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MICROLOCALISATION OF $\mathcal{D}$-MODULES ALONG A SUBMANIFOLD

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RÉSUMÉ. — Soit X une variété analytique complexe. Dans [K-S1], Kashiwara et Schapira ont défini et étudié un bifoncteur μhom dans $D^b(X)$ qui généralise le foncteur de microlocalisation de [SKK]. A peu près au même moment, Kashiwara et Kawai on introduit dans [K-K3] un bifoncteur dans la catégorie des systèmes holonomes réguliers. Si l'on se donne un couple $(\mathcal{M}, \mathcal{N})$ de tels systèmes, ce foncteur consiste à microlocaliser le produit formel $\mathcal{M} \underline{\boxtimes} \mathcal{N}$ le long de la diagonale de $X \times X$, considérée comme un $\mathcal{D}_{T^*X}$-module.

Le but principal de cet article est de mettre en rapport les fonctorialités de la spécialisation et de la microlocalisation ; nous montrons en particulier que le bifoncteur de [K-K3], que nous notons $\underline{\mu\text{hom}}$, est l'analogue du bifoncteur de [K-S1] via le foncteur de De Rham (Théorème 3.2) dans le cadre des $\mathcal{D}$-modules. Nous n'exigeons pas que $\mathcal{M}$ et $\mathcal{N}$ soient holonomes réguliers puisque la propriété essentielle de $\underline{\mu\text{hom}}(\mathcal{M}, \mathcal{N})$ est la régularité de $\mathcal{M} \underline{\boxtimes} \mathcal{N}$ le long de Δ.

ABSTRACT. — The analogues of specialisation and Fourier-Sato transform for sheaves were introduced in the framework of systems of holomorphic differential equations ($\mathcal{D}$-Modules) by Kashiwara, Hotta, Malgrange, Verdier, Brylinsky et al., with a special insight for regular holonomic systems.

With these tools we study a bifunctor on a category of $\mathcal{D}$-Modules which satisfy a regularity condition and prove that it is the analogue of the bifunctor μhom of Kashiwara-Schapira. This category is larger than that of regular holonomic systems.

Introduction

Let X be a complex analytic manifold. In [K-S1] Kashiwara-Schapira defined and studied a bifunctor μhom in $D^b(X)$ which generalised the microlocalisation functor of [SKK]. Around the same time, in [K-K3] Kashiwara-Kawai introduced a bifunctor in the category of regular holonomic systems. Given a pair of such systems $(\mathcal{M}, \mathcal{N})$, it consists of the

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microlocalisation of the formal product $\mathcal{M} \underline{\boxtimes} \mathcal{N}$ along the diagonal of $X \times X$ viewed as a $\mathcal{D}_{T^*X}$-module.

The main purpose of this paper is to study the functorial properties of specialisation and the microlocalisation relating both points of view; in particular we will show that the bifunctor of [K-K3] which we will denote by $\underline{\mu\text{hom}}$, is the analogue of the bifunctor of [K-S1], via the De Rham functor (THEOREM 3.2) in the framework of $\mathcal{D}$-modules. Here we will not ask $\mathcal{M}$, $\mathcal{N}$ to be regular holonomic, since the main point of the definition of $\underline{\mu\text{hom}}(\mathcal{M},\mathcal{N})$ is the regularity of $\mathcal{M} \underline{\boxtimes} \mathcal{N}$ along Δ.

Furthermore, a simple example of the non holonomic case is given.

So we will keep throughout this paper the regularity point of view. More precisely, we will stay in the following situation.

The manifold X is an n-dimensional complex manifold, Y is a d-codimensional smooth submanifold. The functor of microlocalisation is defined in the category of specialisable $\mathcal{D}$-modules along Y (noted B_Y). The category R_Y is the subcategory of B_Y of regular modules along Y, defined by Kashiwara in [K2].

Given $\mathcal{M}$ in B_Y, one defines with [K2] a $\mathcal{D}_{T_YX}$-module $\underline{\nu_Y}(\mathcal{M})$ which satisfies the fundamental relation $\text{Sol}(\underline{\nu_Y}(\mathcal{M})) \simeq \nu_Y(\text{Sol}(\mathcal{M}))$. The ν_Y in the right term denotes the geometrical specialisation in $D^b(X)$ (*cf.* [SKK], also [Ve1], [Ve2]).

The first section is devoted to the study of $\underline{\nu_Y}$ in the category B_Y and its further relations with ν_Y in $D^b(X)$. Although the 1-codimensional case has been thoroughly studied in [Me], [Sa], we found useful to develop here the higher codimensional case, and, as a main tool, we study the behaviour of specialisation under normal deformation along Y. However we don't treat here the specialisation for complexes.

The reduction of the proofs to the so called elementary modules (*cf.* [K2], [Sa]) allows great simplification. THEOREM 1.6 concerning the induced system in Y was obtained in codim 1 by other method (see [Ma2]).

In THEOREM 1.8 we prove that in the case of a smooth hypersurface Y, the complex of nearby cycles $\psi_Y(\mathcal{M})$ (of $\mathcal{D}_Y$-modules) is the inverse image of $\underline{\nu_Y}(\mathcal{M})$ (in the sense of $\mathcal{D}$-modules) by the section $s : Y \to T_YX$ associated to a local equation defining Y. This is the analogue of a theorem of [K-S2], in the framework of $D^b(X)$.

We also study the behaviour of $\underline{\nu_Y}$ under smooth inverse image. The behaviour under proper direct image was studied by SABBAH-MEBKHOUT in [Me].

Since the specialisation of a $\mathcal{D}$-module in B_Y is monodromic as a $\mathcal{D}_{T_YX}$-module one may define $\underline{\mu_Y}(\mathcal{M})$ as the Fourier transform of $\underline{\nu_Y}(\mathcal{M})$

(*cf.* [K-H], [Br-Ma-V]). The section 2 is devoted to the study of $\underline{\mu_Y}$ and its relation with the microlocalisation functor in $D^b(X)$.

In particular we prove that the restriction of $\underline{\mu_Y}(\mathcal{M})$ to Y (Y viewed as the zero section of $T^*_Y X$) coincides with $i^!\mathcal{M}$, where $i : Y \to X$ is the inclusion.

We also prove that although the projection $\pi : T^*_Y X \to Y$ is not proper the direct image of $\underline{\mu_Y}(\mathcal{M})$ has coherent cohomologies given by ${}^{\mathbb{L}}i^*\mathcal{M}$; the proper direct image is isomorphic to ${}^{\mathbb{L}}i^!\mathcal{M}$; of course the coherency is also a consequence of [H-S].

The last section is devoted to the study of $\underline{\mu\mathrm{hom}}(\cdot\,,\cdot)$. Here we stay in the differential case, but of course all the constructions would work for microdifferential systems, with slight adaptation, that is, replacing the V-filtration on $\mathcal{D}_X$ by the ring $\mathcal{E}_\wedge$ of microdifferential operators defined by [K-K2], [K-O], where $\wedge = T^*_Y X$ (see also [MF1] and [MF2]).

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1. Specialisation of $\mathcal{D}$-modules along a submanifold

Let X be an n-dimensional complex analytic manifold and let $Y \subset X$ be a d-codimensional submanifold. Let $\mathcal{D}_X$ denote the sheaf over X of linear holomorphic differential operators of finite order.

We will call $\mathcal{D}_X$-*module*, or $\mathcal{D}$-*module* for short, any sheaf of left modules over $\mathcal{D}_X$ and note $\mathrm{Mod}(\mathcal{D}_X)$ the abelian category whose objects are $\mathcal{D}$-modules.

Let f be a holomorphic map from the manifold X to the manifold Z.

- Let $\mathcal{D}_{X\to Z}$ denote the $(\mathcal{D}_X, f^{-1}\mathcal{D}_Z)$-bimodule $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}\mathcal{D}_Z$ and
- $\mathcal{D}_{Z\leftarrow X}$ denote the $(f^{-1}\mathcal{D}_Z, \mathcal{D}_X)$-bimodule

$$(f^{-1}(\mathcal{D}_Z) \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}\Omega_Z^{\otimes^{-1}}) \underset{\mathcal{O}_X}{\otimes} \Omega_X$$

(for a detailed study of these sheaves see [S], [SKK] and [K3]).

By definition given $\mathcal{M}$ a $\mathcal{D}_X$-module and $\mathcal{L}$ a $\mathcal{D}_Z$-module :

- the inverse image is ${}^{\mathbb{L}}f^*\mathcal{L} = \mathcal{D}_{X\to Z} \overset{\mathbb{L}}{\underset{f^{-1}\mathcal{D}_Z}{\otimes}} f^{-1}\mathcal{L}$,

- the direct image is $\int_f \mathcal{M} = \mathbb{R}f_*\Big(\mathcal{D}_{Z\leftarrow X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}\Big)$
- the proper direct image is $\overset{c}{\int_f} \mathcal{M} = \mathbb{R}f_!\Big(\mathcal{D}_{Z\leftarrow X} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{M}\Big)$.

Let $\mathcal{L}$ and $\mathcal{L}'$ be two left-$\mathcal{D}_Z$-modules. Then we have a natural morphism

$$f^{-1}\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{L},\mathcal{L}')\big) \longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}\big({}^{\mathbb{L}}f^*\mathcal{L}, {}^{\mathbb{L}}f^*\mathcal{L}'\big).$$

Moreover, if f is smooth one obtains an isomorphism

$$f^{-1}\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_Z}(\mathcal{L},\mathcal{O}_Z)\big) \longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}\big({}^{\mathbb{L}}f^*\mathcal{L}, \mathcal{O}_X\big)$$

(*cf.* [S] and [K-S2] for a detailed study of these functors).

Let $Y \subset X$ be a d-codimensional submanifold of X. As usual, $\{\mathcal{D}_X(m)\}_{m\in\mathbb{N}}$ will denote the filtration of $\mathcal{D}_X$ by the order and $V_Y^*(\mathcal{D}_X)$ (or $V^*(\mathcal{D}_X)$ for short, once Y is fixed) the filtration

$$V_Y^k(\mathcal{D}_X) \underset{\text{def}}{=} \Big\{P \in \mathcal{D}_X\ ;\ P(I^j) \subset I^{j+k},\ \text{for every } j \text{ such that } j,\ j+k \geq 0\Big\}.$$

(Here I denotes the defining ideal of Y.) Let

- $\mathrm{gr}_V(\mathcal{D}_X) = \bigoplus_{k\in\mathbb{Z}} \dfrac{V^k(\mathcal{D}_X)}{V^{k+1}(\mathcal{D}_X)}$ and
- $\tau : T_YX \to Y$ be the projection of the normal bundle to Y.

Then $\mathrm{gr}_V(\mathcal{D}_X) \simeq \tau_*\mathcal{D}_{[T_YX]}$ where $\mathcal{D}_{[T_YX]}$ denotes the sheaf of homogeneous differential operators over T_YX; θ will denote the Euler operator which acts by the identity on I/I^2 as well as any of its local representatives in $V^0(\mathcal{D}_X)$.

Remark that $V^0(\mathcal{D}_X)$ is a subring of $\mathcal{D}_X$ containing $\mathcal{O}_X$.

Now let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module.

A filtration $\mathcal{M} = \bigcup_{j\in\mathbb{Z}} \mathcal{M}^j$ is a *good-V filtration* if it satisfies :

i) $V^k(\mathcal{D}_X)\mathcal{M}^j \subset \mathcal{M}^{j+k}$ for every j and k;

ii) $V^k(\mathcal{D}_X)\mathcal{M}^j = \mathcal{M}^{j+k}$ either for $j \gg 0$ and $k \geq 0$ or $j \ll 0$ and $k \leq 0$;

iii) $\mathcal{M}^j$ is a coherent $V^0(\mathcal{D}_X)$-module, for every $j \in \mathbb{Z}$.

DEFINITION 1.0. — A coherent $\mathcal{D}_X$-module is *specialisable along* Y if for every good-V filtration $U^*(\mathcal{M})$ on $\mathcal{M}$ there is locally a non zero polynomial $b \in \mathbb{C}[s]$ such that

$$b(\theta - k)\, U^k(\mathcal{M}) \subset U^{k+1}(\mathcal{M}),$$

also classically called a Bernstein-Sato polynomial or a b-function of $U^*(\mathcal{M})$.

Let us denote by G any section of the canonical morphism $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. We fix on $\mathbb{C}$ a total order $\leq$ such that $u \leq v$ entails $u + m \leq v + m$ for every $m \in \mathbb{C}$, and such that $m \geq 0$, for $m \in \mathbb{N}$; it is well known (see [K2], [Sa]) that if $\mathcal{M}$ is specialisable there exists a unique good-V filtration $V_G(\mathcal{M})$ admitting a b-function whose zeros are contained in G. Moreover KASHIWARA has shown in [K2] that for two such sections G and G' the modules

$$\mathcal{D}_{T_Y X} \underset{\mathcal{D}_{[T_Y X]}}{\otimes} \tau^{-1} \operatorname{gr}_{V_G}(\mathcal{M}) \quad \text{and} \quad \mathcal{D}_{T_Y X} \underset{\mathcal{D}_{[T_Y X]}}{\otimes} \tau^{-1} \operatorname{gr}_{V_{G'}}(\mathcal{M})$$

are isomorphic over $\mathcal{D}_{T_Y X}$, which entails :

DEFINITION 1.1 (see [K2]). — The specialised of $\mathcal{M}$ along Y, $\underline{\nu_Y}(\mathcal{M})$, is the $\mathcal{D}_{T_Y X}$-coherent module

$$\mathcal{D}_{T_Y X} \underset{\mathcal{D}_{[T_Y X]}}{\otimes} \tau^{-1} \operatorname{gr}_{V_G}(\mathcal{M}).$$

Hereafter, G will stand for

$$\{z \in \mathbb{C}\,;\ 0 \leq z < 1\}$$

and if $\mathcal{M}$ is a specialisable $\mathcal{D}$-module,

$$V^*(\mathcal{M}) = V_G^*(\mathcal{M}).$$

We also denote $G = [0, 1[$.

