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HOLOMORPHIC FOLIATIONS IN CERTAIN HOLOMORPHICALLY CONVEX DOMAINS OF $\mathbb{C}^2$

BY

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ABSTRACT. — We consider the problem of describing holomorphic foliations with singularities which are transversal to the boundary of certain domains of $\mathbb{C}^2$.

RÉSUMÉ. — On décrit les feuilletages holomorphes avec singularités définis dans certains domaines de $\mathbb{C}^2$ et transverses au bord.

0. Introduction

We prove in this paper a theorem which supports the idea that imposing geometric constraints to holomorphic foliations on the boundary of their domains of definition leads to strong conclusions regarding the dynamics. We are interested mainly in the following situation: let $\Omega \subset \mathbb{C}^2$ be a bounded domain with smooth boundary, diffeomorphic to the unit ball, and $\mathcal{F}$ a holomorphic foliation with singularities defined in a neighborhood of $\overline{\Omega}$ and transverse to $\partial\Omega$; what can be said about the behaviour of $\mathcal{F}$ inside Ω?

We shall be concerned here with a variant of a such a question, where the domain Ω is a «generalized bidisc». We give the precise definition of generalized bidisc in § 1, following the work of F. FORSTNERIČ [For] on nonlinear Riemann-Hilbert problem. For the moment, it is sufficient to say that a generalized bidisc $\Omega \subset \mathbb{C}^2$ is a holomorphically convex domain whose boundary $\partial\Omega$ decomposes as $\partial\Omega = Y \cup T \cup \Sigma$, where Y and Σ are

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Levi flat smooth solid tori and $T = \partial Y = \partial\Sigma \simeq \mathbb{T}^2$. As the name suggests, the standard bidisc

$$\{(z,w) \in \mathbb{C}^2 \,;\ |z| < 1,\ |w| < 1\}$$

is the most trivial example of generalized bidisc.

If $\mathcal{F}$ is a holomorphic foliation defined in a neighborhood of a generalized bidisc $\overline{\Omega}$, we shall say that $\mathcal{F}$ is *transverse to* $\partial\Omega$ if it is transverse to the two Levi flat smooth solid tori Y and Σ and also to the 2-torus T.

Our result is the following. We denote by $\mathcal{L}_\lambda$ the linear hyperbolic foliation in $\mathbb{C}^2$ given by $x\,\mathrm{d}y + \lambda y\,\mathrm{d}x = 0$, where $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

THEOREM. — *Let $\Omega \subset \mathbb{C}^2$ be a generalized bidisc and let $\mathcal{F}$ be a holomorphic foliation defined in a neighborhood of $\overline{\Omega}$ and transverse to $\partial\Omega$. Then there exists a locally injective holomorphic map ϕ which sends a neighborhood of $\overline{\Omega}$ to a neighborhood of 0 in $\mathbb{C}^2$ and such that $\mathcal{F} = \phi^*(\mathcal{L}_\lambda)$ for some $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Furthermore ϕ is injective as a map between spaces of leaves, i.e. for every leaf $L \in \mathcal{L}_\lambda$ the preimage $\phi^{-1}(\phi(\overline{\Omega}) \cap L)$ is exactly one leaf of $\mathcal{F}_{|\overline{\Omega}}$.*

In other words, the dynamics of $\mathcal{F}$ in $\overline{\Omega}$ is well known [Arn]: there is a hyperbolic singularity p with two separatrices F_1 and F_2 (we shall see that $\overline{F}_1$ and $\overline{F}_2$ are discs), and all the other leaves of $\mathcal{F}$ accumulate to and only to $\overline{F}_1 \cup \overline{F}_2$. In particular, $\overline{\Omega}$ is contained in the basin of the hyperbolic singularity p, that is, the union of leaves that accumulate to $p \in \mathbb{C}^2$.

Our proof does not give a globally injective holomorphic map ϕ and we don't know if it is possible to obtain a map with this property. However, the injectivity of ϕ as a map between spaces of leaves shows that the transverse dynamics of $\mathcal{F}$ is the same as that of $\mathcal{L}_\lambda$.

