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NON-COMPACT COHOMOGENEITY ONE

EINSTEIN MANIFOLDS

BY CHRISTOPH BÖHM (*)

ABSTRACT. — We describe dynamical properties of the cohomogeneity one Einstein equation. For instance we obtain a new Lyapunov function and a decoupling in the Ricci flat case. By applying these results we get complete cohomogeneity one Einstein metrics with negative and zero Einstein constant for trivial vector space bundles over products of isotropy irreducible homogeneous spaces. These Einstein metrics appear in high-dimensional families. The geometry of the Ricci flat examples is especially well understood.

RÉSUMÉ. — DYNAMIQUE DES ÉQUATIONS D'EINSTEIN EN COHOMOGÉNÉITÉ 1. On décrit des propriétés de la dynamique des équations d'Einstein en cohomogénéité 1. En particulier, on obtient une nouvelle fonction de Lyapunov et un découplage dans le cas Ricci plat. Comme application de ces résultats, on construit des métriques d'Einstein complètes, de cohomogénéité 1, sur l'espace total de fibrés triviaux sur des produits d'espaces homogènes à isotropie irréductible, avec des constantes d'Einstein négatives ou bien nulles. Ces métriques d'Einstein appartiennent à des familles de grande dimension. Le cas Ricci plat est particulièrement bien compris.

A Riemannian metric \hat{g} on \widehat{M} is called *Einstein* if the Ricci tensor is a multiple of the metric. By the Theorem of Bonnet-Myers the Einstein constant is non-positive if \widehat{M} is non-compact.

If (\widehat{M}, \hat{g}) is a homogeneous non-compact Ricci flat manifold, then \hat{g} is flat [3], hence \widehat{M} is the product of a torus by a Euclidean space (cf. [5, 7.61]). If $(\widehat{G}/\widehat{K}, \hat{g})$ is a homogeneous Einstein manifold with negative scalar curvature, then it has been conjectured by D.V. Alekseevskii, that \widehat{K} is a maximal compact subgroup of \widehat{G} (cf. [5, 7.57], cp. [2]). In particular \widehat{G}/\widehat{K} is homeomorphic to \mathbb{R}^{n+1} [26]. Concerning homogeneous Einstein metrics with negative scalar curvature we refer to the very recent work of Heber [25].

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In this article we consider only complete non-compact cohomogeneity one Einstein manifolds even though non-complete ones have been studied, see *e.g.* [20]. Furthermore, many of the following examples are first of all Kähler, hyperkähler or they have special holonomy. In 1975 Calabi [11] described cohomogeneity one Kähler-Einstein metrics with negative scalar curvature. The $U(2)$ -invariant Taub-NUT metric, discovered by Hawking [24], is hyperkähler. Then Eguchi and Hanson [18] described a hyperkähler metric on T^*S^2 . This was generalized by Calabi [12] who proved that $T^*\mathbb{C}P^n$ carries a cohomogeneity one hyperkähler metric. More general the cotangent bundle of a compact symmetric space of rank one carries a Ricci flat Kähler metric of cohomogeneity one [35].

Cohomogeneity one Kähler-Einstein metrics and hermitian Einstein metrics on holomorphic line bundles over a product of Kähler manifolds can be found among the bundle constructions in [12], [4], [33], [36], [37] and [17]. A hyperkähler cohomogeneity one 4-manifold was described by Atiyah and Hitchin [1]. Finally we mention the construction of explicit cohomogeneity one Einstein metrics with holonomy type G_2 and $Spin(7)$ (*cf.* [10], [23]).

Now we turn to our main results. Let G be a compact Lie group acting on \widehat{M}^{n+1} with cohomogeneity one. Let $P = G/K$ be the principal orbit type and let \hat{g} be a G -invariant metric on \widehat{M}^{n+1} . We can write

$$\hat{g} = dt^2 + g(t)$$

where $g(t)$ is a smooth curve of G -invariant metrics on P . The cohomogeneity one Einstein equation for \hat{g} is given by an ordinary differential equation for $g(t)$ (see [20]). We obtain a Lyapunov function whose critical points correspond to G -invariant Einstein metrics on P (*cf.* Section 2, (8)). Now let

$$\bar{g}(t) = V^{-\frac{2}{n}}(t)g(t)$$

denote the “unimodular part” of $g(t)$, *i.e.* $\bar{g}(t)$ has volume one. In the Ricci flat case the cohomogeneity one equation for \hat{g} decouples with respect to \bar{g} and V (see Section 3).

Now we turn to our main application. In advance we remark that $\mathbb{R}^3 \times S^2$ and $\mathbb{R}^3 \times \mathbb{R}P^2$ are the lowest-dimensional examples which Theorem A yields.

THEOREM A. — *Let $r \geq 0$, let $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ be non-flat compact isotropy irreducible homogeneous spaces and let $k \geq 2$. Then*

$$\widehat{M} = \mathbb{R}^{k+1} \times G_1/K_1 \times G_2/K_2 \times \cdots \times G_{r+1}/K_{r+1}$$

carries an $(r + 1)$ -dimensional family of Einstein metrics with negative scalar curvature and an r -dimensional family of Ricci flat metrics.

These metrics are invariant under the cohomogeneity one action of $G = \text{SO}(k + 1) \times G_1 \times \cdots \times G_{r+1}$ on \widehat{M} , however the full isometry group does not act transitively. If the conjecture of D.V. Alekseevskii turns out to be true, then these manifolds cannot carry any homogeneous Einstein metric. The principal orbit

$$P = S^k \times G_1/K_1 \times G_2/K_2 \times \cdots \times G_{r+1}/K_{r+1}$$

carries an explicit G -invariant Einstein metric with positive scalar curvature $n(n - 1)$, denoted by g_E . Therefore

$$dt^2 + \sinh^2(t)g_E \quad \text{and} \quad dt^2 + t^2g_E$$

are cohomogeneity one Einstein metrics on \widehat{M} which are smooth outside the singular orbit

$$Q = \{0\} \times G_1/K_1 \times G_2/K_2 \times \cdots \times G_{r+1}/K_{r+1}.$$

We obtain sequences of Einstein metrics with fixed Einstein constant which converge (restricted to a compact set not containing the singular orbit Q) to the above cone solution in the C^∞ -topology (cf. [15, 7.3]). The singular orbit Q is precisely the set of points where the sectional curvature blows up. In the Ricci flat case the geometry of these metrics is even better understood, namely the geometry of the principal orbits tends to the geometry of g_E for $t \rightarrow \infty$, i.e. $\lim_{t \rightarrow \infty} \bar{g}(t) = \bar{g}_E$. Since these metrics have Euclidean volume growth, their geometry at infinity is expected in view of the very general results of Cheeger and Tian [16] and Cheeger and Colding [15]. Supported by numerical results we remark that in the case of a negative Einstein constant we do not expect $\lim_{t \rightarrow \infty} \bar{g}(t) = \bar{g}_E$ (even though in some cases there seems to be a limit).

In the Ricci flat case the above mentioned sequence can be obtained by rescaling a single Ricci flat cohomogeneity one metric \hat{g} . We conjecture that $(\widehat{M}, q, r_j^{-2}\hat{g})$ converges to the above Ricci flat cone in the pointed Gromov-Hausdorff distance for every sequence $r_j \rightarrow \infty$ ($q \in Q$). In the case of a fixed negative Einstein constant these sequences can of course not be obtained by rescaling. However, as in the Ricci flat case these metrics have maximal volume growth, that is there exists $c > 0$ such that $\text{vol}(B_r(\hat{m})) \geq c \cdot \exp(nr)$ for large r .

The most important examples of compact isotropy irreducible homogeneous spaces are the compact irreducibles symmetric spaces, classified

by E. Cartan [13], [14]. But there exists many other isotropy irreducible homogeneous spaces, classified in [41] and [39]. For special choices of $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ the Einstein metrics of Theorem A admit many free isometric actions of finite groups. We just mention one example (cf. Theorem 6.2).

THEOREM B. — *Let $r \geq 0$ and let $k, k_1, k_2, \dots, k_{r+1} \geq 2$. Let Γ be a finite subgroup of $\mathrm{SO}(k+1) \times \mathrm{SO}(k_1+1) \times \dots \times \mathrm{SO}(k_{r+1}+1)$ acting freely on $S^k \times S^{k_1} \times \dots \times S^{k_{r+1}}$ and on $\{0\} \times S^{k_1} \times \dots \times S^{k_{r+1}}$. Then the quotient space*

$$\widehat{M}_\Gamma = (\mathbb{R}^{k+1} \times S^{k_1} \times S^{k_2} \times \dots \times S^{k_{r+1}}) / \Gamma$$

carries an $(r+1)$ -dimensional family of Einstein metrics with negative scalar curvature and an r -dimensional family of Ricci flat metrics.

A concrete example is \mathbb{R}^{k+1} times a product of spherical space forms. These space forms are classified by Wolf [40] and in odd dimensions there exist plenty of them (for instance lens spaces are amongst them).

In order to prove Theorem A, one makes the following general observation: Let g_E be a G -invariant Einstein metric on P , such that the scalar curvature functional, restricted to the G -invariant metrics on P of volume $\mathrm{vol}(P, g_E)$, attains at g_E a local non-degenerate minimum. Then the corresponding cone solution of the cohomogeneity one Einstein equation is a local attractor (for any Einstein constant) thanks to the above mentioned Lyapunov function. Furthermore in case of a non-positive Einstein constant λ , solutions which enter an attracting region, remain in finite distance to the cone solution. In particular they have an infinite interval of existence. We assume now the existence of a singular orbit Q . Since the cases under consideration fit into the more general framework of Eschenburg and Wang [20], there exist local solutions of the Einstein equation, which correspond to smooth G -invariant Einstein metrics on a tubular neighbourhood of Q . We are able to find such local solutions which converge to the cone solution extending results in [6].

The content of this paper is as follows: In Section 1 we derive the cohomogeneity one Einstein equation following [20]. In Section 2 we describe dynamical properties of this equation and the above mentioned Lyapunov function. The Ricci flat case is treated in Section 3. In Section 4 these methods are applied to the stable cone case. In Section 5 the cohomogeneity one manifolds under investigation are described. Our main applications are stated in Section 6. In Section 7 we investigate the initial value problem for the cohomogeneity one manifolds described in Theorem A. In Section 8 we give technical preliminaries for the

proof of Convergence Theorem 9.7, provided in Section 9. In Section 10 entire families of complete Einstein metrics are obtained. Curvature computations show that two members of a family are not isometric. Furthermore the above mentioned convergence is described. The proof of Theorem 6.3 and Theorem 6.4 is given in Section 11.

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1. Einstein condition

Let G be a compact Lie group acting smoothly on a connected, $(n + 1)$ -dimensional manifold \widehat{M} with cohomogeneity one, *i.e.* the orbit space \widehat{M}/G has dimension one ($n > 1$). Let \hat{g} be a G -invariant metric on \widehat{M} , let $P = G/K$ be the principal orbit type and let \widehat{M}_0 denote the union of the principal orbits in \widehat{M} . We can identify \widehat{M}_0 with $I \times P$, where $I = \text{int}(\widehat{M}/G)$. Therefore, we can think of $\hat{g}|_{\widehat{M}_0}$ as

$$dt^2 + g(t)$$

where $g(t)$ is a smooth curve of G -invariant metrics on P . We will consider $g(t)$ as a smooth curve of G -equivariant isomorphisms on TP by fixing a G -invariant background metric g_b on P . Next, let $L(t)$ be the shape operator of $P_t = \{t\} \times P$ with respect to the outer unit normal $N_t = \partial/\partial t$ of P_t . We will think of $L(t)$ as a one-parameter family of G -equivariant endomorphisms on TP . We have

$$(1) \quad L(t) = \frac{1}{2}g^{-1}(t)g'(t).$$

Let Ric_t and $\widehat{\text{Ric}}$ denote the Ricci tensors of $(P_t, g(t))$ and (M, \hat{g}) respectively. By

$$\text{Ric}_t(\cdot, \cdot) = g(t)(r(t)\cdot, \cdot)$$

we can define the Ricci endomorphism $r(t)$ of TP_t and we will consider $r(t)$ as a one-parameter family of G -equivariant endomorphisms on TP .

REMARK. — The shape operator $L(t)$ and the Ricci endomorphism $r(t)$ are of course symmetric with respect to the G -invariant metric $g(t)$ but not in general with respect to the background metric g_b .

Due to [20], Section 2 the Einstein condition for $\hat{g}|_{\widehat{M}_0}$ with respect to the Einstein constant λ is given by

$$(2) \quad g''(t) = g'(t)g^{-1}(t)g'(t) - \frac{1}{2}(\text{tr } g^{-1}(t)g'(t)) \cdot g'(t) + 2g(t)r(t) - 2\lambda \cdot g(t),$$

$$(3) \quad \text{tr } L'(t) = -\text{tr}(L^2(t)) - \lambda,$$

$$(4) \quad \widehat{\text{Ric}}(N_t, TP_t) = 0.$$

The equations (2), (3) and (4) constitute a system which is not overdetermined.

PROPOSITION 1.1 (see [20]). — *If $g(t)$ is a solution of (2), such that $dt^2 + g(t)$ satisfies (3) and (4) in $t_0 > 0$, then $dt^2 + g(t)$ is Einstein on $(t_0 - a, t_0 + a) \times P$ for some $a > 0$.*

Now let us assume that there exists a connected singular orbit Q , that is an orbit whose dimension is strictly smaller than $\dim P$. In particular \widehat{M}/G is homeomorphic to $[0, \infty)$ or to a compact interval (cf. [9, p. 206] and [4]).

PROPOSITION 1.2 (see [20]). — *If $g(t)$ is a solution of (2), such that $dt^2 + g(t)$ can be extended to a C^3 -metric \hat{g} on a tubular neighbourhood of the singular orbit Q , then \hat{g} is a C^∞ -Einstein metric on this tubular neighbourhood.*

2. Dynamical properties of the Einstein equation

Let $\text{Sym}(TP, G)$ denote the space of G -equivariant and with respect to g_b symmetric endomorphisms on TP , let

$$\text{Sym}_+(TP, G) := \{g \in \text{Sym}(TP, G) \mid g \text{ positive definite}\}$$

and let

$$\mathcal{F} := \text{Sym}_+(TP, G) \times \text{Sym}(TP, G).$$

Vice versa we think of $g \in \text{Sym}_+(TP, G)$ as a G -invariant metric on P via g_b . Since the Ricci endomorphism $r(g)$ of g is a rational function of $g \in \text{Sym}_+(TP, G)$ (cf. [5, p. 185]), (2) is a second order differential equation. We consider solutions of (2) as integral curves of the vector field

$$X_\lambda(g, h) = \left(\begin{array}{c} h \\ hg^{-1}h - \frac{1}{2}(\text{tr } g^{-1}h) \cdot h + 2gr(g) - 2\lambda \cdot g \end{array} \right),$$

where $(g, h) \in \mathcal{F}$ and $\lambda \in \mathbb{R}$. By (1), equation (2) is equivalent to

$$L'(t) = -(\text{tr } L(t)) \cdot L(t) + r(t) - \lambda \cdot I_n.$$

If we take the trace of this equation, then we obtain with the help of (3)

$$(5) \quad (\text{tr } L(t))^2 - \text{tr } L^2(t) = s(t) - (n - 1)\lambda$$

where $s(t)$ denotes the scalar curvature of $(P_t, g(t))$. Let

$$e_\lambda : \mathcal{F} \rightarrow \mathbb{R}; (g, h) \mapsto \frac{1}{4}(\text{tr } g^{-1}h)^2 - \frac{1}{4} \text{tr}(g^{-1}hg^{-1}h) - s(g) + \lambda(n - 1)$$

where $s(g)$ denotes the scalar curvature of $g \in \text{Sym}_+(TP, G)$.

