\sim} B$ such that $f'_i \circ \varphi_{|U_i} = f_i$ for $i \in \{1,2\}$.

(6) For the definition of a proper stack see [12]. A proper stack $\mathcal{A}$ and the associated prestack of bounded derived categories $\mathrm{D^b}(\mathcal{A})$ satisfy the hypothesis of Lemma B.5.3. Moreover for each open subset $U \subset X$, the abelian category $\mathcal{A}(U)$ admits filtered exact colimits and the restriction functors commute to such colimits.

that $\mathcal{C}$ satisfies the patching condition for any covering of type $U = U_1 \cup U_2$. Using the fact that $\mathcal{C}$ is separated, we can easily verify by induction that the patching condition is satisfied for finite coverings. Therefore, using again the fact that $\mathcal{C}$ is separated, it is sufficient to prove that objects may be patched in $\mathcal{C}$ for open coverings of type $U = \bigcup_{n\in\mathbb{N}} U_n$ where $U_n \subset U_{n+1}$.

Consider a family of objects $A_n \in \operatorname{Ob} \mathrm{D^b}(\mathcal{A}(U_n))$ and isomorphisms $f_{n-1} : A_{n-1} \xrightarrow{\sim} A_{n|U_{n-1}}$ (the other isomorphisms are uniquely determined by the cocycle condition). Denote by $i_n : U_n \hookrightarrow U$ the inclusion map. Then we lift the morphisms of the system to

$$g_{n-1} : i_{n-1!}A_{n-1} \xrightarrow[\sim]{i_{n-1!}(f_{n-1})} i_{n-1!}A_{n|U_{n-1}} \longrightarrow i_{n!}A_n$$

in $\mathrm{D^b}(\mathcal{A}(U))$. Hence we get a family of morphisms

$$\big\{g_{n-1} : i_{n-1!}A_{n-1} \to i_{n!}A_n\big\}_{n\geqslant 1}$$

in $\mathrm{D^b}(\mathcal{A}(U))$. Note that $g_{n-1|U_{n-1}}$ is an isomorphism by the base change theorem.

By hypothesis $\mathcal{A}(U)$ has enough injective objects and therefore we have an equivalence of categories $\mathrm{K}^b(\mathrm{Inj}(\mathcal{A}(U))) \simeq \mathrm{D^b}(\mathcal{A}(U))$. Here $\mathrm{Inj}(\mathcal{A}(U))$ denotes the full additive subcategory of $\mathcal{A}(U)$ whose objects are the injective objects of $\mathcal{A}(U)$ and $\mathrm{K}^b(\mathrm{Inj}(\mathcal{A}(U)))$ is the triangulated category of bounded complexes of $\mathrm{Inj}(\mathcal{A}(U))$ where morphisms of complexes are considered up to homotopy. Hence there exist objects I_n and morphisms $h_{n-1} : I_{n-1} \to I_n$ in $\mathrm{C}^b(\mathrm{Inj}(\mathcal{A}(U)))$ such that the diagram $\{h_{n-1} : I_{n-1} \to I_n\}_{n\geqslant 1}$ is isomorphic (in $\mathrm{D^b}(\mathcal{A}(U))$) to the diagram $\{g_{n-1} : i_{n-1!}A_{n-1} \to i_{n!}A_n\}_{n\geqslant 1}$. Let $A = \varinjlim I_n$ and consider A as an object in $\mathrm{D^b}(\mathcal{A}(U))$. Then $A_{|U_n}$ is quasi-isomorphic in $\mathrm{C}^b(\mathcal{A}(U))$ to I_n because for $m \geqslant n$ the morphism $I_n \to I_{m|U_n}$ is a quasi-isomorphism. Hence there are natural isomorphisms $A_{|U_n} \simeq A_n$ in $\mathrm{D^b}(\mathcal{A}(U))$, and a simple diagram chase shows that they are compatible with the morphisms f_n. □

Remark B.5.6. — Note that in Proposition B.5.5 the hypothesis that the categories $\mathrm{D^b}(\mathcal{A}(U))$ possess enough injective objects can be weakened. During the proof, we actually only use the fact that any diagram in $\mathrm{D^b}(\mathcal{A}(U))$ of type $\{A_n \to A_{n+1}\}_{n\geqslant 0}$ can be lifted to a diagram $\{I_n \to I_{n+1}\}_{n\geqslant 0}$ in $\mathrm{C}^b(\mathcal{A}(U))$. We do not use the fact that the objects I_n are injective.

