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ON THREE-DIMENSIONAL VORTEX PATCHES

PAR

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RÉSUMÉ. — Nous étudions, en dimension trois d'espace, l'existence et la régularité de la solution du système d'Euler incompressible pour une donnée analogue aux poches de tourbillon définies en dimension deux d'espace par A. Majda [7]. Nos résultats sont comparables à ceux obtenus dans le cas bidimensionnel par J.-Y. Chemin [4], mais l'existence de la solution est seulement locale en temps (globale cependant dans le cas axisymétrique).

ABSTRACT. — We study in three space dimensions the existence and smoothness of the solution of the incompressible Euler system for data analogous to the patches of vorticity defined in two space dimensions by A. Majda [7]. Our results are similar to those obtained in the two-dimensional case by J.-Y.Chemin [4], but the existence of the solution is only local in time (global in the axisymmetric case).

Introduction

The movement in $\mathbb{R}^d$ of an ideal incompressible fluid is described by the so-called incompressible Euler system. For this system, the short time existence of a solution of the Cauchy problem with smooth data has been known for a while. In his survey paper [7], MAJDA shows that this elementary result leads to several important problems as the global existence of a solution for smooth data or the (short time or global) existence of a solution for singular data. Here, we consider the Cauchy problem for merely Lipschitzian data.

In the two-dimensional problem, MAJDA [7] introduced *constant patches of vorticity*, which remain such constant patches thanks to a result of YUDOVITCH [9], and asked whether the boundary of such a patch remains

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smooth when it is initially smooth. This question was recently solved by CHEMIN [4], and we also refer to the survey of GÉRARD [6] for an account on recent two-dimensional results.

In this paper, we still consider the problem of patches of vorticity, but for higher space dimensions. Actually we chose the space dimension $d = 3$ for the sake of simplicity, but it is clear that similar results hold when $d > 3$. It is easy to see that, as soon as $d > 2$, compact patches of vorticity cannot be constant patches, and therefore we introduce some spaces of vorticity naturally related to the geometry of compact patches.

Adapting the method of CHEMIN [4], we establish the same results as when $d = 2$, but we get only a short time existence theorem as could be expected. However, our results are also global in time when the initial velocity field is *axisymmetric* as in MAJDA [7]. Finally, we have been informed that a chapter of SERFATI's thesis [8] is also devoted to this problem of multi-D vortex patches, but it is considered there from a Lagrangian point of view.

1. Notation and statement of the main result

1.a. Vectorial notation.

In this paper, we call *vector field* any $\mathbb{R}^3$-valued distribution defined on $\mathbb{R}^3$. The components of the vector field v are denoted by v_1, v_2 and v_3. When the products of the components are well defined, the scalar product of the two vector fields v and w is

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

while their vector and tensor products are respectively :

$$v \wedge w = \begin{pmatrix} v_2 w_3 & - & v_3 w_2 \\ v_3 w_1 & - & v_1 w_3 \\ v_1 w_2 & - & v_2 w_1 \end{pmatrix}, \quad v \otimes w = \begin{pmatrix} v_1 w_1 & v_2 w_1 & v_3 w_1 \\ v_1 w_2 & v_2 w_2 & v_3 w_2 \\ v_1 w_3 & v_2 w_3 & v_3 w_3 \end{pmatrix}.$$

Using the notation $\partial_j = \partial / \partial x_j$ and the formal «vector» ∇ with «components» ∂_1, ∂_2 and ∂_3, the expressions $\langle \nabla, v\rangle$, $\nabla \wedge v$ and $\nabla \otimes v$ defined formally as above will denote respectively the divergence, the curl and the gradient (i.e. the Jacobian matrix) of the vector field v. Similarly, we will use

$$\langle v, \nabla\rangle\, w = \sum_j v_j \partial_j w \quad \text{and} \quad \langle \nabla, v \otimes w\rangle = \sum_j \partial_j (v_j w)$$

which satisfy, when all the products are well defined,

$$\langle \nabla, v \otimes w \rangle = \langle \nabla, v \rangle\, w + \langle v, \nabla \rangle\, w.$$

Finally, in expressing the Biot-Savart law (LEMMA 2.2), we will use the notation :

$$(v * w)(x) = \int v(x-y) \wedge w(y)\,\mathrm{d}y = \int v(y) \wedge w(x-y)\,\mathrm{d}y.$$

With this notation, we can write *the incompressible Euler system*, which provides a model for the movement of a non-viscous liquid in the space $\mathbb{R}^3$, as follows :

$$\begin{cases} \partial_t v + \langle v, \nabla \rangle v = -\nabla p, \\ \langle \nabla, v \rangle = 0, \\ v_{|t=0} = v^0, \end{cases}$$

where the unknown v is a function of the time variable $t \in \mathbb{R}_+$ valued in the space of vector fields (for short, we will say that v *is* a vector field, even when it is time-dependent), and v^0 is a divergence free data belonging to the Lebesgue space L^p for some $1 < p < \infty$. The equation

$$\partial_t v + \langle v, \nabla \rangle v = -\nabla p$$

simply means that the curl of the left side is identically zero, or equivalently, that the left side is the gradient of a scalar distribution $-p \in \mathcal{S}'$, but it can be proved that this distribution p (called the *pressure*) is completely determined by the problem up to a function of t only, and our first step will be to get rid of it.

Our paper discusses existence and uniqueness results for the solution v of this incompressible Euler system. To be able to state precise results, we now introduce the functional spaces where we will look for these solutions.

1.b. Hölder spaces and dyadic analysis.

For all the objects and estimates we describe here, we refer to BONY [2] and CHEMIN [4]. When $s \in \mathbb{R} \setminus \mathbb{Z}_+$ (resp. when $s \in \mathbb{Z}_+$), we denote by C^s (resp. by C^s_*) the Hölder space with exponent s, and in both cases the corresponding norm is denoted by $\|v\|_s$. *The letter r will always denote a real number from the open interval* (0,1), so that the space C^r is the usual space of bounded functions v satisfying

$$\big|v(x) - v(y)\big| \le C|x-y|^r$$

for some constant C and all $x, y \in \mathbb{R}^3$. More generally, if Ω is an open subset of $\mathbb{R}^3$, the space $C^r(\Omega)$, with the norm $\|v\|_{r(\Omega)}$, is the space of all $v \in L^\infty(\Omega)$ satisfying the previous estimate for all $x, y \in \Omega$ (we take this unusual definition — without requiring that x and y stay in the same component of Ω — just to simplify the proofs below : actually, we could have used everywhere the standard definition).

We have the standard interpolation estimate

$$\|v\|_{\mu s+(1-\mu)t} \leq \|v\|_s^\mu \, \|v\|_t^{1-\mu} \quad \text{for} \quad 0 \leq \mu \leq 1,$$

and the L^∞ norm can also be estimated by interpolating between C^0_* and C^s, $s > 0$: it is the *logarithmic interpolation estimate*

$$\|v\|_{L^\infty} \leq C_s L\big(\|v\|_0, \|v\|_s\big) \quad \text{for} \quad s > 0,$$

where the function

$$L(a,b) = a \operatorname{Log}\Big(2 + \frac{b}{a}\Big)$$

is an increasing function of both variables a and $b \in \mathbb{R}_+$.

When v and w are two Hölder distributions, we denote by $T_v w$ the paraproduct of w by v, for which we have the estimate

$$\|T_v w\|_s \leq C_{s,t} \|v\|_{-t} \, \|w\|_{s+t} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

which is still true for $t = 0$ provided that $\|v\|_{-t}$ is replaced with $\|v\|_{L^\infty}$. When $t > 0$, $v \in C^s$ and $w \in C^{t-s}$, the product of the two distributions v and w is well defined and we have

$$vw = T_v w + T_w v + R(v,w),$$

where the remainder operator R satisfies

$$\big\|R(v,w)\big\|_t \leq C_{s,t} \|v\|_s \, \|w\|_{t-s} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

so that we have the useful estimate

$$\big\|(v - T_v)w\big\|_{\min(s,t)} \leq C_{s,t} \|v\|_s \, \|w\|_{t-s} \quad \text{for } s \in \mathbb{R} \text{ and } t > 0,$$

where $\|w\|_{t-s}$ must be replaced with $\|w\|_{L^\infty}$ when $s = t$.

Next, we will say that the pseudodifferential operator $a(D)$ has a *homogeneous symbol* if $a \in C^\infty(\mathbb{R}^3)$ satisfies, for $\mu \geq 1$ and large $\xi \in \mathbb{R}^3$,

$$a(\mu\xi) = \mu^m a(\xi).$$

If $a(D)$ is such a pseudodifferential operator with homogeneous symbol of order 0, then it is continuous on the Lebesgue spaces L^p provided that $1 < p < \infty$ (see COIFMAN and MEYER [5]). If $a(D)$ is an m-th order pseudodifferential operator with homogeneous symbol, we have the estimate

$$\|a(D)w\|_s \leq C_{a,s}\|w\|_{s+m} \quad \text{for } s \in \mathbb{R},$$

and the commutator $[a(D), T_v] = a(D)T_v - T_v a(D)$ satisfies :

$$\|[a(D), T_v]\, w\|_s \leq C_{a,s,r}\|v\|_r\, \|w\|_{s+m-r} \quad \text{for } s \in \mathbb{R},\ 0 < r < 1.$$

This last estimate is still true when $r = 1$ provided that $\|v\|_r$ is replaced with

$$\|v\|_{\mathrm{Lip}} = \|v\|_{L^\infty} + \sum_j \|\partial_j v\|_{L^\infty},$$

which is the norm associated with the space

$$\mathrm{Lip} = \{v \in L^\infty\, ;\ \nabla \otimes v \in L^\infty\}.$$

We will often use the elliptic pseudodifferential operators $\Lambda^s = \lambda^s(D)$ for $s \in \mathbb{R}$ defined by

$$\lambda^s(\xi) = \big(\chi(\xi) + |\xi|^2\big)^{s/2}$$

where $\chi \in C_0^\infty(\mathbb{R}^3)$ is a nonnegative function with value 1 near 0. They obviously have homogeneous symbols and satisfy $\Lambda^s\Lambda^t = \Lambda^{s+t}$ for all s and $t \in \mathbb{R}$.

Finally, our time dependent vector fields will be taken in the spaces

$$L^\infty\big([0,T]\, ;\ L^p \text{ or } C^s \text{ or } \mathrm{Lip}\big)$$

of bounded functions of $t \in [0,T]$ valued in L^p or C^s or Lip, and similarly we will write $v \in \mathrm{Lip}([0,T]\, ; L^p)$ when $v \in L^\infty([0,T])\, ; L^p)$ and $\partial_t v \in L^\infty([0,T]\, ; L^p)$.

1.c. Patches of vorticity.

As explained in section 2 below (see THEOREM 2.9), there is a classic existence result when the initial velocity field is smooth enough, precisely when $v^0 \in C^s$ for some $s > 1$. But there are serious motivations (see MAJDA [7]) to study the case of more singular data.

In the two-dimensional case, YUDOVITCH [9] proved the existence of a solution for initial velocity fields v^0 satisfying

$$\omega^0 = \partial_1 v_2^0 - \partial_2 v_1^0 \in L^\infty_{\mathrm{comp}}.$$

The scalar distribution $\omega = \partial_1 v_2 - \partial_2 v_1$ is called the *vorticity*, and it is clear that when the initial vorticity ω^0 is a *constant patch of vorticity*, i.e. satisfies

$$\omega^0(x) = \begin{cases} \bar{\omega} & \text{for } x \in \Omega^0, \\ 0 & \text{for } x \notin \Omega^0, \end{cases}$$

where $\bar{\omega}$ is a constant and Ω^0 is a bounded region of $\mathbb{R}^2$, then $\omega(t)$ remains a constant patch of vorticity at any time $t > 0$, i.e.

$$\omega(t,x) = \begin{cases} \bar{\omega} & \text{for } x \in \Omega(t), \\ 0 & \text{for } x \notin \Omega(t), \end{cases}$$

where $\Omega(t)$ is still a bounded region of $\mathbb{R}^2$. Then, answering a question of Majda [7], Chemin [4] proved that $\Omega(t)$ has a smooth boundary at any time $t > 0$ as soon as Ω^0 has a smooth boundary.

Now in the three-dimensional case, the vorticity $\omega = \nabla \wedge v$ is a divergence free vector field, and the situation is rather different : first, we do not have constant patches supported in a *bounded* region Ω^0 because $\langle \nabla, \omega^0 \rangle = 0$ implies that the constant $\bar{\omega}$ must be tangent to the boundary of Ω^0, and second, we have the three-dimensional phenomenon of stretching of the vorticity (see Majda [7]). Therefore, we must find some more suitable vorticity patterns, and our choice is motivated by the result of Chemin [4].

Definition 1.1.—Let $0 < r < 1$, and let Σ be a C^{1+r}, two-dimensional, compact submanifold of $\mathbb{R}^3$. For all $\varepsilon > 0$, we set

$$\Sigma_\varepsilon = \big\{ x \in \mathbb{R}^3 \,;\ \mathrm{dist}(x, \Sigma) \le \varepsilon \big\}.$$

Then we say that the vector field ω *belongs to the space* $C^{r,\Sigma}$ if $\omega \in L^\infty$, if $\langle \nabla, w \otimes \omega \rangle \in C^{r-1}$ for all C^r, divergence free vector fields w tangent to Σ, and if for some constant C

$$\|\omega\|_{r(\mathbb{R}^3 \setminus \Sigma_\varepsilon)} \le C\varepsilon^{-r} \quad \text{for all } 0 < \varepsilon < 1.$$

Here, since $\langle \nabla, w \rangle = 0$, the vector field $\langle \nabla, w \otimes \omega \rangle$ is a substitute for the w-directional derivative of ω (indeed, $\langle w, \nabla \rangle \omega$ cannot be directly written since the products of $\partial_j \omega$ by w_j are not well defined), and therefore $\omega \in C^{r,\Sigma}$ means that ω has some conormal smoothness. We will see below that it is very easy to construct sufficiently many C^r, divergence free vector fields w tangent to any C^{1+r}, two-dimensional, compact submanifold of $\mathbb{R}^3$ (see Proposition 3.2). Finally, our space $C^{r,\Sigma}$ generalizes the two-dimensional situation of constant patches of vorticity since the corresponding two-dimensional definition with the boundary of the patch in the place of Σ would clearly allow initial data of that form.

The main result of the paper is then the following existence theorem.

THEOREM 1.2. — *With $1 < p < \infty$, $1 \leq q < 3$ and $0 < r < 1$, let Σ^0 be a C^{1+r}, two-dimensional, compact submanifold of $\mathbb{R}^3$, and v^0 be an L^p, divergence free initial velocity field with vorticity $\omega^0 = \nabla \wedge v^0 \in L^q \cap C^{r,\Sigma^0}$. Then, the incompressible Euler system*

$$\begin{cases} \partial_t v + \langle v, \nabla \rangle v = -\nabla p, \\ \langle \nabla, v \rangle = 0, \\ v_{|t=0} = v^0 \end{cases}$$

has a unique solution $v \in L^\infty\big([0,T]\,; \mathrm{Lip}\big) \cap \mathrm{Lip}\big([0,T]\,; L^p\big)$ for some $T > 0$. Moreover, if Ψ_t is the associated flow, i.e. the solution of

$$\partial_t \Psi_t(x) = v\big(t, \Psi_t(x)\big), \quad \Psi_0(x) = x,$$

then for all $t \in [0,T]$, the set $\Sigma(t) = \Psi_t(\Sigma^0)$ is a C^{1+r}, two-dimensional, compact submanifold of $\mathbb{R}^3$, and $\omega(t) \in L^q \cap C^{r,\Sigma(t)}$.

1.d. Comments, example and organization of the paper.

(i) Our result is only local in time (i.e. the solution exists in $[0,T]$ for some $T > 0$) : indeed, this is linked to the unsolved question on the global existence in three space dimensions of a smooth solution for any smooth data. However, when the initial velocity field has some symmetry, it is possible to improve the previous result : *in the axisymmetric case* (see DEFINITION 2.10 below), *the results of THEOREM 1.2 are true for all $T > 0$* (see THEOREMS 5.4, 6.1 and 6.4 below).

(ii) We can probably also improve THEOREM 1.2 by considering smoother initial submanifolds Σ^0, but we will not discuss this question in this paper to avoid too long developments.

(iii) The conditions $\langle \nabla, w \otimes \omega^0 \rangle \in C^{r-1}$ for all C^r, divergence free vector fields w tangent to Σ^0 (see DEFINITION 1.1) already imply an estimate

$$\|\omega^0\|_{r(\mathbb{R}^3 \setminus \Sigma^0_\varepsilon)} \leq C\,\varepsilon^{-1}.$$

Therefore, our assumption $\leq C\varepsilon^{-r}$ (contained in DEFINITION 1.1) is only a strengthened form of this estimate. It actually plays no role in the proof of our existence result (see THEOREM 5.4 below) : in the paper, we use this estimate only to prove the propagation of the $C^{r,\Sigma}$ smoothness in THEOREM 6.4.

Indeed, in stating THEOREM 1.2, we have preferred the $C^{r,\Sigma}$ smoothness because it is more intrinsicly related to Σ than the $C^{r,W}$ smoothness considered in sections 3 through 5. We also point out that our proofs establish the $C^{r,\Sigma}$ smoothness of the scalar vorticity *in the two-dimensional problem*, and therefore this completes the result of CHEMIN [4] who proved its $C^{r,W}$ smoothness only.

(iv) As a conclusion of this presentation, we now describe an example that is very close to the two-dimensional problem of constant patches of vorticity. Choose a bounded region D^0 of $\mathbb{R}_+ \times \mathbb{R}$ with smooth boundary such that $\overline{D^0} \subset \mathbb{R}_+^* \times \mathbb{R}$, and set

$$\omega^0(x) = \begin{pmatrix} x_2\,\bar\omega \\ -x_1\,\bar\omega \\ 0 \end{pmatrix} \quad \text{for } x \in \Omega^0, \quad \omega^0(x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } x \notin \Omega^0,$$

where $\bar\omega$ is a constant, and $\Omega^0 = \{x \in \mathbb{R}^3\,;\ ((x_1^2+x_2^2)^{1/2}, x_3) \in D^0\}$. Then, for all $T > 0$, the incompressible Euler system has a unique solution

$$v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}\Big([0,T]\,; \bigcap_{p>3/2} L^p\Big)$$

with $\nabla \wedge v(0) = \omega^0$ (THEOREM 5.4). Moreover, this solution satisfies for all $t \geq 0$

$$\omega(t,x) = \begin{pmatrix} x_2\,\bar\omega \\ -x_1\,\bar\omega \\ 0 \end{pmatrix} \quad \text{for } x \in \Omega(t), \quad \omega(t,x) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{for } x \notin \Omega(t),$$

where $\Omega(t) = \{x \in \mathbb{R}^3; \big((x_1^2+x_2^2)^{1/2}, x_3\big) \in D(t)\}$ and $D(t)$ is a bounded region of $\mathbb{R}_+ \times \mathbb{R}$ with smooth boundary (THEOREM 6.1).