Now let us recall that $\mathcal{M}$ is regular along Y in the sense of Kashiwara if there exists locally a coherent $\mathcal{O}_X$-module $\mathcal{M}_0 \subset \mathcal{M}$ and a non-zero polynomial $b(s) \in \mathbb{C}[s]$ of degree p such that :

1) $\mathcal{M}$ is generated by $\mathcal{M}_0$, that is, $\mathcal{M} = \mathcal{D}_X\, \mathcal{M}_0$.

2) $b(\theta)\, \mathcal{M}_0 \subset \big[\mathcal{D}_X(p) \cap V^1(\mathcal{D}_X)\big]\, \mathcal{M}_0$.

In particular, if $\mathcal{M}$ is regular along Y, then $\mathcal{M}$ is specialisable. Furthermore, the category of regular (specialisable) $\mathcal{D}$-modules along Y is a full abelian subcategory of the category of $\mathcal{D}_X$-modules and the functor $\underline{\nu_Y}(*)$ is exact.

We will denote R_Y (resp. B_Y) the category of regular (resp. specialisable) $\mathcal{D}$-modules along Y. The following result will be an essential tool :

PROPOSITION 1.2. — *Let $\mathcal{M} \in B_Y$. Then, locally on X, there exists $k \in \mathbb{N} \cup \{0\}$ and a surjective morphism $\mathcal{L} \to \mathcal{M} \to 0$ where $\mathcal{L}$ is the cokernel of a matrix $(N+q) \times (N+q)$ of the form*

$$\underbrace{\begin{bmatrix} R(\theta)I & S \\ 0 & \tilde{b}(\theta)I + Q \end{bmatrix}}_{q \qquad N} \begin{matrix} \}q \\ \}N, \end{matrix}$$

where I is the identity matrix, resp. of order N and q, $b(s), R(s) \in \mathbb{C}[s]$ and :

a) $R(\theta) = (\theta + \operatorname{codim} Y)^\ell$, *for some* $\ell \in \mathbb{N} \cup \{0\}$;

b) $\tilde{b}^{-1}(0) \in \{z \in \mathbb{C} ;\ k \leq z < k+1\}$;

c) *The entries of Q belong to $V^1(\mathcal{D}_X)$;*

d) *S is a (q, N)-matrix with entries in $\mathcal{D}_X$.*

NOTATION. — Such a $\mathcal{D}$-module $\mathcal{L}$ is called *elementary.*

Proof. — Let $b(s)$ be the Bernstein-Sato polynomial associated to G. Let k_0 be an integer, $k_0 \geq 0$, such that $V^{k+1}(\mathcal{M}) = V^1(\mathcal{D}_X)\, V^k(\mathcal{M})$, for all $k \geq k_0$. Consider $\mathcal{M}'' = \mathcal{D}_X V^{k_0}(\mathcal{M})$ and choose $v_1, \ldots, v_N \in V^{k_0}(\mathcal{M})$ a familly of local generators of $V^{k_0}(\mathcal{M})$ over $V^0(\mathcal{D}_X)$. Since $\operatorname{supp} \mathcal{M}'$ is contained in Y we may choose $u_1, \ldots, u_q$ in $\mathcal{M}$ such that the images $\bar{u}_1, \ldots, \bar{u}_q$ generate $\mathcal{M}$ and that $I_Y \bar{u}_j = 0$, hence $(\theta + \operatorname{codim} Y)^\ell u_j \in \mathcal{M}''$ for some ℓ. So we may choose $S_{ij} \in \mathcal{D}_X$ such that

$$(\theta + \operatorname{codim} Y)^\ell u_j = \sum_{j=1}^{N} S_{ij}\, v_j, \qquad i = 1, \ldots, q.$$

Also,

$$b(\theta - k_0)\, v_i = \sum_{\ell=1}^{N} Q_{i\ell}\, v_\ell, \quad \text{for} \quad i = 1, \ldots, N,$$

with $Q_{j\ell} \in V^1(\mathcal{D}_X)$, since $V^{k_0+1}(\mathcal{M}) = V^1(\mathcal{D}_X)\, V^{k_0}(\mathcal{M})$. The proposition follows because $\mathcal{M}$ is generated by $v_1, \ldots, v_N$ and $u_1, \ldots, u_q$. □

Therefore, an elementary module is an extension

$$0 \to \mathcal{D}_X^N/\mathcal{D}_X^N(b(\theta)I+Q) \longrightarrow \mathcal{L} \longrightarrow \mathcal{D}_X^q/\mathcal{D}_X^q(R(\theta)I) \to 0,$$

with $b^{-1}(0) \subset \{z \in \mathbb{C};\ k \leq z < k+1\}$ and $R^{-1}(0) \subset \mathbb{Z}^-$. Moreover, if $\mathcal{L}$ is in R_Y we may take $Q_{ij} \in \mathcal{D}_X(p)$ where p is the degree of $b(s)$.

LEMMA 1.3. — *Let $\mathcal{L}$ be of the form $\mathcal{D}^N/\mathcal{D}^N(b(\theta)I+Q)$ without assumption on $b^{-1}(0)$. Then the right multiplication by $b(\theta)I+Q$ defines an injective morphism $\mathcal{D}_X^N \to \mathcal{D}_X^N$ and so $\mathcal{L}$ is quasi-isomorphic to the complex*

$$\mathcal{D}_X^N \xrightarrow{\ b(\theta)I+Q\ } \mathcal{D}_X^N.$$

Proof. — Let $P_1,\dots,P_N \in \mathcal{D}^N$ with $P_i \neq 0$ for some i and let k be the highest order of $P_1,\dots,P_N$ in the V-filtration; let $P_j \in \{P_1,\dots,P_N\}$ be of exactly order k, that is, $P_j \in V^k(\mathcal{D}_X)$ and $P_j \notin V^{k+1}(\mathcal{D}_X)$. If $[P_1 \dots P_N]\,[b(\theta)I+Q] = 0$, then

$$P_j\, b(\theta) = -\sum_{\ell=1}^{N} P_\ell\, Q_{\ell j}$$

is an element of $V^{k+1}(\mathcal{D}_X)$, which is absurd. □

Denote by $\widetilde{X}^{\mathbb{R}}$ the real normal deformation of X along Y, with the canonical projection $p : \widetilde{X}^{\mathbb{R}} \to X$ and canonical morphism $c : \widetilde{X}^{\mathbb{R}} \to \mathbb{R}$ such that $c^{-1}(0) \simeq T_Y X$ (see [K-S2] for details.) Let us recall that we may consider local coordinates (x,y) in X such that Y is given by $x = (x_1,\dots,x_d) = 0$ and a system of local coordinates (x',y',c) in $\widetilde{X}^{\mathbb{R}}$, $c \in \mathbb{R}$ such that $p(x',y',c) = (x'c, y')$ and c is the projection on $\mathbb{R}$. Remark that $c^{-1}(\mathbb{R}) = \widetilde{\chi}^{\mathbb{R}}$. Let us consider the open set $\Omega = c^{-1}(\mathbb{R}^+) \subset \widetilde{X}^{\mathbb{R}}$ and the commutative diagram of morphisms

A)
$$\begin{array}{ccccc} T_Y X & \xhookrightarrow{\ s\ } & \widetilde{X}^{\mathbb{R}} & \xhookleftarrow{\ j\ } & \Omega \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle\tilde{p}} & \\ Y & \xhookrightarrow[\ i\ ]{} & X & & \end{array}$$

Then, for $F^\bullet \in \operatorname{Obj} D^b(X)$ one defines Sato's specialisation

$$\nu_Y(F^\bullet) := s^{-1}\,\mathbb{R}j_*\,\tilde{p}^{\,-1}\,F^\bullet.$$

If we consider the complex normal deformation (the construction is analogue with $\mathbb{R}$ replaced by $\mathbb{C}$), denoted by $\widetilde{X}$, of X along Y, one defines

Verdier's specialisation functor in $D^b(X)$, which we will denote by $\nu_Y^{\mathbb{C}}(\cdot)$ (see [V1]) :

a) First we need to recall Deligne's nearby-cycle functor associated to a holomorphic function $f : X \to \mathbb{C}$

$$\psi_f(\cdot) = i^* \,\mathbb{R}\tilde{p}_* \,\tilde{p}^{-1}(\cdot),$$

where i and $\tilde{p}$ are defined as follows :

Let $(\widetilde{\mathbb{C}}, p)$ be a universal covering of $\mathbb{C}\backslash\{0\}$ and $[\widetilde{X}', (\tilde{p}, \tilde{f})]$ the fiber product $\widetilde{X}' = X \underset{\mathbb{C}}{\times} \widetilde{\mathbb{C}}$ with $\tilde{p} : \widetilde{X}' \to X$ and $\tilde{f} : \widetilde{X}' \to \widetilde{\mathbb{C}}$ the canonical projections

$$\text{B)}\qquad \begin{array}{ccccc} & & \widetilde{X}' & \xrightarrow{\ \tilde{f}\ } & \widetilde{\mathbb{C}} \\ & & {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xhookrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & \mathbb{C} \end{array}$$

b) Let us now consider the commutative diagram associated to the complex deformation $\widetilde{X}$:

$$\text{C)}\qquad \begin{array}{ccccc} c^{-1}(0) = T_Y X & \xhookrightarrow{\ s\ } & \widetilde{X} & \xrightarrow{\ c\ } & \mathbb{C} \\ {\scriptstyle \tau}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle p} & & \\ Y & \xhookrightarrow{\ i\ } & X & & \end{array}$$

Then, by definition,

$$\nu_Y^{\mathbb{C}}(F^\bullet) = \psi_c(p^* F^\bullet)$$

and it is easy to check that there is a natural morphism

$$\nu_Y^{\mathbb{C}}(F^\bullet) \longrightarrow \nu_Y(F^\bullet).$$

This morphism is injective when $F^\bullet = \mathcal{O}_X$ and an isomorphism when $F^\bullet$ is $\mathbb{C}$-constructible (*cf.* [V1]).

Remark 1. — For any $\mathcal{D}_X$-module $\mathcal{M}$ one has natural isomorphisms

$$\text{I)}\quad \nu_Y\big(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\big) \xrightarrow{\ \sim\ } \big(\mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y(\mathcal{O}_X))\big),$$

$$\text{II)}\quad \nu_Y^{\mathbb{C}}\big(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\big) \xrightarrow{\ \sim\ } \mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}\big(\pi^{-1}\mathcal{M}, \nu_Y^{\mathbb{C}}(\mathcal{O}_X)\big).$$

To prove these assertions, we have to construct the morphisms, and, by taking a locally free resolution of $\mathcal{M}$, we then reduce to $\mathcal{M} = \mathcal{D}_X$.

Let us define the first so morphism I) and let II) as an exercise to the reader. Keeping the notations of diagram A) above we get :

$$
\begin{aligned}
s^{-1}\mathbb{R}j_*\, \tilde{p}^{-1}(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X))
&\xrightarrow{\sim} s^{-1}\mathbb{R}j_*(\mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{p}^{-1}\mathcal{D}_X}(\tilde{p}^{-1}\mathcal{M}, \tilde{p}^{-1}\mathcal{O}_X))\\
&\xrightarrow{\sim} s^{-1}\mathbb{R}\mathcal{H}\mathrm{om}_{p^{-1}\mathcal{D}_X}(p^{-1}\mathcal{M}, \mathbb{R}j_*\, \tilde{p}^{-1}\mathcal{O}_X)\\
&\xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y(\mathcal{O}_X)).
\end{aligned}
$$

REMARK 2. — Let $\mathcal{M}$ be a regular $\mathcal{D}_X$-module along Y. Then, by Theorem 7.2 of [K-K3], the natural morphism

$$
\mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y^{\mathbb{C}}(\mathcal{O}_X)) \longrightarrow \mathbb{R}\mathcal{H}\mathrm{om}_{\pi^{-1}\mathcal{D}_X}(\pi^{-1}\mathcal{M}, \nu_Y(\mathcal{O}_X))
$$

is an isomorphism.

Consider the complex normal deformation $\widetilde{X}$ along Y. For a complex of left $\mathcal{D}_{\widetilde{X}}$-modules $F^\bullet$ set

$$
F^\bullet[c^{-1}] := \mathcal{O}_{\widetilde{X}}[c^{-1}] \underset{\mathcal{O}_{\widetilde{X}}}{\otimes} F^\bullet
$$

— this is the localised of $F^\bullet$ along $\widetilde{Y} := c^{-1}(0)$. Remark that the localisation functor is exact since $\operatorname{codim} \widetilde{Y} = 1$.