LEMMA 2.1. — *If the principal orbit type P is not a torus or if $\lambda \neq 0$, then*

$$\mathcal{E}_\lambda := e_\lambda^{-1}(0)$$

is a smooth hypersurface in \mathcal{F} invariant under the flow of X_λ . Furthermore X_λ has no zeros on \mathcal{E}_λ .

Proof. — Let $(\tilde{g}, \tilde{h}) \in \mathcal{E}_\lambda$. We have

$$\frac{\partial e_\lambda}{\partial h}(\tilde{g}, \tilde{h}) = \frac{1}{2}(\text{tr}(\tilde{g}^{-1}\tilde{h}) \cdot \text{tr}(\tilde{g}^{-1} \cdot) - \text{tr}(\tilde{g}^{-1}\tilde{h}\tilde{g}^{-1} \cdot)).$$

Hence $\partial e_\lambda / \partial h(\tilde{g}, \tilde{h}) = 0$ implies

$$\text{tr}(\tilde{g}^{-1}\tilde{h}) \cdot \text{tr}(\tilde{g}^{-1}\tilde{g}) - \text{tr}(\tilde{g}^{-1}\tilde{h}\tilde{g}^{-1}\tilde{g}) = 0.$$

By $n > 1$ we get $\text{tr}(\tilde{g}^{-1}\tilde{h}) = 0$. Thus

$$\frac{\partial e_\lambda}{\partial h}(\tilde{g}, \tilde{h}) = -\frac{1}{2} \text{tr}(\tilde{g}^{-1}\tilde{h}\tilde{g}^{-1} \cdot) = 0,$$

in particular $\text{tr}(\tilde{g}^{-1}\tilde{h}\tilde{g}^{-1}\tilde{h}) = 0$. We conclude $\tilde{h} = 0$. Since

$$\frac{\partial e_\lambda}{\partial g}(\tilde{g}, 0) = \text{tr}(r(\tilde{g})\tilde{g}^{-1} \cdot)$$

we obtain $(e_\lambda)_*(\tilde{g}, \tilde{h}) = 0$ if and only if $\tilde{h} = 0$, \tilde{g} is a (homogeneous) Ricci flat metric on P and $\lambda = 0$ by $(\tilde{g}, \tilde{h}) \in \mathcal{E}_\lambda$. Hence P is a torus by [3] and [5, 7.71].

In order to show that \mathcal{E}_λ is invariant under the flow of X_λ , just derive (5) and use $d/dt s(t) = -2 \text{tr}(r(t)L(t))$.

If $(\tilde{g}, \tilde{h}) \in \mathcal{F}$ is a zero of X_λ , then $\tilde{h} = 0$. Thus $r(\tilde{g}) = \lambda \cdot I_n$. If $(\tilde{g}, \tilde{h}) \in \mathcal{E}_\lambda$, then $\lambda = 0$, hence $r(\tilde{g}) = 0$. \square

REMARK. — If $P = G/K$ is homeomorphic to a torus T^n and G_0 (the connected component of G which contains the identity) acts effectively on P , then G_0 acts freely and is as a group isomorphic to T^n [31]. Furthermore, (2) simplifies since any G -homogeneous metric on P is flat [30]. In this case the Einstein equation is explicitly solvable (cf. (12) and (13)), hence a classification should be possible.

Now we describe a new Lyapunov function of the vector field X_λ restricted to \mathcal{E}_λ . For $g \in \text{Sym}_+(TP, G)$ let

$$V(g) := \sqrt{\det g}$$

and let

$$(6) \quad \bar{g} := V^{-\frac{2}{n}}(g) \cdot g$$

be the unimodular part of g . For $L \in \text{Sym}(TP, G)$ let

$$(7) \quad L_0 := L - \frac{1}{n} \text{tr } L \cdot \mathbf{I}_n$$

denote the trace free part of L . Now we can define the Lyapunov function

$$(8) \quad \kappa : \mathcal{E}_\lambda \rightarrow \mathbb{R}; (g, h) \mapsto \frac{1}{4} V^{\frac{2}{n}}(g) \text{tr}((g^{-1}h)_0)^2 + s(\bar{g}).$$

Let $g(t) \in \text{Sym}_+(TP, G)$ be a smooth curve and let

$$L(g(t)) := \frac{1}{2} g^{-1}(t) g'(t)$$

(cf. (1)). Then

$$(9) \quad \frac{\frac{d}{dt} V(g(t))}{V(g(t))} = \text{tr } L(g(t)).$$

PROPOSITION 2.2. — *Let $(g(t), h(t)) \in \mathcal{E}_\lambda$ be an integral curve of X_λ . Then*

$$\frac{d}{dt} \kappa(g(t), h(t)) = -2 \frac{n-1}{n} \cdot V^{\frac{2}{n}}(g(t)) \cdot \text{tr } L(g(t)) \cdot \text{tr}(L(g(t))_0)^2.$$

Proof. — By

$$\text{tr } L(g(t))^2 = \text{tr}(L(g(t))_0)^2 + \frac{1}{n} (\text{tr } L(g(t)))^2$$

we can rewrite (5) as

$$V^{\frac{2}{n}}(g(t)) \left[\frac{n-1}{n} (\text{tr } L(g(t)))^2 + \lambda(n-1) \right] = \kappa(g(t), h(t)).$$

By (3) and (9) the claim follows. \square

Since $L(g(t))_0$ is symmetric with respect to the metric $g(t)$, we have $\text{tr}(L(g(t))_0)^2 \geq 0$. Therefore, we investigate the levels of κ in the domain

$$(10) \quad \mathcal{I}_\lambda := \{(g, h) \in \mathcal{E}_\lambda \mid s(g) - \lambda(n - 1) > 0, \text{tr } g^{-1}h > 0\}$$

because κ decreases along integral curves of X_λ in \mathcal{I}_λ .

PROPOSITION 2.3. — *Suppose that P is not a torus. Let $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$. Then (\tilde{g}, \tilde{h}) is a critical point of κ if and only if \tilde{g} is Einstein and $(\tilde{g}^{-1}\tilde{h})_0 = 0$.*

Proof. — Let $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$. If \tilde{g} is Einstein and $(\tilde{g}^{-1}\tilde{h})_0 = 0$, then (\tilde{g}, \tilde{h}) is a critical point of κ (cf. [5, 4.23]). Vice versa let (\tilde{g}, \tilde{h}) be a critical point of κ . Since $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$, Proposition 2.2 yields $\text{tr}((\tilde{g}^{-1}\tilde{h})_0)^2 = 0$, thus $(\tilde{g}^{-1}\tilde{h})_0 = 0$. However $\tilde{h} \neq 0$ by $s(\tilde{g}) - \lambda(n - 1) > 0$. Now let B_ϵ be a small neighbourhood of (\tilde{g}, \tilde{h}) in \mathcal{I}_λ . Since $\partial e_\lambda / \partial h(\tilde{g}, \tilde{h}) \neq 0$ by $\tilde{h} \neq 0$ (see the proof of Lemma 2.1) the canonical projection of $B_\epsilon \subset \mathcal{I}_\lambda \subset \mathcal{F}$ onto $\text{Sym}_+(TP, G)$ constitutes a small neighbourhood of \tilde{g} . Therefore \tilde{g} has to be Einstein by [5, 4.23]. \square

Now suppose that g_E is an Einstein metric on P with Ricci curvature $(n - 1) > 0$. Then

$$(11) \quad \begin{cases} \gamma_n(t) = (\sin^2(t)g_E, \sin(2t)g_E) & t \in (0, \pi), \\ \gamma_0(t) = (t^2g_E, 2tg_E) & t > 0, \\ \gamma_{-n}(t) = (\sinh^2(t)g_E, \sinh(2t)g_E) & t > 0, \end{cases}$$

are explicit integral curves of X_λ on \mathcal{E}_λ for $\lambda = n, \lambda = 0$ and $\lambda = -n$. By rescaling we obtain explicit integral curves γ_λ of X_λ for $\lambda \in \mathbb{R}$. These solutions are called *cone solutions*.

COROLLARY 2.4. — *Suppose that P is not a torus. Let $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$. Then (\tilde{g}, \tilde{h}) is a critical point of κ if and only if (\tilde{g}, \tilde{h}) lies on a cone solution.*

Proof. — If $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$ lies on a cone solution, then it is a critical point of κ by Proposition 2.3. Vice versa let $(\tilde{g}, \tilde{h}) \in \mathcal{I}_\lambda$ be a critical point of κ . We can restrict ourselves to the cases $\lambda = n, 0, -n$. By Proposition 2.3

$$\tilde{g} = \alpha^2 g_E \quad \text{and} \quad \tilde{h} = 2\beta g_E$$

where g_E is Einstein with scalar curvature $n(n - 1)$, $\alpha > 0$ and $\beta > 0$ by (10). With the help of (5) we obtain $\beta^2 = \alpha^2 - \text{sign}(\lambda)\alpha^4$. Hence in case $\lambda = n$ we find $t_0 \in (0, \frac{1}{2}\pi)$ and in case $\lambda = 0, -n$ we find $t_0 > 0$ such that $(\tilde{g}, \tilde{h}) = \gamma_\lambda(t_0)$. \square

REMARK. — The Lyapunov function $K(w, w', h)$ described in [6], (24) is different from κ . However, by refining the methods provided in Section 3 one can obtain $K(w, w', h)$ in a similar way as κ .

3. The Ricci flat case

As mentioned earlier (2) is equivalent to $L'(t) + (\text{tr } L(t))L(t) - r(t) = \lambda \cdot I_n$. Let $r_0(g(t))$ and $L_0(g(t))$ denote the trace free part of $r(t)$ and $L(g(t))$ respectively. Hence (2) and (3) are equivalent to

$$(12) \quad L'_0(g(t)) + \frac{\frac{d}{dt}V(g(t))}{V(g(t))}L_0(g(t)) - r_0(\bar{g}(t))V^{-\frac{2}{n}}(g(t)) = 0,$$

$$(13) \quad \frac{d^2}{dt^2}V(g(t)) = V(g(t))(s(\bar{g}(t))V^{-\frac{2}{n}}(g(t)) - \lambda n),$$

$$(14) \quad \frac{n-1}{n} \frac{\left(\frac{d}{dt}V(g(t))\right)^2}{V^2(g(t))} - \text{tr}(L_0(g(t)))^2 = s(g(t)) - \lambda(n-1),$$

(see [7]). Let τ be an anti-derivative of $V^{-\frac{1}{n}} \circ g$, *i.e.*

$$d\tau = (V^{-\frac{1}{n}} \circ g) dt.$$

Consider τ as the new time variable, that is think of $V \circ g$ and \bar{g} as functions depending on τ . By $d/dt = (V^{-1/n} \circ g) d/d\tau$ and $L_0(g(t)) = \frac{1}{2} \bar{g}^{-1}(t) \bar{g}'(t)$ the system (12), (13) and (14) is equivalent to

$$(15) \quad \bar{g}'' - \bar{g}' \bar{g}^{-1} \bar{g}' = -\frac{n-1}{n} \frac{V'}{V} \bar{g}' + 2\bar{g}r_0(\bar{g}),$$

$$(16) \quad \frac{V''}{V} - \frac{1}{n} \frac{V'^2}{V^2} + n\lambda V^{\frac{2}{n}} = s(\bar{g}),$$

$$(17) \quad \frac{n-1}{n} \frac{V'^2}{V^2} - \frac{1}{4} \text{tr}(\bar{g}^{-1} \bar{g}' \bar{g}^{-1} \bar{g}') = s(\bar{g}) - \lambda(n-1)V^{\frac{2}{n}},$$

where we have now dropped the argument τ or $g(\tau)$. Recall ' denotes now $d/d\tau$!

Let

$$\text{Uni}_+ := \{g \in \text{Sym}_+(TP, G) \mid \det(g) = 1\}$$

denote the unimodular endomorphisms in $\text{Sym}_+(TP, G)$. Uni_+ is a smooth manifold, actually a symmetric space (see below). Next we will show that (15) is a differential equation for $\bar{g}(\tau)$. This is not clear since in general $\bar{g}''(\tau) \in T_{\bar{g}(\tau)}\text{Uni}_+$ does not hold.

Let K be the isotropy group of a point $p \in P = G/K$. We can think about $\text{Sym}_+(TP, G)$ as the space of $Ad(K)$ -invariant symmetric positive definit matrices on an $Ad(K)$ -invariant complement \mathcal{P} of T_1K in T_1G . Let $P^K(n)$ denote this space. The space of symmetric positive definit matrices

on the n -dimensional vector space \mathcal{P} , denoted by $P(n)$, endowed with the following Riemannian metric is a symmetric space (see [19]):

$$\langle v, w \rangle_g = \frac{1}{n} \operatorname{tr} g^{-1} v g^{-1} w$$

where $g \in P(n)$ and $v, w \in T_g P(n)$. The group $\operatorname{Gl}(n, \mathbb{R})$ acts on $P(n)$ by $A(g) := A^T g A$. This action is transitive and isometric and $(\operatorname{Gl}(n, \mathbb{R}), O(n))$ is the corresponding symmetric pair.

REMARK. — K acts on $(P(n), \langle \cdot, \cdot \rangle)$ by isometries such that $P^K(n)$ is the fix point set of this action. Hence $P^K(n)$ is totally geodesic and again a symmetric space. Furthermore $P(n)$ splits as symmetric space into the product of

$$P_1(n) := \{g \in P(n) \mid \det g = 1\}$$

with \mathbb{R} . This yields an orthogonal splitting $P^K(n) = P_1^K(n) \times \mathbb{R}$ where $P_1^K(n) := P_1(n) \cap P^K(n)$ is again a symmetric space.

Let D denote the Levi-Civita connection of $(P(n), \langle \cdot, \cdot \rangle)$ and let $D/d\tau$ denote the covariant derivative of vector fields along a curve $g(\tau)$ in $P(n)$ (see [22], 2.68). On the other hand side $P(n)$ can be endowed with the canonical flat metric. Let ∂_τ denote the derivative of vector fields along a curve $g(\tau)$ with respect to this flat connection.

