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POINCARÉ DUALITY FOR

k-A LIE SUPERALGEBRAS

BY

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RÉSUMÉ. — Soit A une k-superalgèbre associative supercommutative avec unité et soit $\mathfrak{L}$ une k-A-superalgèbre de Lie. A partir de ces données, on peut construire une superalgèbre d'opérateurs différentiels $\mathcal{V}(A,\mathfrak{L})$ (généralisant la superalgèbre enveloppante d'une superalgèbre de Lie). Supposons que le corps de base soit de caractéristique 0 et que $\mathfrak{L}$ soit un A-module projectif de type fini. Le but de cet article est d'étudier la dualité de Poincaré pour les complexes de $V(A,\mathfrak{L})$-modules à gauche. Nous verrons que la dualité de Poincaré est satisfaite pour les complexes qui admettent une résolution projective finie. En utilisant notre résultat, nous démontrons des propriétés de dualité pour les représentations induites de superalgèbres de Lie. En particulier, nous montrons que, sous certaines hypothèses de finitude, le Ext-dual d'une représentation induite est une représentation induite.

ABSTRACT. — Let A be a supercommutative associative k-superalgebra with unity and let $\mathfrak{L}$ be a k-A Lie superalgebra. From these data, one can construct a superalgebra of differential operators $V(A,\mathfrak{L})$ (generalizing the enveloping superalgebra of a Lie superalgebra). Assume that the ground field is of characteristic 0 and that $\mathfrak{L}$ is a finitely generated projective A-module. The goal of this article is to study Poincaré duality for complexes in the derived category of left $V(A,\mathfrak{L})$-modules. We will see that Poincaré duality holds for complexes which are quasi-isomorphic to a bounded complex consisting of projective modules. Applying our result, we prove some duality properties for induced representations of Lie superalgebras. In particular, under some finiteness conditions, we show that the Ext-dual of an induced representation is an induced representation.

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1. Introduction

We will denote by k a field of characteristic 0. Let A be a supercommutative k-superalgebra associative with unity. Let $\mathfrak{L}$ be a k-A-Lie superalgebra. By definition, $\mathfrak{L}$ is endowed with a k-Lie superalgebra structure and an A-module structure. Moreover, these two structures are compatible in a certain sense. These data allow us to define a superalgebra $\mathcal{V}(A,\mathfrak{L})$ of differential operators generalizing the enveloping superalgebra of a Lie superalgebra (see [F, p. 125]). The description of $\mathcal{V}(A,\mathfrak{L})$ by generators and relations is similar to the description of a superalgebra of differential operators on a supermanifold. Assume that $\mathfrak{L}$ is a finitely generated projective A-module. I show that there is a correspondence (analogous to the Bernstein's correspondence for D-modules) (see [P]) between complexes of left $\mathcal{V}(A,\mathfrak{L})$-modules and complexes of right $\mathcal{V}(A,\mathfrak{L})$-modules. This correspondence involves the Berezinian complex of $\mathfrak{L}^*$ which we denote by $\Omega^\bullet$. Let $X^\bullet$ be a bounded below complex of left $\mathcal{V}(A,\mathfrak{L})$-modules. Then, with standard notations (see [Bo]), we define $(X^\bullet \otimes_A \Omega^\bullet) \overset{L}{\underset{\mathcal{V}(A,\mathfrak{L})}{\otimes}} A$ and $\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}(A,\mathfrak{L})}(A, X^\bullet)$. Using a resolution of A (analogous to the Koszul resolution), we construct a functorial morphism $\Psi(X^\bullet)$ from $(X^\bullet \otimes_A \Omega^\bullet) \overset{L}{\underset{\mathcal{V}(A,\mathfrak{L})}{\otimes}} A$ to $\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}(A,\mathfrak{L})}(A, X^\bullet)$. If $X^\bullet$ is quasi-isomorphic to a bounded complex of projective modules, then $\Psi(X^\bullet)$ is an isomorphism which establishes the Poincaré duality.

Using our result, we prove some duality properties for induced representations of Lie superalgebras. More precisely, we get the following statements.

