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NOTE ON PULL-BACK AND LELONG NUMBER OF CURRENTS

BY CHARLES FAVRE (*)

ABSTRACT. — We prove a uniform estimate of the Lelong number of the pull-back of a plurisubharmonic function by a holomorphic map in term of the original Lelong number of this function.

RÉSUMÉ. — NOTE SUR LE NOMBRE DE LELONG DES PULL-BACK DE COURANTS. Cet article est consacré à l'étude du nombre de Lelong $\nu(f^*u,0)$ du pull-back d'une fonction plurisousharmonique u par une application holomorphe $f\colon (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ génériquement de rang maximal. Nous prouvons l'estimée $\nu(f^*u,0) \le C_f \times \nu(u,0)$ avec une constante C_f uniforme en u.

1. Statement of the main result

Fix $f\colon (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ a holomorphic germ, and T a positive closed current of bidegree $(1,1)$ defined in a neighborhood of the origin in $(\mathbb{C}^n,0)$. Let $u \in \mathrm{PSH}(\mathbb{C}^n,0)$ be a plurisubharmonic (psh) potential for T such that $T = \mathrm{dd}^c u$. One can set

$$f^*T \;:= \mathrm{dd}^c(u \circ f)$$

as soon as the psh function $u \circ f$ is not identically $-\infty$.

DEFINITION 1 (Lelong number, see [LG86]). — Let $u \in \mathrm{PSH}(\mathbb{C}^n,0)$. The function $r \mapsto \sup_{|z|=r} u(z)$ is an increasing convex function of $\log r$.

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• We can hence define the *Lelong number* of u at 0 by setting

$$\nu(u,0) := \max\{c \geq 0;\ \text{such that } u(z) \leq c\log|z| + O(1)\}$$

which is a finite non-negative real number.

• For a positive closed (1,1) current T in $(\mathbb{C}^n,0)$, the *Lelong number* of T at 0 is

$$\nu(T,0) := \nu(u,0)$$

for any psh potential $T = \mathrm{dd}^c u$.

For a given positive closed current T of bidegree $(1,1)$ so that f^*T exists, we are interested in estimating the Lelong number of the pull-back $\nu(f^*T,0)$ in terms of $\nu(T,0)$. Our theorem can be stated as follows.

THEOREM 2. — *Let $f\colon (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ be a holomorphic map. Then the following conditions are equivalent:*

1) *the map f has generic (maximal) rank equal to n;*
2) *for any positive closed current T of bidegree $(1,1)$ f^*T is well defined, and the operator f^* is continuous for the weak topology of currents;*
3) *the range of f is not pluripolar;*
4) *for any positive closed current T of bidegree $(1,1)$ f^*T is well defined, and there exists a constant $C > 0$ (depending only on f) such that one has the inequality*

$$\nu(T,0) \leq \nu(f^*T,0) \leq C \cdot \nu(T,0)$$

between Lelong numbers of T and f^*T at the origin.

REMARK 3. — The proof gives an estimate for the constant C above. Assume $n = m$ and 1) is satisfied. Then 4) holds with

$$C = 1 + 2\big(\mu(Jf,0) + n - 1\big),$$

where $\mu(Jf,0)$ is the order of vanishing of the Jacobian determinant of f at 0.

Using this remark, we also have a semi local version of Theorem 2.

COROLLARY 4. — *Let X and Y be two connected complex manifolds, and $f\colon X \to Y$ be a holomorphic map whose generic rank is maximal equal to* $\dim(Y)$. *Then for any compact set $K \subset X$, there exists a constant $C_K > 0$ such that for all positive closed current T of bidegree $(1,1)$ and all $p \in K$, one has the inequality*

$$\nu(T,p) \leq \nu(f^*T,p) \leq C_K \cdot \nu\big(T,f(p)\big)$$

between Lelong numbers.

Before giving a proof of this theorem and of its corollary, we will make some remarks about the stated results.

The main result of Theorem 2 is contained in the implication 1) $\Rightarrow$ 4). All the others are either obvious, or were known before.

The second assertion is contained in [M96]. We also refer the reader to this article for more general problems concerning pull-back of positive closed currents by holomorphic mappings.

The upper estimate given in 4) was already known in several different cases (the other inequality is easy to prove).