Proposition B.5.7. — *Let $\mathcal{S}$ be a small diagram in $\mathrm{D^b}(\mathrm{Ind}(\mathcal{A}))$, i.e., $\mathcal{S} \subset \operatorname{Mor} \mathrm{D^b} \mathrm{Ind}(\mathcal{A})$ is a set of morphisms. Then there exists an essentially small full abelian subcategory $\mathcal{B} \subset \mathcal{A}$ such that*

(i) *$\mathcal{B}$ is stable by subobject, quotient and extension in $\mathcal{A}$;*

(ii) *$\mathrm{Ind}\,\mathcal{B} \subset \mathrm{Ind}\,\mathcal{A}$ is stable by subobject, quotient and extension;*

(iii) *$\mathrm{Ind}\,\mathcal{B}$ has enough injectives;*

(iv) *$\mathcal{S}$ is contained in the image of the natural functor*

$$\mathrm{D^b}\big(\mathrm{Ind}(\mathcal{B})\big) \longrightarrow \mathrm{D^b}\big(\mathrm{Ind}(\mathcal{A})\big).$$

Proof. — Follows easily from Theorem 11.2.6 of [12]. □

Combining Proposition B.5.5 with Remark B.5.6 and Proposition B.5.7 we get:

THEOREM B.5.8. — *Let X be a paracompact locally compact topological space with a countable base of open sets and consider a separated full subprestack $\mathcal{C} \subset \mathrm{D^b}(I(k_*))$ that is defined by a local property. Then $\mathcal{C}$ is a stack.*

B.6. Limits and colimits in stacks. — Recall that if $\mathcal{C}$ is a prestack on a topological space X, we denote by ρ_{VU} the restriction functor for two open subsets $V \subset U \subset X$ and $\rho_p^U : \mathcal{C}(U) \to \mathcal{C}_p$ the canonical functor into the stalk at p.

DEFINITION B.6.1. — Let $\mathcal{I}$ be a small category. We say that $\mathcal{C}$ admits limits (resp. colimits) indexed by $\mathcal{I}$ if for every open subset $U \subset X$ the category $\mathcal{C}(U)$ admits limits (resp. colimits) indexed by $\mathcal{I}$ such that the restriction functors commute to these limits (resp. colimits).

Let $\mathcal{I}$ be a finite category. It is easy to see that if $\mathcal{C}$ admits limits (resp. colimits) indexed by $\mathcal{I}$, then for every $p \in X$ the category $\mathcal{C}_p$ admits limits (resp. colimits) indexed by $\mathcal{I}$ and the functor ρ_p^U commutes to such limits (resp. colimits).

However, the converse is not true. We cannot know simply by looking at the stalks whether or not a prestack admits limits or colimits indexed by $\mathcal{I}$ (even if $\mathcal{C}$ is a stack).

If $\mathcal{C}$ is separated we can at least see from the stalks whether or not a given object represents a limit or colimit indexed by a finite category. By duality we only need to consider the case of finite colimits.

LEMMA B.6.2. — *Let $\mathcal{C}$ be a separated prestack on a topological space X. Consider a finite category $\mathcal{I}$, an open subset $U \subset X$ and a functor $\alpha : \mathcal{I} \to \mathcal{C}(U)$. Suppose given an object $L \in \mathrm{Ob}\,\mathcal{C}(U)$ and morphisms $\sigma_i : \alpha(i) \to L$ such that for any morphism $s : i \to j$ of $\mathrm{Mor}\,\mathcal{I}$ we have $\sigma_j \circ \alpha(s) = \sigma_i$. Then the two following assertions are equivalent:*

(i) *$(L, \{\sigma_i\}_{i\in I})$ is a colimit of α in $\mathcal{C}(U)$ and for any open subset $V \subset U$ the pair $(L_{|V}, \{\sigma_{i|V}\}_{i\in I})$ is a colimit of $\rho_{VU}\alpha$ in $\mathcal{C}(V)$;*