The several steps of the proof of the theorem may be grouped according to three main ideas:

(i) To represent the foliation $\mathcal{F}$ by a closed meromorphic 1-form in $\overline{\Omega}$ (§ 2).

(ii) To simplify this closed 1-form to a logarithmic 1-form

$$\lambda_1 \frac{\mathrm{d}f_1}{f_1} + \lambda_2 \frac{\mathrm{d}f_2}{f_2}$$

and to study the polar divisor $\{f_1 = 0\} \cup \{f_2 = 0\}$ (§ 3).

(iii) To modify $(f_1, f_2) \colon \overline{\Omega} \to \mathbb{C}^2$ in order to get a locally injective map ϕ with $\phi^*(\mathcal{L}_\lambda) = (f_1, f_2)^*(\mathcal{L}_\lambda) = \mathcal{F}$ (§ 4).

According to DOUADY-ITO (private communication), a similar theorem can be proved when the foliation is transverse to the boundary of the balls

$$B_p = \{(x,y) \in \mathbb{C}^2\,;\ |x|^p + |y|^p = 1\} \ \text{ for } \ p \geq 2.$$

Since the standard bidisc can be seen as a limit of the balls B_p as $p \to \infty$, we get an alternative proof, in this domain, for our Theorem. Also, in [Bru] the same result is proven for strictly convex compact domains of $\mathbb{C}^2$ with smooth boundary. Because of that, we choosed to state the Theorem for generalized bidiscs: we feel that perhaps the important condition is pseudoconvexity. Anyway, the reader can follow this paper keeping in mind the standard bidisc with the natural horizontal and vertical fibrations; the sole extra difficulty in the more general setting is to produce a sort of «horizontal» fibration (LEMMAS 3 and 4).

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1. Generalized bidiscs

Generalized bidiscs are described in [For] as polynomially convex hulls of certain totally real 2-tori in $\mathbb{C}^2$.

Let $T \subset \mathbb{C}^2$ be a smoothly embedded torus contained in

$$\{(z,w) \in \mathbb{C}^2\,;\ |z| = 1\}$$

and such that the vertical projection of T on the circle $\{|z| = 1,\ w = 0\}$ is a smooth fibration with fibre $\mathbb{S}^1$. Then

$$T_\theta = \{z = \mathrm{e}^{i\theta}\} \cap T$$

is a smooth simple closed curve bounding a disc $Y_\theta \subset \{z = \mathrm{e}^{i\theta}\}$, and T is the boundary of the solid torus

$$Y = \bigcup_{\theta \in [0,2\pi]} Y_\theta.$$

Remark that Y is Levi flat and that $\{Y_\theta\}_{\theta \in [0,2\pi]}$ are the leaves of its charateristic foliation.

Assume that each Y_θ contains the origin of the line

$$\{(z,w) \in \mathbb{C}^2\,;\ z = \mathrm{e}^{i\theta}\}.$$

Then a theorem of Forstnerič says that T is the boundary of a second Levi flat smooth solid torus Σ, which is «almost horizontal». More precisely:

THEOREM (see [For]). — *There exists an embedded smooth solid torus* $\Sigma \subset \mathbb{C}^2$ *with* $\partial\Sigma = T$ *such that*:

(i) Σ *is Levi flat*;

(ii) *the characteristic leaves of* Σ *are discs of the form*

$$\{w = f(z);\, |z| < 1\},$$

f holomorphic and zero-free in $\{|z| < 1\}$ *and smooth in* $\{|z| \leq 1\}$; *the characteristic foliation of* Σ *is smooth, and it is a smooth fibration over the circle*;

(iii) $\Sigma \cup T \cup Y$ *is the boundary of a polynomially convex domain* Ω; *the set* $\overline{\Omega}$ *is a manifold with boundary and corners, and it is the polynomially convex hull of* T.