For a coherent $\mathcal{D}_X$-module let us denote

$$
\mathcal{M}^0 = \mathcal{H}^0(^{\mathbb{L}}p^*)\mathcal{M} = \mathcal{H}^0(\mathcal{O}_{\widetilde{X}} \overset{\mathbb{L}}{\underset{p^{-1}\mathcal{O}_X}{\otimes}} \mathcal{M}) = \mathcal{H}^0(\mathcal{D}_{\widetilde{X}\to X} \overset{\mathbb{L}}{\underset{p^{-1}\mathcal{D}_X}{\otimes}} \mathcal{M}).
$$

THEOREM 1.4. — *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then :*

1) $\mathcal{H}^k(^{\mathbb{L}}p^*\mathcal{M}[c^{-1}]) = 0$ *for all* $k \neq 0$;

2) $\mathcal{M}^0[c^{-1}] = \mathcal{H}^0(^{\mathbb{L}}p^*\mathcal{M}[c^{-1}])$ *is coherent and regular along* $\widetilde{Y}$;

3) *Suppose that $\mathcal{M}$ is specialisable along Y and consider $\mathcal{M}^0[c^{-1}]$ endowed with the canonical V-filtration. Then, the $\mathcal{D}_{\widetilde{Y}}$-modules $\underline{\nu_Y}(\mathcal{M})$ and $\mathrm{gr}^0(\mathcal{M}^0[c^{-1}])$ are naturally isomorphic.*

Proof.

Assertions 1) and 2). Since the morphism p restricted to $\widetilde{X}-\widetilde{Y}$ is smooth, $\mathcal{O}_{\widetilde{X}}[c^{-1}]$ is flat over $p^{-1}\mathcal{O}_X$ so, for all $k\neq 0$,

$$\mathcal{H}^k\big({}^{\mathbb{L}}p^*\mathcal{M}[c^{-1}]\big)=\mathcal{H}^k\big(\mathcal{O}_{\widetilde{X}}[c^{-1}]\overset{\mathbb{L}}{\underset{p^{-1}\mathcal{O}_X}{\otimes}}\mathcal{M}\big).$$

Moreover $\mathcal{D}_{\widetilde{X}\to X}[c^{-1}]$ is isomorphic to $D_{\widetilde{X}}/\mathcal{D}_{\widetilde{X}}\,cD_c[c^{-1}]$ and so it is a coherent left $\mathcal{D}_{\widetilde{X}}$-module and regular along $\widetilde{Y}$. By taking a local free resolution of $\mathcal{M}$ it then follows that $\mathcal{M}^0[c^{-1}]$ is also regular along $\widetilde{Y}$.

Assertion 3). Let us recall that $\mathcal{M}^0$ is endowed with a structure of left $\mathcal{D}_{\widetilde{X}}$-module given in local coordinates (x',y',c) such that $p(x',y',c)=(x'c,y')$, by

$$\text{(A)}\qquad\begin{cases} D_{y'_i}(u\otimes m)=u\otimes D_{y_i}m+D_{y'_i}u\otimes m,\\ D_{x'_i}(u\otimes m)=(D_{x'_i}u)\otimes m+c\,u\otimes D_{x_i}m,\\ D_c(u\otimes m)=D_cu\otimes m+\displaystyle\sum_i x'_iu\otimes D_{x_i}m,\end{cases}$$

(see [K-1]). In particular

$$c\,D_c(u\otimes m)=(c\,D_c)u\otimes m+u\otimes\theta m.$$

We will note $\tilde{\theta}=c\,D_c$.

Let us now remark that the canonical filtration along $\widetilde{Y}$ of $\mathcal{D}_{\widetilde{X}\to X}[c^{-1}]$ is the quotient of the $V_{\widetilde{Y}}$-filtration on $\mathcal{D}_{\widetilde{X}}$. Let us note

- $V^k(\mathcal{M})$ the canonical filtration on $\mathcal{M}$ and
- $\mathcal{M}^0_k[c^{-1}]$ the image in $\mathcal{M}^0[c^{-1}]$ of $\bigoplus_{\ell\in\mathbb{Z}}c^\ell\,\mathcal{O}_{\widetilde{X}}\otimes_{p^{-1}\mathcal{O}_X}p^{-1}\big(V_{k-\ell}(\mathcal{M})\big)$.

The action of $\mathcal{D}_{\widetilde{X}}$ entails that if $b(\theta)$ is the Bernstein-Sato polynomial of $\{V^k(\mathcal{M})\}_{k\in\mathbb{Z}}$ then $b(cD_c-k)\,\mathcal{M}^0_k[c^{-1}]\subset\mathcal{M}^0_{k+1}[c^{-1}]$.

Hence $\mathcal{M}^0_k[c^{-1}]$ is contained in $V^k(\mathcal{M}^0[c^{-1}])$ where $\{V^k(\mathcal{M}^0[c^{-1}])\}$ is the canonical $V_{\widetilde{Y}}$-filtration on the regular $\mathcal{D}_{\widetilde{X}}$-module $\mathcal{M}^0[c^{-1}]$.

Let us now define a canonical morphism $\underline{\nu_Y}(\mathcal{M})\xrightarrow{\psi}\operatorname{gr}^0(\mathcal{M}^0[c^{-1}])$: for every $i\in\mathbb{Z}$ let us denote φ_i the canonical isomorphism

$$\mathcal{O}_{\tilde{Y}}\ \underset{\sim}{\xrightarrow{\varphi_i}}\ \frac{c^i\,\mathcal{O}_{\tilde{X}}}{c^{i+1}\,\mathcal{O}_{\tilde{X}}}.$$

Then one has

$$\begin{aligned}\underline{\nu_Y}(\mathcal{M}) &\simeq \mathcal{D}_{\tilde{Y}} \underset{\mathcal{D}_{[\tilde{Y}]}}{\otimes} p^{-1}\operatorname{gr}(\mathcal{M})\mathcal{O}_{\tilde{Y}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} p^{-1}\operatorname{gr}(\mathcal{M}) \\ &\simeq \bigoplus_{i\in\mathbb{Z}} \frac{c^i\,\mathcal{O}_{\tilde{X}}}{c^{i+1}\,\mathcal{O}_{\tilde{X}}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} p^{-1}(\operatorname{gr}^{-i}(\mathcal{M}))\end{aligned}$$

and to get the morphism ψ we compose with the morphism

$$\bigoplus_{i\in\mathbb{Z}} \frac{c^i\,\mathcal{O}_{\tilde{X}}}{c^{i+1}\,\mathcal{O}_{\tilde{X}}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} p^{-1}\operatorname{gr}^{-i}(\mathcal{M}) \longrightarrow \operatorname{gr}^0(\mathcal{M}^0[c^{-1}]).$$

Now if we consider the V-filtration on $\mathcal{D}_X$ shifted by $\ell \in \mathbb{Z}$, i.e., $\widetilde{V}^k(\mathcal{D}_X) = V^{k+\ell}(\mathcal{D}_X)$, we may define the morphism ψ_ℓ analogue to ψ :

$$\begin{aligned}\mathcal{D}_{\tilde{Y}} &= \mathcal{O}_{\tilde{Y}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} \operatorname{gr}(\mathcal{D}_X) \\ &\xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \frac{\mathcal{O}_{\tilde{X}}\,c^i}{\mathcal{O}_{\tilde{X}}\,c^{i+1}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} p^{-1}(\operatorname{gr}^{-i+\ell}(\mathcal{D}_X)) \\ &\qquad\qquad \longrightarrow \operatorname{gr}^\ell(\mathcal{D}_{\tilde{X}\to X}[c^{-1}]).\end{aligned}$$

Hence $\psi_\ell = c^\ell\,\psi_0$.

Finally, to prove that ψ is an isomorphism, let us remark that, since $\mathcal{M} \mapsto \operatorname{gr}^0(\mathcal{M}^0[c^{-1}])$ and $\mathcal{M} \mapsto \underline{\nu_Y}(\mathcal{M})$ are exact functors, it is now enough to consider $\mathcal{M}$ an elementary module, and therefore, by PROPOSITION 1.2, consider the case where $\mathcal{M}$ is defined by a square matrix $B(\theta)I + Q$, where Q is an element of $M_N(V^1(\mathcal{D}_X))$, such that $B^{-1}(0) = \{-\operatorname{codim} Y\}$ or $B^{-1}(0) \subset [k, k+1]$, for some $k \in \mathbb{N}$. In both cases $\mathcal{M}$ admits a filtered free presentation (with a shift ℓ) :

$$\mathcal{D}_X^N \longrightarrow \mathcal{D}_X^N \longrightarrow \mathcal{M} \to 0,$$

such that $\widetilde{V}^k(\mathcal{D}_X)^N \to \widetilde{V}^k(\mathcal{D}_X)^N \to V^k(\mathcal{M}) \to 0$ is exact for all $k \in \mathbb{Z}$. (Here $V^k(\mathcal{M})$ is again the canonical V-filtration on $\mathcal{M}$).

Hence we get a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathcal{D}_{\tilde{Y}}^N & \xrightarrow[B(\bar{\theta})]{} & \mathcal{D}_{\tilde{Y}}^N & \longrightarrow & \underline{\nu_Y}(\mathcal{M}) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \psi_\ell} & & \downarrow{\scriptstyle \psi_\ell} & & & & \\
\operatorname{gr}^\ell(\mathcal{D}_{\tilde{X}\to X}^N[c^{-1}]) & \xrightarrow[B(\bar{\theta})]{} & \operatorname{gr}^\ell(\mathcal{D}_{\tilde{X}\to X}^N[c^{-1}]) & & \downarrow{\scriptstyle \psi} & & \\
\downarrow{\scriptstyle c^{-\ell}} & & \downarrow{\scriptstyle c^{-\ell}} & & & & \\
\operatorname{gr}^0(\mathcal{D}_{\tilde{X}\to X}^N[c^{-1}]) & \xrightarrow[B(\bar{\theta})]{} & \operatorname{gr}^0(\mathcal{D}_{\tilde{X}\to X}^N[c^{-1}]) & \longrightarrow & \operatorname{gr}^0(\mathcal{M}^0[c^{-1}]) & \longrightarrow & 0
\end{array}$$

Here $\bar\theta = \sum_{i=1} x'_i D_{x'_i}$ denotes the Euler field on $T_{\tilde Y}\widetilde X$.

Hence we only have to check that ψ_ℓ is an isomorphism, which is clear since, for all $\ell \in \mathbb{Z}$,

$$\mathrm{gr}^\ell(\mathcal{D}_{\tilde X \to X}[c^{-1}]) \cong \mathrm{gr}^\ell(\mathcal{O}_{\tilde X}[c^{-1}])[D_{x'}] = \mathcal{O}_{\tilde Y}\, c^\ell[D_{x'}] = c^\ell\, \mathcal{D}_{\tilde Y}. \quad \square$$

Let $\mathcal{M}$ be a specialisable $\mathcal{D}_X$-module along Y. Let us endow $\mathcal{M}^0$ with the filtration $V^k(\mathcal{M}^0)$ analogous to that defined in the proof of the preceding theorem, that is, the image of

$$\bigoplus_{i\geq 0} c^i\,\mathcal{O}_{\tilde X} \underset{p^{-1}\mathcal{O}_X}{\otimes} p^{-1}(V^{k-i}(\mathcal{M})),$$

and consider the natural morphisms

$$\mathcal{M}^0 \longrightarrow \mathcal{M}^0[c^{-1}], \quad \mathcal{M}^0_k \longrightarrow V^k(\mathcal{M}^0[c^{-1}]),$$

and hence

$$\mathcal{M}^0_0/\mathcal{M}^0_1 \xrightarrow{\ \phi\ } \mathrm{gr}^0(\mathcal{M}^0[c^{-1}]).$$

It is clear that $V^1(\mathcal{D}_{\tilde X})\,\mathcal{M}^0_k \subset \mathcal{M}^0_{k+1}$. Moreover

$$\mathbb{L}j^*\,\mathcal{M}^0 = (\mathcal{M}^0 \xrightarrow{\ c\ } \mathcal{M}^0) \longleftarrow (\mathcal{M}^0_{-1} \xrightarrow{\ c\ } \mathcal{M}^0_0)$$

is a quasi-isomorphism.

REMARK 1. — Let $\mathcal{M}$ be an elementary $\mathcal{D}$-module. Since $\mathcal{D}_{\tilde X \to X}$ satisfies the unique continuation principle as a sheaf on $\widetilde X$ and since the right multiplication by a square-(N,N)-matrix of the form $B(\theta) = b(\theta)\,I + Q$ is injective (*cf.* LEMMA 1.3) on $\mathcal{D}^N_{\tilde X \to X|c\neq 0}$ because $p_{|c\neq 0}$ is smooth, we conclude that $B(\theta)$ is injective on $\mathcal{D}_{\tilde X \to X}$ and hence $\mathbb{L}p^*\,\mathcal{M} \underset{\mathrm{QIS}}{\xrightarrow{\ \sim\ }} \mathcal{M}^0$.