(v) Our paper is organized as follows.

In section 2, we collect all the classic material on the incompressible Euler system that is related to our problem. The specialist will not need to read it, but it is a convenient reference for us.

In sections 3 through 5, we prove a variant of THEOREM 1.2 where the spaces $C^{r,\Sigma}$ of conormal distributions are replaced with larger spaces $C^{r,W}$ of striated distributions. Section 3 is devoted to a static estimate, while dynamic estimates are obtained in section 4. These results are put together in section 5 to prove the existence THEOREM 5.4.

Finally, section 6 is devoted to complements : smoothness of the submanifold $\Sigma(t)$ and conormal smoothness of the vorticity field $\omega(t)$.

2. Elementary results concerning the incompressible Euler system

The results of this section are somewhat classic. However we collect them here because we do not know any complete reference for them. As stated here, they will provide a convenient base for the rest of the article. Some proofs are given, but the other ones are only sketched.

We begin with a couple of technical lemmas.

LEMMA 2.1. — *Let v be a vector field satisfying*

$$\langle \nabla, v \rangle = 0, \quad \nabla \wedge v = 0 \quad \textit{and} \quad v = \sum_{\nu \leq N} v^\nu,$$

with $v^\nu \in L^{p^\nu}$, $p^\nu < \infty$ for all ν. Then $v = 0$.

Proof. — We have

$$0 = \nabla \wedge (\nabla \wedge v) = \nabla \langle \nabla, v \rangle - \Delta v = -\Delta v$$

and therefore, v is a harmonic polynomial. But the only polynomial satisfying

$$v \in \sum_{\nu \leq N} L^{p^\nu}$$

is $v = 0$: indeed, $v = 0$ is the only polynomial satisfying

$$\operatorname{meas}\big\{x\,;\ |v(x)| \geq \varepsilon\big\} < \infty$$

for all $\varepsilon > 0$, and here we have

$$\operatorname{meas}\big\{x\,;\ |v(x)| \geq \varepsilon\big\} \leq \sum_{\nu \leq N} \operatorname{meas}\Big\{x\,;\ |v^\nu(x)| \geq \frac{\varepsilon}{N}\Big\} < \infty$$

since $v^\nu \in L^{p^\nu}$ with $p^\nu < \infty$. □

This result immediately gives the classic «Biot-Savart law».

COROLLARY 2.2. — *Let $p < \infty$ and $q < 3$. Then, if we have $v \in L^p$, $\langle \nabla, v \rangle = 0$ and $\omega = \nabla \wedge v \in L^q$, it follows that*

$$v = \omega * \nabla F$$

where $F(x) = -1/4\pi\,|x|$ is the standard fundamental solution of the laplacian.

Proof. — Let us set $\tilde{v} = \omega * \nabla F$. Since

$$\nabla F = \chi \nabla F + (1 - \chi) \nabla F$$

with $\chi \nabla F \in L^1$ and $(1-\chi)\nabla F \in L^{4q/(3q-1)}$, it is easy to check that $\tilde{v}$ is an element of $L^q + L^{4q/(3-q)}$. Moreover, writing that ω is the limit of a sequence $\omega_n \in L^q_{\text{comp}}$, we see that we have $\langle \nabla, \tilde{v} \rangle = 0$ and $\nabla \wedge \tilde{v} = \omega$. Therefore $(v - \tilde{v})$ satisfies the assumptions of LEMMA 2.1, and we have $v = \tilde{v}$. □

Next, we examine the effect of a useful change of variables.

PROPOSITION 2.3. — *Let $v \in L^\infty([0,T]\,;\mathrm{Lip})$ be a divergence free vector field and Ψ_t be the corresponding flow, i.e. the solution of*

$$\partial_t \Psi_t(x) = v\big(t, \Psi_t(x)\big), \quad \Psi_0(x) = x.$$

Then the transformation $w \mapsto w^*$ defined by $w^*(t,x) = w(t, \Psi_t(x))$ maps $L^\infty([0,T]\,;L^1_{\mathrm{loc}})$ onto itself, and we have

$$\partial_t\{w^*\} = \big\{\partial_t w + \langle \nabla, v \otimes w\rangle\big\}^*$$

whenever w and $\partial_t w + \langle \nabla, v \otimes w\rangle$ belong to $L^\infty([0,T]\,;L^1_{\mathrm{loc}})$. Moreover, this transformation maps $L^\infty([0,T]\,;L^p)$ onto itself for all $1 \le p \le \infty$ and also $L^\infty([0,T]\,;C^r)$ onto itself for all $0 < r < 1$, with the estimates

$$\big\|w^*(t)\big\|_{L^p} = \big\|w(t)\big\|_{L^p},$$

$$\mathrm{e}^{-rV(t)}\big\|w(t)\big\|_r \le \big\|w^*(t)\big\|_r \le \mathrm{e}^{rV(t)}\big\|w(t)\big\|_r,$$

where $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$.

Proof. — Since $\langle \nabla, v\rangle \equiv 0$, the Lebesgue measure is invariant under the transformation Ψ_t for every t, so that $w \mapsto w^*$ maps $L^\infty([0,T]\,;L^1_{\mathrm{loc}})$ onto itself and also $L^\infty([0,T]\,;L^p)$ onto itself with $\|w^*(t)\|_{L^p} = \|w(t)\|_{L^p}$. To see that

$$\partial_t\{w^*\} = \big\{\partial_t w + \langle \nabla, v \otimes w\rangle\big\}^*,$$

we first observe that it is obviously true when w is smooth with respect to x (it is the chain rule); then the result follows by taking the limit of a convolution (in x) with a Friedrichs mollifier thanks to a Friedrichs lemma. Finally, the flow Ψ_t classically satisfies

$$\big|\Psi_t(x) - \Psi_t(y)\big| \le \mathrm{e}^{V(t)}\,|x-y|$$

(see e.g. BAHOURI and DEHMAN [1]) so that $w \mapsto w^*$ maps $L^\infty([0,T]\,;C^r)$ onto itself with the given estimate. □

COROLLARY 2.4. — *Let v be as in Proposition 2.3 and $\mathcal{A}$ be a L^∞ function of $t \in [0,T]$ valued in the space of continuous linear operators on L^p (resp. on C^r). Then the problem*

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w\rangle = \mathcal{A}w, \\ w_{|t=0} = w^0 \in L^p \quad (\textit{resp.} \in C^r) \end{cases}$$

has a unique solution $w \in L^\infty([0,T]\,;L^p)$ (resp. $w \in L^\infty([0,T]\,;C^r)$).

Moreover if $a(s)$ is the norm of the continuous linear operator $\mathcal{A}(s)$ and $A(t) = \int_0^t a(s)\,\mathrm{d}s$, we have the estimate

$$\|w(t)\|_{L^p} \le e^{A(t)} \|w^0\|_{L^p}$$

$$\Big(\textit{resp.}\ \|w(t)\|_r \le \mathrm{e}^{A(t)+rV(t)} \|w^0\|_r,\ \textit{where}\ V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s\Big).$$

Proof. — Thanks to PROPOSITION 2.3, the given problem is equivalent to the following

$$\begin{cases} \partial_t\{w^*\} = \mathcal{A}^* w^*, \\ w^*{}_{|t=0} = w^0 \end{cases}$$

where the operator $\mathcal{A}^*$, defined by $\mathcal{A}^* w^* = \{\mathcal{A}w\}^*$, is still a L^∞ function of t valued in the space of continuous linear operators on L^p (resp. on C^r). Then the existence and uniqueness of a solution follows from the classic theory of ordinary differential equations.

Since the transformation $w \mapsto w^*$ is an isometry on $L^\infty([0,T]\,;L^p)$, the estimate in L^p norms also follows from the ordinary differential equation structure. In the case of C^r norms, we can write

$$w^*(t,x) = w^0(x) + \int_0^t \partial_s\{w^*\}(s,x)\,\mathrm{d}s = w^0(x) + \int_0^t \{\mathcal{A}w\}^*(s,x)\,\mathrm{d}s$$

so that

$$w(t,x) = w^*\big(t,\Psi_t^{-1}(x)\big) = w^0\big(\Psi_t^{-1}(x)\big) + \int_0^t \{\mathcal{A}w\}\big(s,\Psi_s\circ\Psi_t^{-1}(x)\big)\,\mathrm{d}s.$$

This expression gives the estimate

$$\big\|w(t)\big\|_r \le \mathrm{e}^{rV(t)}\|w^0\|_r + \int_0^t \mathrm{e}^{r(V(t)-V(s))} a(s)\big\|w(s)\big\|_r\,\mathrm{d}s.$$

Next we define

$$f(t) = \mathrm{e}^{-A(t)-rV(t)}\big\|w(t)\big\|_r,$$

and the multiplication of the previous estimate by $\mathrm{e}^{-A(t)-rV(t)}$ gives

$$\begin{aligned} f(t) &\le \mathrm{e}^{-A(t)} f(0) + \int_0^t \mathrm{e}^{A(s)-A(t)} a(s) f(s)\,\mathrm{d}s \\ &\le \mathrm{e}^{-A(t)} f(0) + \big(1-\mathrm{e}^{-A(t)}\big)\sup_{[0,T]} f. \end{aligned}$$

We now multiply by $e^{A(t)}$ to get for any $t \in [0,T]$

$$\sup_{[0,T]} f \le f(0) + e^{A(t)}\big\{\sup_{[0,T]} f - f(t)\big\}.$$

Finally, we can use this estimate for a sequence t_n such that $f(t_n)$ tends to $\sup_{[0,T]} f$, and this gives $\sup_{[0,T]} f \le f(0)$ which simply means

$$\|w(t)\|_r \le e^{A(t)+rV(t)}\|w^0\|_r$$

for all $t \in [0,T]$. □

This result directly implies the following classic estimates on the vorticity field.

COROLLARY 2.5. — *Let $v \in L^\infty([0,T];\mathrm{Lip})$ be a solution of the incompressible Euler system (see section 1). If $\omega^0 = \nabla \wedge v_{|t=0}$ belongs to some L^q with $1 \le q \le \infty$, then $\omega = \nabla \wedge v \in L^\infty([0,T];L^q)$ with*

$$e^{V(s)-V(t)} \le \frac{\|\omega(s)\|_{L^q}}{\|\omega(t)\|_{L^q}} \le e^{V(t)-V(s)}$$

for all $0 \le s \le t \le T$ (here $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$ as above).

Proof. — It is known (see e.g. BAHOURI and DEHMAN [1]) that the vorticity field ω of a solution of the incompressible Euler system satisfies the equation

$$\partial_t\omega + \langle\nabla, v\otimes\omega\rangle = \langle\omega,\nabla\rangle v.$$

The operator $\mathcal{A} : \omega \mapsto \langle\omega,\nabla\rangle v$ is a L^∞ function of t valued in the space of continuous linear operators on L^∞ and on $L^q \cap L^\infty$ since for all q

$$\|\mathcal{A}\omega(s)\|_{L^q} = \|\langle\omega(s),\nabla\rangle v(s)\|_{L^q} \le \|v(s)\|_{\mathrm{Lip}}\|\omega(s)\|_{L^q}.$$

Therefore it follows from COROLLARY 2.4 that ω is the unique L^∞ solution of the previous equation with $\omega_{|t=0} = \omega^0$. If $\omega^0 \in L^q$, COROLLARY 2.4 also implies the existence of a solution $\omega \in L^\infty([0,T];L^q\cap L^\infty)$ which must satisfy $\omega = \nabla \wedge v$ thanks to the uniqueness of the L^∞ solution. Finally, the estimate immediately follows from the estimate given in COROLLARY 2.4. □

Our last technical step before giving the main existence and uniqueness results of this section consists in the construction of bilinear operators which will allow us to find a « pressure free » form of the incompressible Euler system and to write easily the estimates needed in the next sections.

With the notation introduced in section 1.b, we set for $v, w \in L^\infty$,

$$D(v,w) = \langle \nabla, v \otimes w\rangle - T_{\langle \nabla, v\rangle} w - T_w \langle \nabla, v\rangle,$$
$$K(v,w) = \langle \nabla, v \otimes w\rangle - \Lambda^{-1}\langle \nabla, v \otimes \Lambda w\rangle,$$
$$\Pi(v,w) = \Pi^1(v,w) + \Pi^2(v,w) + \Pi^3(v,w)$$

where

$$\Pi^1(v,w) = -\nabla \Lambda^{-2}\big\langle \nabla, \langle \nabla, v \otimes w\rangle\big\rangle$$
$$\Pi^2(v,w) = \nabla \chi(D)\, \Lambda^{-2}\big\langle \nabla, \langle \nabla, \chi F * (v \otimes w)\rangle\big\rangle$$
$$\Pi^3(v,w) = \chi(D)\, \Lambda^{-2} \sum_{jk} \Big\{\big(\nabla \partial_j \partial_k (1-\chi) F\big) * (v_j w_k)\Big\}.$$

(Here $F(x) = -1/4\pi\,|x|$ is the standard fundamental solution of the laplacian, and Π^2 and Π^3 are well defined since χF and $\nabla \partial_j\, \partial_k (1-\chi) F$ are elements of L^1.) The operator Π is just a variant of that introduced by CHEMIN [3], [4], and we will also use the following other variant

$$\pi(v,w) = \Pi(v,w) + \nabla \Lambda^{-2}\big\langle \nabla, v \otimes \langle \nabla, w\rangle\big\rangle$$

which satisfies $\pi(v,w) = \pi^1(v,w) + \Pi^2(v,w) + \Pi^3(v,w)$ with

$$\pi^1(v,w) = -\nabla \Lambda^{-2}\big\langle \nabla, \langle w, \nabla\rangle v\big\rangle.$$

The following statement lists some properties of these operators.

PROPOSITION 2.6. — *We have :*

- $D(v,w) = \langle \nabla, v \otimes w\rangle$ *as soon as* $\langle \nabla, v\rangle = 0$;
- $\Pi(v,w) = \pi(v,w)$ *as soon as* $\langle \nabla, w\rangle = 0$.

The operator Π *satisfies :*

- $\Pi(v,w) = \Pi(w,v)$;
- $\nabla \wedge \Pi(v,w) = 0$;
- $\langle \nabla, \Pi(v,w)\rangle = \langle \nabla, \langle \nabla, v \otimes w\rangle\rangle$.

Finally, there are constants C depending only on $0 < r < 1$ and on the subscripts s or p such that for all $v, w \in L^\infty$:

(i) *For all $r < s < 1$,*

$$\big\|D(v,w)\big\|_{r-1} \le C_s \|v\|_{r-s}\, \|w\|_s.$$

This estimate is still true for $s = r$ provided that $\|v\|_{r-s}$ is replaced with $\|v\|_{L^\infty}$, and still true for $s = 1$ provided that $\|w\|_s$ is replaced with $\|w\|_{\mathrm{Lip}}$.

(ii) $\langle \nabla, v\rangle = 0$ *implies, for all* $0 < s < 1$,

$$\big\|K(v,w)\big\|_r \leq C_s \|v\|_s \, \|w\|_{r+1-s}$$

and this estimate is still true for $s = 1$ *provided that* $\|v\|_s$ *is replaced with* $\|v\|_{\mathrm{Lip}}$.

(iii) *For all* $1 < p < \infty$,

$$\big\|\pi(v,w)\big\|_{L^p} \leq C_p \|v\|_{\mathrm{Lip}} \, \|w\|_{L^p}$$

and $\langle \nabla, v\rangle = 0$ *implies*

$$\big\|\pi(v,w)\big\|_r \leq C \|v\|_{\mathrm{Lip}} \|w\|_r.$$

(iv) $\langle \nabla, w\rangle = 0$ *implies*

$$\big\|\Pi(v,w)\big\|_{r-1} \leq C\, \|v\|_{\mathrm{Lip}} \|w\|_{r-1}.$$

Proof. — All the algebraic identities easily follow from the definitions. To get the estimate (i), we just rewrite $D(v,w)$ as

$$D(v,w) = \sum_j \Big\{ T_{v_j} \partial_j w + T_{\partial_j w} v_j + \partial_j \, R(v_j, w) \Big\}$$

and use the estimates of section 1.b. Similarly, when $\langle \nabla, v\rangle = 0$ we can write

$$K(v,w) = \langle v - T_v, \nabla \rangle \, w + \Lambda^{-1} \big\langle \nabla, [\Lambda, T_v] \otimes w \big\rangle + \Lambda^{-1} \big\langle \nabla, (T_v - v) \otimes \Lambda w \big\rangle$$

to get the estimate (ii).

The estimates on Π^2 and Π^3 are easy to obtain, and therefore are left to the reader. We have

$$\big\|\pi^1(v,w)\big\|_{L^p} \leq C_p \big\|\langle w, \nabla \rangle v\big\|_{L^p} \leq C_p \|v\|_{\mathrm{Lip}} \, \|w\|_{L^p}$$

which proves the estimates (iii) in L^p norms. When $\langle \nabla, v\rangle = 0$, we also have $\langle \nabla, \partial_j v\rangle = 0$ and

$$\pi^1(v,w) = -\nabla \Lambda^{-2} \sum_j \langle \nabla, \partial_j v \otimes w_j \rangle = -\nabla \Lambda^{-2} \sum_j D(\partial_j v, w_j)$$

so that (iii) follows from the estimate (i) with $s = r$. Finally, when $\langle \nabla, w\rangle = 0$ we can write

$$\Pi^1(v,w) = -\nabla \Lambda^{-2} \big\langle \nabla, \langle \nabla, w \otimes v\rangle \big\rangle = -\nabla \Lambda^{-2} \big\langle \nabla, D(w,v) \big\rangle$$

and (iv) now follows from the estimate (i) with $s = 1$. $\square$

Now, we are ready to transform the incompressible Euler system into a pressure free system.

THEOREM 2.7. — *Let $1 < p < \infty$ and $v^0 \in \mathrm{Lip} \cap L^p$ be a divergence free data. Then the vector field v is a $L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$ solution of the incompressible Euler system*

$$\begin{cases} \partial_t v + \langle v, \nabla \rangle v = -\nabla p, \\ \langle \nabla, v \rangle = 0, \\ v_{|\,t=0} = v^0 \end{cases}$$

if and only if it is a $L^\infty([0,T]\,;\mathrm{Lip})$ solution of the pressure free system

$$\begin{cases} \partial_t v + \langle \nabla, v \otimes v \rangle = \Pi(v,v), \\ v_{|\,t=0} = v^0. \end{cases}$$

Moreover, if such a solution v satisfies $v \in L^\infty([0,T]\,;C^s)$ for some $s \notin \mathbb{Z}_+$, then we also have $v \in C^s([0,T] \times \mathbb{R}^3)$.

Proof. — In three parts.