PROPOSITION 5 (see [De93]). — *Let f be a finite holomorphic germ $(\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ of local degree d and T a positive closed current (of any bidegree). Then*

$$\nu(f^*T,0) \leq d \times \nu(T,0).$$

C. Kiselman also proved 1) $\Rightarrow$ 4) for monomial morphisms.

PROPOSITION 6 (see [K87]). — *Let $M = [a_{ij}] \in M(n,\mathbb{N})$ be an $n \times n$ matrix with non-negative integer coefficients. We assume that* $\det M \neq 0$. *If*

$$f(z) = \Big(\prod_{j=1}^{n} z_j^{a_{1j}}, \dots, \prod_{j=1}^{n} z_j^{a_{nj}}\Big)$$

then for any positive closed $(1,1)$ *current T*

$$\nu(f^*T,0) \leq \max_i \Big\{ \sum_j a_{ij} \Big\} \cdot \nu(T,0).$$

Diller in [D98] also proved the main estimate 4) for birational mappings of $\mathbb{P}^2$.

A warning concerning the implication 1) $\Rightarrow$ 3). When f does not have generic maximal rank, it is not true in general that the image of f is contained in a countable union of hypersurfaces. It is contained in a countable union of polydisks of dimension strictly less than n.

EXAMPLE 7 (see [H73, 4.2]). — *Define $f\colon (\mathbb{C}^3,0) \to (\mathbb{C}^3,0)$ by*

$$f(z,w,t) = (z, z\,\mathrm{e}^{w}, z\,\mathrm{e}^{\mathrm{e}^{w}}).$$

Note that f is independent of the last variable t. Then the set $f(\mathbb{C}^3,0)$ is pluripolar, but it is not included in a countable union of hypersurfaces.

Proof. — We give a short proof of these facts. We begin proving that $f(\mathbb{C}^3,0)$ is pluripolar. Decompose the mapping $f=\pi\circ g\circ p$ with

$$p(z,w,t)=(z,w),$$
$$g(x,y)=\big(y,\mathrm{e}^x,\mathrm{e}^{\mathrm{e}^x}\big),$$
$$\pi(z,w,t)=(z,zw,zt).$$

The range of g is included in the hypersurface $g(\mathbb{C}^2,0)\subset\{\mathrm{e}^w=t\}$, hence is pluripolar. The morphism π is an isomorphism outside $\{z=0\}$. As countable union of pluripolar sets remains pluripolar, we see that the image

$$\begin{aligned} f(\mathbb{C}^3,0) &= \pi\circ g\circ p(\mathbb{C}^3,0)=\pi\big(g(\mathbb{C}^2,0)\big)\\ &= \{0\}\bigcup_{k\geq 0}\pi\big(g(\mathbb{C}^2,0)\cap\{|z|>1/k\}\big)\end{aligned}$$

is also pluripolar.

For the second fact, we proceed as follows. Assume first that $f(\mathbb{C}^3,0)$ is included in an hypersurface defined by a non identically zero holomorphic map h. We thus have the identity

$$h\big(z,z\,\mathrm{e}^w,z\,\mathrm{e}^{\mathrm{e}^w}\big)=0$$

for every z,w in a neighborhood of $0\in\mathbb{C}$. Expand h in power series $h=\sum_{k\geq 0}h_k$ where h_k is a homogeneous polynomial of degree k in three variables. Take an index $k_0\in\mathbb{N}$ such that $h_{k_0}\not\equiv 0$. Then

$$h_{k_0}\big(z,z\,\mathrm{e}^w,z\,\mathrm{e}^{\mathrm{e}^w}\big)=z^{k_0}h_{k_0}\big(1,\mathrm{e}^w,\mathrm{e}^{\mathrm{e}^w}\big)\equiv 0.$$

This would contradict the fact that the three functions $(1,\mathrm{e}^w,\mathrm{e}^{\mathrm{e}^w})$ are algebraically independent.

Now assume $f(\mathbb{C}^3,0)\subset\bigcup_{n\in\mathbb{N}}H_n$ is included in a countable union of hypersurfaces. For each $n\in\mathbb{N}$, the complex space $f^{-1}H_n$ is also an hypersurface by what preceeds. But we have

$$(\mathbb{C}^3,0)\subset f^{-1}f(\mathbb{C}^3,0)\subset\bigcup_{n\in\mathbb{N}}f^{-1}H_n,$$

which can not contain any open subset of $(\mathbb{C}^3,0)$. □

Finally, a word about the motivations of this article. The author came to the problem of estimating Lelong numbers of pull-back of positive closed $(1,1)$ current while working on dynamics of rational maps of the projective space $f\colon \mathbb{P}^k \to \mathbb{P}^k$ with maximal generic rank. Let us give a simple application of Theorem 2 in this context. We first recall some well-known facts which can be found for instance in [Si99].