THEOREM 7.0.1. — *Let $\mathfrak{g}$ be a k-Lie superalgebra and let $\mathfrak{h}$ and $\mathfrak{k}$ be two finite dimensional Lie subsuperalgebras of $\mathfrak{g}$. Let V (respectively W) be a finite dimensional $\mathfrak{h}$-module (respectively $\mathfrak{k}$-module). Put* $\dim \mathfrak{h}_{\bar 0} = h_0$ *and* $\dim \mathfrak{k}_{\bar 0} = s_0$. *Then, if V^* (resp. W^*) denotes the contragredient module of V (resp. W), for all n in $\mathbb{Z}$, we have*

$$\begin{aligned}&\operatorname{Ext}^{n-s_0}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} W\big)\\ &\simeq \operatorname{Ext}^{n-h_0}_{U(\mathfrak{g})}\Big(\big(W^* \otimes \operatorname{Ber}(\mathfrak{k}^*)\big) \otimes_{U(\mathfrak{k})} U(\mathfrak{g}), \big(V^* \otimes \operatorname{Ber}(\mathfrak{h}^*)\big) \otimes_{U(\mathfrak{h})} U(\mathfrak{g})\Big),\end{aligned}$$

where in the left hand side (res. right hand side) the Ext *is taken over left (resp. right) $U(\mathfrak{g})$-modules.*

A. GYOJA [G] and G. ZUCKERMAN [B-C] showed a particular case of this theorem under some strong assumptions over $\mathfrak{g}$ and $\mathfrak{h}$. D. H. COLLINGWOOD and B. SHELTON proved also a duality property of this type but in a slightly different context (see [C-S]). Moreover, the THEOREM 7.0.1 allows us to recover a duality result of M. DUFLO [D2].

If $\mathfrak{k} = \{0\}$ and $W = \{0\}$, we have a more precise result.

THEOREM 7.0.5. — *Let $\mathfrak{g}$ be a k-Lie superalgebra and let $\mathfrak{h}$ be a finite dimensional Lie subsuperalgebra of $\mathfrak{g}$. Let V be a finite dimensional $U(\mathfrak{h})$-module. Put $h_0 = \dim \mathfrak{h}_{\bar{0}}$.*

(a) *If $n \neq h_0$, then* $\mathrm{Ext}^n_{U(\mathfrak{g})}(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, U(\mathfrak{g})) = \{0\}$.

(b) *The right $U(\mathfrak{g})$-modules*

$$\mathrm{Ext}^{h_0}_{U(\mathfrak{g})}\big(U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V, U(\mathfrak{g})\big) \quad \textit{and} \quad \big(V^* \otimes \mathrm{Ber}(\mathfrak{h}^*)\big) \otimes_{U(\mathfrak{h})} U(\mathfrak{g})$$

are isomorphic.

This result has been proved by K.A. BROWN and T. LEVASSEUR [B-L, p. 410] and by G.R. KEMPF [K] in the case where $\mathfrak{g}$ is a finite dimensional semi simple Lie algebra and $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$ is a Verma module.

Notations

For most of the definitions about supermathematics, we refer the reader to [L1]. Here k will be a field of characteristic zero. We will denote by $\bar{0}$ and $\bar{1}$ the elements of $\mathbb{Z}/2\mathbb{Z}$. We will call superspace a k-vector space graded over $\mathbb{Z}/2\mathbb{Z}$, $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Let V and W be two superspaces. If f is a morphism of degree i from V to W and if v is in V_j, we put $\langle v, f\rangle = (-1)^{ij} f(v)$. If V is a superspace, one defines the superspace ΠV which, as a vector space, is equal to V but whose grading is $(\Pi V)_{\bar{0}} = V_{\bar{1}}$ and $(\Pi V)_{\bar{1}} = V_{\bar{0}}$. Let us introduce the map $\pi : V \to \Pi V$ which, as a morphism of vector spaces, is equal to the identity. It is of degree $\bar{1}$. The symmetric superalgebra of V will be denoted by $S(V)$.