We shall say that Ω is a *generalized bidisc.*

Remark that Σ is uniquely defined by T, and the proof of Forstnerič shows that a smooth variation of T will produce a smooth variation of Σ. This means that if $e\colon \mathbb{T}^2 \times [0,1] \to \mathbb{C}^2$ is a smooth embedding such that every $e(\mathbb{T}^2 \times \{t\})$ is a torus as above, then there exists a smooth embedding

$$\hat{e}\colon \mathbb{S}^1 \times \mathbb{D}^2 \times [0,1] \longrightarrow \mathbb{C}^2$$

such that $\hat{e}(\mathbb{S}^1 \times \mathbb{D}^2 \times \{t\})$ is a solid torus with boundary $e(\mathbb{T}^2 \times \{t\})$ as in the theorem above (the fact that $\hat{e}$ is an embedding follows from an index-type argument, see *e.g.* page 885 of [For] where this argument is used to prove that Σ is embedded). The image of $\hat{e}$ admits consequently a smooth foliation by holomorphic discs, obtained glueing together the characteristic foliations of the solid tori $\hat{e}(\mathbb{S}^1 \times \mathbb{D}^2 \times \{t\})$.

In particular, if Ω is a generalized bidisc then the characteristic foliation of Σ can be extended to a collar of Σ (as a smooth foliation by holomorphic discs). We shall see later how to extend this foliation to all of Ω.

Remark also that the projection of Ω on $\{|z| < 1,\ w = 0\}$ gives a holomorphic foliation by holomorphic discs, extending the characteristic foliation of Y. Such a foliation is holomorphically trivial if and only if Ω is biholomorphic to the standard bidisc.

2. Construction of special closed 1-forms

Let Ω and $\mathcal{F}$ be as in the statement of the theorem, with $\partial\Omega = \Sigma \cup T \cup Y$. Let $e\colon \mathbb{T}^2 \times [0,1] \to \mathbb{C}^2$ be a embedding such that $e(\mathbb{T}^2 \times \{0\}) = T$

and $T_t = e(\mathbb{T}^2 \times \{t\})$ is a torus satisfying the hypothesis of Forstnerič theorem and, moreover, transverse to $\mathcal{F}$, for every $t \in [0,1]$. Then, as explained in § 1, every T_t is the boundary of a Levi flat solid torus Σ_t, where $\Sigma_0 = \Sigma$. Because Σ is transverse to $\mathcal{F}$, we may assume that every Σ_t is also transverse to $\mathcal{F}$.

Let $\mathcal{L}_t$ be the one dimensional foliation induced by $\mathcal{F}$ on Σ_t. The property

$$T_x\mathcal{F} \pitchfork T_x\Sigma_t \quad \forall x \in \Sigma_t$$

implies that $T_x\mathcal{F}$ is also transverse to $T_x^{\mathbb{C}}\Sigma_t$ (the complex line tangent to Σ_t at x), *i.e.* $\mathcal{L}_t$ is transverse to the fibration of Σ_t by holomorphic discs. The transversality of $\mathcal{F}$ to T_t means that $\mathcal{L}_t$ is transverse to $\partial\Sigma_t = T_t$.

We orient $\mathcal{L}_t$ along $\partial\Sigma_t$ in such a way that it «enters» into Σ_t, and then this orientation is extended to $\mathcal{L}_t$ inside Σ_t. In order to see why this can be done, let $p_0 \in \partial\Sigma_t$ be fixed and $x \in \Sigma_t$. We choose a smooth path α in Σ_t joining p_0 to x, and extend continuously the orientation of $\mathcal{L}_t$ from p_0 to x. If β is another such path, then $\alpha * \beta$ is homotopic (using p_0 as base point) to a closed path contained in $\partial\Sigma_t$ (where the orientation is already well defined). So that using this homotopy we see that either α or β induce the same orientation of $\mathcal{L}_t$ at x.

Following then $\mathcal{L}_t$ along its leaves allows us to define the first return map

$$h_t : \overline{\Gamma} \longrightarrow \operatorname{int}\Gamma$$

for every holomorphic disc Γ in Σ_t. This map is holomorphic because it is defined between transversal holomorphic sections to the holomorphic foliation $\mathcal{F}$, following continuously paths contained in the leaves.

By Schwarz lemma, h_t has a unique fixed point which is a hyperbolic attractor, so that $\mathcal{L}_t$ has a unique closed orbit γ_t which is also a hyperbolic attractor. Every other orbit of $\mathcal{L}_t$ starts form $\partial\Sigma_t$ and accumulates to γ_t.