We let $\pi\colon \mathbb{C}^{k+1} - \{0\} \to \mathbb{P}^k$ be the natural projection onto $\mathbb{P}^k$, and take $F = (F_0, \cdots, F_k)\colon \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ a polynomial lift of f so that

$$F \circ \pi = \pi \circ f.$$

We assume that the $k+1$ polynomials $\{F_i\}_{0 \le i \le k}$ do not contain any common factors. The indeterminacy set of f is equal to

$$I(f) := \pi\Big(\bigcap_{i=0}^{k} F_i^{-1}\{0\}\Big).$$

Given any positive closed current T of bidegree $(1,1)$ on $\mathbb{P}^k$, one can find a psh function G on $\mathbb{C}^{k+1}$, called its *potential*, such that

1) there exists a constant $c > 0$ for which for all $Z \in \mathbb{C}^{k+1}$ and for all $\lambda \in \mathbb{C}$,

$$G(\lambda Z) = c \log |\lambda| + G(Z);$$

2) $\pi^* T = \mathrm{dd}^c G$.

Conversely, given a psh function G on $\mathbb{C}^{k+1}$ satisfying the homogeneity condition 1), one can find a unique positive closed current T of bidegree $(1,1)$ on $\mathbb{P}^k$ such that 2) holds.

Definition 8. — Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a rational map of maximal generic rank k, and T be a positive current of bidegree (1,1) with potential G. We define f^*T to be the positive closed current of bidegree (1,1) whose potential is $G \circ F$.

The study of the operator f^* turns out to give many interesting informations on f and on its dynamics (see [Si99]). When f is not holomorphic, for any positive closed current T of bidegree $(1,1)$, the current f^*T admits singularity points even if T has a smooth potential. The computation of Lelong numbers of f^*T can be viewed as a quantitative measure of how bad the singularities of this current are. The estimate 4) allows us to extend a result of [D98].

PROPOSITION 9. — *Let $f\colon \mathbb{P}^k \to \mathbb{P}^k$ be a rational map with maximal generic rank and T be a positive closed current of bidegree* (1,1). *Then $\nu(f^*T,p) > 0$ if and only if either $p \in I(f)$ or $\nu(T,f(p)) > 0$.*

Proof. — Assume that $p \notin I(f)$. As f has generic maximal rank, we can apply Theorem 2. This yields a constant $C_f > 0$ such that

$$\nu\big(T, f(p)\big) \leq \nu(f^*T, p) \leq C_f \cdot \nu\big(T, f(p)\big).$$

And it follows that $\nu(f^*T, p) > 0$ if and only if $\nu(T, f(p)) > 0$. It remains to check that if p belongs to $I(f)$, then $\nu(f^*T, p) > 0$. Choose σ a local section of π around p, and $G \in \mathrm{PSH}(\mathbb{C}^{k+1})$ a potential for T. One can find a constant $A > 0$ so that

$$\big|F(\sigma(z))\big| \leq A|z - p|$$

for points z near p. As the function G satisfies an homogeneity relation, one can bound it by

$$G(Z) \leq B \log|Z| + O(1),$$

with $B > 0$. We thus have

$$G(F(\sigma(z)) \leq B \log|z - p| + O(1) \quad \text{and} \quad \nu(f^*T, p) \geq B > 0,$$

which concludes the proof. □

NOTE. — The main theorem has been proved independently by C.Kiselman (see [K99]) with a different method. His proof relies on volume estimates of sublevel sets of psh functions.

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2. Proof of the main theorem

We shall first prove the equivalence between the first three assertions. We conclude by proving 4) $\Rightarrow$ 3), and 1) $\Rightarrow$ 4).