(ii) *$(L, \{(\sigma_i)_p\}_{i\in I})$ is a colimit of $\rho_p^U\alpha$ in $\mathcal{C}_p$ for all $p \in U$.*

Proof. — Let $V \subset U$ be an open subset. The object $L \in \operatorname{Ob} \mathcal{C}(U)$ and the morphisms $\sigma_{i|V}$ define a natural morphism of sheaves ($\mathcal{C}$ is separated) for any object $A \in \operatorname{Ob} \mathcal{C}(V)$

$$\mathcal{H}om_{\mathcal{C}_{|V}}(L_{|V}, A) \longrightarrow \varprojlim_{i \in I} \mathcal{H}om_{\mathcal{C}_{|V}}\big(\alpha(i)_{|V}, A\big). \tag{B.6.1}$$

Since $\mathcal{I}$ is a finite category we have for every $p \in V$

$$\big(\varprojlim_{i \in I} \mathcal{H}om_{\mathcal{C}_{|V}}\big(\alpha(i)_{|V}, A\big)\big)_p \simeq \varprojlim_{i \in I} \mathcal{H}om_{\mathcal{C}_{|V}}\big(\alpha(i)_{|V}, A\big)_p.$$

Hence the morphism (B.6.1) induces in the stalks

$$\mathcal{H}om_{\mathcal{C}_p}(L, A) \simeq \mathcal{H}om_{\mathcal{C}_{|V}}(L_{|V}, A)_p \longrightarrow \varprojlim_{i \in I} \mathcal{H}om_{\mathcal{C}_{|V}}\big(\alpha(i)_{|V}, A\big)_p. \tag{B.6.2}$$

Assertion (i) is clearly equivalent to the fact that the morphism (B.6.1) is an isomorphism for all $V \subset U$ and any $A \in \mathcal{C}(V)$.

Assertion (ii) is equivalent to the fact that the morphism (B.6.2) is an isomorphism for all $V \subset U$, $A \in \mathcal{C}(V)$ and $p \in V$.

Since $\mathcal{C}$ is separated the morphism (B.6.1) is an isomorphism if and only if for every $p \in V$ the morphism (B.6.2) is an isomorphism, which proves the lemma. □

Now suppose that we are given a stack $\mathcal{C}$, a finite category $\mathcal{I}$ and a functor $\alpha : \mathcal{I} \to \mathcal{C}(U)$. In order to check that there exists a colimit of α in $\mathcal{C}(U)$ we can apply Lemma B.6.2. However, in practical situations (as in Section 7.2) it is often difficult to establish the existence of an object L defined on U that verifies condition (ii) of Lemma B.6.2. Therefore we will use a refinement of Lemma B.6.2 adapted to stacks which states that it is sufficient to prove the existence of the object L locally on U.

Proposition B.6.3. — *Let $\mathcal{C}$ be a stack on a topological space X and $\mathcal{I}$ be a finite category. Suppose that for every open subset $U \subset X$ and every functor $\alpha : \mathcal{I} \to \mathcal{C}(U)$ there exists an open covering $U = \bigcup_{j \in J} U_j$, objects $L_j \in \operatorname{Ob} \mathcal{C}(U_j)$ and morphisms $\sigma_i^j : \alpha(i)_{|U_j} \to L_j$ verifying condition* (ii) *of Lemma* B.6.2. *Then $\mathcal{C}$ admits colimits indexed by $\mathcal{I}$.*

Proof. — Consider an open subset $U \subset X$ and a functor $\alpha : \mathcal{I} \to \mathcal{C}(U)$. By hypothesis and Lemma B.6.2 there exists an open covering $U = \bigcup_{j \in J} U_j$ such that $\rho_{U_j U} \alpha$ is representable in $\mathcal{C}(U_j)$ by an object L_j and the restriction to any smaller open subset $W \subset U_j$ commutes to these colimits. The conditions clearly imply that we may patch together the colimits $L_j \in \operatorname{Ob} \mathcal{C}(U_j)$ to an object $L \in \operatorname{Ob} \mathcal{C}(U)$. Now applying again Lemma B.6.2 we see that L is a colimit of α and that all restrictions commute to this colimit. □