LEMMA 1.4.1. — *Let $\mathcal{M}$ be an elementary module. Then :*

(a) *ϕ is an isomorphism;*

(b) *$\mathcal{M}^0 \underset{c}{\longrightarrow} \mathcal{M}^0$ is isomorphic to $\mathcal{M}^0_{-1}/\mathcal{M}^0_0 \underset{c}{\to} \mathcal{M}^0_0/\mathcal{M}^0_1$ that is, the natural morphism*

$$\begin{array}{ccc} \mathcal{M}^0_{-1} & \xrightarrow{\ \ c\ \ } & \mathcal{M}^0_0 \\ \downarrow & & \downarrow \\ \dfrac{\mathcal{M}^0_{-1}}{\mathcal{M}^0_0} & \underset{c}{\longrightarrow} & \dfrac{\mathcal{M}^0_0}{\mathcal{M}^0_1} \end{array}$$

is a quasi-isomorphism.

Proof. — We may argue similarly to [Me]. Let us write $\mathcal{M} = \mathcal{D}_X^N/\mathcal{D}_X^N L$. By the preceding remark we get

$$\mathcal{M}_k^0 = \frac{V^k(\mathcal{D}^N_{\tilde{X}\to X})}{V^k(\mathcal{D}^N_{\tilde{X}\to X}L)},$$

where $V^k(\mathcal{D}_{\tilde{X}\to X})$ is the image of $\bigoplus_{i\geq 0} \mathcal{O}_{\tilde{X}}\, c^i \underset{p^{-1}\mathcal{O}_X}{\otimes} p^{-1}(V^{k-i}(\mathcal{D}_X))$.

Let us now prove that, for all $k \geq 0$,

$$V^k(\mathcal{D}_{\tilde{X}\to X}) \xrightarrow{\sim} V^k(\mathcal{D}_{\tilde{X}\to X}[c^{-1}]).$$

This morphism is injective since left multiplication by c is injective on $\mathcal{D}_{\tilde{X}\to X}$ (recall that $\mathcal{D}_{\tilde{X}\to X}$ is a flat left $\mathcal{O}_{\tilde{X}}$-module).

To see it is surjective we may consider local coordinates (x', y', c) in $\widetilde{X}^{\mathbb{C}}$ and (y, x) in X such that $p(x', y', c) = (y', x'c)$. Then we may write :

$$V^k(\mathcal{D}_{\tilde{X}\to X}) = \Big\{P \in \mathcal{D}_{\tilde{X}\to X}\,;\; P = \sum_{0\leq|\beta_i|+|\alpha_i|\leq m} f_i\, c^{|\alpha_i|+k}\, D_x^{\alpha_i}\, D_y^{\beta_i},\quad f_i \in \mathcal{O}_{\tilde{X}},\ m \in \mathbb{N}\Big\}.$$

Similarly,

$$V^k(\mathcal{D}_{\tilde{X}\to X}[c^{-1}]) = \Big\{P \in \mathcal{D}_{\tilde{X}\to X}[c^{-1}]\,;\; P = \sum_{0\leq j+\alpha_i+\beta_i\leq m} f_i\, c^{i+k}\, D_{x'}^{\alpha_i}\, D_y^{\beta_i},\quad f_i \in \mathcal{O}_{\tilde{X}},\ m \in \mathbb{N}\Big\}$$

and in $\mathcal{D}_{\tilde{X}\to X}[c^{-1}]$

$$f_i\, c^{i+k}\, D_{x'}^{\alpha_i} = f_i\, c^{i+k+|\alpha_i|}\, D_x^{\alpha_i}.$$

Finally, remark that

$$(*)\qquad \mathcal{D}^N_{\tilde{X}\to X} B(\theta) \cap V^k(\mathcal{D}^N_{\tilde{X}\to X}) = V^k(\mathcal{D}^N_{\tilde{X}\to X})\, L.$$

Hence ϕ is an isomorphism.

Now let us prove that $\mathcal{M}^0 \xrightarrow{c} \mathcal{M}^0$ is quasi-isomorphic to

$$\frac{\mathcal{M}^0_{-1}}{\mathcal{M}^0_0} \xrightarrow{c} \frac{\mathcal{M}^0_0}{\mathcal{M}^0_1}.$$

We have to prove that $\mathcal{M}_0^0 \xrightarrow{c} \mathcal{M}_1^0$ is bijective. In that case, it follows from that $V^k(\mathcal{D}_{\tilde{X}\to X}) = cV^0(\mathcal{D}_{\tilde{X}\to X})$, for all $k \geq 0$, and from the relation (*) above. □

For the purpose of this paper the main result of [K2] is an essential tool :

THEOREM 1.5. — *Let $\mathcal{M}$ be regular along Y. Then one has canonical isomorphisms in $D^b(Y)$:*

$$(*) \qquad \nu_Y\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big) \xleftarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{T_YX}}\big(\underline{\nu_Y}(\mathcal{M}),\mathcal{O}_{T_YX}\big),$$

$$(**) \qquad \nu_Y\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{M})\big) \xleftarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{T_YX}}\big(\mathcal{O}_{T_YX},\underline{\nu_Y}(\mathcal{M})\big).$$

Given the scope of this paper, we think useful to explain here these morphisms. First of all, when Y is a smooth hypersurface defined by an equation f, SABBAH in [Me] defined and proved the isomorphisms, for $\mathcal{M}\in R_Y$:

$$\psi_f\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}\big(\mathrm{gr}^0(\mathcal{M}),\mathcal{O}_Y\big),$$

$$\psi_f\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X,\mathcal{M})\big) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}\big(\mathcal{O}_Y,\mathrm{gr}^0(\mathcal{M})\big),$$

and this was the main difficulty.

In fact, from REMARKS 1 and 2 we have

$$\begin{aligned}\nu_Y\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big) &\simeq \nu_Y^{\mathbb{C}}\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big)\\ &\simeq \psi_c\big(p^{-1}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big)\\ &\simeq \psi_c\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_{\widetilde{X}}}({}^{\mathbb{L}}p^*\mathcal{M}[c^{-1}],\mathcal{O}_{\widetilde{X}})\big)\end{aligned}$$

(this last isomorphism holds because $\psi_c(F^\bullet)$ only depends on the behaviour of $F^\bullet$ out of $c=0$ and $\tilde{p}$ is smooth). It entails a natural isomorphism by the results in [Me] and THEOREM 1.4

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_{T_YX}}\big(\mathrm{gr}^0(\mathcal{M}^0[c^{-1}]),\mathcal{O}_{T_YX}\big) \xrightarrow{\sim} \mathbb{R}\mathcal{H}om_{\mathcal{D}_{T_YX}}\big(\underline{\nu_Y}(\mathcal{M}),\mathcal{O}_{T_YX}\big),$$

which entails isomorphism (*).

The isomorphism (**) is deduced in the same way.

The following THEOREM 1.6 will also be a main tool : let $i: Y\hookrightarrow X$ denote the inclusion and let $\mathcal{D}_{Y\to X}$ denote the $(\mathcal{D}_Y, i^{-1}\mathcal{D}_X)$-bimodule $\mathcal{O}_Y \underset{i^{-1}\mathcal{O}_X}{\otimes} i^{-1}\mathcal{D}_X$. It is a well known fact (*cf.* [L-S]) that when $\mathcal{M}\in B_Y$, the induced system

$${}^{\mathbb{L}}i^*\mathcal{M} = \mathcal{M}_Y := \mathcal{D}_{Y\to X} \overset{\mathbb{L}}{\underset{i^{-1}\mathcal{D}_X}{\otimes}} i^{-1}\mathcal{M}$$

has $\mathcal{D}_Y$-coherent cohomologies.

Let us denote by ℓ the inclusion of Y in $T_Y X$ by the zero section. It is clear that if $\mathcal{M} \in B_Y$ then $\underline{\nu_Y}(\mathcal{M})$ is a $\mathcal{D}_{T_Y X}$-module specialisable along Y so ${}^{\mathbb{L}}\ell^* \underline{\nu_Y}(\mathcal{M})$ has $\mathcal{D}_Y$-coherent cohomology.

LEMMA 1.5.1. — *Let $\mathcal{M} \in B_Y$. Then, for all $i \leq -1$*

$$ {}^{\mathbb{L}}\ell^* \Big(\frac{\mathcal{M}_i^0}{\mathcal{M}_{i+1}^0} \Big) = 0. $$

Proof. — The question being local, we may assume that we are given coordinates (c, x', y') on $\widetilde{X}$ and (x, y) on X, such that $Y = \{(x, y), x = 0\}$, $p(c, x', y') = (cx', y')$ and (y', x') are the coordinates on $T_Y X = c^{-1}(0)$. Then $\ell(y) = (y, 0)$.

For the sake of simplicity, we will assume $\operatorname{codim} Y = 1$, the calculations being easily generalised to $d > 1$.

So we want to prove that $\mathcal{M}_i^0 / \mathcal{M}_{i+1}^0 \xrightarrow{x'} \mathcal{M}_i^0 / \mathcal{M}_{i+1}^0$, where x' acts by left multiplication, is an isomorphism.

Let $b(\theta)$ be the Bernstein-Sato polynomial associated to the canonical filtration on $\mathcal{M}$, hence satisfying $b^{-1}(0) \subset [0, 1[$. Then θ is a $\mathcal{D}_Y$-isomorphism on $\operatorname{gr}^{-j}(\mathcal{M})$ for every $j \geq 1$ since $b(\theta + j) \operatorname{gr}^{-j}(\mathcal{M}) = 0$ and $b(j) \neq 0$.

Denote $\mathcal{I}^i = \mathcal{O}_{\widetilde{X}}\, c^i$. We have an epimorphism :

$$ \frac{\mathcal{M}_j^0}{\mathcal{M}_{j+1}^0} \longleftarrow \bigoplus_{i \geq 0} \frac{\mathcal{I}^i}{\mathcal{I}^{i+1}} \underset{p^{-1}\mathcal{O}_Y}{\otimes} p^{-1}(\operatorname{gr}^{j-i} \mathcal{M}). $$

We finally define

$$ x'^{-1} = \sum_{i \geq 0} (c \otimes D_x) \circ (1 \otimes \theta_i^{-1}), $$

where θ_i^{-1} is the inverse of θ in $\operatorname{gr}^{j-i}(\mathcal{M})$ which satisfies $x' x'^{-1} = x'^{-1} x' = \mathrm{id}$. $\square$

We can prove the analogue of Theorem 4.2.3 (iv) of [K-S] :

THEOREM 1.6. — *For every $\mathcal{D}_X$-module in B_Y one has a natural isomorphism in $D^b(\mathcal{D}_Y)$*

$$ {}^{\mathbb{L}}i^* \mathcal{M} \xleftarrow{\sim} {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(\mathcal{M}). $$

Proof. — Let $\widetilde{X}$ denote the complex normal deformation of X along Y and $p : \widetilde{X} \to X$, $c : \widetilde{X} \to \mathbb{C}$ the canonical morphisms such that $T_Y X$ is isomorphic to the hypersurface $c^{-1}(0)$. So we have a commutative diagram

$$\begin{array}{ccc} T_Y X & \overset{j}{\hookrightarrow} & \widetilde{X} \\ {\scriptstyle \ell}\big\uparrow & & \big\downarrow{\scriptstyle p} \\ Y & \underset{i}{\hookrightarrow} & X \end{array}$$

and hence ${}^{\mathbb{L}}i^* \mathcal{M} \simeq {}^{\mathbb{L}}\ell^* \, {}^{\mathbb{L}}j^* \, {}^{\mathbb{L}}p^* \mathcal{M}$, for any $\mathcal{D}_X$-module $\mathcal{M}$. So we have to define, for $\mathcal{M} \in B_Y$, a natural morphism

$$(*) \qquad \theta(\mathcal{M}) : {}^{\mathbb{L}}\ell^* \, {}^{\mathbb{L}}j^* \, {}^{\mathbb{L}}p^* \, \mathcal{M} \longleftarrow {}^{\mathbb{L}}\ell^* \, \underline{\nu_Y}(\mathcal{M}).$$

The construction of this morphism and the proof that it is an isomorphism will relie on the preceding lemma and on THEOREM 1.4. We have

$${}^{\mathbb{L}}j^* \, \mathcal{M}^0 \xrightarrow[\text{QIS}]{\sim} (\mathcal{M}^0_{-1} \xrightarrow{c} \mathcal{M}^0_0) \longrightarrow \Big(\frac{\mathcal{M}^0_{-1}}{\mathcal{M}^0_0} \xrightarrow{c} \frac{\mathcal{M}^0_0}{\mathcal{M}^0_1} \Big)$$

and this defines a morphism in

$$D^b(\mathcal{D}_{\widetilde{Y}}) : {}^{\mathbb{L}}j^* \mathcal{M}^0 \xrightarrow{\sim} \Big(\frac{\mathcal{M}^0_{-1}}{\mathcal{M}^0_0} \xrightarrow{c} \frac{\mathcal{M}^0_0}{\mathcal{M}^0_1} \Big).$$