Let

$$V = \bigcup_{t\in[0,1]} \Sigma_t.$$

Then the previous analysis that $\mathcal{F}_{|\overline{V}}$ has an embedded cylindrical leaf

$$L = \bigcup_{t\in[0,1]} \gamma_t$$

with hyperbolic holonomy, and the other leaves are simply connected and accumulate only to L.

Every leaf of $\mathcal{F}_{|\overline{V}}$ is transverse to the fibres of the fibration of V by holomorphic discs (as usual, this is the fibration induced by the characteristic foliations of Σ_t, $t \in [0,1]$).

LEMMA 1. — *There exists a closed meromorphic* 1*-form* ω_1, *defined in* $\overline{V}$, *which defines* $\mathcal{F}_{|\overline{V}}$ *and such that its polar divisor* $(\omega_1)_\infty$ *is* L.

Proof (*compare* [CLS]). — Let η be a holomorphic coordinate on a holomorphic disc of the fibration of V, such that $\eta = 0$ corresponds to the cylindrical leaf L and η makes the holonomy of L linear. We may extend η to a neighborhood of the disc in such a way that it is constant on leaves of $\mathcal{F}$, so that $\mathcal{F}$ is represented by $\mathrm{d}\eta/\eta = 0$ and $\eta = 0$ corresponds to L. Any other coordinate $\tilde{\eta}$ which makes the holonomy linear is related to η as $\tilde{\eta} = c\eta$ for some $c \in \mathbb{C}^*$, so that $\mathrm{d}\tilde{\eta}/\tilde{\eta} = d\eta/\eta$ and $d\eta/\eta$ is intrisically defined. Moving the holomorphic disc in V, we obtain a collection of closed meromorphic 1-forms which glue together and give the required ω_1. $\square$

Let now
$$Y_t = \Omega \cap \{|z| = t\}, \quad t \in (0,1].$$

Clearly Y_t is a Levi flat solid torus, with characteristic foliation given by the vertical discs $\Omega \cap \{z = c\}$, $|c| = t$. Because $Y = Y_1$ is transverse to $\mathcal{F}$, there exists $\delta \in (0,1)$ such that Y_t is transversal to $\mathcal{F}$ for all $t \in [\delta, 1]$. If
$$U = \bigcap_{t \in [\delta,1]} Y_t,$$

the same arguments as above show that $\mathcal{F}_{|\overline{U}}$ has a cylindrical leaf L' with hyperbolic holonomy and all the other leaves of $\mathcal{F}_{|\overline{U}}$ accumulate to L'. Moreover, there exists in $\overline{U}$ a closed meromorphic 1-form ω_2 which defines $\mathcal{F}_{|\overline{U}}$ and whose polar divisor $(\omega_2)_\infty$ is L'.

We remark that ω_1 and ω_2 are uniquely defined up to a multiplicative constant.

LEMMA 2. — *There exists a closed meromorphic* 1*-form* ω, *defined in a neighborhood of* $\overline{\Omega}$, *which represents* $\mathcal{F}$ *and such that*
$$(\omega)_\infty \cap (\overline{U} \cup \overline{V}) = L \cup L'.$$

Proof. — Let α be a holomorphic 1-form in $\overline{\Omega}$ which represents $\mathcal{F}$ and let H, H' be holomorphic functions in $\overline{V}, \overline{U}$ defined by
$$\alpha = H \cdot \omega_1, \ \alpha = H' \cdot \omega_2.$$

We have:
$$\{H = 0\} = L, \quad \{H' = 0\} = L'.$$

Observe that $\overline{U} \cap \overline{V}$ is disjoint from $L \cup L'$, consequently H/H' is a nonvanishing holomorphic function in $\overline{U} \cap \overline{V}$. This function, as the ratio of two integrating factors, is a first integral for $\mathcal{F}$: from $\omega_2 = H/H'\,\omega_1$ one obtains

$$\mathrm{d}\Big(\frac{H}{H'}\Big) \wedge \omega_1 = 0, \quad \textit{i.e.} \quad \mathrm{d}\Big(\frac{H}{H'}\Big) \wedge \alpha = 0.$$