Let A be an associative supercommutative superalgebra with unity and let M be an A-module. A basis of M is a family $(m_i)_{i \in I \amalg J} \in M_{\bar{0}}^I \times M_{\bar{1}}^J$ such that each element of M can be expressed in a unique way as a linear combination of the $(m_i)_{i \in I \amalg J}$. If I and J are finite, their cardinalities are independent of the basis of the A-module M. Then, the dimension of M over A is the element $|I| + \epsilon |J|$ of $\mathbb{Z}[\epsilon]/(\epsilon^2 - 1)$. If $(e_1, \ldots, e_n)$ is a basis of the A-module M, then the family $(e^1, \ldots, e^n)$ where $\langle e_i, e^j\rangle = \delta_{i,j}$ for $i \neq j$ is a basis of $\mathrm{Hom}_A(M, A)$ called the dual basis of $(e_1, \ldots, e_n)$. Moreover, if M is an A-module, then ΠM has a natural A-module structure defined by :

$$\forall m \in M,\ \forall a \in A, \quad a \cdot \pi m = (-1)^{|a|} \pi(a \cdot m).$$

We will only consider localization with respect to even multiplicative systems. Let S be a multiplicative system of $A_{\bar{0}}$, then M_S will denote the localized module with respect to S. If $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ is a prime ideal (resp. f an element of $A_{\bar{0}}$), then $M_{\mathfrak{p}}$ (resp. M_f) will denote the localization of M with respect to the multiplicative system $A_{\bar{0}} - \mathfrak{p}_{\bar{0}}$ (resp. $\{f^n \mid n \in \mathbb{N}\}$).

Let $\mathcal{A}$ be an abelian category of objects graded over $\mathbb{Z}/2\mathbb{Z}$. We will adopt the following conventions for the complexes : we require that the differentials defining the complexes be odd whereas the morphisms between complexes have to be even. If u is a morphism from $X^\bullet$ to $Y^\bullet$, we will denote by $H^i(u)$ the morphism it induces from $H^i(X^\bullet)$ to $H^i(Y^\bullet)$. In the diagrams, a quasi-isomorphism will be denoted by the sign « $\sim$ » on the arrows « $\xrightarrow{\sim}$ » whereas an isomorphism will be denoted by a double arrow « $\leftrightarrow$ ». In a cohomology module, we will denote by « $[m]$ » the class of the element m. If M is an object of $\mathcal{A}$ and n an integer, $M[-n]$ will be the complex concentrated in degree n and whose n-th component is M. We will denote by $\mathcal{D}(\mathcal{A})$ the derived category of $\mathcal{A}$. One can also define $\mathcal{D}^-(\mathcal{A})$.

If $\mathcal{A}$ is an abelian category, $\mathcal{A}^\circ$ will denote the opposite category. If B is an associative superagebra with unity, then ${}_B\mathcal{M}$ (resp. $\mathcal{M}_B$) will be the category of graded left (resp. right) B-modules.

If $((C^{p,q})_{p,q}, \mathrm{d}_1, \mathrm{d}_2)$ is a double complex, then $(\mathrm{Tot}(C^{\bullet\bullet}), \mathrm{d})$ will denote the total complex associated to $C^{\bullet\bullet}$. So we have :

$$\mathrm{Tot}(C^{\bullet\bullet})^n = \bigoplus_{i+j=n} C^{i,j} \quad \text{and} \quad \mathrm{d} = \mathrm{d}_1 + \mathrm{d}_2.$$

Let $\mathfrak{a}$ be a k-Lie superalgebra. We will write $U(\mathfrak{a})$ for its enveloping superalgebra and Δ for the coproduct in $U(\mathfrak{a})$. If M is a left $U(\mathfrak{a})$-module, then M^* will be the contragredient module. Let now $\mathfrak{g}$ be a Lie superalgebra and $\mathfrak{h}$ be a Lie subsuperalgebra. Let V (resp. W) be a left (resp. right) $U(\mathfrak{h})$-module. We will denote by $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ (resp. $\mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}(W)$) be the left (resp. right) $U(\mathfrak{g})$-module $U(\mathfrak{g})\otimes_{U(\mathfrak{h})}V$ (resp. $W\otimes_{U(\mathfrak{h})}U(\mathfrak{g})$).