Hence a morphism

$${}^{\mathbb{L}}\ell^* \, {}^{\mathbb{L}}j^* \, {}^{\mathbb{L}}p^* \, \mathcal{M} \longrightarrow {}^{\mathbb{L}}\ell^* \, {}^{\mathbb{L}}j^* \, \mathcal{M}^0 \longrightarrow {}^{\mathbb{L}}\ell^* \Big(\frac{\mathcal{M}^0_{-1}}{\mathcal{M}^0_0} \xrightarrow{c} \frac{\mathcal{M}^0_0}{\mathcal{M}^0_1} \Big)$$

in $D^b(\mathcal{D}_Y)$. By LEMMA 1.5.1 the right term is quasi-isomorphic to ${}^{\mathbb{L}}\ell^*(\mathcal{M}^0_0/\mathcal{M}^0_1)$. To get the morphism $\theta(\mathcal{M})$ we compose with

$${}^{\mathbb{L}}\ell^* (\mathcal{M}^0_0/\mathcal{M}^0_1) \longrightarrow {}^{\mathbb{L}}\ell^* \big(\mathrm{gr}^0(\mathcal{M}^0[c^{-1}])\big) \cong {}^{\mathbb{L}}\ell^* \big(\underline{\nu_Y}(\mathcal{M})\big).$$

Notice that $\theta(\cdot)$ is natural in the following sense. Let $f : \mathcal{M} \to \mathcal{N}$ be a $\mathcal{D}_X$-linear morphism of modules in B_Y such that f is compatible with the canonical filtrations, i.e., $f(V^*(\mathcal{M})) \subset V^*(\mathcal{N})$. Then the diagram

$$\begin{array}{ccc} {}^{\mathbb{L}}i^* \, \mathcal{M} & \longrightarrow & {}^{\mathbb{L}}i^* \, \mathcal{N} \\ {\scriptstyle \theta(\mathcal{M})}\big\downarrow & & \big\downarrow{\scriptstyle \theta(\mathcal{N})} \\ {}^{\mathbb{L}}\ell^* \, \underline{\nu_Y}(\mathcal{M}) & \longrightarrow & {}^{\mathbb{L}}\ell^* \, \underline{\nu_Y}(\mathcal{N}) \end{array}$$

is commutative in $D^b(\mathcal{D}_Y)$. In fact $1 \otimes f$ induces natural morphisms

$$\mathcal{M}_k^0 \longrightarrow \mathcal{N}_k^0, \qquad \forall k \in \mathbb{Z}.$$

By LEMMA 1.4.1 and THEOREM 1.4 with REMARK 1, $\theta(\mathcal{M})$ is an isomorphism when $\mathcal{M}$ is an elementary module. We will now argue by reduction to this case. So we have to prove that $\theta(\mathcal{M})_y$ is an isomorphism for any $y \in Y$ so we may consider a (filtered) resolution of $\mathcal{M}$ by elementary modules L_i, $i \leq 0$, which we denote by $L_\bullet$:

$$\cdots \longrightarrow L_i \longrightarrow L_{i-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \mathcal{M} \to 0.$$

Hence we get a commutative diagram

$$\begin{array}{ccccccccc}
(\cdots \longrightarrow & {}^{\mathbb{L}}i^* L_i & \xrightarrow{\Delta_i} & {}^{\mathbb{L}}i^* L_{i-1} & \longrightarrow \cdots \longrightarrow & {}^{\mathbb{L}}i^* L_0)_y & \xrightarrow{\text{QIS}} & {}^{\mathbb{L}}i^* \mathcal{M}_y \\
& \downarrow{\scriptstyle \theta(L_i)} & & \downarrow{\scriptstyle \theta(L_{i-1})} & & \downarrow{\scriptstyle \theta(L_0)} & & \downarrow{\scriptstyle \theta(\mathcal{M})} \\
(\cdots \longrightarrow & {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(L_i) & \underset{\tilde{\Delta}_i}{\rightarrow} & {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(L_{i-1}) & \rightarrow \cdots \rightarrow & {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(L_0))_y & \underset{\text{QIS}}{\longrightarrow} & {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(\mathcal{M})_y
\end{array}$$

in $D^b(Y)$, where $\theta(L_i)$ are isomorphisms and Δ_i and $\widetilde{\Delta}_i$ are morphisms of complexes. We want to prove that $\theta(\mathcal{M})$ induces isomorphisms in the cohomology, and this is equivalent to proving that

$$\theta(L_\bullet)_y : {}^{\mathbb{L}}i^* L_{\bullet y} \longrightarrow {}^{\mathbb{L}}\ell^* \underline{\nu_Y}(L_\bullet)_y$$

induces an isomorphism of the cohomology of the single complexes $S({}^{\mathbb{L}}i^* L_\bullet)_y$ and $S({}^{\mathbb{L}}\ell^* \underline{\nu_Y}(L_\bullet))_y$.

We will use the following result :

LEMMA 1.6.1. — *Let $\mathcal{O}$ be an abelian category and*

$$L_\bullet = \big(\cdots \longrightarrow L_i \xrightarrow{\Delta_i} L_{i-1} \longrightarrow \cdots \longrightarrow L_0 \to 0\big),$$

$$G_\bullet = \big(\cdots \longrightarrow G_i \xrightarrow{\tilde{\Delta}_i} G_{i-1} \longrightarrow \cdots \longrightarrow G_0 \to 0\big)$$

be two double complexes in $\mathcal{O}$, that is, L_i and G_i are complexes in $\mathcal{O}$, $\widetilde{\Delta}_i \widetilde{\Delta}_{i+1} = 0$, $\Delta_i \Delta_{i+1} = 0$ and Δ_i, $\widetilde{\Delta}_i$ are morphisms of complexes.

Let $\theta_\bullet : L_\bullet \to G_\bullet$ be a morphism of double complexes, such that $\theta_i : L_i \to G_i$ is a quasi-isomorphism for all $i \in \mathbb{Z}$. Then $\theta_\bullet$ induces a quasi-isomorphism of the single complexes $S(L_\bullet) \to S(G_\bullet)$.

Proof. — This is an immediate consequence of Thm 1.9.3 of [K-S]. □

We shall consider the following commutative diagram :

$$
\begin{array}{cccc}
 & {}^{\mathbb{L}}\ell^*\, {}^{\mathbb{L}}j^*\, {}^{\mathbb{L}}p^*\, L_\bullet & \xrightarrow{\ \mathrm{QIS}\ } & {}^{\mathbb{L}}\ell^*\, {}^{\mathbb{L}}j^*\, {}^{\mathbb{L}}p^*\, \mathcal{M} \\
 & \downarrow{\scriptstyle \mathrm{QIS}} & & \\
 & {}^{\mathbb{L}}\ell^*\, {}^{\mathbb{L}}j^*\, L^0_\bullet & & \\
 & \uparrow{\scriptstyle \mathrm{QIS}} & & \\
 & {}^{\mathbb{L}}\ell^*\Big(\cdots \to (L^0_{i,-1} \xrightarrow{c} L^0_{i,0}) \to \cdots\Big) & & \\
 & \downarrow{\scriptstyle \mathrm{QIS}} & & \\
\Big\downarrow{\scriptstyle \theta(L_\bullet)} & {}^{\mathbb{L}}\ell^*\Big(\cdots \to \Big(\dfrac{L^0_{i,-1}}{L^0_{i,0}} \xrightarrow{c} \dfrac{L^0_i}{L^0_{i,1}}\Big) \to \cdots\Big) & & \Big\downarrow{\scriptstyle \theta(\mathcal{M})} \\
 & \downarrow{\scriptstyle \mathrm{QIS}} & & \\
 & {}^{\mathbb{L}}\ell^*\Big(\cdots \to \dfrac{L^0_i}{L^0_{i,1}} \to \cdots\Big) & & \\
 & \downarrow{\scriptstyle \mathrm{QIS}} & & \\
 & {}^{\mathbb{L}}\ell^*\Big(\cdots \to \mathrm{gr}^0(L^0_i[c^{-1}]) \to \cdots\Big) & & \\
 & \uparrow{\scriptstyle \mathrm{QIS}} & & \\
 & {}^{\mathbb{L}}\ell^*\, \underline{\nu_Y}(L_\bullet) & \xrightarrow{\ \mathrm{QIS}\ } & {}^{\mathbb{L}}\ell^*\big(\underline{\nu_Y}(\mathcal{M})\big).
\end{array}
$$

Here, the left vertical arrows are morphisms of double complexes which satisfy the assumption of LEMMA 1.6.1. Hence the theorem. □

As an easy application, we recover the following Cauchy-Kowalewski Theorem, proved by [L-MF] in the larger class of fuchsian systems along Y, by other methods.

COROLLARY 1.7. — *Let $\mathcal{M} \in R_Y$. Then the natural morphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Y} \longrightarrow \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}({}^{\mathbb{L}}i^*\mathcal{M}, \mathcal{O}_Y)$$

is an isomorphism.

Proof. — By the properties of $\nu_Y(\cdot)$ and because $\mathcal{M} \in R_Y$, we have

$$\begin{aligned}\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)_{|Y} &\simeq \nu_Y\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\big)_{|Y} \\ &\simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_{T_Y X}}\big(\underline{\nu_Y}(\mathcal{M}), \mathcal{O}_{T_Y X}\big)_{|Y} \\ &\xrightarrow{(*)} \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}\big({}^{\mathbb{L}}\ell^* \,\underline{\nu_Y}(\mathcal{M}), \mathcal{O}_Y\big) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{D}_Y}({}^{\mathbb{L}}i^*\mathcal{M}, \mathcal{O}_Y).\end{aligned}$$

Since the morphism (*) is defined, to prove it is an isomorphism we may suppose that $\mathcal{M}$ is elementary and, after specialisation, it is enough to consider

$$\underline{\nu_Y}(\mathcal{M}) \simeq \frac{\mathcal{D}_{T_Y X}}{\mathcal{D}_{T_Y X}(\tilde{\theta} - \alpha)^p} =: \widetilde{\mathcal{M}}_\alpha, \qquad \alpha \in \mathbb{C}.$$

It is an easy exercise to verify that for a module $\widetilde{\mathcal{M}}_\alpha$ of that form one has :

$$\mathcal{H}om_{\mathcal{D}_{T_Y X}}\big(\widetilde{\mathcal{M}}_\alpha, \mathcal{O}_{T_Y X}\big)_{|Y} \simeq \begin{cases} 0 & \text{if } \alpha \notin \mathbb{N}^* = \{0,1,\dots\}, \\ \mathcal{O}_Y^M & \text{if } \alpha \in \mathbb{N}^*,\ M = \#\{\beta \in \mathbb{N}^d,\ |\beta| = \alpha\}, \end{cases}$$

$$\mathrm{Ext}^1_{\mathcal{D}_{T_Y X}}\big(\widetilde{\mathcal{M}}_\alpha, \mathcal{O}_{T_Y X}\big)_{|Y} \simeq \begin{cases} 0 & \text{if } \alpha \notin \mathbb{N}^*, \\ \mathcal{O}_Y^M & \text{if } \alpha \in \mathbb{N}^*. \end{cases}$$

Similarly,

$$\widetilde{\mathcal{M}}_{\alpha Y} \simeq \begin{cases} 0 & \text{if } \alpha \notin \mathbb{N}^*, \\ \mathcal{D}_Y^M \xrightarrow[0]{} \mathcal{D}_Y^M & \text{if } \alpha \in \mathbb{N}^*. \quad \square \end{cases}$$

The case of a hypersurface.

We now will supose that Y is a smooth hypersurface defined by $f = 0$, $\mathrm{d}f_{|Y} \neq 0$. In this case $\mathrm{d}f$ defines a function $\tilde{f} : T_Y X \to \mathbb{C}$ and we denote by s the section of $T_Y X \to Y$ given by $\tilde{f}^{-1}(1)$ (*cf.* [K-S2]). Let $\mathcal{M}$ be a $\mathcal{D}$-module on B_Y and consider $\psi_f(\mathcal{M})$ the coherent $\mathcal{D}_Y$-module of nearby-cycles defined by KASHIWARA [K2] (when $\mathcal{M}$ is endowed with the good V-filtration associated to $G = \{z \in \mathbb{C};\ 0 \leq z < 1\}$ we have $\psi_f(\mathcal{M}) \simeq \mathrm{gr}^0_G(\mathcal{M})$).

We prove here the analogue of [K-S2, p. 352]. Let $h : T_Y X \to \mathbb{C}$ be given by

$$h = \tilde{f} - 1.$$

Then $h=0$ is an equation for the image of Y by s. So

$$^{\mathbb{L}}s^*\,\underline{\nu_Y}(\mathcal{M}) \underset{\text{QIS}}{\simeq} \{\underline{\nu_Y}(\mathcal{M}) \xrightarrow{h} \underline{\nu_Y}(\mathcal{M})\},$$

where h multiplies to the left. It is easy to see that h is injective and we get a commutative diagram

$$\begin{array}{ccc} \underline{\nu_Y}(\mathcal{M}) & \xrightarrow{\quad h \quad} & \underline{\nu_Y}(\mathcal{M}) \\ \uparrow & & \uparrow \\ 0 & \xrightarrow{\qquad} & \mathrm{gr}^0(\mathcal{M}) \end{array}$$

which defines a morphism

$$\mathrm{gr}^0(\mathcal{M}) \longrightarrow {}^{\mathbb{L}}s^*\,\underline{\nu_Y}(\mathcal{M}).$$

THEOREM 1.8. — *Let $\mathcal{M}$ be in B_Y. Then the morphism*

$$\mathrm{gr}^0(\mathcal{M}) \longrightarrow {}^{\mathbb{L}}s^*\,\underline{\nu_Y}(\mathcal{M}) \quad (\textit{depending on } f)$$

is an isomorphism.