(i) Assume that $v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$ is a solution of the incompressible Euler system. Since $\langle \nabla, v \rangle = 0$, we have

$$\langle v, \nabla \rangle v = \langle \nabla, v \otimes v \rangle \quad \text{and} \quad \Pi(v,v) = \pi(v,v),$$

and thanks to the smoothness of v and to PROPOSITION 2.6 (iii), we see that $-\nabla p$ and $\Pi(v,v)$ belong to $L^\infty([0,T]\,;L^p)$.

Next we find

$$\langle \nabla, -\nabla p \rangle = \big\langle \nabla \langle \nabla, v \otimes v \rangle \big\rangle = \big\langle \nabla, \Pi(v,v) \big\rangle$$

by taking the divergence of the Euler equation and using PROPOSITION 2.6, and since $\nabla \wedge (-\nabla p) = \nabla \wedge \Pi(v,v) = 0$, it follows that $\Pi(v,v) + \nabla p$ satisfies the assumptions of LEMMA 2.1. We can therefore conclude that $-\nabla p = \Pi(v,v)$ and that v is a solution of the pressure free system.

(ii) Assume that $v \in L^\infty([0,T]\,;\mathrm{Lip})$ is a solution of the pressure free system. We find $\partial_t \langle \nabla, v \rangle = 0$ by taking the divergence of the equation, so that $\langle \nabla, v^0 \rangle = 0$ implies $\langle \nabla, v \rangle \equiv 0$. Thus this equation can also be written

$$\partial_t v + \langle v, \nabla \rangle v = \Pi(v,v)\,,$$

and $\Pi(v,v)$ is a gradient since its curl vanishes. Therefore, v is a solution of the incompressible Euler system.

Now the property $\langle \nabla, v \rangle = 0$ has the following two consequences : first that $\Pi(v,v) = \pi(v,v)$ so that $w = v$ is a $L^\infty([0,T]\,;\mathrm{Lip})$ solution of the problem

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w \rangle = \pi(v,w), \\ w_{|\,t=0} = v^0 \in \mathrm{Lip} \cap L^p\,; \end{cases}$$

and second, that the operator $\mathcal{A} : w \mapsto \pi(v,w)$ is continuous on C^r and on $C^r \cap L^p$ thanks to PROPOSITION 2.6 (iii), so that it follows from COROLLARY 2.4 that the problem written above has at most one solution $w \in L^\infty([0,T]\,;C^r)$ and at least one solution $w \in L^\infty([0,T]\,;C^r \cap L^p)$. Since v is a $L^\infty([0,T]\,;C^r)$ solution of this problem, we get that v is an element of $L^\infty([0,T]\,;L^p)$, and this also implies that

$$\partial_t v = \pi(v,v) - \langle v, \nabla \rangle v \in L^\infty\big([0,T]\,;L^p\big).$$

(iii) Finally, the proof of $v \in C^s([0,T]\times\mathbb{R}^3)$ is in CHEMIN [3] : it follows from the fact that $\partial_t^{k+1} v \in L^\infty([0,T]\,;C^{s-k-1})$ for all $k < s$, which is easily obtained by differentiating the equation. $\square$

Thanks to this theorem, we will always study the incompressible Euler system through its pressure free form. For example, we have the following uniqueness result.

THEOREM 2.8. — *For any divergence free data $v^0 \in \mathrm{Lip} \cap L^p$, where $1 < p < \infty$, the incompressible Euler system has at most one solution $v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$.*

Proof. — We take the incompressible Euler system in its pressure free form. If v and $\tilde{v}$ are two solutions with the same data v^0, we get by subtraction

$$\partial_t(v - \tilde{v}) + \big\langle \nabla, v \otimes (v - \tilde{v}) \big\rangle = \Pi(v + \tilde{v}, v - \tilde{v}) - \big\langle \nabla, (v - \tilde{v}) \otimes \tilde{v} \big\rangle$$

so that $u = \Lambda^{-1}(v - \tilde{v})$ is a solution of

$$\begin{cases} \partial_t u + \langle \nabla, v \otimes u \rangle = \Lambda^{-1}\Pi(v + \tilde{v}, \Lambda u) - \Lambda^{-1} D(\Lambda u, \tilde{v}) + K(v,u), \\ u_{|\,t=0} = 0, \end{cases}$$

and since the right side is a continuous linear expression of $u \in C^r$ (see PROPOSITION 2.6), it follows from COROLLARY 2.4 that $u = 0$ is the only solution of this problem. This just means that $v \equiv \tilde{v}$. $\square$

Next, we establish the classic existence result for smooth data.

THEOREM 2.9. — *Let $v^0 \in C^s \cap L^p$ for some $s > 1$ and $1 < p < \infty$, with $\langle \nabla, v^0 \rangle = 0$. Then the incompressible Euler system has a (unique) solution*

$$v \in L^\infty([0,T]\,;C^s) \cap \mathrm{Lip}\big([0,T]\,;L^p\big)$$

for some $T > 0$. Moreover, if $v \in L^\infty_{\mathrm{loc}}([0,T)\,;C^s) \cap \mathrm{Lip}_{\mathrm{loc}}([0,T)\,;L^p)$ for a $T > 0$ such that

$$\int_0^T \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s < \infty\,,$$

then this solution can be continued as a solution

$$v \in L^\infty\big([0,T+\varepsilon]\,;C^s\big) \cap \mathrm{Lip}\big([0,T+\varepsilon]\,;L^p\big)$$

for some $\varepsilon > 0$.

Proof. — The proof is in CHEMIN [3]. However, since our operator Π is slightly different from that of CHEMIN, we now sketch this proof.

We take $v_0(t) \equiv v^0$, then we solve

$$\begin{cases} \partial_t v_{n+1} + \langle \nabla, v_n \otimes v_{n+1} \rangle = \pi(v_n, v_{n+1}), \\ v_{n\,|\,t=0} = v^0. \end{cases}$$

Assuming that $\langle \nabla, v_n \rangle = 0$ and that v_n is in $L^\infty([0,T]\,;C^s)$, it follows from PROPOSITION 2.6 (iii) and COROLLARY 2.4 that this problem has a solution $v_{n+1} \in L^\infty([0,T]\,;C^r)$ for any $r < 1$. Next, we apply the operator $\Lambda^{-1}\langle \nabla, \cdot \rangle$ to the equation and find that $u = \Lambda^{-1}\langle \nabla, v_{n+1} \rangle$ satisfies

$$\begin{cases} \partial_t u + \langle \nabla, v_n \otimes u \rangle = \chi(D)\Lambda^{-3}\langle \nabla, v_n \otimes \Lambda u \rangle + K(v_n, u), \\ u_{|\,t=0} = \Lambda^{-1}\langle \nabla, v^0 \rangle = 0. \end{cases}$$

Since the right side is a continuous linear expression of $u \in C^r$ (see PROPOSITION 2.6 (ii)), it follows again from COROLLARY 2.4 that $\langle \nabla, v_{n+1} \rangle = \Lambda u \equiv 0$. We also show that $v_{n+1} \in L^\infty([0,T]\,;C^s)$ by differentiating the equation with respect to x as many times as needed, and by using $\langle \nabla, v_n \rangle = \langle \nabla, v_{n+1} \rangle = 0$ to get the right estimates, and thus the sequence v_n is well defined by induction.

From these estimates, it is easy to see that for a sufficiently small $T > 0$ depending only on $\|v^0\|_s$, v_n is a bounded sequence in $L^\infty([0,T]\,;C^s)$ and a Cauchy sequence in $L^\infty([0,T]\,;C^{r-1})$ for some $r < 1$. It follows (see e.g. LEMMA 5.1 below) that this sequence has a limit $v \in L^\infty([0,T]\,;C^s)$ which is a solution of the pressure free system. Thanks to THEOREM 2.7 it is also a $\mathrm{Lip}([0,T]\,;L^p)$ solution of the incompressible Euler system as claimed.

Finally, any $L^\infty_{\mathrm{loc}}([0,T)\,;C^s)$ solution of the pressure free system can be estimated on $[0,T)$ by

$$\|v(t)\|_s \le \|v^0\|_s \, \mathrm{e}^{CV(t)}$$

where $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$, and this implies the last part of the statement (see CHEMIN [3] for more complete details). $\square$

We end this section with MAJDA's global existence result for axisymmetric flows that we now define.

DEFINITION 2.10. — We say that the vector field v has an *axisymmetric structure* if it satisfies

$$x_2v_1 - x_1v_2 \equiv \partial_2v_1 - \partial_1v_2 \equiv (x_2\partial_1 - x_1\partial_2)v_3 \equiv 0$$

and if the function

$$\alpha(x) = (x_1^2 + x_2^2)^{-1}(x_2\omega_1(x) - x_1\omega_2(x))$$

is bounded, compactly supported and has a constant sign (here, the ω_j's are the components of the vorticity $\omega = \nabla \wedge v$). Introducing cylindrical coordinates (ρ, θ, x_3), and at $x = (\rho\cos\theta, \rho\sin\theta, x_3)$ the three base vectors

$$e_\rho = (\cos\theta, \sin\theta, 0), \quad e_\theta = (\sin\theta, -\cos\theta, 0), \quad e_3 = (0,0,1),$$

then our conditions on v mean that

$$v = v_\rho e_\rho + v_3e_3 \quad \text{and} \quad \omega = \omega_\theta\, e_\theta$$

where v_ρ, v_3 and $\omega_\theta = \partial_\rho v_3 - \partial_3 v_\rho$ are functions of ρ and x_3 only (independent of θ), while our conditions on ω (i.e. on the function α) mean that $\omega_\theta/\rho = \alpha \in L^\infty_{\mathrm{comp}}$ and has a constant sign.

For flows of such axisymmetric vector fields, we have the following variant of a result of MAJDA (see [7, p. S 202–S 203]).

LEMMA 2.11. — *If $v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$ for some $p < \infty$ is a solution of the incompressible Euler system such that v^0 has an axisymmetric structure, then $v(t)$ has an axisymmetric structure for all t in $[0,T]$. Moreover, the functions*

$$\omega(t) = \nabla \wedge v(t) \quad \textit{and} \quad R(t) = \big\| |x|_{|\operatorname{supp}\omega(t)} \big\|_{L^\infty}$$

satisfy for all $1 \le q \le \infty$ and all $t \in [0,T]$

$$\big\|\omega(t)\big\|_{L^q} \le R(t)\|\alpha^0\|_{L^q} \quad \textit{and} \quad R(t) \le R(0)\big(1 + R(0)\|\alpha^0\|_{L^\infty}t\big)^{3/2}$$

where $\alpha^0(x) = (x_1^2+x_2^2)^{-1}\big(x_2\,\omega_1^0(x) - x_1\omega_2^0(x)\big)$.

Proof. — The proof that $v(t)$ has an axisymmetric structure is left to the reader : it suffices to use the symmetry of the equation, and the equation for ω/ρ we give below. With $\rho = (x_1^2 + x_2^2)^{-1/2}$ as above, we have

$$\partial_t\Big(\frac{\omega}{\rho}\Big) + \Big\langle \nabla, v \otimes \frac{\omega}{\rho}\Big\rangle = 0$$

so that

$$\Big\|\frac{\omega(t)}{\rho}\Big\|_{L^q} = \Big\|\frac{\omega^0}{\rho}\Big\|_{L^q} = \|\alpha^0\|_{L^q}$$

thanks to COROLLARY 2.4, and therefore we can write

$$\big\|\omega(t)\big\|_{L^q} \leq R(t)\Big\|\frac{\omega(t)}{\rho}\Big\|_{L^q} = R(t)\|\alpha^0\|_{L^q}.$$

On the other hand, the Biot-Savart law $v(t) = \omega(t) * \nabla F$ (LEMMA 2.2) implies that

$$R'(t) \leq \big\|v(t)\big\|_{L^\infty} \leq \sup_{x\in\mathbb{R}^3} \int \big|\omega(t, x-y)\big| \times \big|\nabla F(y)\big| \,\mathrm{d}y.$$

With $r(t) = R(0)^{5/3} R(t)^{-2/3}$, we have

$$\begin{aligned}\int_{|y|\leq r(t)} &\big|\omega(t,x-y)\big| \times \big|\nabla F(y)\big| \,\mathrm{d}y \\ &\leq \big\|\omega(t)\big\|_{L^\infty} \int_{|y|\leq r(t)} \big|\nabla F(y)\big| \,\mathrm{d}y \leq R(t) r(t) \|\alpha^0\|_{L^\infty}\end{aligned}$$

while by the Cauchy-Schwarz inequality

$$\begin{aligned}\int_{|y|\geq r(t)} &\big|\omega(t,x-y)\big| \times \big|\nabla F(y)\big| \,\mathrm{d}y \\ &\leq \Big\{\int_{|y|\geq r(t)} \frac{|\omega(t,x-y)|}{\rho(x-y)} \,\big|\nabla F(y)\big|^2 \,\mathrm{d}y\Big\}^{1/2} \\ &\qquad\qquad \times \Big\{\int_{|y|\geq r(t)} \rho(x-y)\,|\omega(t,x-y)| \,\mathrm{d}y\Big\}^{1/2} \\ &\leq \Big\{\Big\|\frac{\omega(t)}{\rho}\Big\|_{L^\infty} \int_{|y|\geq r(t)} \big|\nabla F(y)\big|^2 \,\mathrm{d}y\Big\}^{1/2} \Big|\int_{\mathbb{R}^3} y \wedge \omega(t,y) \,\mathrm{d}y\Big|^{1/2} \\ &\leq \Big\{\frac{\|\alpha^0\|_{L^\infty}}{4\pi r(t)}\Big\}^{1/2} \Big|\int_{\operatorname{supp}\omega^0} y \wedge \omega^0(y) \,\mathrm{d}y\Big|^{1/2} \\ &\leq \|\alpha^0\|_{L^\infty} \Big\{\int_{|y|\leq R(0)} \frac{|y|^2 \,\mathrm{d}y}{4\pi r(t)}\Big\}^{1/2} = \Big\{\frac{R(0)^5}{5 r(t)}\Big\}^{1/2} \|\alpha^0\|_{L^\infty}\end{aligned}$$

where we used that $\partial_t\big(\int y \wedge \omega(t,y)\,\mathrm{d}y\big) \equiv 0$. Therefore we get

$$R'(t) \leq \tfrac{3}{2} R(0)^{5/3} \|\alpha^0\|_{L^\infty} R(t)^{1/3}$$

and it follows that $R(t)$ can be estimated as claimed. $\square$

Finally, from THEOREM 2.9 and LEMMA 2.11 we can deduce the following global existence result.

COROLLARY 2.12. — *Keep the same assumptions as in Theorem* 2.9, *and assume that v^0 has an axisymmetric structure. Then, the incompressible Euler system has a* (*unique*) *solution $v \in L^\infty([0,T]\,;C^s) \cap \mathrm{Lip}([0,T]\,;L^p)$ for all $T>0$.*

Proof. — From THEOREM 2.9, we already have a solution defined in $[0,T]$ for some $T>0$. Then it follows from LEMMA 2.11 that the corresponding vorticity field ω satisfies

$$\int_0^T \|\omega(t)\|_{L^\infty} dt \leq \tfrac{2}{5}\big(1 + R(0)\|\alpha^0\|_{L^\infty} T\big)^{5/2} < \infty.$$

Now, it is shown in BAHOURI and DEHMAN [1] that under this condition any solution $v \in L^\infty_{\mathrm{loc}}([0,T)\,;C^s) \cap \mathrm{Lip}_{\mathrm{loc}}([0,T)\,;L^p)$ can be continued as a solution $v \in L^\infty([0,T+\varepsilon]\,;C^s) \cap \mathrm{Lip}([0,T+\varepsilon];\,L^p)$ for some $\varepsilon > 0$. Therefore, we have a solution $v \in L^\infty([0,T]\,;C^s) \cap \mathrm{Lip}([0,T];\mathrm{L}^p)$ for all $T>0$. $\square$

3. Velocity fields with striated vorticity

The goal of sections 3 through 5 is to prove THEOREM 5.4 below which is a variant of THEOREM 1.2 where the spaces $C^{r,\Sigma}$ of conormal distributions are replaced with spaces $C^{r,W}$ of striated distributions that we now define.

DEFINITION 3.1. — *Any system $W = (w^1, w^2, \dots, w^N)$ of N continuous vector fields is said to be admissible if the function*

$$[W]^{-1} = \Big\{\frac{2}{N(N-1)} \sum_{\mu<\nu} |w^\mu \wedge w^\nu|^2\Big\}^{-1/4}$$

is bounded. If $0<r<1$ and if W is an admissible system of C^r vector fields, we define the space $C^{r,W}$ as the space of all vector fields ω such *that $\omega \in L^\infty$ and $\langle \nabla, w^\nu \otimes \omega \rangle \in C^{r-1}$ for all $\nu \leq N$. The associated norm is defined as*

$$
\begin{aligned}
\|\omega\|_{r,W} &= \sum_\nu \Big\{ \big(1 + \big\|[W]^{-1}\big\|_{L^\infty} \|w^\nu\|_r\big) \|\omega\|_{L^\infty} \\
&\qquad\qquad + \big\|[W]^{-1}\big\|_{L^\infty} \big\|\langle \nabla, w^\nu \otimes \omega\rangle\big\|_{r-1} \Big\} \\
&= N\|\omega\|_{L^\infty} \\
&\qquad + \big\|[W]^{-1}\big\|_{L^\infty} \sum_\nu \Big\{ \|w^\nu\|_r \|\omega\|_{L^\infty} + \big\|\langle \nabla, w^\nu \otimes \omega\rangle\big\|_{r-1} \Big\}.
\end{aligned}
$$

REMARK. — We point out that our vector fields w^ν are not assumed here to be divergence free and that all our results in sections 3 through 5 hold without the restriction $\langle \nabla, w^\nu \rangle = 0$. We will use this property in the proof of THEOREM 6.4 below.

Here we can observe that this space $C^{r,W}$ always contains the Hölder space C^r. Our first result shows the link between this space and the space $C^{r,\Sigma}$ introduced in section 1.