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2. The Berezinian complex

In all this paragraph, $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is an associative supercommutative k-superalgebra with unity.

2.1. Recollections.

(References : [C], [L2].) Let us remark that a prime ideal $\mathfrak{p}$ of A is characterized by its intersection with $A_{\bar{0}}$, $A_{\bar{0}} \cap \mathfrak{p}$, which is a prime ideal of $A_{\bar{0}}$ (because $A_{\bar{1}}$ is included in $\mathfrak{p}$). One defines the ringed space, Spec(A), as follows. The underlying topological space X is the set of prime ideals of $A_{\bar{0}}$ endowed with the Zariski topology. The structural sheaf O_X of Spec(A) is defined as in the non graded case. If $f \in A_{\bar{0}}$, let $D(f)$ be the open subset

$$D(f) = \{\mathfrak{p} \in \operatorname{Spec}(A_{\bar{0}}) \mid f \notin \mathfrak{p}\}.$$

Then, $(D(f))_{f \in A_{\bar{0}}}$ form a basis for the Zariski topology on Spec$(A_{\bar{0}})$.

Let M be an A-module. Let us denote by $\widetilde{M}$ the sheaf associated to the presheaf $U \subset X \mapsto M \otimes_A O(U)$. The sheaf $\widetilde{M}$ has the same properties as in the non graded case [H, p. 110].

2.2. Case of a free module.

Let M be a free A-module of dimension $d_0 + \epsilon d_1$. Put $n = d_0 + d_1$. Let $(e_1, ..., e_n)$ be a basis of M such that $(e_1, ..., e_{d_0})$ are even and $(e_{d_0+1}, ..., e_n)$ are odd. Let us denote by $(e^1, ..., e^n)$ the dual basis and let d be left multiplication by

$$\sum_{i=1}^{n} (-1)^{|e_i|+1} \pi e_i \otimes e^i$$

in the superalgebra $S_A(\Pi M \oplus_A M^*)$. The endomorphism d does not depend on the choice of a basis.

PROPOSITION 2.2.1. — *The complex*

$$J(M) = (S_A(\Pi M \oplus M^*) = \Big(\bigoplus_{n \in \mathbb{N}} S^n(\Pi M) \underset{A}{\otimes} S(M^*), \mathrm{d}\Big)$$

has no cohomology except in degree d_0. *The* A*-module* $H^{d_0}(J(M))$ *is free of dimension* 1 *or* ϵ. *More precisely, the element* $\pi e_1 \cdots \pi e_{d_0} \otimes e^{d_0+1} \cdots e^n$ *is a cycle whose class is a basis of* $H^{d_0}(J(M))$.

A proof of PROPOSITION 2.2.1 is given in [M, p. 172].

DEFINITION. — The module $H^{d_0}(J(M))$ is called the *Berezinian module* of M and is denoted ber(M). The complex ber$(M)[-d_0]$ is called the Berezinian complex and will be denoted Ber(M).

If S is a multiplicative system of $A_{\bar{0}}$, then A_S-modules $\mathrm{ber}(M)_S$ and $\mathrm{ber}(M_S)$ are canonically isomorphic.

2.3. Case of a projective A-module.

Let M be a finitely generated projective A-module. Before defining the Berezinian complex of M, we need to recall the following result [Bou, p. 141].

LEMMA 2.3.1. — *Let M be a finitely generated projective A-module and let $\mathfrak{p}$ be a prime ideal of A. There exists $f \notin \mathfrak{p}$ such that M_f is free.*

NOTATION. — We will denote by $\mathfrak{I}(M)$ the set of all elements f of $A_{\bar{0}}$ such that M_f is a free A_f-module.

COROLLARY 2.3.2. — $(D(f))_{f\in\mathfrak{I}(M)}$ *is an open covering of* $\mathrm{Spec}(A)$.

We come now to the theorem which will allow us to define the Berezinian complex of a finitely generated projective A-module.