Proof. — Let $p \in Y$. In local coordinates we may assume that X is an open set in $\mathbb{C}^n$ with the coordinates $(t, y_1, \dots, y_{n-1})$ and Y defined by $t=0$. Let (y,τ) be the induced coordinates in T_YX; in this situation,

$$\mathrm{d}f = \mathrm{d}t, \quad \tilde{f}\,(y,\tau) = \tau, \quad s(y) = (y,1).$$

Once again it is enough to prove the theorem for elementary $\mathcal{D}$-modules and hence for modules defined by a) $B(\theta)$ and b) $R(\theta)I$, following the notation of PROPOSITION 1.2. In such situations, decomposing $\underline{\nu_Y}(\mathcal{M})$ in direct sums we are led to consider $\underline{\nu_Y}(\mathcal{M}) \simeq \mathcal{D}_{T_YX}/\mathcal{D}_{T_YX}(\tilde{\theta} - \alpha)^p$ and we shall study the complex

$$\underset{\text{degree } -1}{\frac{\mathcal{D}_{T_YX}}{\mathcal{D}_{T_YX}(\tau\, D_\tau - \alpha)^p}} \xrightarrow{\ \tau - 1\ } \underset{\text{degree } 0}{\frac{\mathcal{D}_{T_YX}}{\mathcal{D}_{T_YX}(\tau\, D_\tau - \alpha)^p}}.$$

By using the division theorem of Weierstrass we obtain

$$\mathrm{coker}(\tau - 1) = \frac{\mathcal{D}_{T_YX}}{\mathcal{D}_{T_YX}(\tau\, D_\tau - \alpha)^p + (\tau - 1)\,\mathcal{D}_{T_YX}}$$

$$\simeq \mathcal{D}_Y^p \simeq \psi_f(\mathcal{M}). \quad \square$$

Inverse and direct image.

We will study now the behaviour of the specialisation under inverse image recalling the result of [Me] for direct images.

THEOREM 1.9 (Inverse image). — *Let $f : X \to Z$ be a smooth holomorphic map, $Y' \subset Z$ a submanifold, $Y = f^{-1}(Y')$ and let $\mathcal{M} \in B_{Y'}$ (resp. $\mathcal{M} \in R_{Y'}$). Then $f^*\mathcal{M}$ is in B_Y (resp. $f^*\mathcal{M} \in R_Y$) and*

$$\text{(A)} \qquad \underline{\nu_Y}(f^*\mathcal{M}) \simeq (Tf)^* \underline{\nu_{Y'}}(\mathcal{M}),$$

where Tf is the induced map

$$T_Y X \longrightarrow T_{Y'} Z.$$

Proof. — Since f is smooth, $\mathcal{O}_X$ is flat over $\mathcal{O}_Z$ and so

$$\mathcal{H}^0(f^*\mathcal{M}) \underset{\text{QIS}}{\simeq} f^*\mathcal{M} \simeq \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Z}{\otimes} \mathcal{M}.$$

Let $\{\mathcal{M}^j\}$ be a good V-filtration on $\mathcal{M}$ with b-function $b(s) \in \mathbb{C}[s]$. Then

$$(f^*\mathcal{M})^j \underset{\text{def}}{=} \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Z}{\otimes} \mathcal{M}^j$$

are $V^0(\mathcal{D}_X)$-coherent since they are coherent modules over the coherent subring $\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}V^0(\mathcal{D}_Z)$ of $V^0(\mathcal{D}_X)$. Obviously, all conditions of good V-filtration are satisfied by $(f^*\mathcal{M})^j$ with $b(s)$ as b-function. Furthermore,

$$\frac{(f^*\mathcal{M})^j}{(f^*\mathcal{M})^{j+1}} \simeq \mathcal{O}_Y \underset{f^{-1}\mathcal{O}_{Y'}}{\otimes} f^{-1}\operatorname{gr}^k(\mathcal{M}),$$

which implies (A).

Let now $\mathcal{M} \in R_{Y'}$, $\mathcal{M}_0 \subset \mathcal{M}$ be a coherent sub-$\mathcal{O}_Z$-module such that $\mathcal{M} = \mathcal{D}_Z\mathcal{M}_0$ and $b(s) \in \mathbb{C}[s]$ of degree $\mathcal{M}$ such that

$$\text{(B)} \qquad b(\theta)\,\mathcal{M}_0 \subset \big(\mathcal{D}_Z(m) \cap V^1(\mathcal{D}_Z)\big)\,\mathcal{M}_0.$$

It is clear that $\widetilde{\mathcal{M}_0} = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Z}{\otimes} f^{-1}\mathcal{M}_0$ is $\mathcal{O}_X$-coherent and generates $f^*\mathcal{M}$; clearly, $b(s)$ satisfies the regularity condition (B) with respect to $\widetilde{\mathcal{M}_0}$. □

Having THEOREM 1.4 in account we think that THEOREM 1.9 may be generalised to non smooth case (of course with restrictions).

In order to study direct images, let us recall the result by SABBAH-MEBKHOUT in [Me].

THEOREM 1.10. — *Let $f : X \to Z$ a holomorphic map, Y' a hypersurface of Z and $Y = f^{-1}(Y')$. Assume that Y is a smooth hypersurface. Let $\mathcal{M}$ be in B_Y, and assume that $\mathcal{M} = \mathcal{D}_X \mathcal{M}_0$, $\mathcal{M}_0$ a coherent sub-$\mathcal{O}_X$-module and that f is proper on* $\operatorname{supp} \mathcal{M}$. *Then, for every i,*

$$\int_f^i \mathcal{M} \in B_{Y'}.$$

Moreover, the canonical filtration of $\int_f^i \mathcal{M}$ is induced by the canonical filtration of $\mathcal{M}$.

By adapting the proof of the preceding theorem, one may consider a different situation. Let f be transversal to Y', that is the graph of f in $X \times Z$ is transversal to $X \times Y'$. In that case, $Y = f^{-1}(Y')$ is a smooth submanifold of X and $\operatorname{codim} Y = \operatorname{codim} Y'$.

THEOREM 1.11. — *Let $f : X \to Z$ be a proper holomorphic map transversal to Y'. Let $\mathcal{M} \in B_Y$; then we have :*

1) *for all j, $\int_f^j \mathcal{M}$ is specialisable along Y' ;*

2) $\nu_{Y'}\Big(\int_f^j \mathcal{M}\Big) \simeq \int_{T_f}^j \nu_Y(\mathcal{M})$.

2. Microlocalisation of $\mathcal{D}$-modules

In this section we will define the functor of microlocalisation $\underline{\mu_Y}(*)$ in the category B_Y by means of the formal Fourier transform for $\mathcal{D}$-modules (*cf.* [H-K], [Br], [Br-Ma-V]) and obtain fundamental relations with the geometrical microlocalisation in $D^b(X)$ (still denoted by $\mu_Y(*)$).

We begin by recalling the Fourier transform in the category of $\mathcal{D}$-modules over a holomorphic vector bundle.

Let Y be a complex analytic manifold and $E \xrightarrow{\pi} Y$ be a holomorphic vector bundle on Y. Let us denote $\mathcal{D}_{[E]} \subset \pi_* \mathcal{D}_E$ the sheaf of differential operators polynomial in the fiber variables. Let θ denote the Euler field on E. A $\pi_*(\mathcal{D}_E)$ or a $\mathcal{D}_{[E]}$-left coherent module $\mathcal{M}$ is monodromic if $\mathcal{M}$ is generated by local sections satisfying $b(\theta)\, u = 0$ for some non-zero $b(\theta) \in \mathbb{C}[\theta]$. We denote this abelian subcategory by $\operatorname{Mon}(\mathcal{D}_E)$. Obviously, if $\mathcal{M}$ is in B_Y, $\nu_Y(\mathcal{M})$ is monodromic.

Let E' be the holomorphic dual bundle. Let us consider local coordinates y in Y, (y, x) in E and (y, ξ) in E'. Let us consider $\Omega_{E/Y}$ the sheaf of relative differential forms to $\pi : E \to Y$.

Then the Fourier transform $\mathcal{F}$ is canonically defined as an isomorphism of sheaves over Y by

$$\text{(H)} \qquad \Omega_{E/Y} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{[E]} \underset{\mathcal{O}_Y}{\otimes} \Omega_{E/Y}^{\otimes -1} \xrightarrow{\ \mathcal{F}\ } \mathcal{D}_{[E']}$$

by
$$\begin{cases} d\tau \otimes P(y, D_y) \otimes d\tau^{\otimes -1} \longmapsto P(y, D_y), \\ d\tau \otimes \tau_j \otimes d\tau^{\otimes -1} \longmapsto \dfrac{\partial}{\partial \xi_j}, \\ d\tau \otimes \dfrac{\partial}{\partial \tau_j} \otimes d\tau^{\otimes -1} \longmapsto -\xi_j. \end{cases}$$

Let $\mathcal{M}$ be a $\mathcal{D}_{[E]}$-coherent module. Then the isomorphism $\mathcal{F}$ above gives rise to an exact functor from $\mathrm{Mon}(\mathcal{D}_{[E]})$ to $\mathrm{Mon}(\mathcal{D}_{[E']})$ by setting

$$\mathcal{F}(\mathcal{M}) = \Omega_{E/Y} \underset{\mathcal{O}_Y}{\otimes} \mathcal{M},$$

where $\Omega_{E/Y} \underset{\mathcal{O}_Y}{\otimes} \mathcal{M}$ is regarded as $\Omega_{E/Y} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{[E]} \underset{\mathcal{O}_Y}{\otimes} \Omega_{E/Y}^{\otimes -1}$-module and via the isomorphism (H) a $\mathcal{D}_{[E']}$-module.

Let $\mathcal{M}$ be a monodromic $\pi_* \mathcal{D}_E$-module and let $\mathcal{M}'$ be the $\mathcal{D}_{[E]}$-submodule of the sections u such that there exists (locally) a nontrivial polynomial $b(\theta) \in \mathbb{C}[\theta]$, satisfying $b(\theta)\, u = 0$. We can see that $\mathcal{M}'$ is coherent : let us consider local sections $u_1, \ldots, u_p$, with $u_i \in \mathcal{M}$, generating $\mathcal{M}$ over $\mathcal{D}_E$. Set

$$\mathcal{M}'' = \sum_{i=1}^{p} \mathcal{D}_{[E]} u_i.$$

Therefore $\mathcal{M}'' \subset \mathcal{M}'$ and $\mathcal{M}''$ is $\mathcal{D}_{[E]}$-coherent. So we have

$$\mathcal{D}_E \underset{\mathcal{D}_{[E]}}{\otimes} \mathcal{M}'' = \mathcal{M} = \mathcal{D}_E \underset{\mathcal{D}_{[E]}}{\otimes} \mathcal{M}'$$

and since $\mathcal{D}_E$ is faithfully flat over $\mathcal{D}_{[E]}$ we get $\mathcal{M}'' = \mathcal{M}'$. It then makes sense to define

$$\mathcal{F}(\mathcal{M}) = \mathcal{D}_{E'} \underset{\mathcal{D}[E']}{\otimes} \mathcal{F}(\mathcal{M}').$$

A sheaf of $\mathbb{C}$-vector spaces over E is monodromic if it is locally constant along the orbits $\mathbb{C}^* \eta$, where $\eta \in E - Y$.

Remark. — Monodromic sheaves define an abelian full subcategory of the category of sheaves of $\mathbb{C}$-vector spaces.

An object $K \in D^b(E)$ is monodromic if the sheaves $H^i(K)$ are monodromic. We will denote $\mathrm{Mon}(E)$ the subcategory of $D^b(E)$ formed by monodromic complexes.

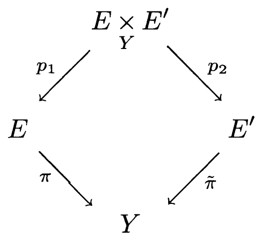
We also keep the notation F for the Fourier-Sato transform

$$\mathrm{Mon}(E) \longrightarrow \mathrm{Mon}(E')$$

(see [Br-Ma-V] and [H-K]). Let us recall how it is defined (*cf.* [K-S] for detailed study). If A is a subset of E, the polar A^0 is

$$A^0 = \{y \in E'\,;\ \tilde{\pi}(y) \in \pi(A) \text{ and } \langle x, y\rangle \geq 0,\ \forall x \in \pi^{-1}\tilde{\pi}(y) \cap A\}.$$

Let us consider p_1, p_2 the projections



and

$$P = \{(x,y) \in E \underset{Y}{\times} E'\,;\ \langle x, y\rangle \geq 0\}.$$

Then the Fourier-Sato transform is the functor

$$F = \mathbb{R}p_{2*} \circ \mathbb{R}\Gamma_P \circ p_1^{-1},$$

which is well defined from $D^+_{\mathbb{R}^+}(E)$ to $D^+_{\mathbb{R}^+}(E')$. Here, $D^+_{\mathbb{R}^+}(E)$ denotes the full subcategory of $D^+(E)$ whose objects have R^+-conic cohomology groups.

In particular, F is defined from $\mathrm{Mon}(E)$ to $\mathrm{Mon}(E')$.

Then the geometrical microlocalisation is just the composition of F with ν_Y in $D^b(X)$. Hereafter we will apply these notions to the following situation : Y is a submanifold of X, $E = T_YX \xrightarrow{\pi} Y$ and $E' = T^*_YX$.