LEMMA 3.1 (J.-H. Eschenburg). — *Let $g(\tau)$ be a smooth curve in $P(n)$. Then*

$$(18) \quad \frac{D^2 g(\tau)}{d\tau^2} = \partial_\tau \partial_\tau g(\tau) - \partial_\tau g(\tau) g^{-1}(\tau) \partial_\tau g(\tau).$$

Proof. — In order to prove (18) we consider

$$\left(\frac{D}{d\tau} - \partial_\tau \right)_{\tau=0}.$$

Since this expression is a tensor we can replace an arbitrary curve $g(\tau)$ by a geodesic $\gamma(\tau)$ of $(P(n), \langle \cdot, \cdot \rangle)$ provided that $g(0) = \gamma(0)$ and $g'(0) = \gamma'(0)$. Such geodesics are given by $\gamma(\tau) = A^T(\tau)\gamma(0)A(\tau)$ where $A(\tau) = \exp(\tau X)$. Here $X \in T_{g(0)}\operatorname{Gl}(n, \mathbb{R})$ is an element of the space of infinitesimal transvections at $g(0)$, *i.e.* we have $g(0)X = X^T g(0)$. Therefore $\gamma'(0) = X^T \gamma(0) + \gamma(0)X = 2\gamma(0)X$. On the other hand side $\gamma(\tau)$ is generated by $\{\exp(\tau X)\}_{\tau \in \mathbb{R}}$, hence we have $\partial_\tau \gamma(\tau) = A^T(\tau)\gamma'(0)A(\tau)$. This yields

$$(\partial_\tau \partial_\tau \gamma(\tau))_{\tau=0} = X^T \gamma'(0) + \gamma'(0)X = \gamma'(0)\gamma^{-1}(0)\gamma'(0)$$

by $X = \frac{1}{2}\gamma^{-1}(0)\gamma'(0)$. \square

Therefore (15) is equivalent to $\frac{D^2\bar{g}}{d\tau^2} + \frac{n-1}{n} \frac{V'}{V} \bar{g}' - 2\bar{g}r_0(\bar{g}) = 0$. Let $\bar{g}(\tau)$ be a smooth curve in Uni_+ and let

$$\bar{s} := s|_{\text{Uni}_+}.$$

Recall

$$\frac{d}{d\tau} \bar{s}(\bar{g}(\tau)) = -\text{tr } r_0(\bar{g}(\tau)) \bar{g}^{-1}(\tau) \bar{g}'(\tau).$$

Thus $-\bar{g}r_0(\bar{g})$ is the gradient of \bar{s} with respect to the symmetric metric $\langle \cdot, \cdot \rangle|_{\text{Uni}_+}$ denoted by $\nabla \bar{s}$. We conclude that (15) is equivalent to

$$\frac{D^2\bar{g}}{d\tau^2} + \frac{n-1}{n} \frac{V'}{V} \bar{g}' + 2\nabla \bar{s}(\bar{g}) = 0.$$

Now suppose $\lambda = 0$. Let

$$\kappa(\bar{g}, \bar{g}') = \frac{1}{4} \text{tr}(\bar{g}^{-1} \bar{g}' \bar{g}^{-1} \bar{g}') + \bar{s}(\bar{g}).$$

With the help of (17) we can replace V'/V by $(\sqrt{n}/\sqrt{n-1})\sqrt{\kappa(\bar{g}, \bar{g}')}$ in the domain $\{V' > 0\}$. Hence (15) is equivalent to

$$(19) \quad \frac{D^2\bar{g}}{d\tau^2} + \sqrt{\frac{n-1}{n}} \cdot \sqrt{\kappa(\bar{g}, \bar{g}')} \cdot \bar{g}' + 2\nabla \bar{s}(\bar{g}) = 0.$$

Next we will show that this equation yields indeed a decoupling of the Ricci flat cohomogeneity one Einstein equation, since the following Proposition shows that a complete Ricci flat cohomogeneity one metric $dt^2 + g(t)$ (which is not flat and does not contain a line) satisfies $\text{tr } L(t) > 0$ for $t > 0$.

PROPOSITION 3.2. — *Let \widehat{M}^{n+1} be a complete Ricci flat cohomogeneity one Einstein manifold which is not flat and does not contain a line. Then there exists no principal orbit P which is minimal.*

Proof. — Since (\widehat{M}, \hat{g}) is not flat, \widehat{M} is non-compact by a theorem of Bochner. Since (\widehat{M}, \hat{g}) does not contain a line we obtain $\widehat{M}/G = [0, \infty)$, hence there exists one non-principal orbit Q . Let $\ell(t) := \text{tr } L(t)$. By (3) we obtain

$$(20) \quad \ell'(t) = \frac{1}{n} \ell^2(t) + \text{tr } L_0^2(t).$$

Since the solutions of $y'(t) = n^{-1}y^2(t)$ are well known, we obtain the following conclusion: If a solution $\ell(t)$ of (20) satisfies $\ell(t_0) < 0$, then $\ell(t)$ blows up in finite time. Hence such a solution does not come from a complete metric $\hat{g} = dt^2 + g(t)$.

If $\dim Q = \dim P$, then Q is a minimal hypersurface of $(\widehat{M}, \widehat{g})$, thus $\ell(0) = 0$. By the above conclusion we obtain $\ell \equiv \text{tr } L_0^2 \equiv 0$ and therefore $(\widehat{M}, \widehat{g})$ would be flat. In case $\dim Q < \dim P$ we get $\lim_{t \rightarrow 0} \ell(t) = +\infty$. Suppose $\ell(t_0) = 0$ for $t_0 > 0$. The case $\text{tr } L_0^2(t_0) > 0$ can be excluded by the above conclusion. But $\ell(t_0) = \text{tr } L_0^2(t_0) = 0$ would imply $L(t_0) = 0$, hence P_{t_0} would be totally geodesic. Therefore $g(t_0 - t) = g(t_0 + t)$. Thus \widehat{M} would be compact. Contradiction! \square

REMARK 3.3. — Equation (19) yields the following Ansatz: Suppose $\bar{\gamma}(\tau)$ is a geodesic in Uni_+ with respect to $\langle \cdot, \cdot \rangle_{|\text{Uni}_+}$ such that $\bar{\gamma}'(\tau) = \beta(\tau)\nabla\bar{s}(\bar{\gamma}(\tau))$. Let $\bar{g}(\tau) := \bar{\gamma}(\alpha(\tau))$ for $\alpha: \mathbb{R} \rightarrow \mathbb{R}$. Then the Ricci flat cohomogeneity one Einstein equation can be reduced to a second order differential equation for α . For instance the Taub-NUT and the Eguchi-Hanson metric are of this type. It would be very interesting to see whether there exists any similar relation between the 2-monopole solution of Atiyah and Hitchin and $\nabla\bar{s}$ and the geometry of $(\text{Uni}_+, \langle \cdot, \cdot \rangle)$.

4. The Lyapunov function close to stable cones

Let g_E be a G -homogeneous metric on P with

$$s(g_E) = n(n - 1).$$

Suppose that g_E is a local non-degenerate minimum of the total scalar curvature functional restricted to the space of G -homogeneous metrics on $P = G/K$ of volume $\text{vol}(P, g_E)$. By [5, 4.23] g_E is Einstein. We will call the corresponding cone solution a *stable cone*.

We already know that a stable cone consists of (degenerate) critical points of κ . We show that the regular levels of κ close to a stable cone are tubes around this cone. Let $\mathcal{D}_{t_0} \subset \mathcal{I}_\lambda$ be any slice of γ_λ at $\gamma_\lambda(t_0) \in \mathcal{I}_\lambda$, that is a smooth hypersurface of \mathcal{I}_λ intersected by γ_λ transversally at $\gamma_\lambda(t_0)$.

PROPOSITION 4.1. — *Suppose that P is not a torus. Let $\bar{\kappa} := \kappa|_{\mathcal{D}_{t_0}} \rightarrow \mathbb{R}$. Then $\gamma_\lambda(t_0) \in \mathcal{I}_\lambda$ is a local non-degenerate minimum of $\bar{\kappa}$, if γ_λ is a stable cone.*

Proof. — Let $(p_1, p_2) := \gamma_\lambda(t_0)$. Obviously, (p_1, p_2) is a local minimum of $\bar{\kappa}$. Thus it is only to show that (p_1, p_2) is a non-degenerate minimum. Let $(g', p_2(L_0 + \beta I_n)) \in T_{(p_1, p_2)}\mathcal{D}_{t_0}$ where $g' \in \text{Sym}(TP, G)$, $L_0 \in \text{Sym}(TP, G)$ is trace free and $\beta \in \mathbb{R}$. Since (p_1, p_2) is a critical point of κ considered as a function from $\mathcal{F} \rightarrow \mathbb{R}$, it is enough to show

$$\frac{d^2}{dt^2} \Big|_{t=0} \kappa(p_1 + tg', p_2 + tp_2(L_0 + \beta I_n)) > 0.$$

If g' is not a multiple of p_1 , then we are done because γ_λ is stable. Thus we can assume $g' = \alpha p_1$ where $\alpha \in \mathbb{R}$. If $L_0 \neq 0$, then we obtain

$$\frac{d^2}{dt^2} \Big|_{t=0} \kappa(p_1(1+t\alpha), p_2 + tp_2(L_0 + \beta I_n)) > 0$$

as well. We claim that $(\alpha p_1, \beta p_2) \in T_{(p_1, p_2)} \mathcal{D}_{t_0}$ forces $\alpha = \beta = 0$.

We can restrict ourselves to the cases $\lambda = n, 0, -n$. If $\lambda = -n$ then

$$\gamma_{-n}(t_0) = (p_1, p_2) = (\sinh^2(t_0)g_E, \sinh(2t_0)g_E)$$

where g_E is Einstein with $s(g_E) = n(n - 1)$. Since $(\alpha p_1, \beta p_2) \in T_{(p_1, p_2)} \mathcal{D}_{t_0} \subset T_{(p_1, p_2)} \mathcal{E}_\lambda$ we investigate

$$\frac{d}{dt} \Big|_{t=0} e_{-n}(\sinh^2(t_0)g_E(1+t\alpha), \sinh(2t_0)g_E(1+t\beta)).$$

By (5) we obtain $\alpha \neq 0$ (otherwise $\alpha = \beta = 0$ and we are done) and

$$\frac{\beta}{\alpha} = \frac{2 \cosh^2(t_0) - 1}{2 \cosh^2(t_0)}.$$

Therefore $(\alpha p_1, \beta p_2)$ is a multiple of $\gamma'_{-n}(t_0)$. But \mathcal{D}_{t_0} has been a slice. Contradiction! The cases $\lambda = n$ and $\lambda = 0$ can be treated in the same manner. \square

By the Lemma of Morse (see [29]) we know that $\gamma_\lambda(t_0) \in \mathcal{D}_{t_0}$ is an isolated critical point of $\bar{\kappa}$ such that the regular levels of κ are “spheres” around this point. Therefore the connected component of a regular level of κ , which is close to $\gamma_\lambda \cap \mathcal{I}_\lambda$, is a tube around γ_λ , at least if we restrict this connected component to a compact domain. Since by Lemma 2.2 any integral curve of X_λ in \mathcal{I}_λ does not intersect these tubes in the outer direction we get the following

COROLLARY 4.2. — *Let g_E be a G -homogeneous metric on the principal orbit $P = G/K$ with $s(g_E) = n(n - 1)$. Suppose that g_E is a local non-degenerate minimum of the total scalar curvature functional restricted to the space of G -homogeneous metrics on P of volume $\text{vol}(P, g_E)$. Then the cone solution γ_λ , which corresponds to g_E , is in \mathcal{I}_λ a local attractor for integral curves of X_λ .*

Proof. — By $s(g_E) > 0$ P cannot be a torus. Hence we obtain the claim by the above discussion. \square

REMARK. — For $\lambda > 0$ we have $\gamma_\lambda(t) \subset \mathcal{I}_\lambda$ for $t \in (0, \frac{1}{2}\pi)$. For $\lambda \leq 0$ we have $\gamma_\lambda(t) \subset \mathcal{I}_\lambda$ for $t > 0$. In the latter case there exists a second explicit integral curve of X_λ , namely $\gamma_\lambda(t)$ where $t < 0$.

In the following part of this section we will assume $\lambda \leq 0$. We will show that the regular levels of κ close to γ_λ are not only local but global tubes around γ_λ . This is the main reason that an integral curve of X_λ which enters such a tube has an infinite interval of existence. We parametrise \mathcal{F} by

$$\mathcal{F} \ni (g, h) = \left(V^{\frac{2}{n}} \bar{g}, V^{\frac{2}{n}} \bar{g}' + \frac{2}{n} V^{\frac{2}{n}-1} V' \bar{g} \right)$$

where $V > 0$, $V' \in \mathbb{R}$, $\bar{g} \in \text{Uni}_+$ and $\bar{g}' \in T_{\bar{g}} \text{Uni}_+$. Thus, we have a diffeomorphism from $\mathbb{R}_+ \times \mathbb{R} \times T \text{Uni}_+$ to \mathcal{F} . By $L = \frac{1}{2} g^{-1} h = \frac{1}{2} \bar{g}^{-1} \bar{g}' + n^{-1} V' / V \cdot I_n$, $\text{tr } \bar{g}^{-1} \bar{g}' = 0$ we get $L_0 = L - n^{-1} \text{tr } L \cdot I_n = \frac{1}{2} \bar{g}^{-1} \bar{g}'$. Thus we can write e_λ and κ in this new coordinates as

$$e_\lambda(V, V', \bar{g}, \bar{g}') = \frac{n-1}{n} \frac{V'^2}{V^2} - \frac{1}{4} \text{tr } \bar{g}^{-1} \bar{g}' \bar{g}^{-1} \bar{g}' - V^{-\frac{2}{n}} s(\bar{g}) + \lambda(n-1)$$

and as

$$\kappa(V, \bar{g}, \bar{g}') = \frac{1}{4} V^{\frac{2}{n}} \text{tr } \bar{g}^{-1} \bar{g}' \bar{g}^{-1} \bar{g}' + s(\bar{g})$$

respectively. Hence

$$(21) \quad e_\lambda(V, V', \bar{g}, \bar{g}') = \frac{n-1}{n} \frac{V'^2}{V^2} - V^{-\frac{2}{n}} (\kappa(V, \bar{g}, \bar{g}') - \lambda(n-1) V^{\frac{2}{n}}).$$

Now let

$$\mathcal{W}_\lambda := \{(V, V', \bar{g}, \bar{g}') \in \mathcal{E}_\lambda \mid V' > 0\}.$$

We recall $\{\gamma_\lambda(t) \mid t > 0\} \subset \mathcal{I}_\lambda \subset \mathcal{W}_\lambda$ by $\kappa(\gamma_\lambda) = s(\overline{gE}) > 0$ and $\lambda \leq 0$. By (21) the projection

$$\mathbb{R}_+ \times \mathbb{R} \times T \text{Uni}_+ \longrightarrow \mathbb{R}_+ \times T \text{Uni}_+; \quad (V, V', \bar{g}, \bar{g}') \longmapsto (V, \bar{g}, \bar{g}')$$

is a global chart of \mathcal{W}_λ . Now let \mathcal{K}_ϵ be the connected component of $\kappa^{-1}(\kappa(\gamma_\lambda) + \epsilon)$ which is (at least locally) in \mathcal{I}_λ a tube around γ_λ ($\epsilon > 0$). With the help of (21) we conclude $\mathcal{K}_\epsilon \subset \mathcal{W}_\lambda$. Therefore we can investigate \mathcal{K}_ϵ in the above chart. For $(V, \bar{g}, \bar{g}') \in \mathcal{K}_\epsilon$ we have

$$\frac{1}{4} V^{\frac{2}{n}} \text{tr } \bar{g}^{-1} \bar{g}' \bar{g}^{-1} \bar{g}' = \kappa(\gamma_\lambda) + \epsilon - s(\bar{g}).$$

We restrict κ to $\{V \equiv V_0\}$ for a suitable $V_0 > 0$. By assumption the connected component of $\mathcal{K}_\epsilon \cap \{V \equiv V_0\}$, which is close to γ_λ , is a sphere around $\gamma_\lambda \cap \{V \equiv V_0\}$. The main point is now that if $(V_0, \bar{g}, \bar{g}') \in \mathcal{K}_\epsilon$ then $(V_0/s, \bar{g}, s^{\frac{1}{n}} \bar{g}') \in \mathcal{K}_\epsilon$ holds as well for $s > 0$! Therefore \mathcal{K}_ϵ is not only locally but globally a tube around γ_λ (and $\mathcal{K}_\epsilon \cap \{V \equiv V_0\}$ is in fact connected).