PROPOSITION 3.2. — *For any C^{1+r}, two-dimensional, compact submanifold Σ of $\mathbb{R}^3$, we can find an admissible system W of five C^r, divergence free vector fields tangent to Σ, and we have $C^{r,\Sigma} \subset C^{r,W}$.*

Proof. — Let $f \in C^{1+r}$ be such that $f_{|\Sigma} = 0$ and $\nabla f_{|\Sigma} \neq 0$. By continuity, this function still satisfies $\nabla f \neq 0$ on Σ_ε for some $\varepsilon > 0$. Then, we choose a function $\chi \in C^\infty$ such that $\chi = 1$ on $\mathbb{R}^3 \setminus \Sigma_\varepsilon$ and $\chi = 0$ near Σ. The five vector fields

$$
w^1 = \begin{pmatrix} 0 \\ -\partial_3 f \\ \partial_2 f \end{pmatrix}, \qquad w^2 = \begin{pmatrix} \partial_3 f \\ 0 \\ -\partial_1 f \end{pmatrix}, \qquad w^3 = \begin{pmatrix} -\partial_2 f \\ \partial_1 f \\ 0 \end{pmatrix},
$$

$$
w^4 = \begin{pmatrix} \partial_3(\chi x_3) \\ 0 \\ -\partial_1(\chi x_3) \end{pmatrix}, \quad w^5 = \begin{pmatrix} -\partial_2(\chi x_1) \\ \partial_1(\chi x_1) \\ 0 \end{pmatrix}
$$

have C^r components and are divergence free and tangent to Σ. Moreover, $|w^4 \wedge w^5|^2 \equiv 1$ on $\mathbb{R}^3 \setminus \Sigma_\varepsilon$ while

$$
|w^1 \wedge w^2|^2 + |w^1 \wedge w^3|^2 + |w^2 \wedge w^3|^2 = \big|\nabla f\big|^4
$$

is positive on the compact set Σ_ε. Thus we have $[W] \geq \delta > 0$, or $[W]^{-1} \leq 1/\delta < \infty$. Finally, since the w^ν's are divergence free and tangent to Σ, we immediately have $C^{r,\Sigma} \subset C^{r,W}$. $\square$

The rest of this section is devoted to the proof of the following result.

PROPOSITION 3.3. — *There exists a constant C_1 depending only on $1 \leq q < 3$ and $0 < r < 1$ such that for all admissible system W of C^r vector fields, and for all $v \in L^p$, $p < \infty$, such that $\langle \nabla, v \rangle = 0$ and $\omega = \nabla \wedge v \in L^q \cap C^{r,W}$, we have $v \in \mathrm{Lip}$ with the estimate*

$$\|v\|_{\mathrm{Lip}} \leq C_1 \|\omega\|_{q,r,W}$$

where we have set

$$\|\omega\|_{q,r,W} = \|\omega\|_{L^q} + \|\omega\|_{L^\infty} \operatorname{Log}\Big(2 + \frac{\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}}\Big).$$

Its proof is based on the following geometric results.

LEMMA 3.4. — *Let w^1 and w^2 be two C^r vector fields and Ω be an open subset of $\mathbb{R}^3$ where $|w^1 \wedge w^2|^{-1/2}$ is bounded. Then we can find functions $a_{jk} \in L^\infty(\Omega)$ and $b^{\ell 1}_{jk}$ and $b^{\ell 2}_{jk} \in C^r(\Omega)$ such that*

$$\xi_j \xi_k - a_{jk}(x)|\xi|^2 = \sum_{\ell,\nu} b^{\ell\nu}_{jk}(x)\, \xi_\ell \langle w^\nu(x), \xi \rangle$$

for $(x, \xi) \in \Omega \times \mathbb{R}^3$, with the estimates $\|a_{jk}\|_{L^\infty(\Omega)} \leq 1$ and

$$\|b^{\ell\nu}_{jk}\|_{r(\Omega)} \leq C\big(\|w^1\|_r + \|w^2\|_r\big)^{15} \big\||w^1 \wedge w^2|^{-1/2}\big\|^{16}_{L^\infty(\Omega)}$$

where the constant C depends only on r.

Proof. — At every point $x \in \Omega$ we define the transformation $\xi \mapsto \eta$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\eta = \begin{pmatrix} \langle w^1 \wedge w^2, \xi \rangle \\ \langle w^1, \xi \rangle \\ \langle w^2, \xi \rangle \end{pmatrix},$$

and we observe that this transformation can be inverted by the formula $\xi = A\eta$ where the matrix A is

$$A = \frac{1}{|w^1 \wedge w^2|^2} \big(w^1 \wedge w^2, w^2 \wedge (w^1 \wedge w^2), (w^1 \wedge w^2) \wedge w^1\big).$$

Then we consider the quadratic form $q_{jk}(\xi) = \xi_j \xi_k$ and we choose the coefficient

$$a_{jk} = \frac{q_{jk}(w^1 \wedge w^2)}{|w^1 \wedge w^2|^2}$$

(this choice clearly gives $\|a_{jk}\|_{L^\infty(\Omega)} \le 1$) so that $\xi = w^1 \wedge w^2$ is an isotropic vector for the quadratic form $q_{jk}(\xi) - a_{jk}|\xi|^2$. In the η variables we get

$$q_{jk}(A\eta) - a_{jk}|A\eta|^2 = {}^t\eta\big({}^tA\,Q_{jk}A - a_{jk}\,{}^tAA\big)\eta$$

where Q_{jk} is the matrix of the quadratic form q_{jk} (in the ξ variables). An elementary computation shows that

$${}^tAA = \frac{1}{|w^1 \wedge w^2|^2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & |w^2|^2 & -\langle w^1, w^2\rangle \\ 0 & -\langle w^1, w^2\rangle & |w^1|^2 \end{pmatrix},$$

and by observing the homogeneity degrees of the columns of the matrix A we see that

$${}^tA\,Q_{jk}A - a_{jk}\,{}^tAA = \frac{1}{|w^1 \wedge w^2|^4}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_4 & \beta_5 \\ \beta_3 & \beta_5 & \beta_6 \end{pmatrix}$$

where the β_j's are homogeneous polynomials in (w^1, w^2) : β_1 of degree 4, β_2 and β_3 of degree 5, and β_4, β_5 and β_6 of degree 6. Moreover, $\beta_1 = 0$ since $\xi = w^1 \wedge w^2$ is an isotropic vector for the quadratic form $q_{jk}(\xi) - a_{jk}|\xi|^2$. Coming back to the ξ variables, we can write

$$\begin{aligned}
\xi_j\xi_k - a_{jk}|\xi|^2 &= \frac{1}{|w^1 \wedge w^2|^4}(2\beta_2\eta_1 + \beta_4\eta_2 + \beta_5\eta_3)\eta_2 \\
&\qquad + \frac{1}{|w^1 \wedge w^2|^4}(2\beta_3\eta_1 + \beta_5\eta_2 + \beta_6\eta_3)\eta_3 \\
&= \frac{2\beta_2\langle w^1 \wedge w^2, \xi\rangle + \beta_4\langle w^1, \xi\rangle + \beta_5\langle w^2, \xi\rangle}{|w^1 \wedge w^2|^4}\,\langle w^1, \xi\rangle \\
&\qquad + \frac{2\beta_3\langle w^1 \wedge w^2, \xi\rangle + \beta_5\langle w^1, \xi\rangle + \beta_6\langle w^2, \xi\rangle}{|w^1 \wedge w^2|^4}\,\langle w^2, \xi\rangle \\
&= (b_{jk}^{11}\xi_1 + b_{jk}^{21}\xi_2 + b_{jk}^{31}\xi_3)\langle w^1, \xi\rangle \\
&\qquad + (b_{jk}^{12}\xi_1 + b_{jk}^{22}\xi_2 + b_{jk}^{32}\xi_3)\langle w^2, \xi\rangle
\end{aligned}$$

where the $b_{jk}^{\ell\nu}$ are homogeneous polynomials of (w^1, w^2) of degree 7 divided by the quantity $|w^1 \wedge w^2|^4$. Therefore we have got the required expression for $\xi_j\xi_k - a_{jk}|\xi|^2$, and it is elementary to check that

$$\|b_{jk}^{\ell\nu}\|_{r(\Omega)} \le C\big(\|w^1\|_r + \|w^2\|_r\big)^{15}\,\big\||w^1 \wedge w^2|^{-1/2}\big\|_{L^\infty(\Omega)}^{16}.$$

Our proof is complete. $\square$

LEMMA 3.5. — *Let W be an admissible system of N C^r vector fields. Then we can find functions $a_{jk} \in L^\infty$ and $b_{jk}^{\ell\nu} \in C^r$ such that*

$$\xi_j \xi_k - a_{jk}(x)|\xi|^2 = \sum_{\ell,\nu} b_{jk}^{\ell\nu}(x)\, \xi_\ell \langle w^\nu(x), \xi\rangle$$

for $(x, \xi) \in \mathbb{R}^6$, with the estimates $\|a_{jk}\|_{L^\infty} \leq 1$ and

$$\|b_{jk}^{\ell\nu}\|_r \leq C N^3 M_{r,W}^{19} \big\|[W]^{-1}\big\|_{L^\infty}$$

where $M_{r,W} = \big\|[W]^{-1}\big\|_{L^\infty} \sum_\nu \|w^\nu\|_r$, and where the constant C depends only on r.

Proof. — On each

$$\Omega^{\mu\nu} = \Big\{x \in \mathbb{R}^3 \,;\ |w^\mu \wedge w^\nu|^{-1/2}(x) < 2\big\|[W]^{-1}\big\|_{L^\infty}\Big\},$$

LEMMA 3.4 gives coefficients a_{jk} in $L^\infty(\Omega^{\mu\nu})$ and $b_{jk}^{\ell\mu}$, $b_{jk}^{\ell\nu}$ in $C^r(\Omega^{\mu\nu})$ solving our problem. Then it suffices to combine these local constructions with a partition of unity $1 = \sum_{\mu<\nu} \varphi^{\mu\nu}$ satisfying :

(i) $\operatorname{supp} \varphi^{\mu\nu} \subset \Omega^{\mu\nu}$ and

(ii) $\|\varphi^{\mu\nu}\|_r \leq C N^2 M_{r,W}^4$.

To construct such a partition of unity, we choose a nonnegative function $\chi \in C_0^\infty$ with $\|\chi\|_{L^1} = 1$ and $\operatorname{supp}\chi$ contained in the unit ball, then we set for $\mu < \nu$

$$\begin{aligned}
F^{\mu\nu} &= \big\{x \in \mathbb{R}^3 \,;\ |w^\mu \wedge w^\nu|^2(x) \geq \inf_{\mathbb{R}^3} [W]^4\big\},\\
F_\varepsilon^{\mu\nu} &= \big\{x \in \mathbb{R}^3 \,;\ \operatorname{dist}(x, F^{\mu\nu}) \leq \varepsilon\big\},\\
\chi_\varepsilon(x) &= \varepsilon^{-3}\chi(x/\varepsilon),\\
\varphi_\varepsilon^{\mu\nu}(x) &= \int_{F_{\varepsilon/2}^{\mu\nu}} \chi_{\varepsilon/2}(x-y)\, \mathrm{d}y, \quad \text{and}\\
\varphi^{\mu\nu} &= \varphi_\varepsilon^{\mu\nu} \prod_{(\bar\mu,\bar\nu)<(\mu,\nu)} (1 - \varphi_\varepsilon^{\bar\mu\,\bar\nu}),
\end{aligned}$$

where $(\bar\mu, \bar\nu) < (\mu, \nu)$ means $\bar\mu < \mu$ or ($\bar\mu = \mu$ and $\bar\nu < \nu$). Since

$$1 - \sum_{\mu<\nu} \varphi^{\mu\nu} = \prod_{\mu<\nu} (1 - \varphi_\varepsilon^{\mu\nu}),$$

since $\varphi_\varepsilon^{\mu\nu} \equiv 1$ on $F^{\mu\nu}$ and since $\mathbb{R}^3 = \bigcup_{\mu<\nu} F^{\mu\nu}$, this is a partition of unity.

Moreover, the estimate

$$\|\varphi^{\mu\nu}\|_r \le \sum_{(\bar\mu,\bar\nu)\le(\mu,\nu)} \|\varphi_\varepsilon^{\bar\mu\bar\nu}\|_r \le CN^2\varepsilon^{-r}$$

implies estimate (ii) above if we chose $\varepsilon = (2M_{r,W})^{-4/r}$. Finally, on $F_\varepsilon^{\mu\nu} \supset \operatorname{supp}\varphi^{\mu\nu}$ we have

$$\begin{aligned}|w^\mu \wedge w^\nu|^2(x) &\ge \inf_{\mathbb{R}^3}[W]^4 - \varepsilon^r \| |w^\mu \wedge w^\nu|^2\|_r \\ &\ge \inf_{\mathbb{R}^3}[W]^4 - 2\,\varepsilon^r\Big(\sum_\nu \|w^\nu\|_r\Big)^4 \\ &\ge \tfrac{7}{8}\inf_{\mathbb{R}^3}[W]^4 \quad \Big(> \tfrac{1}{4}\inf_{\mathbb{R}^3}[W]^4\Big)\end{aligned}$$

thanks to our choice for ε, and this proves that

$$|w^\mu \wedge w^\nu|^{-1/2} \le \sqrt{2}\,\big\|[W]^{-1}\big\|_{L^\infty}$$

on $\operatorname{supp}\varphi^{\mu\nu}$. This implies the inclusion (i) above. □

COROLLARY 3.6. — *With any admissible system W of N C^r vector fields, we can associate functions $a_{jk} \in L^\infty$ with $\|a_{jk}\|_{L^\infty} \le 1$ such that for all $\omega \in C^{r,W}$,*

$$\big\|\partial_j\partial_k\omega - \Delta(a_{jk}\omega)\big\|_{r-2} \le CN^3 M_{r,W}^{19}\|\omega\|_{r,W}$$

where $M_{r,W} = \big\|[W]^{-1}\big\|_{L^\infty}\sum_\nu \|w^\nu\|_r$, and where the constant C depends only on r.

Proof. — Let us quantify the algebraic relation of LEMMA 3.5 by putting the coefficients on the right and the differentiations on the left. We obtain for $\omega \in C^{r,W}$

$$\partial_j\partial_k\omega - \Delta(a_{jk}\omega) = \sum_{\ell,m,\nu} \partial_\ell\partial_m(b_{jk}^{\ell\nu} w_m^\nu \omega)$$

so that it suffices to estimate the C^{r-1} norm of sums $\sum_m \partial_m(bw_m\omega)$. For this, we use the paraproduct operator T_b and write

$$\begin{aligned}\sum_m \partial_m(bw_m\omega) = \sum_m \partial_m\{(b-T_b)w_m\omega\} \\ + \sum_m [\partial_m, T_b]w_m\omega + T_b\sum_m \partial_m(w_m\omega).\end{aligned}$$

The first term can be estimated by

$$\begin{aligned}\big\|\partial_m\big\{(b-T_b)w_m\omega\big\}\big\|_{r-1} &\le \big\|(b-T_b)w_m\omega\big\|_r\\ &\le C\|b\|_r\|w_m\omega\|_{L^\infty}\\ &\le CN^3M_{r,W}^{19}\big\|[W]^{-1}\|_{L^\infty}\|w\|_r\|\omega\|_{L^\infty},\end{aligned}$$

and similarly, the second term can be estimated by

$$\begin{aligned}\big\|[\partial_m,T_b]w_m\omega\big\|_{r-1} &= \|T_{\partial_m b}w_m\omega\|_{r-1}\\ &\le C\|\partial_m b\|_{r-1}\|w_m\omega\|_{L^\infty}\\ &\le CN^3M_{r,W}^{19}\big\|[W]^{-1}\big\|_{L^\infty}\|w\|_r\|\omega\|_{L^\infty}.\end{aligned}$$

Finally, the third term can be estimated simply by

$$\begin{aligned}\Big\|T_b\sum_m\partial_m(w_m\omega)\Big\|_{r-1} &\le C\|b\|_{L^\infty}\big\|\langle\nabla,w\otimes\omega\rangle\big\|_{r-1}\\ &\le CN^3M_{r,W}^{19}\big\|[W]^{-1}\big\|_{L^\infty}\big\|\langle\nabla,w\otimes\omega\rangle\big\|_{r-1},\end{aligned}$$

and this implies our estimate after summation in ℓ and ν. □

Proof of Proposition 3.3. — The elliptic pseudo-differential operators Λ^s constructed in section 1.b satisfy $\Lambda^{-2}\Delta=\chi(D)\Lambda^{-2}-1$, and therefore the elementary formula $\nabla\wedge\omega=-\Delta v$ immediately gives for high frequencies

$$\big(1-\chi(D)\Lambda^{-2}\big)v=\Lambda^{-2}(\nabla\wedge\omega).$$

For low frequencies, we use the Biot-Savart law $v=\omega*\nabla F$ (see LEMMA 2.2) to get the estimate

$$\|v\|_{L^\infty}\le C\big(\|\omega\|_{L^q}+\|\omega\|_{L^\infty}\big)$$

since $\nabla F\in L^{q/(q-1)}+L^1$. Then we can write

$$\begin{aligned}\|v\|_{\mathrm{Lip}} &= \|v\|_{L^\infty}+\sum_j\|\partial_j v\|_{L^\infty}\\ &\le \|v\|_{L^\infty}+\sum_j\big\|\partial_j\chi(D)\Lambda^{-2}v\big\|_{L^\infty}+\sum_j\big\|\partial_j(1-\chi(D)\Lambda^{-2})v\big\|_{L^\infty}\\ &\le C\|v\|_{L^\infty}+\sum_j\big\|\partial_j\Lambda^{-2}(\nabla\wedge\omega)\big\|_{L^\infty}\\ &\le C\Big(\|\omega\|_{L^q}+\|\omega\|_{L^\infty}+\sum_{jk}\|\Lambda^{-2}\partial_j\partial_k\omega\|_{L^\infty}\Big).\end{aligned}$$

Next, we use the a_{jk} coefficients of COROLLARY 3.6 to write

$$\begin{aligned}
&\|\Lambda^{-2}\partial_j\partial_k\omega\|_{L^\infty}\\
&\quad\le \big\|\big(1-\chi(D)\Lambda^{-2}\big)a_{jk}\omega\big\|_{L^\infty}+\big\|\Lambda^{-2}\big(\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big)\big\|_{L^\infty}\\
&\quad\le C\|\omega\|_{L^\infty}+\big\|\Lambda^{-2}\big(\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big)\big\|_{L^\infty}.
\end{aligned}$$

Since the pseudo-differential operators Λ^{-2}, $\Lambda^{-2}\partial_j\partial_k$ and $\Lambda^{-2}\Delta$ have homogeneous symbols, we have the following estimates in Hölder norms

$$\begin{aligned}
&\big\|\Lambda^{-2}\big(\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big)\big\|_0\le C\|\omega\|_{L^\infty},\quad\text{and}\\
&\big\|\Lambda^{-2}\big(\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big)\big\|_r\le \big\|\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big\|_{r-2}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\le CN^3M_{r,W}^{19}\|\omega\|_{r,W}
\end{aligned}$$

thanks to COROLLARY 3.6, and the logarithmic interpolation estimate gives

$$\big\|\Lambda^{-2}\big(\partial_j\partial_k\omega-\Delta(a_{jk}\omega)\big)\big\|_{L^\infty}\le C\|\omega\|_{L^\infty}\operatorname{Log}\Big(2+\frac{N^3M_{r,W}^{19}\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}}\Big).$$

Finally, we conclude by observing that

$$\begin{aligned}
&2+\frac{N^3M_{r,W}^{19}\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}}\\
&\quad=2+N^4M_{r,W}^{19}+N^3M_{r,W}^{20}\\
&\qquad\qquad+N^3M_{r,W}^{19}\big\|[W]^{-1}\big\|_{L^\infty}\sum_\nu\frac{\|\langle\nabla,w^\nu\otimes\omega\rangle\|_{r-1}}{\|\omega\|_{L^\infty}}\\
&\quad\le\Big\{2+N+M_{r,W}+\big\|[W]^{-1}\big\|_{L^\infty}\sum_\nu\frac{\|\langle\nabla,w^\nu\otimes\omega\rangle\|_{r-1}}{\|\omega\|_{L^\infty}}\Big\}^{23}\\
&\quad=\Big\{2+\frac{\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}}\Big\}^{23},
\end{aligned}$$

and this completes the proof of our proposition. □

4. Estimates for smooth solutions

The main result of this section is the following.