DEFINITION 2.1. — Let $\mathcal{M} \in B_Y$; the *microlocalised* of $\mathcal{M}$ along Y, $\underline{\mu_Y}(\mathcal{M})$, is the $\mathcal{D}_{T^*_YX}$-module $\mathcal{F}(\underline{\nu_Y}(\mathcal{M}))$.

REMARK. — In [Br-Ma-V] (*cf.* [H-K] as well) Brylinski-Malgrange-Verdier proved that the functors $\mathrm{Sol}(*) = \mathbb{R}\mathcal{H}om_{\mathcal{D}}(*, \mathcal{O})$ and $\mathrm{DR}(*) = \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathcal{O}, *)$ commute with Fourier-Sato transform in the category of regular holonomic $\mathcal{D}_{[E]}$-modules. Actually, the same result holds for $\underline{\nu_Y}(\mathcal{M})$ with $\mathcal{M} \in B_Y$.

PROPOSITION 2.2. — *Let* $\mathcal{M} \in B_Y$; *then* $\mathrm{Sol}(\underline{\nu_Y}(\mathcal{M}))$ *and* $\mathrm{DR}(\underline{\nu_Y}(\mathcal{M}))$ *are monodromic.*

Proof. — By the preceding remark we may restrict the proof to elementary modules by already classical arguments. After specialisation we may assume $\underline{\nu_Y}(\mathcal{M}) \simeq \mathcal{D}_E/\mathcal{D}_E(\tilde{\theta} - \alpha)^p$, where $\alpha \in \mathbb{C}$ and $p \in \mathbb{N}$, and the proposition follows. ▯

Because $\underline{\nu_Y}(\mathcal{M})$ is monodromic we can use the results of [Ma1], [H-K] and obtain :

THEOREM 2.3. — *Let* $\mathcal{M}$ *be in* B_Y. *Then*

$$F\big(\mathrm{Sol}(\underline{\nu_Y}(\mathcal{M}))\big) \simeq \mathrm{Sol}\big(\underline{\mu_Y}(\mathcal{M})\big)[-\operatorname{codim} Y],$$

$$F\big(\mathrm{DR}(\underline{\nu_Y}(\mathcal{M}))\big) \simeq \mathrm{DR}\big(\underline{\mu_Y}(M)\big)[-\operatorname{codim} Y] \quad \textit{in} \quad D^b(T^*_Y X).$$

By THEOREM 1.5 we also conclude :

COROLLARY 2.4. — *Let* $\mathcal{M} \in R_Y$. *Then*

$$\mathrm{Sol}\big(\underline{\mu_Y}(\mathcal{M})\big) \simeq \mu_Y\big(\mathrm{Sol}(\mathcal{M})\big)[\operatorname{codim} Y],$$

$$\mathrm{DR}\big(\underline{\mu_Y}(\mathcal{M})\big) \simeq \mu_Y\big(\mathrm{DR}(\mathcal{M})\big)[\operatorname{codim} Y] \quad \textit{in} \quad D^b(T^*_Y X).$$

To end this section, our aim is to show that, using the category B_Y, it is possible to obtain the parallelism between geometrical and formal microlocalisation as it was the case between geometrical and formal specialisation.

It is not within our scope to exhaust all possible relations; nevertheless, we find useful to prove two results on inverse images, which are the analogue of Theorem 4.3.2 of [K-S].

Let $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X)$, Y a d-codimensional submanifold of X, $i : Y \hookrightarrow X$ the inclusion, and denote

$$\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)\,[n] \quad (n = \dim X),$$

$$^{\mathbb{L}}i^!\, \mathcal{M} := \mathbb{D}(\,^{\mathbb{L}}i^*\, \mathbb{D}\, \mathcal{M}).$$

Let π denote the projection $T^*_Y X \to Y$ and $\tilde{\ell}$ the immersion of Y in $T^*_Y X$ by the zero section. For $F^\bullet \in D^b(X)$ we have :

$$\mathbb{R}\pi_*\, \mu_Y(F^\bullet) \simeq i^!\, F^\bullet \simeq \tilde{\ell}^{-1}\, \mu_Y(F^\bullet),$$

$$\mathbb{R}\pi_!\, \mu_Y(F^\bullet) \simeq \mathbb{R}\Gamma_Y(\mu_Y(F^\bullet)) \simeq i^{-1}\, F^\bullet \otimes \omega_{Y|X}.$$

Let us note for short

$$\tilde{Y} := T_Y X, \quad \tilde{Y}^* := T_Y^* X.$$

THEOREM 2.5. — *Let $\mathcal{M} \in B_Y$. Then there are natural isomorphisms in* $D^+(\mathrm{Mod}(\mathcal{D}_Y))$:

a) ${}^{\mathbb{L}}i^! \mathcal{M} \xrightarrow{\sim} {}^{\mathbb{L}}\tilde{\ell}^* \underline{\mu_Y}(\mathcal{M})$;

b) $\displaystyle\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) \xleftarrow{\sim} {}^{\mathbb{L}}i^* \mathcal{M}$;

c) $\displaystyle{}^{c}\!\!\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) \xleftarrow{\sim} {}^{\mathbb{L}}i^! \mathcal{M}$.

Proof. — We have to define the morphisms and once this is done, check that they are isomorphisms by reducing to the case $\mathcal{M}$ elementary. Let us start by b). By definition,

$$\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) = \mathbb{R}\pi_*\big(\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \overset{\mathbb{L}}{\underset{\tilde{\mathcal{D}}_Y^*}{\otimes}} \underline{\mu_Y}(\mathcal{M})\big).$$

The isomorphism (H) given by Fourier transform induces a $\mathcal{D}_Y$-linear isomorphism $\ell^{-1}\, \underline{\nu_Y}(\mathcal{M}) \to \tilde{\ell}^{-1}\, \underline{\mu_Y}(\mathcal{M})$ which extends to an isomorphism

$$\ell^{-1}\big(\mathcal{D}_{Y \to \tilde{Y}} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}}}{\otimes}} \underline{\nu_Y}(\mathcal{M})\big) \simeq \tilde{\ell}^{-1}\big(\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}^*}}{\otimes}} \underline{\mu_Y}(\mathcal{M})\big).$$

More precisely, this isomorphism is defined as follows :

$$\begin{aligned}
{}^{\mathbb{L}}\ell^* \nu_Y(\mathcal{M}) &\simeq \ell^{-1}\Big[\big(\mathcal{D}_{Y \to \tilde{Y}} \underset{\mathcal{D}^{\mathbb{L}}_{\tilde{Y}}}{\otimes} \nu_Y(\mathcal{M})\big)\Big] \\
&\simeq \ell^{-1}\Big[\big(\mathcal{D}_{Y \to \tilde{Y}} \underset{\mathcal{O}_Y}{\otimes} \Omega^{\otimes -1}_{\tilde{Y}|Y}\big) \overset{\mathbb{L}}{\underset{\mathcal{O}_Y}{\otimes}} \big(\Omega_{\tilde{Y}|Y} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}}}{\otimes}} \nu_Y(\mathcal{M})\big)\Big] \\
&\simeq \tilde{\ell}^{-1}\Big[(\mathcal{D}_{Y \leftarrow \tilde{Y}^*}) \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}^*}}{\otimes}} \mu_Y(\mathcal{M})\Big].
\end{aligned}$$

The last isomorphism follows from that by Fourier-transform we have

$$\mathcal{D}_{\tilde{Y}^*} \underset{\mathcal{O}_Y}{\otimes} \Omega_{\tilde{Y}|Y} \simeq \Omega_{\tilde{Y}|Y} \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_{\tilde{Y}}$$

as sheaves on Y.

By THEOREM 1.6, the left side is isomorphic to ${}^{\mathbb{L}}i^* \mathcal{M}$. To achieve the proof of b) we will need the following :

LEMMA 1. — *The cohomology of* $\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M})$ *is* $\mathbb{R}^+$*-conic.*

Proof. — Since the statement is of local nature, we may suppose

$$\underline{\mu_Y}(\mathcal{M}) \simeq \frac{\mathcal{D}_{\tilde{Y}^*}}{\mathcal{D}_{\tilde{Y}^*}(\tilde{\theta} - \alpha)^p}, \quad \alpha \in \mathbb{C},$$

where $\tilde{\theta}$ is the Euler field of $T^*_Y X \to Y$. Therefore, in this situation :

$$\mathcal{F}^\bullet := \mathcal{D}_{Y \leftarrow \tilde{Y}^*} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}^*}}{\otimes}} \underline{\mu_Y}(\mathcal{M}) \underset{\text{QIS}}{\simeq} \big(\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \xrightarrow{\tilde{\theta} - \alpha} \mathcal{D}_{Y \leftarrow \tilde{Y}^*}\big).$$

In local coordinates $(x_1, \ldots, x_d, y_1, \ldots, y_{n-d})$ in X such that

$$Y = \big\{(x,y)\, ;\ x_1 = \cdots = x_d = 0\big\},$$

we have $\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \simeq \mathcal{O}_{\tilde{Y}^*}[D_{y_1} \cdots D_{y_{n-d}}]$ and

$$\mathcal{F}^\bullet \underset{\text{QIS}}{\simeq} \mathcal{O}_{\tilde{Y}^*}[D_{y_1} \cdots D_{y_{n-d}}] \xrightarrow{\tilde{\theta} + \alpha + d} \mathcal{O}_{\tilde{Y}^*}[D_{y_1} \cdots D_{y_{n-d}}],$$

where $\tilde{\theta} + \alpha + d$ acts on $\mathcal{O}_{\tilde{Y}^*}$ by the left. Hence the conclusion of the lemma. ∎

Therefore,

$$\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) \simeq \tilde{\ell}^{-1}\Big(\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}^*}}{\otimes}} \underline{\mu_Y}(\mathcal{M})\Big),$$

and b) follows.

Note also that LEMMA 1 entails

$${}^{c}\!\!\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) \simeq \mathbb{R}\Gamma_Y\big(\mathcal{D}_{Y \leftarrow \tilde{Y}^*} \overset{\mathbb{L}}{\underset{\mathcal{D}_{\tilde{Y}^*}}{\otimes}} \underline{\mu_Y}(\mathcal{M})\big).$$

To treat a) and c) we recall that by the results in [Me], [H-K] and [Ma1], duality and specialisation commute in B_Y, and duality and Fourier formal transform commute in $\mathrm{Mon}(\mathcal{D}_{\tilde{Y}})$. With these facts in mind, it is not a too hard task to define the morphisms a) and c). To prove that they are isomorphisms we will prove :

LEMMA 2.

1) *Let $\mathcal{M}$ be given by*

$$\frac{\mathcal{D}_X^N}{\mathcal{D}_X^N(b(\theta)I+Q)},$$

with $b^{-1}(0)\in\{z\in\mathbb{C}\,;\ k\le z<k+1,\ k\in\mathbb{N}\}$ *and* $Q\in V^1(\mathcal{D}_X)^{N\times N}$. *Then*

$$^{\mathbb{L}}i^!\,\mathcal{M}=0.$$

2) *Let $\mathcal{M}$ be given by*

$$\frac{\mathcal{D}_X^N}{\mathcal{D}_X^N(\theta+j)^p},\quad j\in\mathbb{Z}^+.$$

Then $^{\mathbb{L}}i^!\mathcal{M}[d]$ *is quasi-isomorphic to* $\mathcal{D}_Y^M\xrightarrow{\ 0\ }\mathcal{D}_Y^M$, *where*

$$M=\big\{\alpha\in\mathbb{N}^d\,;\ |\alpha|=j-1\big\}.$$

Proof. — This is an exercise of filtrations on $\mathcal{D}_{Y\leftarrow X}$ having in mind the action of $b(\theta)$ on the graded rings. □

Let us now prove c). We may assume

$$\underline{\mu_Y}(\mathcal{M})\simeq\frac{\mathcal{D}_{\tilde{Y}^*}}{\mathcal{D}_{\tilde{Y}^*}(\tilde{\theta}+\alpha+d)^p},\quad\alpha\in\mathbb{C},$$

and we get a commutative diagram

$$\begin{array}{ccc}
\displaystyle\int_{\tilde\pi}^{c}\underline{\mu_Y}(\mathcal{M}) & & \\
\simeq\big\downarrow\text{QIS} & & \\
\Big(\mathbb{R}\Gamma_Y\big(\mathcal{O}_{\tilde{Y}^*}[D_{y_1}\cdots D_{y_{n-d}}]\big) & \xrightarrow{(\tilde{\theta}-\alpha)^p} & \mathbb{R}\Gamma_Y\big(\mathcal{O}_{\tilde{Y}^*}[D_{y_1}\cdots D_{y_{n-d}}]\big)\Big)\\
\big\uparrow\text{QIS} & & \big\uparrow\text{QIS}\\
\Big(\mathbb{R}\Gamma_{[Y]}\big(\mathcal{O}_{\tilde{Y}}[D_{y_1}\cdots D_{y_{n-d}}]\big) & \xrightarrow{(\tilde{\theta}-\alpha)^p} & \mathbb{R}\Gamma_{[Y]}\big(\mathcal{O}_{\tilde{Y}^*}[D_{y_1}\cdots D_{y_{n-d}}]\big)\Big)
\end{array}$$

because of the regularity of the operator $(\tilde{\theta}-\alpha)^p$ along Y. Finally, we see that

$$\begin{aligned}
\int_{\tilde\pi}^{c}\underline{\mu_Y}(\mathcal{M})&\simeq\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_{\tilde{Y}^*}}\Big(\frac{\mathcal{D}_{\tilde{Y}^*}}{\mathcal{D}_{\tilde{Y}^*}(\tilde{\theta}-\alpha)},B_{Y|\tilde{Y}^*}[-d]\Big)\underset{\mathbb{C}}{\otimes}\mathbb{C}[D_{y_1}\cdots D_{y_{n-d}}]\\
&\simeq\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_Y}\Big(\tilde{\ell}^!\Big(\frac{\mathcal{D}_{\tilde{Y}^*}}{\mathcal{D}_{\tilde{Y}^*}(\tilde{\theta}-\alpha)^p}\Big),\mathcal{O}_Y\Big)\underset{\mathbb{C}}{\otimes}\mathbb{C}[D_{y_1}\cdots D_{y_{n-d}}]
\end{aligned}$$