PROPOSITION 4.3. — *Let $\lambda \leq 0$ and let g_E be an Einstein metric as in Corollary 4.2. Then there exist regular levels of κ such that one connected component of this level constitutes a tube around the cone γ_λ which corresponds to g_E . Furthermore, if a solution of (2) and (3) enters such a tube, say at $t_0 > 0$, then it is defined for $t > t_0$.*

Proof. — The existence of such levels follows from the above discussion. Furthermore, the connected components of levels of κ , which lie between such a level and the cone γ_λ itself, are tubes around γ_λ as well and all of them lie in \mathcal{W}_λ . Now let

$$\gamma(t) = (V(t), V'(t), \bar{g}(t), \bar{g}'(t))$$

be a solution of (2) and (3) which enters such a tube, say at $t_0 > 0$. By Proposition 2.2 $\gamma(t)$ is imprisoned within such a tube and remains therefore in \mathcal{W}_λ . Suppose that $[t_0, t_{\max})$ is the maximal interval of existence of $\gamma(t)$ ($t_{\max} \in (t_0, \infty)$). By the special shape of these tubes and by $V'(t) > 0$, the coordinate function $V'(t)$ has to reach $+\infty$ in finite time (otherwise $\gamma(t)$ would be an integral curve of the smooth vector field X_λ , which does not leave a compact domain). But this is impossible by (21), because

$$(22) \quad \begin{aligned} V'(t) &= \frac{\sqrt{n}}{\sqrt{n-1}} V(t)^{\frac{n-1}{n}} \sqrt{\kappa(\gamma(t)) - \lambda(n-1)V(t)^{\frac{2}{n}}} \\ &\leq V(t) \frac{\sqrt{n}}{\sqrt{n-1}} \left(\frac{\sqrt{\kappa(\gamma(t_0))}}{V(t_0)^{\frac{1}{n}}} + \sqrt{-\lambda(n-1)} \right) \end{aligned}$$

is a linear differential inequality. \square

Now let us assume the existence of a singular orbit Q . Suppose $\widehat{M}/G = [0, \infty)$.

COROLLARY 4.4. — *Let the assumption be as in Proposition 4.3. Suppose furthermore, that $\gamma(t)$ is a solution of (2) which defines a C^3 -metric on a tubular neighbourhood of the singular orbit Q . Assume that $\gamma(t)$ enters one of the tubes described in Proposition 4.3. Then $\gamma(t)$ defines a complete C^∞ -Einstein metric on \widehat{M} with maximal volume growth.*

Proof. — By Proposition 1.2 $\gamma(t)$ is a solution of (3) as well and the first part of the claim follows with the help of Proposition 4.3.

Therefore we are left proving that $\text{vol}(B_r(\widehat{m})) \geq c_0 \cdot r^{n+1}$ in the Ricci flat case and $\text{vol}(B_r(\widehat{m})) \geq c_{-n} \cdot \exp(nr)$ for big r in the case $\lambda = -n$ ($\widehat{m} \in \widehat{M}^{n+1}$, $c_0, c_{-n} > 0$). Let $t_0 > 0$. Geometric arguments show that

it is enough to prove $\int_{t_0}^t V(s) ds \geq c'_0 t^{n+1}$ in the Ricci flat case and $\int_{t_0}^t V(s) ds \geq c'_{-n} \exp(nt)$ in the case $\lambda = -n$ for all $t > t_0$ ($c'_0, c'_{-n} > 0$). By Proposition 2.2 and Proposition 4.3 we get in (22) $\lim_{t \rightarrow \infty} \kappa(\gamma(t)) > 0$. Integrating (22) yields the claim. \square

Now it is of course interesting to determine the asymptotic behaviour of such a solution $\gamma(t)$. Numerical investigations show that the cases $\lambda < 0$ and $\lambda = 0$ behave different.

In the last part of this section we turn to the Ricci flat case. Let U be a small neighbourhood of $\overline{g_E}$ in Uni_+ and let $\phi: V^{m-1} \rightarrow U$ be a local parametrisation of Uni_+ with $\phi(0) = \overline{g_E}$. Suppose that $(D\phi)_0 \cdot e_1, \dots, (D\phi)_0 \cdot e_{m-1}$ constitutes an orthonormal basis of $T_{\overline{g_E}} \text{Uni}_+$ with respect to $\langle \cdot, \cdot \rangle$. Let $x(\tau) = (x_1(\tau), \dots, x_{m-1}(\tau))$ be a smooth curve in V^{m-1} and let $\bar{g}(\tau) = \phi(x(\tau))$. Let Γ_{ij}^k denote the Christoffel symbols and let $\nabla \bar{s} = \sum_{i=1}^{m-1} \bar{s}_i \cdot U_i$ where $U_i = D\phi \cdot e_i$. Then (19) is locally equivalent to

$$(23) \quad x''_k + \sum_{i,j=1}^{m-1} \Gamma_{ij}^k(x) x'_i x'_j + x'_k \frac{\sqrt{n-1}}{\sqrt{n}} \sqrt{\kappa(x, x')} + 2\bar{s}_k(x) = 0$$

where $1 \leq k \leq m - 1$ (cf. [22, 2.77]). By $(y(\tau), z(\tau)) = (x(\tau), x'(\tau))$ the system (23) is equivalent to

$$\begin{aligned} y'_k &= z_k, \\ z'_k &= - \sum_{i,j=1}^{m-1} \Gamma_{ij}^k(y) z_i z_j - z_k \frac{\sqrt{n-1}}{\sqrt{n}} \sqrt{\kappa(y, z)} - 2\bar{s}_k(y), \end{aligned}$$

where $1 \leq k \leq m - 1$. We think about these equations as equations satisfied by integral curves of a vector field X . Since $\phi(0) = \overline{g_E}$ the point $(y, z) = (0, 0) \in \mathbb{R}^{m-1} \times \mathbb{R}^{m-1}$ is a zero of X .

LEMMA 4.5. — *The zero $(y, z) = (0, 0)$ of the above defined vector field X is stable.*

Proof. — We have

$$DX_{(0,0)} = \begin{pmatrix} 0 & I_{m-1} \\ -\left(2 \frac{\partial \bar{s}_k}{\partial y_i}(0)\right)_{1 \leq k, i \leq m-1} & -\frac{\sqrt{n-1}}{\sqrt{n}} \sqrt{\kappa(0,0)} I_{m-1} \end{pmatrix}.$$

Let

$$H_{ki} := \langle D_{U_k} \nabla \bar{s}, U_i \rangle_{\overline{g_E}}.$$

Since \bar{g}_E is a non-degenerate minimum of \bar{s} by assumption,

$$2(H_{ki})_{1 \leq k, i \leq m-1}$$

is a symmetric positive definite matrix, denoted by S_+ . By the special choice of the parametrization ϕ we have

$$S_+ = 2 \left(\frac{\partial \bar{s}_k}{\partial y_i}(0) \right)_{1 \leq k, i \leq m-1}.$$

Let $\alpha := \frac{\sqrt{n-1}}{\sqrt{n}} \sqrt{\kappa(0,0)}$ and let v_1, \dots, v_{m-1} be a basis of eigenvectors of S_+ with respect to the eigenvalues $\mu_1, \dots, \mu_{m-1} > 0$. Let

$$\lambda_{\pm}^i := \frac{-\alpha \pm \sqrt{\alpha^2 - 4\mu_i}}{2}.$$

Then

$$\begin{pmatrix} 0 & I_{m-1} \\ -S_+ & -\alpha I_{m-1} \end{pmatrix} \begin{pmatrix} v_i \\ \lambda_{\pm}^i v_i \end{pmatrix} = \lambda_{\pm}^i \begin{pmatrix} v_i \\ \lambda_{\pm}^i v_i \end{pmatrix}$$

for $1 \leq i \leq m-1$. Therefore λ_{\pm}^i is an eigenvalue of $DX_{(0,0)}$. On the other hand side, if δ is an eigenvalue of $DX_{(0,0)}$, then we can conclude $-(\alpha + \delta)\delta = \mu_i$ for an $i \in \{1, \dots, m-1\}$. Thus $\delta = \lambda_+^i$ or $\delta = \lambda_-^i$. Since $-\alpha < 0$ and $\mu_i > 0$ we get the claim by the Stable Manifold Theorem (cf. [34, Section 2.7]). \square

Therefore a solution $\bar{g}(\tau)$ of (19) where $(\bar{g}(\tau_0), \bar{g}'(\tau_0))$ is close enough to $(\bar{g}_E, 0)$ suffices $\lim_{\tau \rightarrow \infty} \bar{g}(\tau) = \bar{g}_E$. With the help of (17) we can compute $V(\bar{g}(\tau))$. Obviously $V \circ g$ does not blow up in finite time. Hence we obtain the desired limiting behaviour summarized in the following

PROPOSITION 4.6. — *Let $\lambda = 0$, let $\gamma(t)$ be a solution of (2) and (3) and let g_E be as in Corollary 4.2. Then there exist tubes around $\gamma_0(t) = t^2 g_E$ described in Proposition 4.3 such that if $\gamma(t)$ enters such a tube, then $\lim_{t \rightarrow \infty} \bar{\gamma}(t) = \bar{g}_E$*

This behaviour was observed in [23] for explicit cohomogeneity one Einstein metrics with special holonomy. Actually all these examples fit into the above frame work.

REMARK. — The Eguchi-Hanson metric is ALE, but the standard metric on S^3 is of course not stable in our sense. It is the common feature of the Eguchi-Hanson metric, the Ricci flat metrics provided by Theorem A and the examples with special holonomy [10], [23] that all these metrics have Euclidean volume growth. Therefore their geometry at infinity is expected in view of [16] and [15].

5. Stable cones

In this section we provide examples of cohomogeneity one manifolds \widehat{M} which admit stable cone solutions. We assume the existence of a singular orbit $Q = G/H$. Let $\widehat{M}_{\leq \epsilon}$ be a tubular neighbourhood of Q . We will restrict ourselves to $\widehat{M}_{\leq \epsilon}$ because we are not going to investigate (the possibly several ways) how $\widehat{M}_{\leq \epsilon}$ can be compactified as a cohomogeneity one manifold. Hence we consider only the case $\widehat{M}/G = [0, \infty)$.

Recall that $P = G/K$ has been the principal orbit type. We can easily arrange $K < H$ and we know that H/K is a k -dimensional homogeneous sphere ($k \geq 1$, cf. [20]). In order to obtain stable cones we have to find compact Lie groups G, H, K such that $K < H < G$, H/K is a H -homogeneous sphere and G/K carries a G -invariant metric which is a local non-degenerate minimum of the total scalar curvature functional restricted to the space of G -invariant metrics of volume 1. Unfortunately there exists no general existence statement for such Einstein metrics (if K would be maximal compact in G see [38] for such a result). Nevertheless there exist several examples.

EXAMPLE 5.1. — *Let $r \geq 0$, let $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ be compact, connected, isotropy irreducible homogeneous spaces with positive scalar curvature and let $k \geq 2$. Then*

$$\widehat{M} = \mathbb{R}^{k+1} \times G_1/K_1 \times G_2/K_2 \times \dots \times G_{r+1}/K_{r+1}$$

considered as a $(\text{SO}(k+1) \times G_1 \times G_2 \times \dots \times G_{r+1})$ -cohomogeneity one manifold admits a unique cone solution, which is stable.

We remark that there exist many compact isotropy irreducible homogeneous spaces (cf. [41] and [39], [5, p.201–205]). In the above case the scalar curvature of any $(\text{SO}(k+1) \times G_1 \times G_2 \times \dots \times G_{r+1})$ -invariant metric on $P = S^k \times G_1/K_1 \times G_2/K_2 \times \dots \times G_{r+1}/K_{r+1}$ with volume 1 is bounded from below (cf. [38, Thm 2.1] for a converse statement). Furthermore P is a trivial S^k -bundle over the singular orbit $Q = G_1/K_1 \times G_2/K_2 \times \dots \times G_{r+1}/K_{r+1}$. But there exist also examples where the sphere bundle $P \rightarrow Q$ is non-trivial.

EXAMPLE 5.2. — *$\mathbb{H}P^{m+1} \setminus \{\text{point}\}$ and $\text{CaP}^2 \setminus \{\text{point}\}$ considered as a $\text{Sp}(1) \times \text{Sp}(m+1)$ -cohomogeneity one manifold and a $\text{Spin}(9)$ -cohomogeneity one manifold respectively admit a stable cone solution ($m \geq 1$).*

For the proof see [5, 9.82, 9.84] or [6] (cf. Section 11). In the first case the principal orbit type is S^{4m+3} (which is a non-trivial S^3 -bundle over the singular orbit $\mathbb{H}P^m$) in the latter case the principal orbit type

is S^{15} (which is a non-trivial S^7 -bundle over S^8). The stable cone of $\mathbb{H}P^{m+1} \setminus \{\text{point}\}$ corresponds to the Jensen metric on S^{4m+3} [27] whereas the stable cone of $\mathbb{C}aP^2 \setminus \{\text{point}\}$ corresponds to the Bourguignon-Karcher metric on S^{15} (see [8]). There exists a second cone solution, which comes from the curvature one metric. But this cone is not stable.

EXAMPLE 5.3. — *The Lie group triple*

$$(G, H, K) = (\text{Sp}(m + 1), \text{Sp}(m) \times \text{Sp}(1), \text{Sp}(m) \times \text{U}(1))$$

gives rise to a non-trivial \mathbb{R}^3 -bundle over $\mathbb{H}P^m$ which admits a stable cone solution for $m \geq 1$

For the proof see [5, 9.83] or [6] (cf. Section 11). The principal orbit type is $\mathbb{C}P^{2m+1}$ (which is a non-trivial S^2 -bundle over $\mathbb{H}P^m$). The stable cone corresponds to the Ziller metric on $\mathbb{C}P^{2m+1}$ (see [42]). There exists a second cone solution, which comes from the symmetric metric on $\mathbb{C}P^{2m+1}$. But this cone is not stable.

REMARK. — Example 5.2 and 5.3 come from Hopf fibrations (cf. [42], [5, p. 257–258]). The dimension of the homogeneous sphere H/K is in all of the above examples bigger or equal than 2. By the O’Neill’s formulas (cf. [5, 9.70d, Fig. 9.72]) this condition is necessary for the existence of a stable cone. This implies for instance, that the cone solution which appears in the bundle constructions [12], [4], [33], [36], [37], [17] is never stable.