PROPOSITION 4.1. — *Let $v \in L^\infty([0,T]\,;C^{1+r})$ be a solution of the incompressible Euler system and w^0 be a C^r vector field. Then the problem*

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w\rangle = \langle w, \nabla\rangle v, \\ w_{|t=0} = w^0 \end{cases}$$

has a unique solution $w \in L^\infty([0,T]\,;C^r)$ and we have

$$\big\|w(t)\big\|_r + \frac{\|\langle \nabla, w(t) \otimes \omega(t)\rangle\|_{r-1}}{\|\omega(t)\|_{L^\infty}} \le \mathrm{e}^{C_2 V(t)} \Big\{ \|w^0\|_r + \frac{\|\langle \nabla, w^0 \otimes \omega^0\rangle\|_{r-1}}{\|\omega^0\|_{L^\infty}} \Big\}$$

where $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$, and C_2 depends only on $0 < r < 1$.

Proof. — The existence and uniqueness of the solution w in the space $L^\infty([0,T]\,;C^r)$ follow from COROLLARY 2.4 since the operator $\mathcal{A} : w \mapsto \langle w, \nabla\rangle v$ is continuous on C^r, and therefore, we only have to prove the estimate given in the statement.

For this, we set $u = \Lambda^{-1}\langle \nabla, w \otimes \omega\rangle \in L^\infty([0,T]\,;C^r)$. Using the transformation $w \mapsto w^*$ described in PROPOSITION 2.3, we observe that

$$\partial_t\{w^*\} = \big\{\langle w, \nabla\rangle v\big\}^* \quad \text{and} \quad \partial_t\{\omega^*\} = \big\{\langle \omega, \nabla\rangle v\big\}^*,$$

so that

$$\begin{aligned} \partial_t\big\{(w\otimes\omega)^*\big\} &= \partial_t\{w^* \otimes \omega^*\} = \big(\partial_t\{w^*\}\big) \otimes \omega^* + w^* \otimes \big(\partial_t\{\omega^*\}\big) \\ &= \big\{\langle w, \nabla\rangle v\big\}^* \otimes \omega^* + w^* \otimes \big\{\langle \omega, \nabla\rangle v\big\}^*, \end{aligned}$$

that is

$$\partial_t(w \otimes \omega) + \big\langle \nabla, v \otimes (w \otimes \omega)\big\rangle = \big\{\langle w, \nabla\rangle v\big\} \otimes \omega + w \otimes \big\{\langle \omega, \nabla\rangle v\big\}.$$

When applying the operator $\Lambda^{-1}\langle \nabla, \cdot\,\rangle$ to this equation, we find

$$\begin{aligned} \partial_t u &= \Lambda^{-1}\big\langle \nabla, \big\{\langle w, \nabla\rangle v\big\} \otimes \omega + w \otimes \big\{\langle \omega, \nabla\rangle v\big\}\big\rangle \\ &\qquad\qquad - \Lambda^{-1}\big\langle \nabla, \big\langle \nabla, v \otimes (w \otimes \omega)\big\rangle\big\rangle \\ &= \Lambda^{-1}\big\langle \nabla, \big\{\langle w, \nabla\rangle v\big\} \otimes \omega + w \otimes \big\{\langle \omega, \nabla\rangle v\big\}\big\rangle \\ &\qquad\qquad - \Lambda^{-1}\big\langle \nabla, \big\langle \nabla, w \otimes (v \otimes \omega)\big\rangle\big\rangle. \end{aligned}$$

Since $\langle \nabla, w \otimes (v \otimes \omega) \rangle = \{ \langle w, \nabla \rangle v \} \otimes \omega + v \otimes \langle \nabla, w \otimes \omega \rangle$, we get

$$\partial_t u = \Lambda^{-1} \big\langle \nabla, w \otimes \big\{ \langle \omega, \nabla \rangle v \big\} \big\rangle - \Lambda^{-1} \big\langle \nabla, v \otimes \langle \nabla, w \otimes \omega \rangle \big\rangle$$

and finally

$$\partial_t u + \langle \nabla, v \otimes u \rangle = \Lambda^{-1} \big\langle \nabla, w \otimes \big\{ \langle \omega, \nabla \rangle v \big\} \big\rangle + K(v, u)$$

where the operator K is that of section 2. Using the operator D of section 2, we now define linear operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ as

$$\begin{aligned}
\mathcal{B}u &= \Lambda^{-1}(\nabla \wedge u), \\
\mathcal{A}w &= \langle w, \nabla \rangle v - \mathcal{B}\Lambda^{-1} \langle \nabla, w \otimes \omega \rangle, \\
\mathcal{D}u &= K(v, u) + \Lambda^{-1} D(\omega, \mathcal{B}u) + \Lambda^{-1} D(\Lambda u, v), \\
\mathcal{C}w &= \Lambda^{-1} \big\langle \nabla, w \otimes \big\{ \langle \omega, \nabla \rangle v \big\} \big\rangle \\
&\qquad\qquad + K\big(v, \Lambda^{-1} \langle \nabla, w \otimes \omega \rangle\big) - \mathcal{D}\Lambda^{-1} \langle \nabla, w \otimes \omega \rangle,
\end{aligned}$$

and observe that, by construction of $\mathcal{A}$ and $\mathcal{C}$, the vector fields w and $u = \Lambda^{-1} \langle \nabla, w \otimes \omega \rangle$ are solutions in $L^\infty([0, T]\,; C^r)$ of the system

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w \rangle = \mathcal{A}w + \mathcal{B}u, \\ \partial_t u + \langle \nabla, v \otimes u \rangle = \mathcal{C}w + \mathcal{D}u. \end{cases}$$

As in Corollary 2.4, our estimate will follow from estimates on these operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. In the statement of Lemma 4.2, the time does not appear because it is assumed to be fixed.

Lemma 4.2. — *There exist constants C depending only on r such that :*

(i) $\|\mathcal{B}u\|_r \leq C \|u\|_r$;

(ii) *The linear operators $\mathcal{A}$ and $\mathcal{D}$ depend linearly on v and satisfy*

$$\|\mathcal{A}w\|_r \leq C \|v\|_{\mathrm{Lip}} \|w\|_r, \qquad \|\mathcal{A}w\|_{r/2} \leq C \|v\|_{1-r/2} \|w\|_r,$$

$$\|\mathcal{D}u\|_r \leq C \|v\|_{\mathrm{Lip}} \|u\|_r, \qquad \|\mathcal{D}u\|_{r/2} \leq C \|v\|_{1-r/2} \|u\|_r.$$

(iii) *The linear operator $\mathcal{C}$ depends bilinearly on (ω, v) and satisfies*

$$\begin{aligned}
&\|\mathcal{C}w\|_r \leq C \|\omega\|_{L^\infty} \|v\|_{\mathrm{Lip}} \|w\|_r, \\
&\|\mathcal{C}w\|_{r/2} \leq C \|\omega\|_{-r/2} \|v\|_{\mathrm{Lip}} \|w\|_r, \\
&\|\mathcal{C}w\|_{r/2} \leq C \|\omega\|_{L^\infty} \|v\|_{1-r/2} \|w\|_r.
\end{aligned}$$

Proof.

(i) The estimate for the operator $\mathcal{B}$ is obvious.

(ii) The operator $\mathcal{A}$ satisfies

$$\begin{aligned}
\mathcal{A}w &= \langle w, \nabla\rangle v - \Lambda^{-2}\nabla \wedge \langle \nabla, w \otimes \omega\rangle \\
&= \langle w - T_w, \nabla\rangle v + \langle T_w, \nabla\rangle\, \chi(D)\Lambda^{-2} v - \langle T_w, \nabla\rangle \Lambda^{-2}\Delta v \\
&\qquad + \Lambda^{-2}\big(\nabla \wedge \langle \nabla, (T_w - w) \otimes \omega\rangle\big) - \Lambda^{-2}\nabla \wedge \langle \nabla, T_w \otimes \omega\rangle.
\end{aligned}$$

The C^r norm (resp. the $C^{r/2}$ norm) of the first, second and fourth terms can be estimated by $C\ (\|v\|_{\mathrm{Lip}} + \|\omega\|_{L^\infty})\|w\|_r$ (resp. by $C(\|v\|_{1-r/2} + \|\omega\|_{-r/2})\|w\|_r$). Moreover, the relation $\Delta v = -\nabla \wedge \omega$ allows to write the other two terms in the form

$$\begin{aligned}
&\langle T_w, \nabla\rangle \Lambda^{-2}(\nabla \wedge \omega) - \Lambda^{-2}\nabla \wedge \langle \nabla, T_w \otimes \omega\rangle \\
&\qquad = \sum_j \left[T_{w_j}, \partial_j \Lambda^{-2} \begin{pmatrix} 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{pmatrix} \right] \omega,
\end{aligned}$$

so that the C^r norm (resp. the $C^{r/2}$ norm) of these terms can be estimated by $C\|w\|_r\|\omega\|_{L^\infty}$ (resp. by $C\|w\|_r\|\omega\|_{-r/2}$) and this gives our estimates for the operator $\mathcal{A}$. The estimates for the operator $\mathcal{D}$ easily follow from PROPOSITION 2.6 (i) and (ii).

(iii) Finally, we can write

$$\begin{aligned}
\big\langle \nabla, w \otimes \{\langle \omega, \nabla\rangle v\}\big\rangle &= \big\langle \nabla, \langle \nabla, \omega \otimes (w \otimes v)\rangle\big\rangle - \big\langle \nabla, \langle \nabla, \omega \otimes w\rangle \otimes v\big\rangle \\
&= \big\langle \nabla, \langle \nabla, w \otimes (\omega \otimes v)\rangle\big\rangle - \big\langle \nabla, \langle \nabla, \omega \otimes w\rangle \otimes v\big\rangle \\
&= \big\langle \nabla, \omega \otimes \{\langle w, \nabla\rangle v\}\big\rangle \\
&\qquad\qquad + \big\langle \nabla, \langle \nabla, w \otimes \omega - \omega \otimes w\rangle \otimes v\big\rangle \\
&= D\big(\omega, \langle w, \nabla\rangle v\big) + D\big(\langle \nabla, w \otimes \omega - \omega \otimes w\rangle, v\big) \\
&= D\big(\omega, \mathcal{A}w + \mathcal{B}u\big) + D\big(\Lambda u - D(\omega, w), v\big)
\end{aligned}$$

since $\langle \nabla, \omega\rangle = \langle \nabla, \langle \nabla, w \otimes \omega - \omega \otimes w\rangle\rangle = 0$. It follows that

$$\begin{aligned}
\mathcal{C}w &= \Lambda^{-1}D(\omega, \mathcal{A}w + \mathcal{B}u) + \Lambda^{-1}D\big(\Lambda u - D(\omega, w), v\big) \\
&\qquad\qquad - \Lambda^{-1}D(\omega, \mathcal{B}u) - \Lambda^{-1}D(\Lambda u, v) \\
&= \Lambda^{-1}D\big(\omega, \mathcal{A}w\big) - \Lambda^{-1}D(D(\omega, w), v\big),
\end{aligned}$$

and this implies the estimates (iii) thanks to PROPOSITION 2.6 (i) and to the estimates obtained above for the operator $\mathcal{A}$. $\square$

End of the proof of Proposition 4.1. — We now get the required estimates as in the proof of COROLLARY 2.4. We can write as there

$$\begin{aligned}\big\|w(t)\big\|_r &\leq \mathrm{e}^{rV(t)}\|w^0\|_r + \int_0^t \mathrm{e}^{r(V(t)-V(s))}\big(\|\mathcal{A}w(s)\|_r + \|\mathcal{B}u(s)\|_r\big)\,\mathrm{d}s \\ &\leq \mathrm{e}^{rV(t)}\|w^0\|_r \\ &\qquad + C\int_0^t \mathrm{e}^{r(V(t)-V(s))}\big\|v(s)\big\|_{\mathrm{Lip}}\Big\{\big\|w(s)\big\|_r + \frac{\|u(s)\|_r}{\|\omega(s)\|_{L^\infty}}\Big\}\,\mathrm{d}s,\end{aligned}$$

and similarly

$$\big\|u(t)\big\|_r \leq \mathrm{e}^{rV(t)}\|u^0\|_r + \int_0^t \mathrm{e}^{r(V(t)-V(s))}\big(\|\mathcal{C}w(s)\|_r + \|\mathcal{D}u(s)\|_r\big)\,\mathrm{d}s$$

which gives

$$\begin{aligned}\frac{\|u(t)\|_r}{\|\omega(t)\|_{L^\infty}} &\leq \mathrm{e}^{rV(t)}\frac{\|\omega^0\|_{L^\infty}}{\|\omega(t)\|_{L^\infty}}\frac{\|u^0\|_r}{\|\omega^0\|_{L^\infty}} \\ &\qquad + C\int_0^t \mathrm{e}^{r(V(t)-V(s))}\frac{\|\omega(s)\|_{L^\infty}}{\|\omega(t)\|_{L^\infty}}\|v(s)\|_{\mathrm{Lip}} \\ &\qquad\qquad \times\Big\{\|w(s)\|_r + \frac{\|u(s)\|_r}{\|\omega(s)\|_{L^\infty}}\Big\}\,\mathrm{d}s \\ &\leq \mathrm{e}^{(r+1)V(t)}\frac{\|u^0\|_r}{\|\omega^0\|_{L^\infty}} + C\int_0^t \mathrm{e}^{(r+1)(V(t)-V(s))}\|v(s)\|_{\mathrm{Lip}} \\ &\qquad\qquad \times\Big\{\|w(s)\|_r + \frac{\|u(s)\|_r}{\|\omega(s)\|_{L^\infty}}\Big\}\,\mathrm{d}s\end{aligned}$$

thanks to the estimate of $\|\omega(s)\|_{L^\infty}/\|\omega(t)\|_{L^\infty}$ obtained in COROLLARY 2.5. If we sum these two estimates, we get

$$\begin{aligned}\Big\{\big\|w(t)\big\|_r + \frac{\|u(t)\|_r}{\|\omega(t)\|_{L^\infty}}\Big\} &\leq \mathrm{e}^{(r+1)V(t)}\Big\{\|w^0\|_r + \frac{\|u^0\|_r}{\|\omega^0\|_{L^\infty}}\Big\} \\ &\quad + C\int_0^t \mathrm{e}^{(r+1)(V(t)-V(s))}\big\|v(s)\big\|_{\mathrm{Lip}}\Big\{\big\|w(s)\big\|_r + \frac{\|u(s)\|_r}{\|\omega(s)\|_{L^\infty}}\Big\}\,\mathrm{d}s,\end{aligned}$$

and we can conclude as in the proof of COROLLARY 2.4. □

This result leads us to the estimate of the $C^{r,W}$ norm of the vorticity.

COROLLARY 4.3. — *Let $v \in L^\infty([0,T]\,;C^{1+r})$ be a solution of the incompressible Euler system and $W^0 = (w^{1,0}, w^{2,0}, \dots, w^{N,0})$ be an admissible system of C^r vector fields. We set $W(t) = (w^1(t), w^2(t), \dots, w^N(t))$ where the vector fields $w^\nu \in L^\infty([0,T]\,;C^r)$ are obtained by solving the problems*

$$\begin{cases} \partial_t w^\nu + \langle \nabla, v \otimes w^\nu \rangle = \langle w^\nu, \nabla \rangle v, \\ w^\nu{}_{|t=0} = w^{\nu,0} \end{cases}$$

as in Proposition 4.1. Then the system $W(t)$ is admissible, and we have the estimates

$$\big\| [W(t)]^{-1} \big\|_{L^\infty} \le \mathrm{e}^{V(t)/2} \big\| [W^0]^{-1} \big\|_{L^\infty},$$

$$\frac{\|\omega(t)\|_{r,W(t)}}{\|\omega(t)\|_{L^\infty}} \le \mathrm{e}^{(C_2+\frac12)\,V(t)} \frac{\|\omega^0\|_{r,W^0}}{\|\omega^0\|_{L^\infty}},$$

where $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$ and C_2 is the constant in Proposition 4.1.