The intersection $\overline{U} \cap \overline{V}$ is naturally fibrated by the tori $Y_t \cap \Sigma_s$, which are transverse to $\mathcal{F}$. Hence the space of leaves of $\mathcal{F}_{|\overline{U}\cap\overline{V}}$ is a compact elliptic curve. It follows that the holomorphic first integral H/H' is constant in $\overline{U} \cap \overline{V}$, and we get

$$\omega_2 = c \cdot \omega_1, \quad c \in \mathbb{C}^*.$$

We may define ω in $\overline{U} \cap \overline{V}$ as $\omega_{|\overline{U}} = \omega_2$, $\omega_{|\overline{V}} = c \cdot \omega_1$, and then extend this closed meromorphic 1-form to all of $\overline{\Omega}$, using the holomorphic convexity of Ω (see [Siu]). It is evident that such extension satisfies

$$(\omega)_\infty \cap (\overline{U} \cup \overline{V}) = L \cup L'. \quad \square$$

3. Structure of the polar divisor

Let ω be the closed meromorphic 1-form constructed in the previous section. Because every irreducible component of $(\omega)_\infty$ must intersect $\partial\overline{\Omega}$ and hence $\overline{U} \cup \overline{V}$, we have only two possibilities:

(a) $(\omega)_\infty$ has only one irreducible component $\{f = 0\}$, $f \in \mathcal{O}(\overline{\Omega})$, and

$$\{f = 0\} \cap \overline{U} = L', \quad \{f = 0\} \cap \overline{V} = L\,;$$

(b) $(\omega)_\infty$ has two irreducible components $\{f_1 = 0\}$, $\{f_2 = 0\}$, with

$$\begin{aligned} &\{f_1 = 0\} \cap \overline{U} = \emptyset, && \{f_1 = 0\} \cap \overline{V} = L, \\ &\{f_2 = 0\} \cap \overline{U} = L', && \{f_2 = 0\} \cap \overline{V} = \emptyset. \end{aligned}$$

The first possibility cannot occur. In fact, if $\lambda = \mathrm{Res}_{\{f=0\}}\,\omega$ we have that $\omega - \lambda \mathrm{d}f/f$ is a closed holomorphic 1-form in $\overline{\Omega}$, and the Poincaré's lemma (see [G-R]) gives:

$$\omega = \lambda \frac{\mathrm{d}f}{f} + \mathrm{d}H \quad \text{for some } H \in \mathcal{O}(\overline{\Omega}).$$

Putting $\phi = e^{\frac{1}{\lambda}H}$, we find that

$$\omega = \lambda \frac{d(\phi f)}{\phi f},$$

so that $\mathcal{F}$ has a nontrivial holomorphic first integral ϕf. But this is incompatible with the dynamics of $\mathcal{F}$ in $\overline{U} \cup \overline{V}$.

Hence $(\omega)_\infty = \{f_1 = 0\} \cup \{f_2 = 0\}$. These two analytic curves are the closures of two leaves $F_1, F_2 \in \mathcal{F}$, and $F_1 \cap \overline{V} = L$, $F_2 \cap \overline{U} = L'$. Let

$$\lambda_1 = \operatorname{Res}_{\overline{F}_1} \omega, \quad \lambda_2 = \operatorname{Res}_{\overline{F}_2} \omega\,;$$

by the same argument as before we get:

$$\omega = \lambda_1 \frac{df_1}{f_1} + \lambda_2 \frac{d(\phi f_2)}{\phi f_2}, \quad \text{with } \phi \in \mathcal{O}^*(\overline{\Omega}).$$

We can replace f_2 by f_2/ϕ, so that

$$\omega = \lambda_1 \frac{df_1}{f_1} + \lambda_2 \frac{df_2}{f_2}.$$

Our aim is to show that $\overline{F}_1$ and $\overline{F}_2$ are biholomorphically equivalent to discs.