1) $\Rightarrow$ 2). — We assume that f has generic maximal rank equal to n. If $u \in \mathrm{PSH}(\mathbb{C}^n, 0)$ is non degenerate, the psh function $u \circ f$ can not be identically $-\infty$ as the range of f contains some open ball. Hence f^*T is well-defined for any closed positive current T of bidegree $(1,1)$. For a sequence of positive closed $(1,1)$ current $T_j \to T$ converging weakly towards T, one can find a sequence u_j of psh potential of T_j converging in L^1_{loc} to u a psh potential for T. It remains to check that $u_j \circ f \to u \circ f$ in L^1_{loc}.

As f has maximal generic rank, $u_j \circ f \to u \circ f$ almost everywhere. Now one can extract a subsequence $u_{j_k} \circ f$ converging in L^1_{loc} to a psh function (see [Ho83] p.94). As any such limit should be equal to $u \circ f$, we infer $u_j \circ f \to u \circ f$ in L^1_{loc}, thus $f^*T_j \to f^*T$ in the weak topology.

2) $\Rightarrow$ 3). — If the range $f(\mathbb{C}^n, 0)$ is pluripolar, one can find $u \in \mathrm{PSH}(\mathbb{C}^n, 0)$ non-degenerate such that $u \circ f \equiv -\infty$. In that case, if $T := \mathrm{dd}^c u$, f^*T is not defined.

We also give an example of a sequence of positive closed currents of bidegree $(1,1)$ so that $T_n \to T$, f^*T_n and f^*T are all well-defined, but for which the sequence f^*T_n fails to converge to f^*T. For this, work in the unit ball, and take $f(z,w) = (0,w)$, $T_n = \mathrm{dd}^c u_n$, with

$$u_n(z,w) = \max\{n^{-1} \log |z|, -2 + |w|^2\}.$$

Then $T_n \to 0$ but $f^*T_n = \mathrm{dd}^c |w|^2$.

3) $\Rightarrow$ 1). — We only sketch the proof. We proceed by induction on m. Assume $f\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ is a holomorphic germ such that $\mathrm{rk}\, Df_z$ the rank of Df_z is smaller than $n-1$ for any $z \in (\mathbb{C}^m, 0)$. Set

$$N := \max\{\mathrm{rk}\, Df_z\} \leq n - 1,$$

and define for each $k \leq N$,

$$V_k := \{z \in (\mathbb{C}^m, 0)\,;\ \mathrm{rk} Df_z \leq k\}.$$

By assumption, V_N contains an open neighborhood of the origin. Define

$$W := V_N - V_{N-1}.$$

The set V_k is the set where all minors of Df_z of size $k+1$ have zero determinant, and hence defines a closed analytic subspace of $(\mathbb{C}^m, 0)$.

Hence W is a Zariski open set of V_N. Now, on W the rank of the differential of f is constant equal to N. We can thus apply locally the constant rank theorem. Take any countable covering $\{U_i\}_{i\in I}$ of W by open subsets such that for each $i \in I$, the set $f(U_i)$ is a (non-closed) analytic subset of $(\mathbb{C}^n, 0)$ of dimension $N < n$. For any $i \in I$ $f(U_i)$ is pluripolar. A countable union of pluripolar sets remains pluripolar, hence $f(W) = \bigcup_{i\in I} f(U_i)$ is pluripolar. As $\dim(V_{N-1}) < m$, we can apply the induction hypothesis to conclude that

$$f(\mathbb{C}^m, 0) = f(W) \cup f(V_{N-1})$$

is pluripolar too.

The implication 4) $\Rightarrow$ 3) follows from 2) $\Rightarrow$ 3).

In fact, we even have that when the range of f is pluripolar, the supremum of $(\nu(T,0))^{-1}\nu(f^*T,0)$ over all positive closed current T of bidegree $(1,1)$ for which f^*T is well-defined, is not finite.

Take $u \in \mathrm{PSH}(\mathbb{C}^n, 0)$ non-degenerate such that $u \circ f \equiv -\infty$. For any $\alpha > 0$, define

$$v_\alpha(z) := \max\bigl\{\alpha \log|z|, u(z) + \log|z|\bigr\}.$$

Then

$$\nu(f^*v_\alpha, 0) = \alpha \cdot \nu(\log|f|, 0), \quad \nu(v_\alpha, 0) = \min\bigl\{\alpha, \nu(u,0) + 1\bigr\}.$$

Hence for $\alpha \geq \nu(u,0) + 1$,

$$\bigl(\nu(\mathrm{dd^c} v_\alpha, 0)\bigr)^{-1}\nu(f^*\mathrm{dd^c} v_\alpha, 0) = C\alpha$$

with $C = (\nu(u,0)+1)^{-1}\nu(\log|f|, 0)$.