Proof. — In order to prove that $W(t)$ is admissible, we just have to prove the estimate of $\|[W(t)]^{-1}\|_{L^\infty}$. Denoting by ${}^t(\nabla \otimes v)$ the transpose of the Jacobian matrix of v, we have for all $\mu < \nu$

$$\begin{aligned} \partial_t(w^\mu \wedge w^\nu) &+ \langle \nabla, v \otimes (w^\mu \wedge w^\nu) \rangle \\ &= \big\{ \langle w^\mu, \nabla \rangle v \big\} \wedge w^\nu + w^\mu \wedge \big\{ \langle w^\nu, \nabla \rangle v \big\} \\ &= \big\{ \langle \nabla, v \rangle - {}^t(\nabla \otimes v) \big\} (w^\mu \wedge w^\nu) \\ &= -{}^t(\nabla \otimes v)(w^\mu \wedge w^\nu). \end{aligned}$$

For every fixed $x \in \mathbb{R}^3$, standard estimates for solutions of ordinary differential equations then give

$$\big| (w^\mu \wedge w^\nu)(t, \Psi_t(x)) \big| \ge \mathrm{e}^{-V(t)} \big| (w^{\mu,0} \wedge w^{\nu,0})(x) \big|,$$

and by summing such estimates, we get

$$\big[W(t)\big]^4 \big(\Psi_t(x)\big) \ge \mathrm{e}^{-2V(t)}\, [W^0]^4(x).$$

Since this is true for every $x \in \mathbb{R}^3$, we have obtained

$$\big\| [W(t)]^{-1} \big\|_{L^\infty} \le \mathrm{e}^{V(t)/2} \big\| [W^0]^{-1} \big\|_{L^\infty}.$$

Finally, the last estimate of the statement follows from this one and that of PROPOSITION 4.1 since

$$\frac{\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}} = \sum_\nu \Big\{ 1 + \big\| [W]^{-1} \big\|_{L^\infty} \Big(\|w^\nu\|_r + \frac{\|\langle \nabla, w^\nu \otimes \omega \rangle\|_{r-1}}{\|\omega\|_{L^\infty}} \Big) \Big\}$$

by definition of $\|\omega\|_{r,W}$. $\square$

We end this section with a second corollary that provides an estimate of the lifetime of the solutions constructed in THEOREM 2.9. As in PROPOSITION 3.3, we set

$$\|\omega\|_{q,r,W} = \|\omega\|_{L^q} + \|\omega\|_{L^\infty} \operatorname{Log}\Big(2 + \frac{\|\omega\|_{r,W}}{\|\omega\|_{L^\infty}}\Big),$$

and we can state :

COROLLARY 4.4. — *There are constants C_0 and C_1 depending only on $1 \le q < 3$ and $0 < r < 1$ such that for all data $v^0 \in C^{1+r} \cap L^p$, $1 < p < \infty$, satisfying $\langle \nabla, v^0 \rangle = 0$ and $\omega^0 = \nabla \wedge v^0 \in L^q$, and all admissible system W^0 of C^r vector fields, the incompressible Euler system has a (unique) solution $v \in L^\infty([0,T]\,; C^{1+r}) \cap \operatorname{Lip}([0,T]\,; L^p)$ with*

$$T = \frac{1}{C_0\|\omega^0\|_{q,r,W^0}} \quad \text{and} \quad \big\|v(t)\big\|_{\mathrm{Lip}} \le C_0C_1\|\omega^0\|_{q,r,W^0} \quad \text{for } t \in [0,T].$$

Proof. — From THEOREM 2.9, we know that the incompressible Euler system has a solution defined in $[0,T_0]$ for some $T_0 > 0$. Then it follows from COROLLARY 2.5 that $\omega = \nabla \wedge v \in L^\infty([0,T_0]\,; L^q \cap C^r)$ with the estimates

$$\big\|\omega(t)\big\|_{L^q \text{ or } L^\infty} \le \mathrm{e}^{V(t)}\|\omega^0\|_{L^q \text{ or } L^\infty}$$

where $V(t) = \int_0^t \|v(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$. Therefore, using the results of PROPOSITION 3.3 and COROLLARY 4.3, we can write

$$\begin{aligned} V'(t) = \|v(t)\|_{\mathrm{Lip}} &\le C_1\big\|\omega(t)\big\|_{q,r,W(t)} \\ &\le C_1\,\mathrm{e}^{V(t)}\big\{\|\omega^0\|_{q,r,W^0} + (C_2 + \tfrac12)V(t)\,\|\omega^0\|_{L^\infty}\big\} \\ &\le C_1\,\mathrm{e}^{C_3V(t)}\|\omega^0\|_{q,r,W^0} \end{aligned}$$

where $C_3 = C_2 + \frac32$. If we also set

$$C_0 = 1 + C_1C_3, \quad T = \big(C_0\|\omega^0\|_{q,r,W^0}\big)^{-1},$$

this estimate implies that when $t \le T_0 < T$, then

$$\begin{aligned} &C_3V'(t)\ \mathrm{e}^{-C_3V(t)} \le C_1C_3\|\omega^0\|_{q,r,W^0} \\ &\quad\Longrightarrow\ 1 - \mathrm{e}^{-C_3V(t)} \le C_1C_3t\|\omega^0\|_{q,r,W^0} \\ &\quad\Longrightarrow\ \mathrm{e}^{C_3V(t)} \le \Big(1 - \frac{t}{T}\frac{C_1C_3}{C_0}\Big)^{-1} \le C_0 \\ &\quad\Longrightarrow\ V(t) \le \frac{\operatorname{Log} C_0}{C_3} \ \text{ and } \ V'(t) \le C_0C_1\|\omega^0\|_{q,r,W^0}. \end{aligned}$$

Therefore, the solution $v \in L^\infty([0,T_0]\,; C^{1+r}) \cap \operatorname{Lip}([0,T_0]\,; L^p)$ can be continued thanks to THEOREM 2.9 as long as $T_0 < T$, and the estimate of $\|v(t)\|_{\mathrm{Lip}} = V'(t)$ is proved. $\square$

5. Construction of solutions for data with striated vorticity

We are going to construct our solutions by taking limits of smooth solutions. These limits are obtained thanks to the following classic consequence of interpolation estimates in Hölder spaces.

LEMMA 5.1. — *Let v_n be a bounded sequence in $L^\infty([0,T]\,;C^s)$ that is a Cauchy sequence in $L^\infty([0,T]\,;C^t)$ for some $t < s$. Then this sequence has a limit $v \in L^\infty([0,T]\,;C^s)$ and the convergence holds in the sense of the norm of $L^\infty([0,T]\,;C^t)$ for all $t < s$. This lemma is still true when $s = 1$ and C^s is replaced with* Lip.

Our next statement is a typical application of this lemma.

PROPOSITION 5.2. — *With $1 < p < \infty$, let $v_n \in L^\infty([0,T]\,;C^{1+r}) \cap \mathrm{Lip}([0,T]\,;L^p)$ be a sequence of solutions of the incompressible Euler system for a sequence v_n^0 of data. We assume that the sequence v_n is bounded in $L^\infty([0,T]\,;\mathrm{Lip})$ and that v_n^0 tends in C^{r-1} to a limit $v^0 \in L^p$. Then the incompressible Euler system with data v^0 has a (unique) solution $v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$.*

Moreover, if w^0 is a C^r vector field and if $\omega_n^0 = \nabla \wedge v_n^0$ is such that the sequence $\langle \nabla, w^0 \otimes \omega_n^0 \rangle$ is bounded in C^{r-1}, then the L^∞ solution of the problem

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w \rangle = \langle w, \nabla \rangle v, \\ w_{|t=0} = w^0 \end{cases}$$

actually satisfies $w \in L^\infty([0,T]\,;C^r)$ and $\langle \nabla, w \otimes \omega \rangle \in L^\infty([0,T]\,;C^{r-1})$ with the same estimate as in Proposition 4.1 where $V(t)$ now stands for a common bound of $\int_0^t \|v_n(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$.

Finally, if W^0 is an admissible system of C^r vector fields and if ω_n^0 is bounded in C^{r,W^0}, then for all $t \in [0,T]$, the system $W(t)$ defined as in Corollary 4.3 is an admissible system of C^r vector fields, and $\omega(t) = \nabla \wedge v(t) \in C^{r,W(t)}$.

Proof. — Our proof will be divided in several steps.

(i) *Construction of the solution v.*

Thanks to THEOREM 2.7, we can consider the incompressible Euler system in its pressure free form. By subtraction, we can write

$$\begin{aligned} \partial_t(v_n - v_m) &+ \big\langle \nabla, v_n \otimes (v_n - v_m) \big\rangle \\ &= \Pi(v_n + v_m, v_n - v_m) + \big\langle \nabla, (v_m - v_n) \otimes v_m \big\rangle \end{aligned}$$

so that the quantity $u_{nm} = \Lambda^{-1}(v_n - v_m)$ satisfies

$$\begin{aligned}\partial_t u_{nm} &+ \langle \nabla, v_n \otimes u_{nm} \rangle \\ &= K(v_n, u_{nm}) + \Lambda^{-1}\Pi(v_n + v_m, \Lambda u_{nm}) - \Lambda^{-1} D(\Lambda u_{nm}, v_m).\end{aligned}$$

Thanks to PROPOSITION 2.6, the right side can be written $\mathcal{A}_{nm} u_{nm}$ where the linear operator $\mathcal{A}_{nm}$ satisfies

$$\|\mathcal{A}_{nm} u\|_r \le C(\|v_n\|_{\mathrm{Lip}} + \|v_m\|_{\mathrm{Lip}})\|u\|_r.$$

Now, the integrals $\int_0^t \|v_n(s)\|_{\mathrm{Lip}}\, \mathrm{d}s$ have a common bound $V(t)$ independent of n by assumption, and therefore it follows from COROLLARY 2.4 that

$$\big\|v_n(t) - v_m(t)\big\|_{r-1} = \big\|u_{nm}(t)\big\|_r \le \mathrm{e}^{(2C+r)V(t)} \|v_n^0 - v_m^0\|_{r-1}.$$

Since v_n^0 is a Cauchy sequence in C^{r-1}, this estimate implies that v_n is a Cauchy sequence in $L^\infty([0,T]\,; C^{r-1})$, and it follows from LEMMA 5.1 that v_n has a limit $v \in L^\infty([0,T]\,; \mathrm{Lip})$ and that the convergence holds in the sense of the norm of $L^\infty([0,T]\,; C^s)$ for all $s < 1$. Using this last property, we can take the limit in the equation

$$\partial_t v_n + \langle \nabla, v_n \otimes v_n \rangle = \Pi(v_n, v_n)$$

to see that v is a solution of the pressure free system with data v^0.

(ii) *A lemma on the flows associated with the sequence v_n.*

If Ψ_{nt} is the flow associated with v_n, we set

$$\Psi_n(t,x) = \big(t, \Psi_{nt}(x)\big).$$

Then, we have the following result.

LEMMA 5.3. — *Assume that v_n is a bounded sequence in $L^\infty([0,T]\,; \mathrm{Lip})$. Call $V(t)$ a common bound of $\int_0^t \|v_n(s)\|_{\mathrm{Lip}}\, \mathrm{d}s$ and set*

$$v_{nm}(t) = \int_0^t \|v_n(s) - v_m(s)\|_{L^\infty}\, \mathrm{d}s.$$

Then for all $w \in L^\infty([0,T]\,; C^r)$ we have

$$\big\|(w \circ \Psi_n) - (w \circ \Psi_m)\big\|_{r/2} \le 2\,\mathrm{e}^{rV} \|w\|_r\, v_{nm}^{r/2}.$$

In particular, this quantity tends to zero when m and n tend to infinity as soon as the sequence v_n is convergent in $L^\infty([0,T]\,; C^s)$ for some s.

Proof. — Since Ψ_{nt} is the solution of the ordinary differential equation $\partial_t \Psi_{nt} = v_n \circ \Psi_n$, $\Psi_{n0} = \mathrm{Id}$, we get the estimate $\|\Psi_n - \Psi_m\|_{L^\infty} \leq \mathrm{e}^V v_{nm}$ by standard arguments. Therefore we have

$$\begin{aligned}
&\big\|(w\circ\Psi_n) - (w\circ\Psi_m)\big\|_{r/2} \\
&\quad \leq \big\|(w\circ\Psi_n) - (w\circ\Psi_m)\big\|_r^{1/2}\big(2\,\big\|(w\circ\Psi_n) - (w\circ\Psi_m)\big\|_{L^\infty}\big)^{1/2} \\
&\quad \leq \big(2\|w\|_r \mathrm{e}^{rV}\big)^{1/2}\big(2\,\|w\|_r\|\Psi_n - \Psi_m\|_{L^\infty}^r\big)^{1/2} \leq 2\,\mathrm{e}^{rV}\|w\|_r v_{nm}^{r/2}
\end{aligned}$$

which is our conclusion. □

(iii) *Smoothness of w and $u = \Lambda^{-1}\langle\nabla, w\otimes\omega\rangle$.*

Using the C^r vector field w^0, we construct a sequence $w_n \in L^\infty([0,T]\,;C^r)$ by solving the problems

$$\begin{cases}
\partial_t w_n + \langle\nabla, v_n\otimes w_n\rangle = \langle w_n, \nabla\rangle v_n, \\
w_{n\,|\,t=0} = w^0
\end{cases}$$

as in PROPOSITION 4.1. Since v_n is bounded in $L^\infty([0,T]\,;\mathrm{Lip})$ and $u_n^0 = \Lambda^{-1}\langle\nabla, w^0\otimes\omega_n^0\rangle$ is bounded in C^r, it follows from PROPOSITION 4.1 that w_n and $u_n = \Lambda^{-1}\langle\nabla, w_n\otimes\omega_n\rangle$ *are bounded sequences in* $L^\infty([0,T]\,;C^r)$. Moreover, these sequences satisfy

$$\begin{cases}
\partial_t w_n + \langle\nabla, v_n\otimes w_n\rangle = \mathcal{A}_n w_n + \mathcal{B} u_n\,, \\
\partial_t u_n + \langle\nabla, v_n\otimes u_n\rangle = \mathcal{C}_n w_n + \mathcal{D}_n u_n\,, \\
w_{n\,|\,t=0} = w^0, \quad u_{n\,|\,t=0} = u_n^0\,,
\end{cases}$$

where $\mathcal{A}_n$, $\mathcal{C}_n$ and $\mathcal{D}_n$ are the operators $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{D}$ (see the proof of PROPOSITION 4.1) associated with v_n.

Thanks to PROPOSITION 2.3, the vector fields $w_n^* = w_n\circ\Psi_n$ and $u_n^* = u_n\circ\Psi_n$ are solutions of the system

$$\begin{cases}
\partial_t w_n^* = (\mathcal{A}_n w_n)\circ\Psi_n + (\mathcal{B} u_n)\circ\Psi_n\,, \\
\partial_t u_n^* = (\mathcal{C}_n w_n)\circ\Psi_n + (\mathcal{D}_n u_n)\circ\Psi_n\,, \\
w^*_{n\,|\,t=0} = w^0\,, \quad u^*_{n\,|\,t=0} = u_n^0\,,
\end{cases}$$

and *we claim that we also have*

$$\begin{cases}
\partial_t(w_n^* - w_m^*) = \mathcal{A}_n^*(w_n^* - w_m^*) + \mathcal{B}_n^*(u_n^* - u_m^*) + \varphi_{nm}\,, \\
\partial_t(u_n^* - u_m^*) = \mathcal{C}_n^*(w_n^* - w_m^*) + \mathcal{D}_n^*(u_n^* - u_m^*) + \psi_{nm}\,, \\
(w_n^* - w_m^*)_{|\,t=0} = 0, \quad (u_n^* - u_m^*)_{|\,t=0} = u_n^0 - u_m^0\,,
\end{cases}$$

where the linear operators $\mathcal{A}_n^*$, $\mathcal{B}_n^*$, $\mathcal{C}_n^*$ and $\mathcal{D}_n^*$ satisfy for some constant C independent of n

$$\|\mathcal{A}_n^* w\|_{r/2} + \|\mathcal{C}_n^* w\|_{r/2} \le C\|w\|_{r/2},$$
$$\|\mathcal{B}_n^* u\|_{r/2} + \|\mathcal{D}_n^* u\|_{r/2} \le C\|u\|_{r/2}$$

and where

$$\lim_{\min(m,n)\to\infty} \big(\|\varphi_{nm}\|_{r/2} + \|\psi_{nm}\|_{r/2}\big) = 0.$$

Indeed, we can write by subtraction

$$\begin{aligned}
\partial_t(w_n^* - w_m^*) &= (\mathcal{A}_n w_n)\circ\Psi_n + (\mathcal{B}u_n)\circ\Psi_n \\
&\qquad\qquad - (\mathcal{A}_m w_m)\circ\Psi_m - (\mathcal{B}u_m)\circ\Psi_m \\
&= \big(\mathcal{A}_n\big\{(w_n^* - w_m^*)\circ\Psi_n^{-1}\big\}\big)\circ\Psi_n \\
&\qquad + \big(\mathcal{B}\big\{(u_n^* - u_m^*)\circ\Psi_n^{-1}\big\}\big)\circ\Psi_n \\
&\qquad\quad + \big(\mathcal{A}_n\big\{(w_m^*\circ\Psi_n^{-1}\big\}\big)\circ\Psi_n - (\mathcal{A}_m w_m)\circ\Psi_m \\
&\qquad\qquad + \big(\mathcal{B}\big\{u_m^*\circ\Psi_n^{-1}\big\}\big)\circ\Psi_n - (\mathcal{B}u_m)\circ\Psi_m \\
&= \mathcal{A}_n^*(w_n^* - w_m^*) + \mathcal{B}_n^*(u_n^* - u_m^*) + \varphi_{nm}^{\mathcal{A}} + \varphi_{nm}^{\mathcal{B}}
\end{aligned}$$

provided that we set

$$\begin{aligned}
\mathcal{A}_n^* w &= \big(\mathcal{A}_n\{w\circ\Psi_n^{-1}\}\big)\circ\Psi_n\,, \quad \mathcal{B}_n^* u = \big(\mathcal{B}\{u\circ\Psi_n^{-1}\}\big)\circ\Psi_n\,, \\
\varphi_{nm}^{\mathcal{A}} &= \big(\mathcal{A}_n\{w_m^*\circ\Psi_n^{-1}\}\big)\circ\Psi_n - (\mathcal{A}_m w_m)\circ\Psi_m\,, \\
\varphi_{nm}^{\mathcal{B}} &= \big(\mathcal{B}\{u_m^*\circ\Psi_n^{-1}\}\big)\circ\Psi_n - (\mathcal{B}u_m)\circ\Psi_m\,.
\end{aligned}$$

For $\mathcal{A}_n^*$, it follows from LEMMA 4.2 that we have the estimate

$$\begin{aligned}
\|\mathcal{A}_n^* w\|_{r/2} &\le \mathrm{e}^{rV/2}\|\mathcal{A}_n\{w\circ\Psi_n^{-1}\}\|_{r/2} \\
&\le C\mathrm{e}^{rV/2}\|v_n\|_{\mathrm{Lip}}\|w\circ\Psi_n^{-1}\|_{r/2} \le C\mathrm{e}^{rV}\|v_n\|_{\mathrm{Lip}}\|w\|_{r/2}\,,
\end{aligned}$$

and similarly

$$\|\mathcal{B}_n^* u\|_{r/2} \le C\mathrm{e}^{rV}\|u\|_{r/2}.$$

For $\varphi_{nm}^{\mathcal{A}}$ we can write

$$\begin{aligned}
\varphi_{nm}^{\mathcal{A}} = \big(\mathcal{A}_n\{w_m^*\circ\Psi_n^{-1}\}\big)\circ\Psi_n - \big(\mathcal{A}_n\{w_m^*\circ\Psi_n^{-1}\}\big)\circ\Psi_m \\
+ \big(\mathcal{A}_n\{(w_m^*\circ\Psi_n^{-1}) - w_m\}\big)\circ\Psi_m + \big(\{\mathcal{A}_n - \mathcal{A}_m\}w_m\big)\circ\Psi_m\,.
\end{aligned}$$

The quantity

$$\left\|\left(\mathcal{A}_n\{w_m^* \circ \Psi_n^{-1}\}\right) \circ \Psi_n - \left(\mathcal{A}_n\{w_m^* \circ \Psi_n^{-1}\}\right) \circ \Psi_m\right\|_{r/2}$$

tends to zero thanks to LEMMA 5.3 since the C^r norm of $\mathcal{A}_n\{w_m^* \circ \Psi_n^{-1}\}$ is bounded. The third term can be estimated in $C^{r/2}$ norm by

$$\begin{aligned} C\,\mathrm{e}^{rV/2}\|v_n\|_{\mathrm{Lip}}&\left\|(w_m^* \circ \Psi_n^{-1}) - w_m\right\|_{r/2} \\ &\leq C\,\mathrm{e}^{rV}\|v_n\|_{\mathrm{Lip}}\left\|(w_m \circ \Psi_m) - (w_m \circ \Psi_n)\right\|_{r/2} \end{aligned}$$

and again tends to zero thanks to LEMMA 5.3. Finally, the $C^{r/2}$ norm of the fourth term can be estimated by

$$\mathrm{e}^{rV/2}\left\|\{\mathcal{A}_n - \mathcal{A}_m\}w_m\right\|_{r/2} \leq C\,\mathrm{e}^{rV/2}\|v_n - v_m\|_{1-r/2}\|w_m\|_r$$

thanks to LEMMA 4.2, and this also tends to zero since v_n is convergent in $L^\infty([0,T]\,;C^{1-r/2})$. All these estimates prove our claim for $\mathcal{A}_n^*$, $\mathcal{B}_n^*$ and $\varphi_{nm}^{\mathcal{A}}$, and it is clear that the other estimates can be obtained along the same lines.