6. Main existence results

Theorem A is precisely Theorem 6.1 and Theorem B is a special case of Theorem 6.2. Theorem 6.1, 6.3, 6.4 are proved by the Convergence Theorems 9.7 and 11.1, Corollary 4.4 and results of Section 10. We remark, that these theorems might be regarded as pure existence results for special solutions of the differential equations (2) and (3).

THEOREM 6.1. — *Let $r \geq 0$, let $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ be non-flat compact isotropy irreducible homogeneous spaces and let $k \geq 2$. Then*

$$\widehat{M} = \mathbb{R}^{k+1} \times G_1/K_1 \times G_2/K_2 \times \dots \times G_{r+1}/K_{r+1}$$

carries a $(r + 1)$ -dimensional family of Einstein metrics with negative scalar curvature and a r -dimensional family of Ricci flat metrics.

For special choices of the isotropy irreducible homogeneous spaces $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ the Einstein metrics of Theorem 6.1 admit many free isometric actions of finite groups Γ .

THEOREM 6.2. — *Let $r \geq 0$, let $G_1/K_1, G_2/K_2, \dots, G_{r+1}/K_{r+1}$ be non-flat compact isotropy irreducible homogeneous spaces and let $k \geq 2$. Let Γ be a finite subgroup of $O(k+1) \times G_1 \times \dots \times G_{r+1}$ acting freely on*

$$P = S^k \times G_1/K_1 \times \dots \times G_{r+1}/K_{r+1}$$

and on

$$Q = \{0\} \times G_1/K_1 \times \dots \times G_{r+1}/K_{r+1}.$$

Then the quotient space

$$\widehat{M}_\Gamma = (\mathbb{R}^{k+1} \times G_1/K_1 \times G_2/K_2 \times \dots \times G_{r+1}/K_{r+1})/\Gamma$$

carries a $(r+1)$ -dimensional family of Einstein metrics with negative scalar curvature and a r -dimensional family of Ricci flat metrics.

One class of examples is obtained in the following way: Γ acts freely on $\{0\} \times G_1/K_1 \times \dots \times G_{r+1}/K_{r+1}$ and trivially on S^{k+1} . For instance $\Gamma = \Gamma_1 \times \dots \times \Gamma_{r+1}$ where Γ_i is a finite subgroup of G_i . Examples of irreducible symmetric spaces which admit many such group actions are odd-dimensional spheres, odd-dimensional Grassmann manifolds and $SU(3)/SO(3)$. In fact such group actions are classified by Wolf [40]. But there exist examples where Γ acts on S^k non-trivially, for instance: Let Γ be the cyclic group of order $s > 1$ and suppose that $k > 2$ is odd. Suppose furthermore, that some of the G_i/K_i admit a non-trivial free \mathbb{Z}_s -action by isometries (contained in G_i). Now let Γ act diagonally on $S^k \times G_1/K_1 \times \dots \times G_{r+1}/K_{r+1}$. Recall that even on S^k there exist plenty of such \mathbb{Z}_s -actions.

In Theorem 6.1 the normal bundle of the singular orbit is trivial. But there exist examples where this is not the case. In the following theorem we think about $\mathbb{H}P^{m+1} \setminus \{\text{point}\}$ as a non-trivial \mathbb{R}^4 -bundle over $\mathbb{H}P^m$ and about $\mathbb{C}aP^2 \setminus \{\text{point}\}$ as a non-trivial \mathbb{R}^8 -bundle over S^8 .

THEOREM 6.3. — *$\mathbb{H}P^{m+1} \setminus \{\text{point}\}$ and $\mathbb{C}aP^2 \setminus \{\text{point}\}$ admit a 1-parameter family of Einstein metrics with negative scalar curvature and a Ricci flat metric ($m \geq 1$).*

For $m = 1$ and $\lambda = 0$ one obtains an explicit solution with holonomy type $Spin(7)$ [10], [23] (in [32] numerical solutions were described for $\mathbb{H}P^{m+1} \setminus \{\text{point}\}$). The metrics on $\mathbb{H}P^{m+1} \setminus \{\text{point}\}$ are $Sp(1) \times Sp(m+1)$ -invariant whereas the metrics on $\mathbb{C}aP^2 \setminus \{\text{point}\}$ are $Spin(9)$ -invariant. However the full isometry group of these metrics does not act transitively. Again these metrics have maximal volume growth and in the same sense

as described in the introduction they can be chosen arbitrary close to the Jensen cone metric and the Bourguignon-Karcher cone metric (see Section 5). In the Ricci flat case the geometry of the principal orbits converges to the geometry of the Jensen metric and the Bourguignon-Karcher metric.

THEOREM 6.4. — *The Lie group triple*

$$(G, H, K) = (\mathrm{Sp}(m + 1), \mathrm{Sp}(m) \times \mathrm{Sp}(1), \mathrm{Sp}(m) \times \mathrm{U}(1))$$

gives rise to a non-trivial \mathbb{R}^3 -bundle over $\mathbb{H}\mathbb{P}^m$ ($m \geq 1$). For $m \geq 3$ this \mathbb{R}^3 -bundle admit a 1-parameter family of Einstein metrics with negative scalar curvature and a Ricci flat metric.

The cases $m = 1$ and $m = 2$ are missing since they are not covered by the general Convergence Theorem 11.1. However the case $m = 1$ and $\lambda = 0$ can be solved explicitly. According to [10], [23] this manifold has holonomy type G_2 . All these metrics are $\mathrm{Sp}(m + 1)$ -invariant however the full isometry group does not act transitively, they have maximal volume growth and they can be chosen arbitrary close to the Ziller cone metric (see Section 5) as described above. In the Ricci flat case the geometry of the principal orbits converges to the geometry of the Ziller metric.

7. The initial value problem

In this section we describe the singular initial value problem for the cohomogeneity one manifolds \widehat{M} of Example 5.1. Of course the existence and uniqueness problem is solved in [20] (Example 5.1 fits into this framework), however the continuous dependence on the initial values has not been investigated in [20]. From now on we will restrict ourselves to the case $r \geq 1$. The case $r = 0$ is dealt with in Section 11.

Let $\ell_1, \ell_2, \dots, \ell_r$ denote the dimensions of $G_1/K_1, G_2/K_2, \dots, G_r/K_r$ and let m denote the dimension of G_{r+1}/K_{r+1} . Let g_b^i be a G_i -invariant background metric on G_i/K_i ($i \in \{1, \dots, r + 1\}$). Suppose that g_b^i is Einstein with Einstein constant $(\ell_i - 1)$ for $i \in \{1, \dots, r\}$ and suppose that g_b^{r+1} is Einstein with Einstein constant $r_h > 0$ which we will specify later on. Let g^{S^k} denote the curvature one metric on S^k ($k > 1$). Let

$$G := \mathrm{SO}(k + 1) \times G_1 \times G_2 \times \dots \times G_{r+1}.$$

Any G -invariant smooth metric \hat{g} on $\widehat{M} \setminus Q$ can be written as

$$(24) \quad \hat{g}|_{\widehat{M} \setminus Q} = dt^2 + f^2(t)g^{S^k} + g_1^2(t)g_b^1 + g_2^2(t)g_b^2 + \dots + g_r^2(t)g_b^r + h^2(t)g_b^{r+1}$$

where f, g_1, \dots, g_r, h are positive smooth functions. The Einstein equa-

tions (3) and (4) are equivalent to the following system

$$(25) \quad k \frac{f''}{f} + \sum_{j=1}^r \ell_j \frac{g_j''}{g_j} + m \frac{h''}{h} = -\lambda,$$

$$(26) \quad \frac{f''}{f} - \frac{f'^2}{f^2} + \frac{f'}{f} \left(k \frac{f'}{f} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + m \frac{h'}{h} \right) - (k-1) \frac{1}{f^2} = -\lambda,$$

$$(27) \quad \frac{g_i''}{g_i} - \frac{g_i'^2}{g_i^2} + \frac{g_i'}{g_i} \left(k \frac{f'}{f} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + m \frac{h'}{h} \right) - (\ell_i - 1) \frac{1}{g_i^2} = -\lambda,$$

$$(28) \quad \frac{h''}{h} - \frac{h'^2}{h^2} + \frac{h'}{h} \left(k \frac{f'}{f} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + m \frac{h'}{h} \right) - r_h \frac{1}{h^2} = -\lambda,$$

where $1 \leq i \leq r$.

Vice versa let $(f(t), g_1(t), \dots, g_r(t), h(t))$ be a solution of (25), (26), (27) and (28). In order to ensure that the metric

$$dt^2 + f^2(t)g^{S^k} + g_1^2(t)g_b^1 + g_2^2(t)g_b^2 + \dots + g_r^2(t)g_b^r + h^2(t)g_b^{r+1}$$

can be extended to a smooth metric on a tubular neighbourhood of Q , according to [20], we have to demand the singular initial value

$$\begin{aligned} (f(0), f'(0), g_1(0), g_1'(0), \dots, g_r(0), g_r'(0), h(0), h'(0)) \\ = \underbrace{(0, 1, \bar{g}_1, 0, \bar{g}_2, \dots, \bar{g}_r, 0, \bar{h}, 0)}_{=: a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}} \end{aligned}$$

where $\bar{g}_1, \dots, \bar{g}_r, \bar{h} > 0$. The main Theorem in [20] provides the existence and uniqueness of a solution $(f(t), g_1(t), \dots, g_r(t), h(t))$ of (25), (26), (27) and (28) with the singular initial value $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$. This gathers a $(r+1)$ -parameter family of Einstein metrics for any Einstein constant λ in $\mathbb{R} \setminus \{0\}$ and a r -parameter family of Ricci flat metrics, defined on a tubular neighbourhood of Q . However, these Einstein metrics may not be complete, *i.e.* in case $\lambda \leq 0$ these solutions may only be defined on $[0, \epsilon)$ for some $\epsilon > 0$ or in case $\lambda > 0$ they may not close up to a smooth metric at the second non-principal orbit.

THEOREM 7.1. — *Consider the differential equation*

$$(29) \quad y'(t) = A(y(t)) + \frac{1}{t} B(y(t))$$

where $A, B: U_a^j \rightarrow \mathbb{R}^j$ are analytic functions on an open neighbourhood U_a^j of $a \in \mathbb{R}^j$ ($j \geq 1$). If $B(a) = 0$ and $I_j - \frac{1}{n} DB_a$ is invertible for $n \in \mathbb{N}$,

then there exists a unique solution $y(t)$ of (29) with $y(0) = a$. Furthermore the solution $y(t)$ depends continuously on $a \in \{b \in U_a^j \mid B(b) = 0\}$.

Proof. — The existence and uniqueness are proved in [20, Section 5] (see also Theorem 9.1 in [28]). The continuous dependence on the initial value follows with the arguments of [21, Section 4]. \square

The equations (26), (27), (28) define a $(2r+4)$ -dimensional vector field. By the same method as in [6, Section 2] (*cf.* the proof of Lemma 8.4) we can apply Theorem 7.1. This yields

THEOREM 7.2. — *For $\bar{g}_1, \dots, \bar{g}_r, \bar{h} > 0$ there exists a unique solution*

$$c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}(t) := (f(t), f'(t), g_1(t), g_1'(t), \dots, g_r(t), g_r'(t), h(t), h'(t))$$

of (25), (26), (27) and (28) with $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}(0) = a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ which depends continuously on $(\bar{g}_1, \dots, \bar{g}_r, \bar{h})$. Moreover the metric which corresponds to $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}(t)$ (*cf.* (24)) can be extended to a C^∞ -Einstein metric on a tubular neighbourhood of Q in \widehat{M} .

Proof. — The existence and uniqueness of a solution of (26), (27) and (28) with initial value $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ and the continuous dependence on $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ follow from Theorem 7.1 and the above discussion. By uniqueness this solution coincides with the earlier mentioned solution provided by [20]. According to [20] (main Theorem) the corresponding metric can be extended to a C^∞ -Einstein metric on a tubular neighbourhood of Q . Therefore, this solution satisfies equation (25) as well. \square

8. The bottom equation

In this section we will provide preliminaries in order to prove the Convergence Theorem 9.7. We investigate the system of differential equations given by (25), (26), (27), (28) in Section 7. The following change of coordinates is suitable (*cf.* [6]):

$$w := \frac{f}{h}.$$

Then (25), (26), (27), (28) are equivalent to

$$(30) \quad \frac{w''}{w} - \frac{w'^2}{w^2} + \frac{w'}{w} \left(k \frac{w'}{w} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + (m+k) \frac{h'}{h} \right) - (k-1) \frac{1}{h^2 w^2} + r_h \frac{1}{h^2} = 0,$$

$$(31) \quad \frac{g_i''}{g_i} - \frac{g_i'^2}{g_i^2} + \frac{g_i'}{g_i} \left(k \frac{w'}{w} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + (m+k) \frac{h'}{h} \right) - (\ell_i - 1) \frac{1}{g_i^2} = -\lambda,$$

$$(32) \quad \frac{h''}{h} - \frac{h'^2}{h^2} + \frac{h'}{h} \left(k \frac{w'}{w} + \sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + (m+k) \frac{h'}{h} \right) - r_h \frac{1}{h^2} = -\lambda,$$

$$(33) \quad k(k-1) \frac{w'^2}{w^2} + 2k \frac{w'}{w} \left(\sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + (m+k-1) \frac{h'}{h} \right) + \left(\sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + (m+k) \frac{h'}{h} \right)^2 - \sum_{j=1}^r \ell_j \frac{g_j'^2}{g_j^2} - (m+k) \frac{h'^2}{h^2} - \left(k(k-1) \frac{1}{h^2 w^2} + \sum_{j=1}^r \ell_j (\ell_j - 1) \frac{1}{g_j^2} + m r_h \frac{1}{h^2} \right) + \lambda(n-1) = 0,$$

where $1 \leq i \leq r$. The singular initial value $(f(0), f'(0)) = (0, 1)$ changes into the singular initial value $(w(0), w'(0)) = (0, \bar{h}^{-1})$ where $h(0) = \bar{h} > 0$.