1) $\Rightarrow$ 4). — Let us first prove the following general result.

LEMMA 10. — *If $f\colon (\mathbb{C}^m,0) \to (\mathbb{C}^n,0)$ is an arbitrary holomorphic germ, and T is a positive closed current of bidegree $(1,1)$ so that f^*T is well-defined, one has the inequality*

$$\nu(f^*T,0) \geq \nu(T,0)$$

between Lelong numbers.

Proof. — We fix $u \in \mathrm{PSH}(\mathbb{C}^n, 0)$ a local potential for T. We always have $|f(Z)| \leq A|Z|$ for some constant, so that the estimate

$$u(Z) \leq \nu(T,0)\log|Z| + O(1)$$

implies

$$u\bigl(f(Z)\bigr) \leq \nu(T,0)\log|Z| + O(1)$$

which gives us the stated inequality. ☐

We now proceed with the proof of the upper bound for $\nu(f^*T,0)$ given in 4). As before, u will denote a local potential for T.

Let us show how to reduce the proof of this estimate to the equidimensional case *i.e.* when $n=m$.

We assume the estimate has already been proved for $n=m$. By assumption, the rank of the Jacobian derivative of f is generically equal to n. We can therefore find a closed embedding

$$i\colon L=(\mathbb{C}^n,0)\hookrightarrow(\mathbb{C}^m,0)$$

of a piece of n-plane into $(\mathbb{C}^m,0)$ such that the rank of the Jacobian derivative of the restriction

$$\bar{f}\ :=f\circ i$$

to $(\mathbb{C}^n,0)$ is also generically equal to n. We can now apply the estimate to $\bar{f}$ and use Lemma 10. We get

$$\nu(f^*T,0)\le\nu(i^*\circ f^*T,0)\le\nu(\bar{f}^*T,0)\le C_{\bar{f}}\cdot\nu(T,0).$$

Let us deal now with the equidimensional case. The assumption on f can be rewritten as its Jacobian derivative does not vanish identically on a neighborhood of the origin.

Take a line L passing through 0 intersecting $\mathrm{Crit}(f)$ the critical set of f only at 0, and not tangent to any irreducible component of $\mathrm{Crit}(f)$. We can assume it is given in coordinates $z=(z_1,\cdots,z_n)$ by

$$L\ :=\{z_2=\cdots=z_n=0\}.$$

We can find an open cone around this line L

$$\mathcal{C}\ :=\big\{z\in U\,;\ \mathrm{dist}(z,L)<\varepsilon|z|\big\}$$

such that $\mathcal{C}\cap\mathrm{Crit}(f)=\emptyset$.

Instead of working in this cone, it is more convenient to work on an open set. We thus consider the blow-up π of the origin 0, and replace the germ f by the composition $g\ :=f\circ\pi$. In coordinates,

$$\pi(z)=(z_1,z_1z_2,\ldots,z_1z_n).$$

We look at g in the open set $\overline{\pi^{-1}\{\mathcal{C}\}}$. Define

$$E=\pi^{-1}\{0\}=\{z_1=0\}.$$

Let us point out some special properties of the map g.

1) $\mathrm{Crit}(g)=E$.

2) $g^{-1}\{0\}=E$.

We can thus write the Jacobian determinant of g under the form

$$Jg(z) = z_1^N \psi(z)$$

for some integer $N \in \mathbb{N}$ and some holomorphic function ψ which does not vanish at any point of E. In a sufficiently small neighborhood V of the origin, we can find a constant $C > 0$ such that for all $z \in V$

$$\big|Jg(z)\big| \geq C|z_1|^N. \tag{1}$$

For the proof of Remark 3 and Corollary 4, we will need the following estimation on the integer N. It gives precisely a control on the constant C of assertion 4) of the theorem.

LEMMA 11. — *The integer N introduced above can be chosen as*

$$N = \mu(Jf,0) + n - 1,$$

where $\mu(Jf,0)$ is the order of vanishing of the holomorphic function Jf at the point 0.