THEOREM 2.3.3. — *Let M be a finitely generated projective A-module. There is a unique complex of A-modules (up to isomorphism) denoted* $\mathrm{Ber}(M)$ *such that, for all f in $\mathfrak{I}(M)$,* $\mathrm{Ber}(M)_f$ *is canonically isomorphic to* $\mathrm{Ber}(M_f)$.

DEFINITION. — The complex $\mathrm{Ber}(M)$ constructed by the previous theorem is called the *Berezinian complex* of M.

Proof of the theorem 2.3.3. — Put $\mathrm{Spec}(A) = (X, O_X)$. We know that

$$X = \bigcup_{f\in\mathfrak{I}(M)} D(f).$$

For all (f, f') in $\mathfrak{I}(M)^2$, the complexes of $A_{ff'}$-modules $(\mathrm{Ber}(M_f))_{f'}$ and $\mathrm{Ber}(M_{ff'})$ are canonically isomorphic. This remark proves that we can define a complexes of sheaves $\mathrm{Ber}(M)$ over X such that :

$$\mathrm{BER}(M)_{|D(f)} = \widetilde{\mathrm{Ber}(M_f)}$$

for all f in $\mathfrak{I}(M)$. We put :

$$\mathrm{Ber}(M) = \Gamma\big(X, \mathrm{BER}(M)\big).$$

This finishes the proof of the THEOREM 2.3.3.

REMARKS. — In the case where M is a finitely generated projective A-module, the Berezinian complex is not necessarily concentrated in one degree. If for all f in $\mathcal{I}(M)$, M_f has the same dimension $d_0 + \epsilon d_1$, then we will say that the rank of M is $d_0 + \epsilon d_1$. In this case, the Berezinian complex $\mathrm{Ber}(M)$ has only one non zero component (namely in degree d_0), hence we can define the Berezinian module $\mathrm{ber}(M)$ by $\mathrm{Ber}(M) = \mathrm{ber}(M)[-d_0]$.

3. Ring of differential operators defined by a k-A Lie superalgebra

3.1. Recollections.

Let A be a supercommutative associative k-superalgebra with unity and let $\mathfrak{L}$ be a k-Lie superalgebra that is also an A-module. Assume that we are given $\sigma : \mathfrak{L} \to \mathrm{Der}(A)$ a morphism of Lie superalgebras and of A-modules. Assume moreover that for all D and Δ belonging to $\mathfrak{L}$ and all a in A, we have :

$$(*) \qquad [\Delta, aD] = a\,[\Delta, D]\,(-1)^{|a|\cdot|\Delta|} + \sigma(\Delta)(a)D.$$

Then $\mathfrak{L}$ is called a *k-A-Lie superalgebra.* Let $\mathcal{V}(A,\mathfrak{L})$ be the superalgebra of differential operators generated by A and $\mathfrak{L}$ (see [F], [R]). It can be described as follows : $\mathcal{V}(A,\mathfrak{L})$ is the k-superalgebra generated by the elements of A, the elements of $\mathfrak{L}$ and the following relations

$$(**) \qquad \begin{cases} a \cdot b = (ab), \\ D \cdot a - (-1)^{|a|\cdot|D|} a \cdot D = \sigma(D)(a), \\ D \cdot \Delta - (-1)^{|\Delta|\cdot|D|} \Delta D = [D, \Delta]\,, \\ a \cdot D = (aD). \end{cases}$$

Let $\mathcal{V}(A,\mathfrak{L})_n$ be the left A-submodule of $\mathcal{V}(A,\mathfrak{L})$ generated by products of at most n elements of $\mathfrak{L}$. We define thus a filtration on $\mathcal{V}(A,\mathfrak{L})$. If $\mathfrak{L}$ is A-projective, then the graded A-superalgebra $\mathrm{Gr}\,\mathcal{V}(A,\mathfrak{L})$ (with respect to this filtration) is isomorphic to the symmetric superalgebra $S_A(\mathfrak{L})$ (see [R, p. 198]).