REMARK 8.1. — *The solution $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}(t)$ and the singular initial value $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ (cf. Theorem 7.2) change under the above coordinate change, however we will not change notation.*

In order to investigate (30), (31), (32), (33), we go into charts. (In the following we will sometimes drop the argument of a function.) Since (33) is a quadratic equation for w'/w we get

$$(34) \quad \frac{w'}{w} = \frac{1}{h} \frac{-2k \left(\sum_{j=1}^r \ell_j \frac{g_j' h}{g_j} + (m+k-1)h' \right) \pm \sqrt{Dh^2}}{2k(k-1)},$$

where

$$D(w, g_1, g_1', \dots, g_r, g_r', h, h') := 4k \left(\sum_{j=1}^r \ell_j \frac{g_j'}{g_j} + m \frac{h'}{h} \right)^2 + 4k(k-1) \sum_{j=1}^r \ell_j \frac{g_j'^2}{g_j^2} + 4k(k-1) \left\{ m \frac{h'^2}{h^2} + k(k-1) \frac{1}{h^2 w^2} + \sum_{j=1}^r \ell_j (\ell_j - 1) \frac{1}{g_j^2} + m r_h \frac{1}{h^2} - \lambda(n-1) \right\}.$$

Let

$$V^{2r+3} := \{(w, g_1, g'_1, \dots, g_r, g'_r, h, h') \in R^{2r+3} \mid w, g_1, \dots, g_r, h > 0, D > 0\}$$

and let

$$(35) \quad N : V^{2r+3} \longrightarrow \mathbb{R};$$

$$(w, g_1, \dots, h') \longmapsto \frac{-2k \left(\sum_{j=1}^r \ell_j \frac{g'_j h}{g_j} + (m+k-1)h' \right) + \sqrt{Dh^2}}{2k(k-1)}.$$

We think of $(\tilde{g}, \tilde{h}) \in \mathcal{F} = \text{Sym}_+(TP, G) \times \text{Sym}(TP, G)$ as

$$(\tilde{g}, \tilde{h}) = (w^2 h^2, g_1^2, \dots, g_r^2, h^2, 2(ww'h^2 + w^2hh'), 2g_1g'_1, \dots, 2g_rg'_r, 2hh'),$$

that is we can parametrise \mathcal{F} by $w, g_1, \dots, g_r, h > 0$ and $w', g'_1, \dots, g'_r, h' \in \mathbb{R}$. By

$$\mathcal{U}_\lambda := \left\{ (w, w', g_1, \dots, h, h') \in \mathcal{E}_\lambda \mid (w, g_1, \dots, h') \in V^{2r+3}, w' = \frac{w}{h} N \right\}.$$

we get a chart $\phi : \mathcal{U}_\lambda \rightarrow V^{2r+3}$ defined by

$$(w, w', g_1, \dots, h') \longmapsto (w, g_1, \dots, h').$$

LEMMA 8.2. — For $\bar{g}_1, \dots, \bar{g}_r, \bar{h} > 0$ with $(\ell_i - 1)/\bar{g}_i^2 - \lambda > 0$ and $r_h/\bar{h}^2 - \lambda > 0$ let $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}^{\mathcal{I}_\lambda}$ denote the first connected component of $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}} \cap \mathcal{I}_\lambda$. Then

$$c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}^{\mathcal{I}_\lambda} \subset \mathcal{U}_\lambda.$$

Proof. — If $(w, w', g_1, \dots, h, h') \in \mathcal{I}_\lambda$ (see (10)), then we obtain $(w, g_1, \dots, h') \in V^{2r+3}$. By (31), (32) and the above choice of $\bar{g}_1, \dots, \bar{g}_r, \bar{h}$ we conclude $g''_1(0), \dots, g''_r(0), h''(0) > 0$. Finally the claim follows with the help of (34) and $w'(0) = \bar{h}^{-1} > 0$. \square

If we apply the chart ϕ to the system (30), (31), (32), (33), then we obtain the following first order differential equation on V^{2r+3}

$$w' = \frac{1}{h} wN,$$

$$g'_i = x_i,$$

$$x'_i = \frac{x_i^2}{g_i} - x_i \left(\frac{k}{h} N + \sum_{j=1}^r \ell_j \frac{x_j}{g_j} + (m+k) \frac{y}{h} \right) + (\ell_i - 1) \frac{1}{g_i} - \lambda g_i,$$

$$h' = y,$$

$$y' = \frac{y^2}{h} - y \left(\frac{k}{h} N + \sum_{j=1}^r \ell_j \frac{x_j}{g_j} + (m+k) \frac{y}{h} \right) + r_h \frac{1}{h} - \lambda h,$$

where $1 \leq i \leq r$. We change coordinates again:

$$w_i := \frac{g_i}{h}.$$

The above differential equation takes now the form

$$(36) \quad w' = \frac{1}{h} wN,$$

$$(37) \quad w'_i = \frac{1}{h} (x_i - w_i y),$$

$$(38) \quad x'_i = \frac{1}{h} \left\{ \frac{x_i^2}{w_i} - x_i \left(\frac{kwN}{w} + \sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + (\ell_i - 1) \frac{1}{w_i} - \lambda w_i h \right\},$$

$$(39) \quad h' = \frac{1}{h} y h,$$

$$(40) \quad y' = \frac{1}{h} \left\{ y^2 - y \left(\frac{kwN}{w} + \sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + r_h - \lambda h^2 \right\},$$

where $1 \leq i \leq r$.

DEFINITION 8.3. — Let $\bar{g}_1, \dots, \bar{g}_r, \bar{h} > 0$ with $(\ell_i - 1)/\bar{g}_i^2 - \lambda > 0$ and $r_h/\bar{h}^2 - \lambda > 0$. After applying the chart ϕ and the above coordinate change the singular initial value $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ (cf. Remark 8.1) changes into

$$b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} := \left(0, \bar{w}_1 := \frac{\bar{g}_1}{\bar{h}}, 0, \dots, \bar{w}_r := \frac{\bar{g}_r}{\bar{h}}, 0, \bar{h}, 0 \right),$$

where $\bar{w}_1, \dots, \bar{w}_r, \bar{h} > 0$, $r_h - \lambda \bar{h}^2 > 0$ and $(\ell_i - 1)/\bar{w}_i^2 - \lambda \bar{h}^2 > 0$. The solution $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}^{\mathcal{I}_\lambda}(t)$ (cf. Lemma 8.2) changes into the unique solution

$$d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}(t) := (w(t), w_1(t), x_1(t), \dots, w_r(t), x_r(t), h(t), y(t))$$

of (36), ..., (40) with $d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}(0) = b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$. Furthermore

$$V^{2r+3} = \{ (w, w_1, x_1, \dots, h, y) \in R^{2r+3} \mid w, w_1, \dots, w_r, h > 0, \tilde{D} > 0 \}$$

(for the definition of \tilde{D} see (42)) and

$$\phi(\mathcal{I}_\lambda) = \left\{ (w, \dots, y) \in V^{2r+3} \mid \frac{k}{w^2} (k-1) + \sum_{j=1}^r \ell_j (\ell_j - 1) \frac{1}{w_j^2} + m r_h - \lambda (n-1) h^2 > 0 \right\}.$$

In case $\lambda \leq 0$ we obtain

$$\phi(\mathcal{I}_\lambda) = V^{2r+3} = \{(w, w_1, \dots, h, y) \in \mathbb{R}^{2r+3} \mid w, w_1, \dots, w_r, h > 0\}.$$

Now we are prepared to blow up the singularity $\{h \equiv 0\}$. Before we consider

$$\tilde{N} := wN$$

(cf. (35)) as a function with argument $(w, w_1, x_1, \dots, w_r, x_r, h, y)$ instead of $(w, g_1 = w_1h, \dots, h, y)$. We have

$$(41) \quad \tilde{N}(w, w_1, x_1, \dots, w_r, x_r, h, y) = \frac{-2kw\left(\sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k-1)y\right) + \sqrt{\tilde{D}}}{2k(k-1)}$$

where

$$(42) \quad \begin{aligned} \tilde{D}(w, w_1, x_1, \dots, w_r, x_r, h, y) &= 4kw^2\left(\sum_{j=1}^r \ell_j \frac{x_j}{w_j} + my\right)^2 + 4k(k-1)w^2\sum_{j=1}^r \ell_j \frac{x_j^2}{w_j^2} \\ &\quad + 4k(k-1)w^2\left\{my^2 + k(k-1)\frac{1}{w^2}\right. \\ &\quad \left. + \sum_{j=1}^r \ell_j(\ell_j - 1)\frac{1}{w_j^2} + m\mathfrak{r}_h - \lambda(n-1)h^2\right\}. \end{aligned}$$

Observe that \tilde{N} is a smooth function on

$$V_{\tilde{N}}^{2r+3} = \left\{ (w, w_1, x_1, \dots, h, y) \in \mathbb{R}^{2r+3} \mid w_1, \dots, w_r > 0, w, h \in \mathbb{R}, \tilde{D} > 0 \right\}$$

with

$$(43) \quad \tilde{N}(0, \bar{w}_1, 0, \dots, \bar{w}_r, 0, \bar{h}, 0) = 1$$

where $\bar{h} \in \mathbb{R}$!

We consider the right hand side of (36), ..., (40) as a vector field. In a small neighbourhood of $b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ we can stretch this vector field with

the positive factor h/\tilde{N} without losing information ($\bar{h} > 0$). The integral curves of this new vector field satisfy

$$(44) \quad w' = 1,$$

$$(45) \quad w'_i = \frac{1}{\tilde{N}}(x_i - w_i y),$$

$$(46) \quad x'_i = \frac{1}{\tilde{N}} \left\{ \frac{x_i^2}{w_i} - x_i \left(\sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + (\ell_i - 1) \frac{1}{w_i} - \lambda w_i h \right\} - k \frac{x_i}{w},$$

$$(47) \quad h' = \frac{1}{\tilde{N}} y h,$$

$$(48) \quad y' = \frac{1}{\tilde{N}} \left\{ y^2 - y \left(\sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + r_h - \lambda h^2 \right\} - k \frac{y}{w}.$$

Let

$$\bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} := (0, \bar{w}_1, 0, \dots, \bar{w}_r, 0, \bar{h}, 0) \in \mathbb{R}^{2r+3}$$

for $\bar{w}_1, \dots, \bar{w}_r > 0$ and $\bar{h} \in \mathbb{R}$.

LEMMA 8.4. — For $\bar{w}_1, \dots, \bar{w}_r > 0$ and $\bar{h} \in \mathbb{R}$ with $r_h - \lambda \bar{h}^2 > 0$ and $(\ell_i - 1)/\bar{w}_i^2 - \lambda \bar{h}^2 > 0$ there exists a unique solution $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ of (44), ..., (48) with $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}(0) = \bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$. The solution $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ depends continuously on $\bar{h} \geq 0$. Furthermore

$$\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} \cap \phi(\mathcal{I}_\lambda) = d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}$$

for $\bar{h} > 0$ close to $b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$.

Proof. — Let $\bar{w}_1, \dots, \bar{w}_r, \bar{h}$ be as stated above. If there is a solution

$$y(t) = (w(t), w_1(t), x_1(t), \dots, h(t), y(t))$$

of (44), ..., (48) with $y(0) = \bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$, then $w(t) = t$. For this we can replace the terms $-k x_i(t)/w(t)$ and $-k y(t)/w(t)$ by $-k x_i(t)/t$ and $-k y(t)/t$ respectively. Then $y(t)$ satisfies an equation like (29). It is easy to check that the assumptions of Theorem 7.1 are satisfied in a neighbourhood of $\bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$. Thus we get a unique solution $y(t)$ with $y(0) = \bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ which depends continuously of \bar{h} . For $\bar{h} > 0$ we have $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} \cap \phi(\mathcal{I}_\lambda) = d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}$ by uniqueness. Since \tilde{N} might be zero in $\phi(\mathcal{I}_\lambda)$ this holds only close to $b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$. \square

In Lemma 8.4 we mean by $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} \cap \phi(\mathcal{I}_\lambda) = d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}$ that the images of these integral curves coincide.

We consider now the right hand side of (44), ..., (48) again as a vector field. In a small neighbourhood of $\bar{b}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ we can stretch this vector field with the positive factor \tilde{N} without losing information ($\bar{h} \geq 0!$). The integral curves of this new vector field satisfy

$$(49) \quad w' = \tilde{N},$$

$$(50) \quad w'_i = x_i - w_i y,$$

$$(51) \quad x'_i = \frac{x_i^2}{w_i} - x_i \left(\frac{k\tilde{N}}{w} + \sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + (\ell_i - 1) \frac{1}{w_i} - \lambda w_i h,$$

$$(52) \quad h' = yh,$$

$$(53) \quad y' = y^2 - y \left(\frac{k\tilde{N}}{w} + \sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right) + r_h - \lambda h^2.$$

Let

$$(54) \quad V_{\kappa_{\text{ex}}}^{2r+3} = \left\{ (w, w_1, x_1, \dots, h, y) \in \mathbb{R}^{2r+3} \mid w, w_1, \dots, w_r, > 0, h \in \mathbb{R}, \tilde{D} > 0 \right\}.$$

The vector field given by (49), ..., (53) is smooth on $V_{\kappa_{\text{ex}}}^{2r+3}$ because \tilde{N} is smooth on $V_{\kappa_{\text{ex}}}^{2r+3} \subset V_{\tilde{N}}^{2r+3}$. By (43) we obtain the statements of Lemma 8.4 for this differential equation as well. Even better, the restriction that

$$\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}} \cap \phi(\mathcal{I}_\lambda) = d_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}^{\phi(\mathcal{I}_\lambda)}$$

holds only close to $b_{\bar{w}_1, \dots, \bar{w}_r, \bar{h}}$ can be dropped because the singularity $\{\tilde{N} \equiv 0\}$ is removed.

Now we are going to investigate the *bottom* $\{h \equiv 0\}$ of (49), ..., (53). By (52) any solution of (49), ..., (53), which starts in the bottom, remains in the bottom. Hence we can think of (49), (50), (51) and (53) with $h \equiv 0$ as a $(2r + 2)$ -dimensional differential equation. We will call this equation *bottom equation*.

We investigate constant solutions $(\tilde{w}, \tilde{w}_1, \dots, \tilde{x}_r, \tilde{y})$ of the bottom equation in the domain

$$(55) \quad V_b^{2r+2} := \left\{ (w, w_1, \dots, x_r, 0, y) \in \mathbb{R}^{2r+3} \mid w, w_1, x_1, \dots, w_r, y > 0 \right\}.$$

We have $V_b^{2r+2} \subset V_{\kappa_{\text{ex}}}^{2r+3}$. By (49) we get

$$\tilde{N}(\tilde{w}, \tilde{w}_1, \dots, \tilde{x}_r, 0, \tilde{y}) = 0$$

and (50) gathers $\tilde{x}_i = \tilde{w}_i \tilde{y}$. Thus

$$\tilde{y}^2(1 - n) + r_h = 0$$

by (53), hence $\tilde{y}^2 = r_h/(n - 1)$. In the same way we get $\tilde{x}_i^2 = (\ell_i - 1)/(n - 1)$. Thus $\tilde{w}_i^2 = (\ell_i - 1)/r_h$. Now $\tilde{N} = 0$ yields

$$n(n - 1)\tilde{y}^2 - \left(k(k - 1)\frac{1}{\tilde{w}^2} + \sum_{j=1}^r \ell_j(\ell_j - 1)\frac{1}{\tilde{w}_j^2} + mr_h\right) = 0,$$

thus $\tilde{w}^2 = (k - 1)/r_h$. Vice versa we obtain the following

LEMMA 8.5. — *The bottom equation has a unique constant solution in V_b^{2r+2} , namely*

$$\begin{aligned} & (w(t), w_1(t), x_1(t), \dots, w_r(t), x_r(t), y(t)) \\ \equiv & \left(\sqrt{\frac{k-1}{r_h}}, \sqrt{\frac{\ell_1-1}{r_h}}, \sqrt{\frac{\ell_1-1}{n-1}}, \dots, \sqrt{\frac{\ell_r-1}{r_h}}, \sqrt{\frac{\ell_r-1}{n-1}}, \sqrt{\frac{r_h}{n-1}} \right). \end{aligned}$$

This result is not unexpected as we will see in a moment. For

$$\lambda = n, 0, -n$$

let

$$c_\lambda(t) = \sin(t), t, \sinh(t)$$

respectively. Then the cone solution $\gamma_\lambda(t) \cap \mathcal{I}_\lambda$ has in the coordinates $(w, w_1, x_1, \dots, w_r, x_r, h, y)$ the following form

$$(56) \quad \left(\sqrt{\frac{k-1}{r_h}}, \sqrt{\frac{\ell_1-1}{r_h}}, \sqrt{\frac{\ell_1-1}{n-1}} c'_\lambda(t), \dots, \sqrt{\frac{\ell_r-1}{r_h}}, \sqrt{\frac{\ell_r-1}{n-1}} c'_\lambda(t), c_\lambda(t), \sqrt{\frac{r_h}{n-1}} c'_\lambda(t) \right),$$

where $t \in (0, \frac{1}{2}\pi), t > 0, t > 0$ for $\lambda = n, 0, -n$ respectively. Let

$$(57) \quad \tilde{\gamma}_\lambda(t) = \gamma_\lambda(t)$$

for $t \in (-\frac{1}{2}\pi, \frac{1}{2}\pi), t \in \mathbb{R}, t \in \mathbb{R}$ in case $\lambda = n, 0, -n$ respectively. Thus the constant solution from Lemma 8.5 is nothing but the *foot point* $\tilde{\gamma}_\lambda(0)$ of γ_λ .