Corollary B.6.4. — *Let* $\mathcal{C}$ *be a prestack,* $\eta : \mathcal{C} \to \mathcal{C}^{\ddagger}$ *the natural functor into the associated stack and* $\mathcal{I}$ *be a finite category. Suppose that the stalks of* $\mathcal{C}$ *admit colimits indexed by* $\mathcal{I}$*. Moreover we assume that for any open subset* $U \subset X$ *and any functor* $\alpha : \mathcal{I} \to \mathcal{C}(U)$ *the following statement holds:*

For any point $p \in U$ *there exists an open neighborhood* $U_p \subset U$*, an object* $L^p \in \mathcal{C}(U_p)$ *and morphisms* $\sigma_i^p : \alpha(i)_{|U_p} \to L^p$ *such that condition* (ii) *of Lemma* B.6.2 *is verified. Then* $\mathcal{C}^{\ddagger}$ *admits colimits indexed by* $\mathcal{I}$*.*

Proof. — Let $\alpha : \mathcal{I} \to \mathcal{C}^{\ddagger}(U)$ be a functor. Consider the functors $\rho_p^{\ddagger U}\alpha$ for all $p \in X$. Since $\mathcal{I}$ is finite there exists an open neighborhood U_p of p such that $\rho_p^{\ddagger U}\alpha$ factors through $\mathcal{C}(U_p)$. Hence we can apply Proposition B.6.3. ☐

In particular we get the much weaker statement that if a prestack $\mathcal{C}$ admits colimits indexed by a finite category $\mathcal{I}$ then $\mathcal{C}^{\ddagger}$ admits colimits indexed by $\mathcal{I}$.

B.7. A criterion for abelian stacks. — We can apply the results of the last paragraph to additive prestacks with abelian stalks. First recall that if $\mathcal{C}$ is an additive prestack then $\mathcal{C}^{\ddagger}$ is additive.

Theorem B.7.1. — *Let* $\mathcal{C}$ *be an additive prestack with abelian stalks. Suppose that for every* $p \in X$ *and every morphism* $f : A \to B$ *in* $\mathcal{C}_p$ *there exists an open neighborhood* U *of* p *such that* f *may be represented by a morphism* $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ *in* $\mathcal{C}(U)$ *and there are morphisms* $K \to \widetilde{A}$*,* $\widetilde{B} \to K'$ *such that* $K \to \widetilde{A}$ *is a kernel in* $\mathcal{C}_q$ *and* $\widetilde{B} \to K'$ *is a cokernel in* $\mathcal{C}_q$ *for any* $q \in U$*. Then* $\mathcal{C}^{\ddagger}$ *is an abelian stack.*

Proof. — Clearly the conditions of Proposition B.6.3 and Corollary B.6.4 are satisfied for cokernels and kernels. Hence $\mathcal{C}^{\ddagger}$ admits cokernels and kernels.

Let $f : A \to B$ be a morphism of $\mathcal{C}^{\ddagger}(U)$ and consider the natural morphism $\operatorname{coim} f \to \operatorname{im} f$. Since the categories of germs are abelian this morphism is an isomorphism in the stalks. Since $\mathcal{C}^{\ddagger}$ is separated it is also an isomorphism in $\mathcal{C}^{\ddagger}(U)$. ☐

Corollary B.7.2. — *Let* $\mathcal{C}$ *be an additive stack on* X *such that all stalks are abelian categories. Then* $\mathcal{C}$ *is an abelian stack if and only if for every morphism* $f : A \to B$ *in* $\mathcal{C}_p$ *there is an open neighborhood* U *of* p *such that* f *may be represented by a morphism* $\widetilde{f} : \widetilde{A} \to \widetilde{B}$ *in* $\mathcal{C}(U)$ *and there are morphisms* $K \to \widetilde{A}$*,* $\widetilde{B} \to K'$ *such that* $K \to \widetilde{A}$ *is a kernel in* $\mathcal{C}_q$ *and* $\widetilde{B} \to K'$ *is a cokernel in* $\mathcal{C}_q$ *for any* $q \in U$*.*

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