LEMMA 3. — *The set $\overline{F}_2$ is a disc, given by the graph of a holomorphic function v: $\overline{F}_2 = \{(z,w) \in \mathbb{C}^2;\, |z| \leq 1,\ w = v(z)\}$.*

Proof. — Define $\pi: \overline{F}_2 \to \{|z| \leq 1\}$ as the vertical projection. Since $\overline{F}_2 \cap \Sigma = \phi$, the map π is onto and proper. Let $\widetilde{F}_1 \xrightarrow{\rho} \overline{F}_1$ be the normalization of $\overline{F}_1$ and let $\widetilde{\pi} = \pi \circ \rho$. If $1-\delta \leq |z| \leq 1$, δ small, then $\#\widetilde{\pi}^{-1}(z) = 1$ because $\overline{F}_2 \cap \overline{U} = L'$ and L' is transverse to the vertical discs. It follows that $\#\widetilde{\pi}^{-1}(z) = 1$ for every $|z| \leq 1$, *i.e.* $\overline{F}_2$ is smooth and transverse to the vertical discs. The existence of v is then evident. $\square$

To prove an analogue of LEMMA 3 for $\overline{F}_1$ we need to project along an « almost horizontal » fibration. It is helpful the following construction, based on § 1.

We consider a smooth embedding j of $\mathbb{D}^2 \times \mathbb{S}^1$ onto $\overline{Y}$ such that, denoting by S_t the torus $\{(x,y,\theta) \in \mathbb{D} \times S^1\,;\ x^2 + y^2 = t\}$, $t \in (0,1]$, and by S_0 the circle $\{(x,y,\theta) \in \mathbb{D}^2 \times S^1\,;\ x = y = 0\}$ we have:

(i) $j(S_1) = T$, $j(S_0) = \{(z,w) \in \mathbb{C}^2,\ w = v(z)\}$;

(ii) $j(S_t)$ is transverse to $\mathcal{F}$ and to the vertical fibration of $\overline{Y}$, for all $t \in (0,1]$.

It is not difficult to see that such an embedding exists (recall the structure of the foliation induced by $\mathcal{F}$ on Y). Since $j(S_t)$ surrounds the circle $\{|z| = 1,\ w = v(z)\}$ and v is holomorphic in $\{|z| \leq 1\}$ we may apply Forstnerič theorem in order to obtain a smooth foliation $\mathcal{H}$ by closed holomorphic discs of $\overline{\Omega}$, which extends the characteristic foliation of $\overline{\Sigma}$ and contains $\overline{F}_2$ as a leaf. The property (ii) ensures that every disc of such foliation, except $\overline{F}_2$, is transverse to $\mathcal{F}$ near its boundary. Obviously, this foliation is in fact a smooth fibration over the closed disc.

The next lemma says that this foliation (or fibration) is well adapted to $\mathcal{F}$.

LEMMA 4. — *Every disc of $\mathcal{H}$ different from $\overline{F}_2$ is transverse to $\mathcal{F}$. In particular $\overline{F}_1$ is a disc, transverse to every leaf of $\mathcal{H}$.*

Proof. — Let K be the set of points of $\overline{\Omega}\setminus\overline{F}_2$ where $\mathcal{F}$ is not transverse to $\mathcal{H}$ (K includes the singular points of $\mathcal{F}$). Then K is closed in $\overline{\Omega}\setminus\overline{F}_2$. Let Q be the space of leaves of $\mathcal{H}_{|\overline{\Omega}\setminus\overline{F}_2}(Q \simeq \mathbb{S}^1 \times (0,1])$, and let $q\colon \overline{\Omega}\setminus\overline{F}_2 \to Q$ be the projection. Then $q(K)$ is closed, because the fibres of q are compact.

On the other hand, $q(K)$ is open. For if $q^{-1}(a)$ is a fibre tangent to $\mathcal{F}$, then a holomorphic 1-form α defining $\mathcal{F}$ restricted to $q^{-1}(a)$ has an isolated singular point in a point $p \in q^{-1}(a)$, not belonging to $\partial[q^{-1}(a)]$ by the choice of $\mathcal{H}$. Then if $\tilde{a}$ is near $a \in Q$ the form α restricted to $q^{-1}(\tilde{a})$ will have a singular point $\tilde{p} \in q^{-1}(\tilde{a})$ near p, *i.e.* $\mathcal{F}$ is tangent to $\mathcal{H}$ at $\tilde{p}$ (isolated zeroes of holomorphic functions are stable under continuous perturbations).