Proof. — Set $N_0 := \mu(Jf,0)$. We first check that for a (generic) suitable choice of line L, one has in a small cone $\mathcal{C}$ around L as above

$$\big|Jf(z)\big| \geq C|z|^{N_0}. \tag{2}$$

Expand the holomorphic jacobian determinant Jf in power series

$$Jf = \sum_{k \geq N_0} h_k$$

where h_k is a homogeneous polynomial of degree k and h_{N_0} is not identically zero. Let $\mathbb{P}^{n-1}$ be the set of complex lines in $\mathbb{C}^n$ passing through the origin, and for a point $z \in \mathbb{C}^n$ set $L_z = \mathbb{C}z$. By homogeneity of h_{N_0}, one can define the continuous function $H \colon \mathbb{P}^{n-1} \to \mathbb{R}_+$ by

$$H(L_z) = |z|^{-N_0}\big|h_{N_0}(z)\big|.$$

Take a generic line L such that $H(L) > 0$. Then for all lines L' close to L, one has $H(L') \geq \frac{1}{2}H(L)$. Hence in a small cone $\mathcal{C}$ around L, one has $H(L_z) \geq \frac{1}{2}H(L)$.

We infer for all $z \in \mathcal{C}$,

$$\begin{aligned}|f(z)| &\geq \Big|h_{N_0}(z) - \sum_{k\geq N_0+1} h_k(z)\Big| \\ &\geq |h_{N_0}(z)| - \Big|\sum_{k\geq N_0+1} h_k(z)\Big| \\ &\geq 2^{-1}H(L)|z|^{N_0} - C'|z|^{N_0+1} \geq C|z|^{N_0},\end{aligned}$$

for some constants $C, C' > 0$.

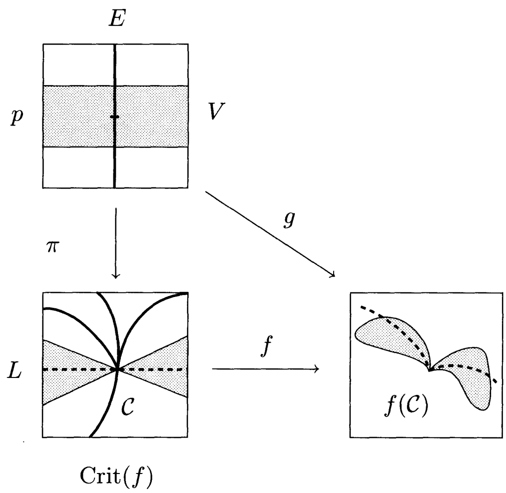
Now a direct computation yields

$$\det(D\pi_z) = z_1^{n-1}.$$

Therefore, if we have chosen a line L so that equation (2) applies, we get for all $z \in \mathcal{C}$,

$$\begin{aligned}\big|\det(Dg_z)\big| &= \big|\det(D\pi_z) \cdot \det(Df_{\pi(z)})\big| \\ &\geq |z_1|^{n-1} \cdot C|z_1|^{\mu(Jf,0)},\end{aligned}$$

which concludes the proof of Lemma 11. ∎



In the sequel, we will assume that V is a small ball in $\mathbb{C}^n$ endowed with the usual euclidean metric. If $r > 0$ and K is a compact set, we set

$$B(K, r) := \{z\,;\ \mathrm{dist}(z, K) < r\}.$$

The key lemma is:

LEMMA 12. — *There exists two integers* $N_0, N_1 \in \mathbb{N}^*$, *and two positive constants* $C_0, C_1 > 0$ *such that for all* $z \in V$,

$$g\big(B(z, C_0|z_1|^{N_0})\big) \supset B\big(g(z), C_1|z_1|^{N_1}\big).$$

Moreover, we can choose $N_0 = N + 1$, *and* $N_1 = 2N + 1$ (*with the above notations*).

Proof. — The idea is to approximate the range of $g(B(z,r))$ by $Dg_z(B(z,r))$ and estimate the size of the latter.