9. Convergence Theorem

Following the ideas of [6] we will show that the solution $\bar{d}_{\bar{w}_1, \dots, \bar{w}_r, 0}$ (for a suitable choice of $(\bar{w}_1, \dots, \bar{w}_r)$) converges to the constant solution of Lemma 8.5. This is the main step in order to proof the Convergence Theorem 9.7.

We make the following Ansatz: The equations (49) and (53) define a 2-dimensional vector field, if we demand $w_i \equiv \sqrt{(\ell_i - 1)/r_h}$ and $x_i = w_i y$, namely

$$Z(w, y) = \begin{pmatrix} \tilde{N}_2(w, y) \\ y^2(1 - n) - y \frac{k}{w} \tilde{N}_2(w, y) + r_h \end{pmatrix}$$

where

$$\begin{aligned} \tilde{N}_2(w, y) &:= \tilde{N}(w, w_1, w_1 y, \dots, w_r, w_r y, 0, y) \\ &= w \frac{-(n - 1)y + \frac{1}{\sqrt{k}} \sqrt{\Upsilon}}{k - 1} \end{aligned}$$

and

$$\Upsilon = y^2(n - 1)(n - k) + k(k - 1)^2 \frac{1}{w^2} + (k - 1)(n - k)r_h$$

by (41). If we apply Theorem 7.1 combined with the methods of the proof of Lemma 8.4 once more we get the following

LEMMA 9.1. — *There exists a unique integral curve $\bar{b}(t) = (\bar{b}_w(t), \bar{b}_y(t))$ of $Z(w, y)$ with $\bar{b}(0) = (0, 0)$.*

The solution $\bar{b}(t)$ is of course not constant by $\tilde{N}_2(0, 0) = 1$. The surprising fact is now, that this solution defines a solution of the entire system (49), ..., (53).

PROPOSITION 9.2. — *Let $\bar{b}(t)$ be the integral curve of Z defined in Lemma 9.1. Then*

$$\begin{aligned} b(t) = \left(\bar{b}_w(t), \sqrt{\frac{\ell_1 - 1}{r_h}}, \sqrt{\frac{\ell_1 - 1}{r_h}} \bar{b}_y(t), \dots, \right. \\ \left. \sqrt{\frac{\ell_r - 1}{r_h}}, \sqrt{\frac{\ell_r - 1}{r_h}} \bar{b}_y(t), 0, \bar{b}_y(t) \right) \end{aligned}$$

is a solution of (49), ..., (53). Hence $b = \bar{d}_{\sqrt{(\ell_1 - 1)/r_h}, \dots, \sqrt{(\ell_r - 1)/r_h}, 0}$.

Proof. — Equations (49) and (53) are fulfilled by means of the above Ansatz. By $w_i \equiv \sqrt{(\ell_i - 1)/r_h}$ and $x_i = w_i y$ we obtain (50). Thus it is only to show that (51) is satisfied. By the definition of Z and $x_i = w_i y$ we get

$$\bar{b}'_y = \bar{b}^2_y - \bar{b}_y \left(\frac{k}{\bar{b}_w} \tilde{N}(\bar{b}_w, w_1, w_1 \bar{b}_y, \dots, w_r, w_r \bar{b}_y, 0, \bar{b}_y) + \sum_{j=1}^r \ell_j \frac{w_j \bar{b}_y}{w_j} + (m + k) \bar{b}_y \right) + r_h$$

hence multiplying this equation with the constant w_i yields

$$(w_i \bar{b}_y)' = \frac{(w_i \bar{b}_y)^2}{w_i} - (w_i \bar{b}_y) \left(\frac{k}{\bar{b}_w} \tilde{N}(\bar{b}_w, w_1, (w_1 \bar{b}_y), \dots, w_r, (w_r \bar{b}_y), 0, \bar{b}_y) + \sum_{j=1}^r \ell_j \frac{(w_j \bar{b}_y)}{w_j} + (m + k) \bar{b}_y \right) + \frac{1}{w_i} \underbrace{w_i^2 r_h}_{=(\ell_i - 1)}$$

By uniqueness the images of b and $\bar{d}_{\sqrt{(\ell_1 - 1)/r_h}, \dots, \sqrt{(\ell_r - 1)/r_h}, 0}$ coincide. \square

DEFINITION 9.3. — Let $r_h := (n - 1)$.

In order to simplify the discussion, we smooth the vector field Z with the factor w . We obtain the vector field

$$\bar{Z}(w, y) = \begin{pmatrix} w \tilde{N}_2(w, y) \\ (n - 1)w(1 - y^2) - k \tilde{N}_2(w, y)y \end{pmatrix}.$$

The integral curve \bar{b} of Z becomes an integral curve of \bar{Z} just by changing the time parameter. We show that the integral curve \bar{b} of the vector field \bar{Z} ends in $(\sqrt{(k - 1)/(n - 1)}, 1)$. In order to do that we observe that the vector field $\bar{Z}(w, \tilde{h}')$ in [6, Section 5, eq. (37)] is precisely the above vector field $\bar{Z}(w, y)$ with the following setup: In [6] one has to set

$$\|\mathcal{A}\| = 0, \text{ Ric}^Q = (d_P - 1), d_P = n, d_Q = n - k, d_S = k, \tilde{h}' = y.$$

By $\|\mathcal{A}\| = 0$ and $\text{Ric}^Q = d_P - 1$ we get $B(w) = b_1 = 1$ in [6] (cf. equation (11), (14) in [6]). By $b_1 = 1$, the function $\tilde{N}(w, 0, \tilde{h}')$ in [6], (33) coincides with $\tilde{N}_2(w, y)$. Even more, if we stretch \bar{Z} by $1/\tilde{N}_2 w$, then the integral curves of this stretched vector field are precisely the solutions of the restriction of the differential equation (36) in [6] to $\{y_3 \equiv 0\}$. Thus, the image of the integral curve \bar{b} of \bar{Z} and the image of the solution \bar{c}_0 of Lemma 5.4 in [6] coincide. Now remark, that the constant w_1 in [6] (cf. Lemma 4.1 in [6]) gets $\sqrt{(k - 1)/(n - 1)}$. Thus, we can apply the Convergence Theorem 5.7 in [6] and we obtain

PROPOSITION 9.4. — *Let $r_h = n - 1$. Then the integral curve $\bar{b}(t)$ of the vector field Z , defined in Lemma 9.1, satisfies*

$$\lim_{t \rightarrow \infty} \bar{b}(t) = \left(\sqrt{\frac{k-1}{n-1}}, 1 \right).$$

COROLLARY 9.5. — *Let $r_h = n - 1$. Then the solution $b(t)$ of (49), ..., (53), defined in Proposition 9.2, satisfies $\lim_{t \rightarrow \infty} b(t) = \bar{\gamma}_\lambda(0)$ for any Einstein constant $\lambda \in \mathbb{R}$.*

By continuous dependence on the initial value $\bar{h} \geq 0$ we know now (cf. Lemma 8.4, Definition 8.3) that $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$ comes for small $\bar{h} > 0$ arbitrary close to the foot point $\bar{\gamma}_\lambda(0)$ of the cone solution γ_λ (see (57)).

LEMMA 9.6. — *The solution*

$$d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$$

does not leave $\phi(\mathcal{I}_\lambda)$ for very small $\bar{h} > 0$ before it comes close to $\bar{\gamma}_\lambda(0)$.

Proof. — For $\lambda \leq 0$ we have $\phi(\mathcal{I}_\lambda) = V^{2r+3}$ (see Definition 8.3). The proof of Lemma 5.6 in [6] shows, that the integral curve $\bar{b}(t) = (\bar{b}_w(t), \bar{b}_y(t))$ satisfies $\bar{b}_w(t), \bar{b}_y(t) > 0$ for $t > 0$. Furthermore \bar{b} remains in a rectangle with positive distance to $\{w \equiv 0\}$ for $t > \epsilon$ (for any $\epsilon > 0$). Thus the solution $b(t)$ has positive distance to the singularities $\{w \equiv 0\}, \{w_1 \equiv 0\}, \dots, \{w_r \equiv 0\}$ of (49), ..., (53) for $t > \epsilon$. Since $\tilde{D} > 0$ along $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$ by Definition 8.3 we obtain the claim for $\lambda \leq 0$.

For $\lambda > 0$, we have only to show that $\tilde{D} > 0$. Since by continuous dependence on the initial value the h -coordinate of $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$ between the singular initial value and the point which is arbitrary close to $\bar{\gamma}_\lambda(0)$ is arbitrary small, we obtain the claim. \square

Now we will prove that the function κ (cf. (8)) defined on \mathcal{E}_λ and therefore on $\phi(\mathcal{I}_\lambda)$ (actually on V^{2r+3}) can be extended to a smooth function on $V_{\kappa_{\text{ex}}}^{2r+3}$ (cf. (54)). This extension will be denoted by κ_{ex} .

We compute κ in the coordinates $(w, w_1, x_1, \dots, w_r, x_r, h, y)$ using the chart ϕ to replace w' by \tilde{N}/h . By (6), (7) and (8) we obtain

$$\begin{aligned} \kappa(w, w_1, x_1, \dots, h, y) &= (w^k w_1^{\ell_1} \dots w_r^{\ell_r})^{\frac{2}{n}} \\ &\times \left\{ k \frac{\tilde{N}^2}{w^2} + 2k \frac{\tilde{N}}{w} y + \sum_{j=1}^r \ell_j \frac{x_j^2}{w_j^2} + (m+k)y^2 \right. \\ &\quad - \frac{1}{n} \left(k \frac{\tilde{N}}{w} + \sum_{j=1}^r \ell_j \frac{x_j}{w_j} + (m+k)y \right)^2 \\ &\quad \left. + k(k-1) \frac{1}{w^2} + \sum_{j=1}^r \ell_j (\ell_j - 1) \frac{1}{w_j^2} + m r_h \right\} \end{aligned}$$

(for the definition of \tilde{N} see (41)). Since \tilde{N} is a smooth function on $V_{\tilde{N}}^{2r+3} \supset V_{\kappa_{\text{ex}}}^{2r+3}$ we can extend κ to a smooth function κ_{ex} defined on $V_{\kappa_{\text{ex}}}^{2r+3}$.

Next we will show, that the levels of κ_{ex} around $\bar{\gamma}_\lambda(t)$ (see (57)) are tubes around this degenerate level of κ_{ex} . For this we consider the case $\lambda = 0$ first. For a fixed $t_0 > 0$ let \mathcal{D}_{t_0} be a small disk around $\gamma_0(t_0)$ in $\{h \equiv t_0\}$. Of course \mathcal{D}_{t_0} is a slice of $\gamma_0(t)$ at $\gamma_0(t_0)$. By Proposition 4.1 the levels of $\bar{\kappa} := \kappa|_{\mathcal{D}_{t_0}}$ are spheres around the non-degenerate minimum $\gamma_0(t_0)$ of $\bar{\kappa}$. But the function κ_{ex} does not depend on h because \tilde{D} does not depend on h for $\lambda = 0$ (cf. (42))! So we are done in case $\lambda = 0$.

We consider the remaining cases $\lambda = n$ and $\lambda = -n$. Let

$$\mathcal{D}_0 \subset V_b^{2r+2} \subset V_{\kappa_{\text{ex}}}^{2r+3}$$

be a small disk around $\bar{\gamma}_0(0) = \bar{\gamma}_\lambda(0)$ (see (55)). In the case $\lambda = 0$ we know that the levels of κ_{ex} restricted to $\mathcal{D}_0 \subset V_b^{2r+2}$ are spheres around the non-degenerate minimum $\bar{\gamma}_0(0)$ of

$$\bar{\kappa}_{\text{ex}} := \kappa_{\text{ex}}|_{\mathcal{D}_0}.$$

But the function κ_{ex} restricted to $\mathcal{D}_0 \subset \{h \equiv 0\}$ does not depend on λ by (42). Hence we get also in the cases $\lambda = n$ and $\lambda = -n$ that the levels of $\bar{\kappa}_{\text{ex}}$ are spheres around the non-degenerate minimum $\bar{\gamma}_\lambda(0)$ of $\bar{\kappa}_{\text{ex}}$. In particular $\bar{\gamma}_\lambda(0)$ is an isolated critical point of the $\bar{\kappa}_{\text{ex}}$. Thus the regular levels of κ_{ex} intersect \mathcal{D}_0 transversally. Furthermore, κ_{ex} is invariant under the reflection $(w, w_1, x_1, \dots, w_r, x_r, h, y) \mapsto (w, w_1, x_1, \dots, w_r, x_r, -h, y)$. Hence $\partial \kappa_{\text{ex}} / \partial h|_{\mathcal{D}_0} = 0$ but $(\kappa_{\text{ex}})_*|_{\mathcal{D}_0 \setminus \bar{\gamma}_\lambda(0)} \neq 0$. We conclude that the regular levels of κ_{ex} in $V_{\kappa_{\text{ex}}}^{2r+3}$ around the singular level $\bar{\gamma}_\lambda$ are tubes around $\bar{\gamma}_\lambda$.

CONVERGENCE THEOREM 9.7. — *Let $\bar{h} > 0$. The solution*

$$c \sqrt{(\ell_1 - 1)/r_h} \bar{h}, \dots, \sqrt{(\ell_r - 1)/r_h} \bar{h}, \bar{h}$$

of the differential equation (25),(26),(27),(28) (see Theorem 7.2) converges for $\bar{h} \rightarrow 0$ to the cone solution γ_λ (cf. (11),(56)) for any Einstein constant $\lambda \in \mathbb{R}$.