Because Σ is transverse to $\mathcal{F}$, $q(K) \neq Q$ and consequently $q(K) = \phi$. This proves the first affirmative, the second one follows from the first and from $\overline{F}_1 \cap Y = \phi$. $\square$

Let us summarize what we have proven so far. The foliation $\mathcal{F}$ is given by

$$\omega = 0, \quad \omega = \lambda_1 \frac{\mathrm{d}f_1}{f_1} + \lambda_2 \frac{\mathrm{d}f_2}{f_2}\cdot$$

LEMMAS 3 and 4 guarantee that $f_1(f_2)$ has no critical points along its zero set $\overline{F}_1(\overline{F}_2)$, which is a disc. The discs $\overline{F}_1$ and $\overline{F}_2$ intersect each other transversally at exactly one point $p \in \Omega$, which has to be a hyperbolic singularity of $\mathcal{F}$ because its separatrices F_1 and F_2 have hyperbolic holonomies. Moreover, LEMMA 4 implies $p \in \Omega$ is the sole singularity of $\mathcal{F}$. The disc $\overline{F}_1$ intersects $\partial\Omega$ along a circle in Σ, wheres $\overline{F}_2$ intersects $\partial\Omega$ along a circle in Y; these two circles are the limit set of $\mathcal{F}_{|\partial\Omega}$.

4. Construction of the conjugation

The holomorphic map $\phi_0 = (f_1, f_2)\colon \overline{\Omega} \to \mathbb{C}^2$ satisfies $\phi_0^*(\mathcal{L}_\lambda) = \mathcal{F}$, for $\lambda = \lambda_1/\lambda_2$, and moreover it is a biholomorphism between a neighborhood of p and a neighborhood of $(0,0)$. If $u\colon \overline{\Omega} \to \mathbb{C}$ is any holomorphic function, then

$$\phi \stackrel{\text{def}}{=} (\mathrm{e}^u f_1, \mathrm{e}^{-\lambda u} f_2)$$

enjoys the same properties. A straightforward computation gives

$$\mathrm{Jac}(\phi) = \mathrm{e}^{(1-\lambda)u}\left[\mathrm{Jac}\,(\phi_0) + Z \cdot u\right]$$

where Z is the holomorphic vector field

$$\Big(f_1\frac{\partial f_2}{\partial w} + \lambda f_2 \frac{\partial f_1}{\partial w}\Big)\frac{\partial}{\partial z} - \Big(f_1\frac{\partial f_2}{\partial z} + \lambda f_2 \frac{\partial f_1}{\partial z}\Big)\frac{\partial}{\partial w} = P(z,w)\frac{\partial}{\partial z} + Q(z,w)\frac{\partial}{\partial w}$$

which is tangent to $\mathcal{F}$ and has a hyperbolic singularity at $p = (z_p, w_p)$. We look for $u \in \mathcal{O}(\overline{\Omega})$ in order to have

$$\mathrm{Jac}(\phi)(z,w) \neq 0$$

for all $(z,w) \in \overline{\Omega}$. For example, we can try to solve

$$\mathrm{Jac}(\phi) = \mathrm{e}^{(1-\lambda)u} \cdot \mathrm{Jac}(\phi_0)(z_p, w_p).$$

Putting $g(z,w) = \mathrm{Jac}(\phi_0)(z_p,w_p) - \mathrm{Jac}(\phi_0)(z,w)$, we have to solve the equation

$$(*) \qquad\qquad Z \cdot u = g$$

where $g \in \mathcal{O}(\overline{\Omega})$ and $g(z_p, w_p) = 0$.

The structure of $\mathcal{F}$ near $\overline{F}_1 \cup \partial\Omega$ implies the existence of a neighborhood W of $\overline{F}_1 \cup \partial\Omega$ such that:

(i) $\overline{F}_2 \cap W$ is the union of a disc D (containing p) and an annulus A (containing $\overline{F}_2 \cap Y$);

(ii) every leaf of $\mathcal{F}_{|W}$ different from F_1 and $F_2 \cap W$ is simply connected and accumulates to p.