(see [L-MF]). Hence, by LEMMA 2 we have

$$
{}^{c}\!\!\int_{\tilde{\pi}} \underline{\mu_Y}(\mathcal{M}) \begin{cases} = 0 & \text{if } \alpha \notin \mathbb{Z}_+, \\ \underset{\text{QIS}}{\simeq} & \begin{matrix} \mathcal{D}_Y^M & \to & \mathcal{D}_Y^M \\ \uparrow & & \uparrow \\ \text{deg. } -1 & & \text{deg. } 0. \end{matrix} \quad \text{if } \alpha \in \mathbb{Z}_+ \end{cases}
$$

The proof of a) is similar. ▯

REMARK 3. — Actually, we obtained here a very useful result : *let $E \xrightarrow{\pi} Y$ be a holomorphic fiber bundle, $\mathcal{M}$ a monodromic $\mathcal{D}_E$-module. Then $\int_\pi \mathcal{M}$ has coherent cohomologies.* Of course this can also be proved using the theorem for smooth direct images of [H-S].

REMARK 4. — As pointed out in REMARK 1 one defines a natural isomorphism

$$
\mu_Y\big(\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)\big) \xrightarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\tilde{\pi}^{-1}\mathcal{D}_X}\big(\tilde{\pi}^{-1}\mathcal{M}, \mu_Y(\mathcal{O}_X)\big)
$$

for any coherent $\mathcal{D}$-module $\mathcal{M}$, in $D^b(T^*X)$.

Let now $f : X \to Z$ be a holomorphic map, $Y' \subset Z$ a smooth manifold such that $Y = f^{-1}(Y')$ is a smooth manifold of X. Let us consider the associated maps ρ_f and $\bar{\omega}_f$ defined by

$$
T_Y^*X \xrightarrow{\rho_f} Y \times T_{Y'}^*Z \xrightarrow{\bar{\omega}_f} T_{Y'}^*Z.
$$

Then, using THEOREM 1.9 and adapting the proof of THEOREM 2.5, we obtain :

THEOREM 2.6. — *Assume that f is smooth. Let $\mathcal{M} \in B_{Y'}$. Then, in $D^b(\mathcal{D}_{T_Y^*X})$, we have*

$$
\underline{\mu_Y}(f^*\mathcal{M}) \simeq \int_{\rho_f} \bar{\omega}_f^* \, \underline{\mu_{Y'}}(\mathcal{M}).
$$

3. Application to the functor μhom for $\mathcal{D}$-modules

We will consider a subcategory of the category of pairs of $\mathcal{D}_X$-modules satisfying a regularity condition :

Let $p_1 : X \times X \to X$ and $p_2 : X \times X \to X$ be the first and the second projections. One defines an exact bifunctor (i.e., exact in the two variables) in $\mathrm{Mod}(\mathcal{D}_X) \times \mathrm{Mod}(\mathcal{D}_X)$, noted $\underline{\boxtimes}$, by : given $\mathcal{M}$ and $\mathcal{N}$ two $\mathcal{D}$-modules,

$$\mathcal{M} \,\underline{\boxtimes}\, \mathcal{N} = \mathcal{D}_{X\times X} \underset{p_1^{-1}\mathcal{D}_X \underset{\mathbb{C}}{\otimes} P_2^{-1}\mathcal{D}_X}{\otimes} \big(p_1^{-1}\mathcal{M} \underset{\mathbb{C}}{\otimes} p_2^{-1}\mathcal{N}\big)$$

(see [SKK]).

Denote by Δ the diagonal of $X \times X$ and identify X to Δ. Then we may identify $T^*_\Delta(X \times X)$ to T^*X by the first projection. Finally let us denote by $\mathcal{M}^*$ the left $\mathcal{D}_X$-module $\mathbb{D}(\mathcal{M}) \underset{\mathcal{O}_X}{\otimes} \Omega_X^{\otimes -1}$.

Consider the full abelian subcategory of the pairs $(\mathcal{M}, \mathcal{N})$ such that $\mathcal{M}^* \,\underline{\boxtimes}\, \mathcal{N}$ is regular along Δ and denote it by R_Δ. This category is larger than the category of pairs of regular holonomic systems, considered in [K-K3], as shown by the following simple example.

EXAMPLE.

Let $X = X' = \mathbb{C}^n$.

Let $\lambda \in \mathbb{C}$ and $\mathcal{M}$ be the $\mathcal{D}_X$-module regular along $Y = \{0\} \subset \mathbb{C}^n$ given by $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(\theta - \lambda)$, where $\theta = \sum_{i=1}^n x_i\, D_{x_i}$.

Let $\mathcal{N}$ be the $\mathcal{D}_{X'}$-module given by $\mathcal{N} = \mathcal{D}_{X'}/\mathcal{D}_{X'}\, x'_1 + \cdots + \mathcal{D}_{X'}\, x'_n$.

Then $\mathcal{M}^* \,\underline{\boxtimes}\, \mathcal{N} = \mathcal{D}_{X\times X'}\, u$, where the generator u satisfies

$$\Big(\sum_{i=1}^n (x_i - x'_i)\, D_{x_i} + \lambda + 1\Big) u = 0.$$

Remark that $\tilde\theta = \sum_{i=1}^n (x_i - x'_i)\, D_{x_i}$ is a vector field tangent to Δ and $\tilde\theta \in V^0_\Delta(\mathcal{D}_{X\times X'})$ acting by the identity on I_Δ/I_Δ^2. Hence, $\mathcal{M}^* \,\underline{\boxtimes}\, \mathcal{N}$ is regular along Δ.

DEFINITION 3.1. — Let $(\mathcal{M}, \mathcal{N}) \in R_\Delta$. One sets

$$\underline{\mu\mathrm{hom}}(\mathcal{M}, \mathcal{N}) = \underline{\mu_\Delta}(\mathcal{M}^* \,\underline{\boxtimes}\, \mathcal{N})$$

(regarded as a $\mathcal{D}_{T^*X}$-module).

Let us now consider the following sheaves on T^*X; for further details we refer to [SKK], [KS] :

i) $\mathcal{E}_X$ is the sheaf of microdifferential operators of finite order;

ii) Given Y a submanifold of X, of codimension d,

$$C^{\mathbb{R}}_{Y|X} \underset{\text{def}}{=} \mu_Y(\mathcal{O}_X)[d],$$

iii) $\mathcal{E}^{\mathbb{R}}_X \underset{\text{def}}{=} \mu_\Delta(\Omega_{X\times X})[n]$; this is the sheaf of rings of microlocal operators.

If $\pi : T^*X \to X$ is the projection, then $\pi^{-1}\mathcal{D}_X$ is a subsheaf of rings of $\mathcal{E}^{\mathbb{R}}_X$.

Given a $\mathcal{D}_X$-module $\mathcal{M}$, we will note

$$\mathcal{M}^{\mathbb{R}} = \mathcal{E}^{\mathbb{R}} \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{M}.$$

THEOREM 3.2. — *Let $(\mathcal{M},\mathcal{N}) \in R_\Delta$. Then one has an isomorphism*

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}\big(\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \mathcal{M},\, \mathcal{N}^{\mathbb{R}}\big) \simeq \mathrm{DR}\big(\underline{\mu\mathrm{hom}}(\mathcal{M},\mathcal{N})\big) \ \textit{in}\ D^b(T^*X).$$

Proof. — For the sake of simplicity let us denote $\widetilde{\mathcal{M}}$ the $\mathcal{E}_X$-module

$$\mathcal{E}_X \underset{\pi^{-1}\mathcal{D}_X}{\otimes} \pi^{-1}\mathcal{M}.$$

Since $\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}},\mathcal{N}^{\mathbb{R}}) \simeq \mathbb{R}\mathcal{H}om_{\mathcal{E}_{X\times X}}(\widetilde{(\mathcal{M}\underline{\boxtimes}\mathcal{N}^*)}, C^{\mathbb{R}}_{X|X\times X})$, we are led to prove that if $\mathcal{M}$ is a regular $\mathcal{D}$-module along a submanifold Y of X of codimension d, one has an isomorphism :

$$(*) \qquad \mathbb{R}\mathcal{H}om_{\mathcal{E}_X}(\widetilde{\mathcal{M}}, C^{\mathbb{R}}_{Y|X}) \xrightarrow{\sim} \mathrm{DR}\,\underline{\mu_Y}(\mathcal{M}^*).$$

Let us define this morphism and check it is really an isomorphism.

From REMARK 4, for an arbitrary $\mathcal{D}$-module $\mathcal{M}$ one has an isomorphism

$$\mathbb{R}\mathcal{H}om_{\mathcal{E}_X}\big(\widetilde{\mathcal{M}}, C^{\mathbb{R}}_{Y|X}\big) \simeq \mu_Y\big(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X)\big)[d].$$

The right member is by definition

$$F\big(\nu_Y(\mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M},\mathcal{O}_X))\big)[d],$$

hence

$$\begin{aligned}\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}\big(\widetilde{\mathcal{M}}, C^{\mathbb{R}}_{Y|X}\big) &\simeq F\big(\nu_Y(\mathrm{DR}(\mathcal{M}^*))\big)[d]\\ &\simeq F\big(\mathrm{DR}(\underline{\nu_Y}(\mathcal{M}^*))\big)[d] && \text{by THEOREM 1.5}\\ &\simeq \mathrm{DR}\big(\mathcal{F}(\underline{\nu_Y}(\mathcal{M}^*))\big) && \text{by THEOREM 2.3}\\ &\simeq \mathrm{DR}\,\underline{\mu_Y}(\mathcal{M}^*). \quad \square\end{aligned}$$

REMARK 5. — Actually we proved, in a joint paper (with E. ANDRONIKOF, see [A-MF]), that, for $\mathcal{M} \in R_Y$, one has an isomorphism

$$\mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}\big(\widetilde{\mathcal{M}}, C^{\mathbb{R}}_{Y|X}\big) \xleftarrow{\sim} \mathbb{R}\mathcal{H}\mathrm{om}_{\mathcal{E}_X}\big(\widetilde{\mathcal{M}}, C^{\mathbb{R},f}_{Y|X}\big),$$

where $C^{\mathbb{R},f}_{Y|X}$ is a subsheaf of the sheaf $C^{\mathbb{R}}_{Y|X}$ defined in [A].

Additional Notations

X	: a complex manifold.
$T_Y X$	: the normal bundle to Y, where Y is a submanifold of X.
$T^*_Y X$	: the conormal bundle to Y.
$D^b(X)$	: the full subcategory of the derived category of sheaves of abelian groups on X, formed by the complexes with bounded cohomology.
$D_x = \partial/\partial x$	: partial derivation in the x-variable.
$\mathcal{O}_X$	: the sheaf of holomorphic functions on X.
Ω_X	: the sheaf of differential forms of maximum degree on X.
$\Gamma_S(F)$	: $i_*\, i^!\, F$ where $i : S \to X$ in the inclusion (here S is a locally closed subset of X).
$\mathbb{R}\Gamma_S(F)$	: the right derived functor of Γ_S.
$\Gamma_{[Y]}(F)$	: $\varinjlim \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X/I^m, F)$, where Y is an analytic subset of X and I is the defining ideal of Y.
$\mathbb{R}\Gamma_{[Y]}(*)$	: right derived functor of $\Gamma_{[Y]}(*)$.
B_M	: the sheaf of hyperfunctions on a real analytic variety, i.e., $\mathbb{R}\Gamma_M(\mathcal{O}_X)[\dim M]\otimes \mathcal{O}r_M$, with X the complexification of M and $\mathcal{O}r_M$ the orientation of sheaf on M.

$B_{Y|X}$: $\mathbb{R}\Gamma_{[Y]}(\mathcal{O}_X)[d]$, with $d =$ codimension of Y and where Y is a submanifold of X.

$B^{\infty}_{Y|X}$: $\mathbb{R}\Gamma_Y(\mathcal{O}_X)[d]$.

$\mathcal{H}om_R(A, B)$: the sheaf of R-homomorphisms from A to B, where A and B are sheaves of left R-modules and R is a sheaf of rings.

$A \underset{R}{\otimes} B$: the tensor product of A and B over R, where A (resp. B) is a sheaf of left R-modules (resp. right R-modules).

$\overset{\mathbb{L}}{\underset{R}{\otimes}}$: the left derived functor of $\underset{R}{\otimes}$.

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