Proof. — By Proposition 9.5, Proposition 9.2 and Lemma 8.4

$$d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$$

comes for small $\bar{h} > 0$ arbitrary close to the foot point $\bar{\gamma}_\lambda(0)$ of γ_λ . We can assume that this happens arbitrary close to the bottom $\{h \equiv 0\}$ (see Lemma 9.6). Proposition 4.1 yields that the regular levels of $\kappa: V^{2r+3} \rightarrow \mathbb{R}$ close to $\{\gamma_\lambda(t) \mid \epsilon \leq t \leq 1\}$ are tubes around γ_λ ($\epsilon > 0$ small). The above discussion shows that these tubes can be extended to tubes in $V_{\kappa_{\text{ex}}}^{2r+3}$ around $\bar{\gamma}_\lambda$. By Proposition 2.2 we conclude that $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$ intersects these tubes not in the outer direction. Since by (28) the coordinate function $h'(t)$ of $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$, that is $y(t)$ in the chart ϕ , is positive as long $\frac{r_h}{h^2(t)} - \lambda > 0$, the solution $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$ converges to the cone solution γ_λ for $\bar{h} \rightarrow 0$. By Definition 8.3 and Lemma 8.2 $c_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}, \bar{h}}$ corresponds to $d^{\phi(\mathcal{I}_\lambda)}_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}}$, i.e. $c_{\sqrt{(\ell_1-1)/r_h}, \dots, \sqrt{(\ell_r-1)/r_h}, \bar{h}, \bar{h}}$ converges to the cone solution γ_λ for $\bar{h} \rightarrow 0$. \square

10. Inhomogeneity, moduli and convergence

In this section we will assume $\lambda \leq 0$. In order to prove Theorem 6.1 we will show that solutions $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ (cf. Theorem 7.2) which are close enough to the stable cone solution define complete inhomogeneous Einstein metrics (see Section 7, (24)).

LEMMA 10.1. — *There exists $\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta} > 0$ such that $c_{\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta}}(t)$ is defined for $t > 0$ and such that the same holds for any $(\bar{g}_1, \dots, \bar{g}_r, \bar{h})$ close to $(\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta})$.*

Proof. — By the Convergence Theorem 9.7 there exist $\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta} > 0$ such that $c_{\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta}}(t)$ enters one of the tubes described in Proposition 4.3. With the help of Proposition 4.3 we obtain the claim. If $(\bar{g}_1, \dots, \bar{g}_r, \bar{h})$ is close to $(\bar{\alpha}_1, \dots, \bar{\alpha}_r, \bar{\beta})$ then by continuous dependence, $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ enters the tubes described above as well. Hence $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ is defined for $t > 0$. \square

Corollary 4.4 implies the completeness of the metrics which corresponds to these solutions. In the next step we show that two Einstein metrics defined by distinct solutions $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ are not isometric at least if \bar{h} is very small.

Assume there is an isometry Ψ between (\widehat{M}, \hat{g}) , defined by $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$, and (\widehat{M}', \hat{g}') , defined by $c_{\bar{g}'_1, \dots, \bar{g}'_r, \bar{h}'}$. If Ψ maps orbits onto orbits, then $(\bar{g}_1, \dots, \bar{g}_r, \bar{h}) = (\bar{g}'_1, \dots, \bar{g}'_r, \bar{h}')$ since the geometry of the singular orbit Q is determined by $(\bar{g}_1, \dots, \bar{g}_r, \bar{h})$. Therefore we can assume that there is a principal orbit which is not mapped onto an principal orbit. Thus there exists a $\hat{p}' \in \widehat{M}'$ such that $T_{\hat{p}'}\widehat{M}'$ is generated by Killing fields. Hence \hat{g}' is a homogeneous metric and \hat{g} as well. However, for initial values $a_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ with $\bar{h} > 0$ very small the term $\lim_{t \rightarrow 0} h''(t)/h(t)$ becomes arbitrary large (cf. (28)). But $-h''(t)/h(t)$ is a sectional curvature of \hat{g} by means of the Riccati equation

$$L'(t) + L^2(t) + \widehat{R}_{.N}N = 0$$

(cf. [20]). Thus the sectional curvature at the singular orbit becomes arbitrary large if \bar{h} is very small. On the other hand side a cone metric is outside the singular orbit a perfect C^∞ -metric, i.e. the sectional curvature on a principal orbit P_{t_0} is bounded. We know from the proof of Lemma 10.1, that $c_{\bar{g}_1, \dots, \bar{g}_r, \bar{h}}$ comes arbitrary close to the cone solution γ_λ for a suitable choice of $\bar{g}_1, \dots, \bar{g}_r, \bar{h}$. Therefore the sectional curvature of the corresponding metric comes arbitrary close to the sectional curvature of the cone metric restricted to P_{t_0} . Hence this metric cannot be homogeneous for very small $\bar{h} > 0$! Thus we have proved Theorem 6.1.

In the last part of this section we will prove that the complete Einstein metrics given by

$$c_{\bar{h}} := c_{\sqrt{(\ell_1-1)/r_h \bar{h}}, \dots, \sqrt{(\ell_r-1)/r_h \bar{h}}, \bar{h}}$$

(see Theorem 9.7) converge (restricted to a compact set not containing the singular orbit Q) for $\bar{h} \rightarrow 0$ to the cone metric in the C^∞ -topology. We restrict ourselves to the case $\lambda = 0$ and $\lambda = -n$. Let $T_0 > t_0 > 0$ and let D_{t_0} be any slice of γ_λ at $\gamma_\lambda(t_0)$. Using the flow of the vector field X_λ described in Section 2 we obtain a slice D_{T_0} of γ_λ at $\gamma_\lambda(T_0)$. With the help of the Convergence Theorem 9.7 we know that $c_{\bar{h}}$ intersects D_{t_0} for small \bar{h} as close to $\gamma_\lambda(t_0)$ as we wish. Suppose this happens at the time $t_{\bar{h}} > 0$. In order to prove the desired convergence property we have to show that

$$\lim_{\bar{h} \rightarrow 0} t_{\bar{h}} = t_0.$$

Let $\epsilon > 0$. We go into the $(w, w_1, x_1, \dots, w_r, x_r, h, y)$ -coordinates. Let $\delta \in (0, \frac{1}{4}\epsilon]$ and suppose that δ is smaller than t_0 . Let

$$T_\delta := \left\{ (w, w_1, x_1, \dots, w_r, x_r, h, y) \in V_{\kappa_{\text{ex}}}^{2r+3} \mid h \in [0, c_\lambda(\delta)], \right. \\ \left. \kappa_{\text{ex}}(w, w_1, x_1, \dots, w_r, x_r, h, y) \leq \kappa_{\text{ex}}(\bar{\gamma}_\lambda(0)) + \delta \right\}$$

be the κ_{ex} -tube around $\bar{\gamma}_\lambda|_{[0, \delta]}$ of radius δ (for the definition of c_λ see (56)). Let D_δ denote the top and let D_0 denote the bottom of this tube. Since the y -coordinate of γ_λ equals c'_λ we can assume that the y -coordinate restricted to T_δ is an element of $[1 - \frac{1}{4}\epsilon, 1 + \frac{1}{4}\epsilon]$ by choosing δ small enough. Every solution of (36), ..., (40) which initial value lies in the tube $T_\delta \setminus D_0$ intersects D_δ by Proposition 2.2. By Corollary 9.5 the solution $b(t)$ of (49), ..., (53) (defined in Proposition 9.2) reaches the boundary of D_0 in finite time, say at $s_b > 0$. By continuous dependence on the initial value $c_{\bar{h}}$ considered as a solution of (49), ..., (53)! reaches the boundary of T_δ in less time than $2s_b$ for $\bar{h} > 0$ small enough. Furthermore we can assume that the hight function h along $c_{\bar{h}}$ up to that intersection point is as small as we wish. Recall that (49), ..., (53) is obtained from (36), ..., (40) by scaling with h . Therefore there exists $\bar{h}_1(\delta, \epsilon) > 0$ such that the time needed by $c_{\bar{h}}$ (considered as a solution of (36), ..., (40)!) to reach the boundary of T_δ is less or equal than $\frac{1}{4}\epsilon$ for $\bar{h} \leq \bar{h}_1(\delta, \epsilon)$. With the help of (39) we conclude that the time needed by $c_{\bar{h}}$ to reach D_δ is less or equal than $\frac{1}{4}\epsilon + (1 + \frac{1}{4}\epsilon)\delta$ for $\bar{h} \leq \bar{h}_1(\delta, \epsilon)$. For \bar{h} small enough we can assume that the time which $c_{\bar{h}}$ needs to go from D_δ to D_{t_0} is $\frac{1}{4}\epsilon$ -close to $t_0 - \delta$. Therefore we obtain the above mentioned convergence property.

11. Non-trivial vector bundles

In this section we will extend the Convergence Theorem 5.7 in [6] for non-positive Einstein constants. This yields Theorem 6.3, Theorem 6.4 and Theorem 6.1 for $r = 0$.

In [6] the most simple non-trivial case of the cohomogeneity one Einstein equations is investigated: It is assumed that the space of G -invariant metrics on the principal orbit type P is 2-dimensional and that there exists a connected singular orbit Q with $\dim Q > 0$. For any G -invariant metric \hat{g} on \widehat{M} we obtain

$$\hat{g}|_{\widehat{M}_0} = dt^2 + f^2(t)g^{S^k} + h^2(t)g^Q,$$

where g^{S^k} is the standard metric on S^k and g^Q is a G -invariant metric on Q . The Einstein condition for \hat{g} becomes:

$$(58) \quad k \frac{f''}{f} + d_Q \frac{h''}{h} = -\lambda,$$

$$(59) \quad \frac{f''}{f} + (k - 1) \frac{f'^2}{f^2} + d_Q \frac{f'h'}{fh} - \frac{1}{f^2} (k - 1) - \frac{d_Q}{k} \|\mathcal{A}\|^2 \frac{f^2}{h^4} = -\lambda,$$

$$(60) \quad \frac{h''}{h} + (d_Q - 1) \frac{h'^2}{h^2} + k \frac{f'h'}{fh} - \frac{1}{h^2} Ric^Q + 2\|\mathcal{A}\|^2 \frac{f^2}{h^4} = -\lambda,$$

$$(61) \quad Ric(X, N) = 0.$$

The arising constants are defined as follows:

$$d_Q := \dim Q$$

and $Ric^Q \geq 0$ is the Einstein constant of the isotropy irreducible space (Q, g^Q) . Let D^1 be the Levi-Civita connection on P with respect to the metric

$$g^1 := g^{S^k} + g^Q$$

and let H_1, H_2, \dots, H_{d_Q} form a horizontal orthonormal basis regarding the Riemannian submersion $\pi : (G/K, g^1) \rightarrow (G/H, g^Q)$. Now

$$\|\mathcal{A}\| := \left[\sum_{i=1}^{d_Q} g^{S^k} ((D_{H_1}^1 H_i)|_v, (D_{H_1}^1 H_i)|_v) \right]^{\frac{1}{2}} \geq 0.$$

Here $|_v$ denotes the projection onto the tangent space of the fibers S^{d_S} . The Ricci form of $(Ric, \hat{g}|_{\widehat{M}_0})$ is denoted by \widehat{M}_0 , N is the normal vector of the principal orbits and X is tangential to these. In Table 1 we give some examples of non-trivial vector space bundle defined by the Lie group triple (G, H, K) (cf. [20]), which satisfy the above assumptions [6]. These example are precisely the Hopf fibration describe in [5, p. 257–258].

<i>Description</i>	$\mathbb{C}P^{m+1}$	$\mathbb{H}P^{m+1}$	$\mathbb{Z}P^{m+1}$	$\mathbb{C}aP^2$
G	$U(m + 1)$	$Sp(1) \times Sp(m + 1)$	$Sp(m + 1)$	$Spin(9)$
H	$U(1) \times U(m)$	$Sp(1) \times Sp(1) \times Sp(m)$	$Sp(1) \times Sp(m)$	$Spin(8)$
K	$U(m)$	$Sp(1) \times Sp(m)$	$U(1) \times Sp(m)$	$Spin(7)$
k	1	3	2	7
d_Q	$2n$	$4n$	$4n$	8
$\ \mathcal{A}\ ^2$	1	3	8	7
Ric^Q	$2m + 2$	$4m + 8$	$4m + 8$	28

Table 1: Examples with $\|\mathcal{A}\| > 0$

In [6, Thm 2.3] it is proved, that for $\bar{h} > 0$ there exists a unique solution

$$c_{\bar{h}}(t) := (f(t), f'(t), h(t), h'(t))$$

of (58), (59) and (60) with $c_{\bar{h}}(0) = a_{\bar{h}} = (0, 1, \bar{h}, 0)$ which depends continuously on \bar{h} . Moreover the corresponding metric $\hat{g}|_{\hat{M}_0} = dt^2 + f^2(t)g^{S^k} + h^2(t)g^Q$ can be extended to a C^∞ -Einstein metric on a tubular neighbourhood of Q .

CONVERGENCE THEOREM 11.1. — *Let \widehat{M} be a cohomogeneity one manifold satisfying the above assumptions. Suppose $k \geq 2$ and*

$$\widehat{D} = (\text{Ric}^Q)^2 - 4\|\mathcal{A}\|^2 \frac{k}{k+1} (d_Q + 2k) > 0.$$

Then $c_{\bar{h}}$ converges for $\bar{h} \rightarrow 0$ to the unique stable cone solution of \widehat{M} for any Einstein constant $\lambda \in \mathbb{R}$.

Proof. — Condition $\widehat{D} > 0$ implies of course

$$D = (\text{Ric}^Q)^2 - 4\|\mathcal{A}\|^2 \frac{k-1}{k} (d_Q + 2k) > 0$$

(cf. [5, 9.72 (a)]). By [6], Lemma 4.1 in case $\|\mathcal{A}\| = 0$ there exists a unique cone solution, which is stable, and in case $\|\mathcal{A}\| > 0$ there exist two cone solutions, one of them stable the other not. The main issue of [6, Section 5] is to prove Theorem 11.1 in case $\lambda = d_P = k + d_Q$. Until Lemma 5.2 in [6] there is no change in the proof for $\lambda \leq 0$. After Lemma 5.2 one has to replace h by $b_1 \bar{h}$ in case $\lambda = 0$ and by $b_1 \sinh(\bar{h})$ in case $\lambda = -d_P$ (instead of replacing h by $b_1 \sin(\bar{h})$). For the definition of b_1 see Lemma 4.1 in [6]. Lemma 5.3 and Lemma 5.4 are obtained in the same manner. Defining the vector field \bar{Z} (see [6, eq. (37)]) one gets rid of the Einstein constant λ since $\bar{h} = 0$! Therefore the discussion of the bottom vector field \bar{Z} does not depend on λ ! Since the attractor function \bar{K} can be extended in the same manner, one gets Theorem 11.1. \square

Now we are in the same situation as after proving the Convergence Theorem 9.7. The proof of Theorem 6.3 and Theorem 6.4 is obtained by Corollary 4.4 (completeness) and the discussion in Section 10.

REMARK. — The cases $m = 1, 2$ in Theorem 6.4 are missing since here $\widehat{D} \leq 0$. But one might be able to check by hand that in these cases the

above theorem holds as well. For instance the explicit Ricci flat solution [10], [23] should help a lot in case $m = 1$.

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