LEMMA 5. — *The equation (*) has a solution in W.*

Proof. — The vector field Z defines on every leaf $L \in \mathcal{F}_{|W}$ a closed holomorphic 1-form τ_L which is the «differential of the time»: τ_L is given by

$$\frac{\mathrm{d}z}{P(z,w)}\Big|_L \quad \text{where } P(z,w) \neq 0, \qquad \frac{\mathrm{d}w}{Q(z,w)}\Big|_L \quad \text{where } Q(z,w) \neq 0.$$

The closed holomorphic 1-form $g_{|L} \cdot \tau_L$ is exact for every $L \in \mathcal{F}_{|w \setminus A}$: this is trivial if L is not a separatrix of p ($\pi_1(L) = 0$), and it is a consequence of $g(z_p, w_p) = 0$ if L is a separatrix ($g_{|L} \cdot \tau_L$ extends holomorphically to $\overline{L} = L \cup \{p\}$).

Equation (*) has a solution u_0 in a neighborhood N of p (for instance, we can make Z linear near p and solve (*) by power series methods...). If $q \in W \setminus A$, we consider a path γ contained in a leaf of $\mathcal{F}_{|W \setminus A}$ which joins a point $q_0 \in N$ to q. Then we define

$$u(q) = u_0(q_0) + \int_\gamma g_{|L} \cdot \tau_L.$$

The previous discussion shows that u is a well defined holomorphic function in $W \setminus A$. It is evident that $Z \cdot u = g$.

On the other hand, we may consider a tubular neighborhood $\widetilde{W}$ of $\overline{F}_2$, such that every leaf of $\mathcal{F}_{|\widetilde{W}}$ different from the separatrices is simply connected and such that $\widetilde{W} \cap W$ has two connected components: one containing p and the other containing A. A good choice of W and $\widetilde{W}$ ensures that the component C containing A is homeomorphic to $\mathbb{S}^1 \times \mathbb{D}^3$ and the space of leaves of $\mathcal{F}_{|C \setminus A}$ is a compact elliptic curve.

The same argument as above allows to extend $u_0 \in \mathcal{O}(N)$ to a solution of $Z \cdot u = g$ defined in $\widetilde{W}$; let us denote by $\tilde{u}$ such a solution. The difference $(u - \tilde{u})$ is defined in $C \setminus A$ and it is a first integral of Z. The fact that the space of leaves of $\mathcal{F}_{|C \setminus A}$ is a compact curve implies that $u - \tilde{u}$ is constant in $C \setminus A$. Consequently, u extends holomorphically to A, and hence we have found a solution of (*) in W. $\square$

The usual Hartog's Kugelsatz (see [Siu], [G-R]) allows to extend this solution u to all of $\overline{\Omega}$, giving finally a holomorphic locally injective map

$$\phi = (\mathrm{e}^u f_1, \mathrm{e}^{-\lambda u} f_2)$$

which satisfies $\phi^*(\mathcal{L}_\lambda) = F$.

It remains to be proved that ϕ is injective as a map between spaces of leaves. Every leaf of $\mathcal{F}$ intersects $\partial\Omega$ and hence accumulates to both F_1 and F_2, so to $p \in \mathbb{C}^2$ as well. Assume by way of contradiction that two leaves L_1 and L_2 of $\mathcal{F}$ are mapped by ϕ into the same leaf L of $\mathcal{L}_\lambda$. Let U be a small neighborhood of p where ϕ is 1–1 and $V = \phi(U)$ is a ball around $(0,0) \in \mathbb{C}^2$. Then $L_1 \cap U$ and $L_2 \cap U$ are two different leaves of $\mathcal{F}_{|U}$ mapped by ϕ to a single leaf of $\mathcal{L}_{\lambda|V}$, because for every $L \in \mathcal{L}_\lambda$ the intersection $L \cap V$ is connected. Since $\phi\colon U \to V$ is a biholomorphism which sends the leaves of $\mathcal{F}_{|U}$ to the leaves of $\mathcal{L}_{\lambda|V}$, this is absurd.

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