Let S be a multiplicative system of $A_{\bar{0}}$. We know (see [F, p. 128]) that $\mathfrak{L}_S = A_S \otimes_A \mathfrak{L}$ is endowed with a natural k-A_S Lie superalgebra structure (extending that of $\mathfrak{L}$). Put $\mathcal{V} = \mathcal{V}(A,\mathfrak{L})$ and $\mathcal{V}(S) = \mathcal{V}(A_S,\mathfrak{L}_S)$.

EXAMPLES :

- The simplest example is obtained when σ is 0. Then, $\mathfrak{L}$ is just an A-superalgebra and $\mathcal{V}(A,\mathfrak{L})$ is the enveloping superalgebra of $\mathfrak{L}$.

- Let $\mathcal{X}$ be a paracompact smooth supermanifold (over $\mathbb{R}$ or $\mathbb{C}$) (see [L1], [Ko]). Let X (respectively $O_{\mathcal{X}}$) be the underlying topological space (resp. structural sheaf) of $\mathfrak{X}$. We write $\mathfrak{X} = (X, O_{\mathfrak{X}})$. Put

$$A = O_{\mathfrak{X}}(X), \quad \mathfrak{L} = \operatorname{Der} O_{\mathfrak{X}}(X), \quad \sigma = \mathrm{id}.$$

Then $\mathcal{V}(A, \mathfrak{L})$ is the superalgebra of differential operators over X. Moreover, $\operatorname{Der} O_{\mathfrak{X}}(X)$ is a finitely generated projective $O_{\mathfrak{X}}(X)$-module (see [Hus, p. 31], [S, p. 266] and [We, p. 100]).

• Let A be a k-Poisson superalgebra. Let D_A be the A-module of Kähler differentials for A. Then D_A is naturally endowed with a k-A Lie superalgebra structure [Hu1]. Note that this structure depends on the Poisson bracket on A.

• Let $\mathcal{N} = (N, O_{\mathcal{N}})$ be a real Poisson supermanifold. Put $A = O_{\mathcal{N}}(N)$. Then the A-module of differential forms of degree 1 on N, $\Omega^1(N)$, is naturally endowed with a $\mathbb{R}$-A-Lie superalgebra structure [Hu1]. As previously, this structure depends on the Poisson bracket on A. The natural epimorphism from D_A to $\Omega^1(N)$ is a morphism of $\mathbb{R}$-A Lie superalgebras.

4. Equivalence of category between complexes of left $\mathcal{V}(A, \mathfrak{L})$-modules and complexes of right $\mathcal{V}(A, \mathfrak{L})$-modules

From now on, we assume A, $\mathfrak{L}$ and σ given and, we put $\mathcal{V} = \mathcal{V}(A, \mathfrak{L})$. Moreover, if D is in $\mathfrak{L}$ and a in A, we will write $D(a)$ (instead of $\sigma(D)(a)$) the action of D on a.

PROPOSITION 4.0.1. — *If $\mathfrak{L}$ is a finitely generated projective A-module, then* $\operatorname{Ber}_A(\mathfrak{L}^*)$ *is endowed with a natural right $\mathcal{V}$-module structure. In the case where $\mathfrak{L}$ is free, the action of $\mathfrak{L}$ on* $\operatorname{Ber}_A(\mathfrak{L}^*)$ *is induced by its adjoint action on $J(\mathfrak{L}^*)$.*

a) Proof in the case where $\mathfrak{L}$ is free. — Before starting the proof of this proposition, let us introduce the following definition.

DEFINITION. — An A-module M will be called a *(left) A-$\mathfrak{L}$-module* if it is also a $\mathfrak{L}$-module and if the following relation is satisfied : for all m in M, a in A and D in $\mathfrak{L}$,

$$D \cdot (a \cdot m) - (-1)^{|D| \cdot |a|} a \cdot (D \cdot m) = D(a) \cdot m. \tag{1}$$

An A-$\mathfrak{L}$-module M is a $\mathcal{V}$-module if and only if it satisfies the following : for all m in M, D in $\mathfrak{L}$ and a in A, we have

$$a \cdot (D \cdot m) = (aD) \cdot m. \tag{2}$$

For example, $\mathfrak{L}$ is an A-$\mathfrak{L}$-module but not a left $\mathcal{V}$-module.