We have $|Jg(z)| \geq C|z_1|^N$ for all $z \in V$. In V, all eigenvalues of Dg_z are uniformly bounded by some constant $D > 0$. Therefore for all $z \in V - E$,

$$\big|Dg_z^{-1}\big|^{-1} \geq \inf\big\{|\lambda|\ ;\ \lambda \in \mathrm{Spec}(Dg_z)\big\} \geq \frac{C}{D^{n-1}}|z_1|^N.$$

And for all $z \in V$, for all $r > 0$,

$$Dg_z\big(B(z,r)\big) \supset B\big(g(z), C'|z_1|^N r\big),$$

for some constant $C' > 0$. Now by Taylor's formula, there exists another constant $C'' > 0$ such that for all $z, w \in V$,

$$\big|g(w) - g(z) - Dg_z \cdot (w - z)\big| \leq C''|w - z|^2.$$

If we choose $M > N$ and take $r = |z_1|^M$, we infer for z sufficiently small

$$g\big(B(z, |z_1|^M)\big) \supset B\big(g(z), C'|z_1|^{N+M} - C''|z_1|^{2M}\big),$$

which gives the desired result with $N_1 = N + M$. □

To conclude, we follow Diller [D98]. Define

$$\Delta_r := L \cap \big\{|z| \leq r\big\}.$$

We first apply Lemma 12 to each point of the set $\partial\Delta_r$. We obtain

$$g\big(B(\partial\Delta_r, C_0 r^{N_0})\big) \supset B\big(\partial g(\Delta_r), C_1 r^{N_1}\big). \tag{3}$$

We consider now translated of $g(\Delta_r)$ by vectors z of norm $|z| < C_1 r^{N_1}$. The estimate (3) tells us that $\partial(z + g(\Delta_r))$ is still included in the range of g. We have more precisely for all $|z| \leq C_1 r^{N_1}$,

1) $z \in z + g(\Delta_r)$,
2) $\partial(z + g(\Delta_r)) \subset g\,\big(B(\partial\Delta_r, C_0 r^{N_0})\big)$.

We are now in position to prove the desired inequality. We start with

$$u\big(g(z)\big) \leq \nu(g^*u,0)\log|z| + D$$

for some constant $D \in \mathbb{R}$. We want to prove an analog estimate for u. Fix $z \in V$ and $r > 0$ such that $|z| < C_1 r^{N_1}$. Then the maximum principle applied to u on the analytic disk $z + g(\Delta_r)$ yields

$$\begin{aligned} u(z) &\leq \max_{z+g(\Delta_r)} u \leq \max_{\partial(z+g(\Delta_r))} u \\ &\leq \max_{g(B(\partial\Delta_r, C_0 r^{N_0}))} u \\ &\leq \max_{w\in B(\partial\Delta_r, C_0 r^{N_0}))} u\big(g(w)\big) \\ &\leq \max_{w\in B(\partial\Delta_r, C_0 r^{N_0}))} \nu(g^*u,0)\log|w| + D \\ &\leq \nu(g^*u,0)\log r + D' \end{aligned}$$

for $D' := D + \nu(g^*u,0)\log(\frac{3}{2})$ (by possibly reducing C_0 we can assume that $C_0 r^{N_0-1} \leq \frac{1}{2}$). As this is true for any r satisfying $|z| \leq C_1 r^{N_1}$, we obtain

$$u(z) \leq \frac{1}{N_1}\nu(g^*u,0)\log|z| + D''.$$

Thus $\nu(u,0) \geq N_1^{-1}\nu(g^*u,0)$. To conclude the proof we use the general inequality in Lemma 10

$$\nu(u,0) \geq \frac{1}{N_1}\nu(g^*u,0) \geq \frac{1}{N_1}\nu\big((f\circ\pi)^*u,0\big) \geq \frac{1}{N_1}\nu(f^*u,0).$$

The proof combined with Lemmas 11 and 12 gives more precisely (see Remark 3):

LEMMA 13. — *If $f\colon (\mathbb{C}^n,z) \to (\mathbb{C}^n,f(z))$ is a germ of holomorphic map of maximal generic rank, then for any positive closed current T of bidegree $(1,1)$, one has the inequality*

$$\nu(f^*T,z) \leq \big(2(n-1+\mu(Jf,0))+1\big)\cdot\nu\big(T,f(z)\big)$$

between Lelong numbers.

Proof of Corollary 4. — We localize first the problem and assume that $X = B^m(0,1)$, $Y = B^n(0,1)$ are unit balls respectively in $\mathbb{C}^m$ and $\mathbb{C}^n$. As before, it is sufficient to prove it in the equidimensional case *i.e.* $X = Y = B^n(0,1)$.

As f has maximal generic rank, we can apply Lemma 13 at each point $z \in K$. Now on the compact set K, the function $z \mapsto \mu(Jf, z)$ is upper semi continuous, hence bounded above by a constant C_K. This yields Corollary 4. □

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