Since u_n^0 is bounded in C^r and is obviously convergent in $C^{-r/2}$, it is also convergent in $C^{r/2}$ by interpolation, and it follows from our claim and standard ordinary differential equation estimates that w_n^* and u_n^* are Cauchy sequences in $L^\infty([0,T]\,;C^{r/2})$. But this also implies that w_n and u_n are Cauchy sequences in $L^\infty([0,T]\,;C^{r/2})$ since

$$\begin{aligned} \|w_n - w_m\|_{r/2} &\leq \mathrm{e}^{rV/2}\|w_n^* - w_m \circ \Psi_n\|_{r/2} \\ &\leq \mathrm{e}^{rV/2}\|w_n^* - w_m^*\|_{r/2} \\ &\qquad + \mathrm{e}^{rV/2}\|(w_m \circ \Psi_m) - (w_m \circ \Psi_n)\|_{r/2} \end{aligned}$$

where the last term tends to zero thanks to LEMMA 5.3.

Finally, it follows from LEMMA 5.1 that w_n and u_n have limits w and $u \in L^\infty([0,T]\,;C^r)$ and that the convergence holds in $L^\infty([0,T]\,;C^s)$ for all $s < r$, and this allows us to take the limit in the equations $\partial_t w_n + \langle \nabla, v_n \otimes w_n\rangle = \langle w_n, \nabla\rangle v_n$ and $u_n = \Lambda^{-1}\langle \nabla, w_n \otimes \omega_n\rangle$. We also get the estimate for $\|w\|_r + (\|u\|_r/\|\omega\|_{L^\infty})$ by taking the limit of such estimates for w_n and u_n.

(iv) *Admissible systems.*

Using the previous arguments for each element $w^{\nu,0}$ of the admissible system W^0, we see that we only have to show that the system $W(t)$ is admissible, and this is proved by taking the limit in the estimate

$$\left\|[W_n(t)]^{-1}\right\|_{L^\infty} \leq \mathrm{e}^{V(t)/2}\left\|[W^0]^{-1}\right\|_{L^\infty}$$

which follows from COROLLARY 4.3, where $V(t)$ is a common bound of $\int_0^t \|v_n(s)\|_{\mathrm{Lip}}\,\mathrm{d}s$. □

Then we can prove our main result.

THEOREM 5.4. — *With $1 < p < \infty$, $1 \le q < 3$ and $0 < r < 1$, let W^0 be an admissible system of C^r vector fields, and v^0 be an L^p, divergence free data such that $\omega^0 = \nabla \wedge v^0 \in L^q \cap C^{r,W^0}$. Then the incompressible Euler system has a unique solution $v \in L^\infty([0,T]\,;\mathrm{Lip}) \cap \mathrm{Lip}([0,T]\,;L^p)$ for some $T > 0$. Moreover, for all $t \in [0,T]$, the system $W(t)$ defined as in Corollary 4.3 is an admissible system of C^r vector fields, and we have $\omega(t) = \nabla \wedge v(t) \in L^q \cap C^{r,W(t)}$. Finally, if v^0 has an axisymmetric structure (see Definition 2.10), then the previous results are true for all $T > 0$ (global existence and smoothness result).*

Proof. — Take a nonnegative function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\|\chi\|_{L^1} = 1$ and set

$$\chi_n(x) = n^3 \chi(nx).$$

Then the vector fields $v_n^0 = \chi_n * v^0$ satisfy $v_n^0 \in C^{1+r} \cap L^p$, $\langle \nabla, v_n^0 \rangle = 0$, $\omega_n^0 = \nabla \wedge v_n^0 = \chi_n * \omega^0 \in L^q$ and $\lim_{n\to\infty} \|v_n^0 - v^0\|_s = 0$ for all $s < 1$ since $v^0 \in \mathrm{Lip}$ thanks to PROPOSITION 3.3.

Moreover, since $\omega_n^0 = \chi_n * \omega^0$, we have $\|\omega_n^0\|_{L^q \text{ or } L^\infty} \le \|\omega^0\|_{L^q \text{ or } L^\infty}$. Now, from the elementary estimate

$$\left\| w^{\nu,0} \otimes (\chi_n * \omega^0) - \chi_n * (w^{\nu,0} \otimes \omega^0) \right\|_r \le C \|w^{\nu,0}\|_r \|\omega^0\|_{L^\infty}$$

we can deduce that ω_n^0 is a bounded sequence in $L^q \cap C^{r,W^0}$. Then it follows from COROLLARY 4.4 that the incompressible Euler system with data v_n^0 has a solution $v_n \in L^\infty([0,T]\,;C^{1+r}) \cap \mathrm{Lip}([0,T]\,;L^p)$ where the lifetime $T > 0$ is independent of n, and that the sequence v_n is bounded in $L^\infty([0,T]\,;\mathrm{Lip})$. Therefore the result follows from PROPOSITION 5.2.

When v^0 has an axisymmetric structure, we need a lemma.

LEMMA 5.5. — *There exists a nonnegative function $\chi \in C_0^\infty(\mathbb{R}^3)$ with $\|\chi\|_{L^1} = 1$ such that for all $v \in \mathrm{Lip} \cap L^p$, $p < \infty$, with an axisymmetric structure, the vector fields $v_n = \chi_n * v$, where $\chi_n(x) = n^3\chi(nx)$, have an axisymmetric structure and satisfy, with $|x'| = (x_1^2 + x_2^2)^{1/2}$,*

$$\big\| |x|_{|\operatorname{supp} \omega_n} \big\|_{L^\infty} + \left\| \frac{\omega_n}{|x'|} \right\|_{L^1} + \left\| \frac{\omega_n}{|x'|} \right\|_{L^\infty} \le C$$

for some constant C independent of n.

Proof. — We choose $\chi(x) = \|\varphi\|_{L^1}^{-1} \varphi(x)$ where $\varphi(x) = f(|x|^2 - 1)$ and

$$f(t) = \begin{cases} \mathrm{e}^{1/t} & \text{on } \mathbb{R}_-, \\ 0 & \text{on } \mathbb{R}_+. \end{cases}$$

This function $\chi \in C_0^\infty(\mathbb{R}^3)$ is nonnegative and satisfies

$$\begin{cases} \|\chi\|_{L^1} = 1, \\ |x| \le |y| \;\Rightarrow\; \chi(x) \ge \chi(y)\,, \\ \partial_j \psi(x) = -2x_j \chi(x) \end{cases}$$

where $\psi \in C_0^\infty(\mathbb{R}^3)$ is defined by

$$\psi(x) = \|\varphi\|_{L^1}^{-1} g\big(|x|^2 - 1\big) \quad \text{and} \quad g(t) = \int_t^\infty f(s)\,\mathrm{d}s.$$

Now, assume that v has an axisymmetric structure, and let us show that $v_n = \chi_n * v$ also has an axisymmetric structure. The conditions on v_n (see definition 2.10) are obtained as follows :

$$\begin{aligned} &x_2(\chi_n * v_1) - x_1(\chi_n * v_2) \\ &\quad = (x_2\chi_n) * v_1 + \chi_n * (x_2 v_1) - (x_1\chi_n) * v_2 - \chi_n * (x_1 v_2) \\ &\quad = \chi_n * (x_2 v_1 - x_1 v_2) - \frac{\psi_n}{2n} * (\partial_2 v_1 - \partial_1 v_2) = 0, \end{aligned}$$

$$\begin{aligned} &\partial_2(\chi_n * v_1) - \partial_1(\chi_n * v_2) \\ &\quad = \chi_n * (\partial_2 v_1 - \partial_1 v_2) = 0, \end{aligned}$$

$$\begin{aligned} &(x_2\partial_1 - x_1\partial_2)(\chi_n * v_3) \\ &\quad = \partial_1\big\{x_2(\chi_n * v_3)\big\} - \partial_2\big\{x_1(\chi_n * v_3)\big\} \\ &\quad = \partial_1(x_2\chi_n) * v_3 + \chi_n * \partial_1(x_2 v_3) - \partial_2(x_1\chi_n) * v_3 - \chi_n * \partial_2(x_1 v_3) \\ &\quad = \big\{(x_2\partial_1 - x_1\partial_2)\chi_n\big\} * v_3 + \chi_n * \big\{(x_2\partial_1 - x_1\partial_2) v_3\big\} = 0 \end{aligned}$$

since $(x_2\partial_1 - x_1\partial_2)\chi_n = 0$ by symmetry.

Next, we show that $\omega_n/|x'|$ is bounded. When $|x'| \ge 1/n$, then

$$\omega_n(x) = \int \omega(y)\,\chi_n(x-y)\,\mathrm{d}y = \int_{|y'| \le 2|x'|} \omega(y)\,\chi_n(x-y)\,\mathrm{d}y$$

and therefore we have

$$\Big|\frac{\omega_n(x)}{|x'|}\Big| \le \int_{|y'| \le 2|x'|} \frac{|\omega(y)|}{|y'|}\,\frac{|y'|}{|x'|}\,\chi_n(x-y)\,\mathrm{d}y \le 2\Big\|\frac{\omega}{|x'|}\Big\|_{L^\infty}.$$

On the other hand, at a point x such that $x' = 0$ we have

$$\omega_n(x) = \int \omega(y)\,\chi_n(x-y)\,\mathrm{d}y = 0$$

by symmetry, and therefore we can write for $|x'| \leq 1/n$,

$$\begin{aligned}\Big|\frac{\omega_n(x)}{|x'|}\Big| &\leq \big\|\nabla \otimes \omega_n{}_{||x'|\leq \frac{1}{n}}\big\|_{L^\infty}\\ &\leq \|\nabla\chi_n\|_{L^1}\big\|\omega_{||x'|\leq \frac{2}{n}}\big\|_{L^\infty}\\ &\leq \big(n\|\nabla\chi\|_{L^1}\big)\Big(\frac{2}{n}\Big\|\frac{\omega}{|x'|}\Big\|_{L^\infty}\Big).\end{aligned}$$

Thanks to these two estimates, we have proved

$$\Big\|\frac{\omega_n}{|x'|}\Big\|_{L^\infty} \leq 2\|\nabla\chi\|_{L^1}\Big\|\frac{\omega}{|x'|}\Big\|_{L^\infty}$$

which is a bound *independent of* n.

Then, we mention that ω_n is compactly supported since $\operatorname{supp}\omega_n \subset \operatorname{supp}\omega + \operatorname{supp}\chi_n$, and this gives

$$R_n = \big\||x|_{|\operatorname{supp}\omega_n}\big\|_{L^\infty} \leq \big\||x|_{|\operatorname{supp}\omega}\big\|_{L^\infty} + \frac{1}{n} = R + \frac{1}{n} \leq R+1$$

which is again a bound *independent of* n. This computation also provides a bound *independent of* n for $\|\omega_n/|x'|\|_{L^1}$ since

$$\Big\|\frac{\omega_n}{|x'|}\Big\|_{L^1} \leq \tfrac{4}{3}\pi R_n^3\Big\|\frac{\omega_n}{|x'|}\Big\|_{L^\infty} \leq \tfrac{8}{3}\pi(R+1)^3\,\|\nabla\chi\|_{L^1}\Big\|\frac{\omega}{|x'|}\Big\|_{L^\infty}.$$

Finally, we have to show that if

$$x_2\omega_1(x) - x_1\omega_2(x) \geq 0$$

for all x, then this is also true for ω_n. For this, we set $\bar{x} = (-x', x_3)$ for all $x = (x', x_3) \in \mathbb{R}^3$. Since $|x-y|^2 = |x'-y'|^2 + (x_3-y_3)^2$ while $|x-\bar{y}|^2 = |x'+y'|^2 + (x_3-y_3)^2$, it is clear that $|x-y| \leq |x-\bar{y}|$ if and only if $\langle x', y'\rangle \geq 0$. Now

$$\begin{aligned}x_2(\chi_n * \omega_1)(x) - x_1(\chi_n * \omega_2)(x) &= \int \big(x_2\omega_1(y) - x_1\omega_2(y)\big)\chi_n(x-y)\,\mathrm{d}y\\ &= \int \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\,\chi_n(x-y)\,\mathrm{d}y.\end{aligned}$$

This integral is over the domain $D_n(x) = \big\{y; |x-y| \leq 1/n\big\}$ that we divide in three parts :

$$\begin{aligned}D_n^1(x) &= \Big\{y\,;\ |x-y| \leq \frac{1}{n} \leq |x-\bar{y}|\Big\},\\ D_n^2(x) &= \Big\{y\,;\ |x-y| \leq \frac{1}{n}, |x-\bar{y}| \leq \frac{1}{n} \text{ and } \langle x', y'\rangle \geq 0\Big\},\\ D_n^3(x) &= \Big\{y\,;\ |x-y| \leq \frac{1}{n}, |x-\bar{y}| \leq \frac{1}{n} \text{ and } \langle x', y'\rangle \leq 0\Big\}.\end{aligned}$$

On $D_n^3(x)$ we can use the symmetry $\omega(\bar{y}) = -\omega(y)$ and the change of variables $y \mapsto \bar{y}$ to get

$$\int_{D_n^3(x)} \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\, \chi_n(x-y)\, \mathrm{d}y = -\int_{D_n^2(x)} \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\, \chi_n(x-\bar{y})\, \mathrm{d}y$$

so that we have

$$\begin{aligned}\int \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\, \chi_n(x-y)\, \mathrm{d}y \\ &= \int_{D_n^1(x)} \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\, \chi_n(x-y)\, \mathrm{d}y \\ &\quad + \int_{D_n^2(x)} \langle x', y'\rangle \frac{|\omega(y)|}{|y'|}\big(\chi_n(x-y) - \chi_n(x-\bar{y})\big)\, \mathrm{d}y,\end{aligned}$$

and finally these integrals are nonnegative since on $D_n^1(x)$

$$|x-y| \le |x-\bar{y}| \implies \langle x', y'\rangle \ge 0$$

while on $D_n^2(x)$

$$\langle x', y'\rangle \ge 0 \implies |x-y| \le |x-\bar{y}| \implies \chi_n(x-y) \ge \chi_n(x-\bar{y}).$$

This completes the proof of the lemma. □

End of the proof of Theorem 5.4. — When v^0 has an axisymmetric structure, we choose $v_n^0 = \chi_n * v^0$ as in LEMMA 5.5 and we obtain thus from COROLLARY 2.12 a sequence $v_n \in L^\infty([0,T]\,; C^{1+r})$ of solutions of the pressure free system with data v_n^0 *for all* $T > 0$. Therefore all we need to prove, before using the result of PROPOSITION 5.2, is that this sequence v_n is bounded in $L^\infty([0,T]\,; \mathrm{Lip})$ for all $T > 0$.

From PROPOSITION 3.3 and COROLLARY 4.3, we know that the functions $V_n(t) = \int_0^t \|v_n(s)\|_{\mathrm{Lip}}\, \mathrm{d}s$ satisfy

$$\begin{aligned}V_n'(t) &= \big\|v_n(t)\big\|_{\mathrm{Lip}} \\ &\le C_1\Big[\|\omega_n(t)\|_{L^1} + \big\|\omega_n(t)\big\|_{L^\infty}\Big\{\mathrm{Log}\Big(2 + \frac{\|\omega_n^0\|_{r,W^0}}{\|\omega_n^0\|_{L^\infty}}\Big) + \big(C_2 + \tfrac{1}{2}\big)V_n(t)\Big\}\Big].\end{aligned}$$

On the other hand, it follows from LEMMAS 2.11 and 5.5 that

$$\big\|\omega_n(t)\big\|_{L^1} \le C_3(1+C_3 t)^{3/2} \quad \text{and} \quad \big\|\omega_n(t)\big\|_{L^\infty} \le C_3(1+C_3 t)^{3/2}$$

for some constant C_3 independent of n. Therefore, again for some constants C_4 then C_5 independent of n, we have

$$\begin{aligned} V_n'(t) &\le C_4(1+C_4t)^{3/2}\big(1+V_n(t)\big) \\ &\implies 1+V_n(t) \le \exp \tfrac{2}{5}(1+C_4t)^{5/2} \\ &\implies \big\|v_n(t)\big\|_{\mathrm{Lip}} \le C_5\,\mathrm{e}^{C_5t^{5/2}}, \end{aligned}$$

and this implies that v_n is bounded in $L^\infty([0,T]\,;\mathrm{Lip})$ for all $T>0$ as required. □

6. Other smoothness results

The existence result stated in Theorem 1.2 directly follows from Proposition 3.2 and Theorem 5.4. As a first complement, we establish the smoothness of the submanifold $\Sigma(t)$ described in Theorem 1.2.

Theorem 6.1. — *Under the assumptions of Theorem 1.2, let us set $\Sigma(t) = \Psi_t(\Sigma^0)$ where Ψ_t is the flow associated with the velocity field $v\in L^\infty([0,T]\,;\mathrm{Lip})$ constructed in Theorem 5.4. Then for all $t\in[0,T]$, $\Sigma(t)$ is a C^{1+r}, two dimensional, compact submanifold of $\mathbb{R}^3$.*

Proof. — Let $f\in C^{1+r}$ be such that $f_{|\Sigma^0}=0$ and $\nabla f_{|\Sigma^0}\neq 0$. Then, an equation of $\Sigma(t)$ is $\varphi(t,x)=0$ where φ is the solution of the problem

$$\begin{cases} \partial_t\varphi + \langle\nabla, v\otimes\varphi\rangle = 0,\\ \varphi|_{t=0} = f, \end{cases}$$

and we just have to prove that $\varphi\in L^\infty([0,T]\,;C^{1+r})$ near $\Sigma(t)$.

Thanks to Corollary 2.4, we already know that $\varphi\in L^\infty([0,T]\,;C^s)$ for all $s<1$. By taking the gradient of the equation, we find that $\nabla\varphi$ is a solution of the problem

$$\begin{cases} \partial_t\nabla\varphi + \langle\nabla, v\otimes\nabla\varphi\rangle + {}^t(\nabla\otimes v)\nabla\varphi = 0\\ \nabla\varphi_{|t=0} = \nabla f, \end{cases}$$

and again Corollary 2.4 shows that $\nabla\varphi\in L^\infty([0,T]\,;L^\infty)$.

As in Proposition 3.2, we now introduce the three C^r vector fields

$$w^{1,0} = \begin{pmatrix} 0\\ -\partial_3 f\\ \partial_2 f\end{pmatrix},\quad w^{2,0} = \begin{pmatrix} \partial_3 f\\ 0\\ -\partial_1 f\end{pmatrix},\quad w^{3,0} = \begin{pmatrix} -\partial_2 f\\ \partial_1 f\\ 0\end{pmatrix};$$

they are divergence free and tangent to Σ^0, and therefore the assumption $\omega^0 \in C^{r,\Sigma^0}$ implies that $\langle \nabla, w^{\nu,0} \otimes \omega^0 \rangle \in C^{r-1}$. Then it follows from THEOREM 5.4 that the corresponding $w^\nu(t)$ satisfy $w^\nu \in L^\infty([0,T]\,;C^r)$. Moreover, if (λ,μ,ν) is any circular permutation of $(1,2,3)$, the vector field $u^\lambda = w^\mu \wedge w^\nu$ is a $L^\infty([0,T]\,;C^r)$ solution of the problem

$$\begin{cases} \partial_t u^\lambda + \langle \nabla, v \otimes u^\lambda \rangle = -{}^t(\nabla \otimes v)u^\lambda, \\ u^\lambda{}_{|t=0} = (\partial_\lambda f)\nabla f \end{cases}$$

(see the proof of COROLLARY 4.3). By using the transformation $u \mapsto u^*$ introduced in PROPOSITION 2.3, we can see that $\{(\partial_\lambda f) \circ \Psi^{-1}\}\nabla\varphi$ is a $L^\infty([0,T]\,;L^\infty)$ solution of that very same problem, and therefore we have

$$\{(\partial_\lambda f) \circ \Psi_t^{-1}\}\nabla\varphi(t) \equiv u^\lambda(t)$$

thanks to the uniqueness result of COROLLARY 2.4. Therefore we have

$$\nabla\varphi(t) = \sum_\lambda \frac{(\partial_\lambda f) \circ \Psi_t^{-1}}{|(\nabla f) \circ \Psi_t^{-1}|^2} u^\lambda(t),$$

and finally $\varphi \in L^\infty([0,T]\,;C^{1+r})$ follows from the fact that the right side is a $L^\infty([0,T]\,;C^r)$ vector field (at least near $\Sigma(t)$ where $(\nabla f) \circ \Psi_t^{-1}$ does not vanish). $\square$

To complete the proof of THEOREM 1.2, we just have to establish that $\omega(t) \in L^q \cap C^{r,\Sigma(t)}$ for all $t \in [0,T]$: this is our THEOREM 6.4 below for which we need some preparation. In our next statements, the constants C may depend on various quantities such as the time, *but they are always independent of* $0 < \varepsilon < 1$.

LEMMA 6.2. — *Let* v^0 *be a data such as in Theorem* 1.2, *and* v *in* $L^\infty([0,T]\,;\mathrm{Lip})$ *be the corresponding solution constructed in Theorem* 5.4. *Then for all* $0 < \varepsilon < 1$ *and all vector fields* $w^0 \in C^r$ *vanishing in* Σ^0_ε, *the* L^∞ *solution* w *of the problem*

$$\begin{cases} \partial_t w + \langle \nabla, v \otimes w \rangle = \langle w, \nabla \rangle v, \\ w_{|t=0} = w^0 \end{cases}$$

satisfies $w \in L^\infty([0,T];C^r)$ *and* $\langle \nabla, w \otimes \omega \rangle \in L^\infty([0,T];C^{r-1})$ *with an estimate*

$$\|w(t)\|_r + \frac{\|\langle \nabla, w(t) \otimes \omega(t)\rangle\|_{r-1}}{\|\omega(t)\|_{L^\infty}} \le C\Big\{\|w^0\|_r + \frac{\|w^0\|_{L^\infty}}{\|\omega^0\|_{L^\infty}}\|\omega^0\|_{r(\mathbb{R}^3\setminus\Sigma^0_{\varepsilon/2})}\Big\}.$$

Proof.—Let us define $v_n^0 = \chi_n * v^0$ as in the proof of THEOREM 5.4. Then the solution v constructed in THEOREM 5.4 is the limit of the solutions v_n with data v_n^0, and our lemma simply follows from PROPOSITION 5.2 since we can write for large n

$$\begin{aligned}\big\|\langle\nabla, w^0\otimes\omega_n^0\rangle\big\|_{r-1} &\le \|w^0\otimes\omega_n^0\|_r\\ &\le \|w^0\|_r\,\|\omega_n^0\|_{L^\infty} + \|w^0\|_{L^\infty}\,\|\omega_n^0\|_{r(\mathbb{R}^3\setminus\Sigma_\varepsilon^0)}\\ &\le \|w^0\|_r\,\|\omega^0\|_{L^\infty} + \|w^0\|_{L^\infty}\,\|\omega^0\|_{r(\mathbb{R}^3\setminus\Sigma_{\varepsilon/2}^0)}.\end{aligned}$$

Our proof is complete. □

PROPOSITION 6.3. — *Let v^0 be a data such as in Theorem 1.2, and $v\in L^\infty([0,T]\,;\mathrm{Lip})$ be the corresponding solution constructed in Theorem 5.4. Then, for all vector fields $w^T\in C^r$, the problem*

$$\begin{cases}\partial_t w + \langle\nabla, v\otimes w\rangle = \langle w,\nabla\rangle v,\\ w_{|t=T} = w^T\end{cases}$$

has a unique solution $w\in L^\infty([0,T];L^\infty)$ and we have

$$\big\|w(0)\big\|_{r(\mathbb{R}^3\setminus\Sigma_\varepsilon^0)} \le C\varepsilon^{-r}$$

for all $0<\varepsilon<1$.

Proof.—The existence and uniqueness of a solution $w\in L^\infty([0,T]\,;L^\infty)$ easily follow from COROLLARY 2.4.

For any $0<\varepsilon<1$, let $\chi\in C^\infty$ and $\varphi\in C^\infty$ satisfy

$$\chi=\begin{cases}0 & \text{in } \Sigma_{\varepsilon/2}^0,\\ 1 & \text{in } \mathbb{R}^3\setminus\Sigma_\varepsilon^0,\end{cases}\qquad \|\chi\|_r\le C\varepsilon^{-r}\,;$$

$$\varphi=\begin{cases}0 & \text{in } \Sigma_{\varepsilon/4}^0,\\ 1 & \text{in } \mathbb{R}^3\setminus\Sigma_{\varepsilon/2}^0,\end{cases}\qquad \|\varphi\|_r\le C\varepsilon^{-r}.$$

We set

$$w^{1,0}=\begin{pmatrix}\varphi\\0\\0\end{pmatrix},\quad w^{2,0}=\begin{pmatrix}0\\\varphi\\0\end{pmatrix},\quad w^{3,0}=\begin{pmatrix}0\\0\\\varphi\end{pmatrix},\quad u^T=(\chi\circ\Psi_T^{-1})w^T$$

where Ψ_t is the flow associated with v. The vector fields $w^{\nu,0}$ vanish in $\Sigma_{\varepsilon/4}^0$ so that we can use the result of LEMMA 6.2 and the assumption $\omega^0\in C^{r,\Sigma^0}$

(see DEFINITION 1.1) to get three vector fields $w^\nu \in L^\infty([0,T];C^r)$ with an estimate

$$\|w^\nu(T)\|_r \leq C\{\|\varphi\|_r + \|\omega^0\|_{r(\mathbb{R}^3\setminus\Sigma^0_{\varepsilon/8})}\} \leq C\varepsilon^{-r}.$$

The matrix $W(t) = (w^1(t), w^2(t), w^3(t))$ satisfies

$$\begin{cases} \partial_t(\det W) + \langle\nabla, v \otimes (\det W)\rangle = 0, \\ \det W_{|t=0} = \det W(0) \end{cases}$$

(see the proof of COROLLARY 4.3), and since $W(0) \equiv \mathrm{Id}$ in $\mathbb{R}^3 \setminus \Sigma^0_{\varepsilon/2}$, we have $\det W(T) \equiv 1$ in $\mathbb{R}^3 \setminus \Psi_T(\Sigma^0_{\varepsilon/2})$. The vector field u^T is supported in $\mathbb{R}^3 \setminus \Psi_T(\Sigma^0_{\varepsilon/2})$ and therefore, the vector field $W(T)^{-1}u^T$ has C^r components with the estimates

$$\|W(T)^{-1}u^T\|_{L^\infty} \leq C, \quad \|W(T)^{-1}u^T\|_r \leq C(\|\chi\|_r + \|W(T)\|_r) \leq C\varepsilon^{-r}.$$

Using the transformation $u \mapsto u^*$ considered in PROPOSITION 2.3, we can see that the $L^\infty([0,T];C^r)$ vector field

$$u(t) = W(t)(\{W(T)^{-1}u^T\} \circ \Psi_T \circ \Psi_t^{-1})$$

is the solution of the problem

$$\begin{cases} \partial_t u + \langle\nabla, v \otimes u\rangle = \langle u, \nabla\rangle v, \\ u_{|t=T} = u^T \end{cases}$$

so that we have $u(t) = (\chi \circ \Psi_t^{-1})w(t)$. For $t = 0$, this gives

$$\chi w(0) = u(0) = W(0)(\{W(T)^{-1}u^T\} \circ \Psi_T)$$

which clearly satisfies

$$\begin{aligned} \|\chi w(0)\|_r \leq \|W(0)\|_r \|W(T)^{-1}u^T\|_{L^\infty} \\ + \|W(0)\|_{L^\infty} \|W(T)^{-1}u^T\|_r \|\nabla\Psi_T\|^r_{L^\infty} \\ \leq C(\|\varphi\|_r + \|W(T)^{-1}u^T\|_r) \leq C\varepsilon^{-r}, \end{aligned}$$

and this proves our result since $\chi = 1$ in $\mathbb{R}^3 \setminus \Sigma^0_\varepsilon$. □

THEOREM 6.4. — *Let v^0 be a data such as in Theorem* 1.2, *and v in $L^\infty([0,T]\,;\mathrm{Lip})$ be the corresponding solution constructed in Theorem* 5.4. *Then we have $\omega(t) \in L^q \cap C^{r,\Sigma(t)}$ for all $t \in [0,T]$ where $\Sigma(t)$ is the submanifold considered in Theorem* 6.1.

Proof. — It already follows from COROLLARY 2.5 that ω is an element of $L^\infty([0,T]\,;L^q \cap L^\infty)$. Therefore, we just have to prove that $\langle \nabla, w^t \otimes \omega(t)\rangle$ belongs to C^{r-1} for all C^r, divergence free vector fields w^t tangent to $\Sigma(t)$, and that, for all $0<\varepsilon<1$,

$$\|\omega(t)\|_{r(\mathbb{R}^3\setminus\Sigma(t)_\varepsilon)} \le C\varepsilon^{-r}.$$

(i) *The estimate in* $\mathbb{R}^3 \setminus \Sigma(t)_\varepsilon$.

For a fixed $t \in [0,T]$, we choose a $\chi \in C^\infty$ satisfying $\chi = 0$ in $\Sigma(t)_{\varepsilon/2}$ and $\chi = 1$ in $\mathbb{R}^3 \setminus \Sigma(t)_\varepsilon$, then we set

$$w^{1,t} = \begin{pmatrix}\chi\\0\\0\end{pmatrix}, \quad w^{2,t} = \begin{pmatrix}0\\\chi\\0\end{pmatrix}, \quad w^{3,t} = \begin{pmatrix}0\\0\\\chi\end{pmatrix}.$$

Using the result of PROPOSITION 6.3, the corresponding $w^\nu(0)$ satisfy $w^\nu(0) = 0$ in $\Psi_t^{-1}(\Sigma(t)_{\varepsilon/2}) \supset \Sigma^0_{\varepsilon'/2}$, $\varepsilon' = \varepsilon\,\mathrm{e}^{-V(t)}$, and

$$\|w^\nu(0)\|_r = \|w^\nu(0)\|_{r(\mathbb{R}^3\setminus\Sigma^0_{\varepsilon'/2})} \le C\varepsilon^{-r}.$$

Thanks to LEMMA 6.2, it follows that

$$\|\partial_\nu(\chi\omega(t))\|_{r-1} = \|\langle\nabla, w^{\nu,t}\otimes\omega(t)\rangle\|_{r-1} \le C\,\varepsilon^{-r},$$

and this gives $\chi\omega(t) \in C^r$ with $\|\chi\omega(t)\|_r \le C\varepsilon^{-r}$, which is the required estimate since $\chi = 1$ in $\mathbb{R}^3 \setminus \Sigma(t)_\varepsilon$.

(ii) *The differentiation in the direction of any tangent vector field.*

Again for a fixed $t \in [0,T]$, we consider the C^r vector field $w^{0,t} = \nabla\varphi(t)$ constructed in the proof of theorem 6.1, and the C^r values $\bar w^{1,t}$, $\bar w^{2,t}$ and $\bar w^{3,t}$ at time t of the vector fields with C^r initial values

$$\bar w^{1,0} = \begin{pmatrix}0\\-\partial_3 f\\\partial_2 f\end{pmatrix}, \quad \bar w^{2,0} = \begin{pmatrix}\partial_3 f\\0\\-\partial_1 f\end{pmatrix}, \quad \bar w^{3,0} = \begin{pmatrix}-\partial_2 f\\\partial_1 f\\0\end{pmatrix}$$

as in the proof of PROPOSITION 3.2. These four vector fields generate $\mathbb{R}^3$ at any point of a neighborhood of $\Sigma(t)$, and we can complete this system with the vector fields

$$w^{1,t} = \begin{pmatrix} \chi \\ 0 \\ 0 \end{pmatrix}, \quad w^{2,t} = \begin{pmatrix} 0 \\ \chi \\ 0 \end{pmatrix}, \quad w^{3,t} = \begin{pmatrix} 0 \\ 0 \\ \chi \end{pmatrix},$$

where $\chi = 1$ outside this neighborhood and $\chi = 0$ near $\Sigma(t)$: now $\mathbb{R}^3$ is generated at any point of $\mathbb{R}^3$ by this system of seven C^r vector fields. The initial values $w^{1,0}$, $w^{2,0}$ and $w^{3,0}$ corresponding to $w^{1,t}$, $w^{2,t}$ and $w^{3,t}$ have C^r components thanks to the arguments given in part (i) above, but the initial value $w^{0,0} \in L^\infty$ corresponding to $w^{0,t} = \nabla\varphi(t)$ only satisfies

$$\|w^{0,0}\|_{r(\mathbb{R}^3 \setminus \Sigma^0_\varepsilon)} \leq C\varepsilon^{-r}$$

thanks to PROPOSITION 6.3.

If we now take a C^r, divergence free vector field w^t tangent to $\Sigma(t)$, we can find C^r coefficients a_ν and b_ν such that

$$w^t = a_1\bar{w}^{1,t} + a_2\bar{w}^{2,t} + a_3\bar{w}^{3,t} + b_0 w^{0,t} + b_1 w^{1,t} + b_2 w^{2,t} + b_3 w^{3,t}.$$

Using the transformation $w \mapsto w^*$ considered in PROPOSITION 2.3, it is easy to see that the corresponding initial value of this vector field is then

$$w^0 = \sum_\nu (a_\nu \circ \Psi_t)\bar{w}^{\nu,0} + \sum_\nu (b_\nu \circ \Psi_t) w^{\nu,0}.$$

This vector field w^0 has C^r components. Indeed, it is clear that

$$(a_\nu \circ \Psi_t)\bar{w}^{\nu,0} + (b_\nu \circ \Psi_t) w^{\nu,0} \in C^r \quad \text{for } \nu \geq 1,$$

and this is also true for $(b_0 \circ \Psi_t) w^{0,0}$ thanks to the following arguments : since $w^{0,t}$ is the only non tangent vector field in our system, we have $b_0 = 0$ on $\Sigma(t)$ and this implies that the C^r coefficient $b_0 \circ \Psi_t$ vanishes on Σ^0 ; therefore, if $0 \leq 2\varepsilon = \operatorname{dist}(x, \Sigma^0) \leq \operatorname{dist}(y, \Sigma^0)$, we have

$$\begin{aligned}
&\big|(b_0 \circ \Psi_t) w^{0,0}(x) - (b_0 \circ \Psi_t) w^{0,0}(y)\big| \\
&\qquad \leq \big|(b_0 \circ \Psi_t)(x)\big| \times \big|w^{0,0}(x) - w^{0,0}(y)\big| \\
&\qquad\qquad\qquad + \big|(b_0 \circ \Psi_t)(x) - (b_0 \circ \Psi_t)(y)\big| \times \big|w^{0,0}(y)\big| \\
&\qquad \leq \big(\|b_0 \circ \Psi_t\|_r 2^r \varepsilon^r\big)\big(\|w^{0,0}\|_{r(\mathbb{R}^3 \setminus \Sigma^0_\varepsilon)} |x - y|^r\big) \\
&\qquad\qquad\qquad + \big(\|b_0 \circ \Psi_t\|_r |x - y|^r\big) \|w^{0,0}\|_{L^\infty} \\
&\qquad \leq \|b_0 \circ \Psi_t\|_r \big(2^r C + \|w^{0,0}\|_{L^\infty}\big) |x - y|^r,
\end{aligned}$$

so that $(b_0 \circ \Psi_t) w^{0,0} \in C^r$.

Next, we observe that w^0 is divergence free and tangent to Σ^0. Indeed, when applying the operator $\Lambda^{-1}\langle\nabla,\cdot\rangle$ to the equation

$$\partial_t w + \langle\nabla, v\otimes w\rangle = \langle w,\nabla\rangle v,$$

we find that $u=\Lambda^{-1}\langle\nabla,w\rangle \in L^\infty([0,T]\,;C^r)$ satisfies

$$\partial_t u + \langle\nabla, v\otimes u\rangle = K(v,u),$$

where the operator K is continuous on C^r (see PROPOSITION 2.6 (ii)), and therefore $\langle\nabla,w^t\rangle = 0$ implies $\langle\nabla,w^0\rangle=0$. Similarly, the scalar distribution $u=\langle w,\nabla\rangle\varphi \in L^\infty([0,T]\,;C^r)$, where $\varphi(t)$ is the equation of $\Sigma(t)$ constructed in the proof of THEOREM 6.1, is a solution of

$$\partial_t u + \langle\nabla, v\otimes u\rangle = 0$$

(see the proof of THEOREM 6.1), and therefore w^t tangent to $\Sigma(t)$ implies w^0 tangent to Σ^0.

Finally, the result

$$\big\langle\nabla, w^t\otimes\omega(t)\big\rangle \in C^{r-1}$$

simply follows from THEOREM 5.4 since we can include the C^r, divergence free vector field w^0 tangent to Σ^0 in the admissible system W^0. □

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