If M is an A-$\mathfrak{L}$-module, then M^* has a natural A-$\mathfrak{L}$-module structure defined by the following operations : if a is in A, m in M, ω in M^* and D in $\mathfrak{L}$, then :

$$\langle m, a\cdot\omega\rangle = (-1)^{|a|\cdot|m|}\langle a\cdot m,\omega\rangle,$$
$$\langle m, D\cdot\omega\rangle = -(-1)^{|D|\cdot|m|}\langle D\cdot m,\omega\rangle + (-1)^{|D|\cdot|m|}D\big(\langle m,\omega\rangle\big).$$

In particular, $\mathfrak{L}^*$ is an A-$\mathfrak{L}$-module. Similarly, if M and N are A-$\mathfrak{L}$-modules, then ΠM, $M\underset{A}{\otimes}N, S_A(M), S_A(M^*)$ are naturally endowed with an A-$\mathfrak{L}$-module structure. Hence, $S_A(\Pi M\oplus M^*)$ has a natural A-$\mathfrak{L}$-module structure. As this structure commutes with the differential of $J(M)$ (the complex defining the Berezinian module), we endow $\mathrm{ber}(M)$ with the induced A-$\mathfrak{L}$-module structure.

From now on, we assume $M=\mathfrak{L}^*$.

Let D be an element of $\mathfrak{L}$. Let us call *Lie derivative* of D and denote by L_D the action of D on $\mathrm{ber}(\mathfrak{L}^*)$. On $\mathrm{ber}(\mathfrak{L}^*)$, we define the following right actions of $\mathfrak{L}$ and A : for a in A, D in $\mathfrak{L}$ and ω in $\mathrm{ber}(\mathfrak{L}^*)$,

$$\begin{cases} \omega\cdot D = -(-1)^{|D|\cdot|\omega|}L_D(\omega), \\ \omega\cdot a = (-1)^{|D|\cdot|\omega|}a\omega. \end{cases}$$

As $\mathrm{ber}(\mathfrak{L}^*)$ is an A-$\mathfrak{L}$-module, we know that these actions satisfy the right analog of (1). In order to prove that $\mathrm{ber}(\mathfrak{L}^*)$ is a right $\mathcal{V}$-module, we have to prove the right analog of (2). In other words, we need to prove that the Lie derivative satisfies the relation

$$L_{aD} = (-1)^{|D|\cdot|a|}\mu_{D(a)} + \mu_a\circ L_D, \tag{3}$$

where μ_a is left multiplication by a. Let $(e_1,\dots,e_n)$ be a basis of the A-module $\mathfrak{L}$ such that $(e_1,\dots,e_{d_0})$ are even and $(e_{d_0+1},\dots,e_n)$ are odd and let $(e^1,\dots,e^n)$ be the dual basis. We denote by $\omega_{[e]}$ the class of the element $\pi e^1\dots\pi e^{d_0}\otimes e_{d_0+1}\dots e_n$. Put $D=f_ie_i$ and let us compute L_D :

$$L_D(\omega_{[e]}) = \Big[\sum_{j=1}^{d_0}\pi e^1\cdots\pi(f_ie_i\cdot e^j)\pi e^{j+1}\cdots\pi e^{d_0}\otimes e_{d_0+1}\cdots e_n(-1)^{j|D|}\Big]$$
$$+\Big[\sum_{k=1}^{d_1}\pi e^1\cdots\pi e^{d_0}\otimes e_{d_0+1}\cdots[f_ie_i,e_{d_0+k}]\cdots e_n(-1)^{(d_0+k)|D|}\Big].$$

If ℓ is in $[1,n]$ and j is in $[1,d_0]$, then using (*), one shows that :

$$\langle e_\ell, D\cdot e^j\rangle = \langle e_\ell, f_i(e_i\cdot e^j)\rangle + \delta_{j,i}e_\ell(f_i).$$

- If e_i is odd, then $D\cdot e^j = f_i(e_i\cdot e^j)$. Moreover, we have :

$$[f_ie_i, e_{d_0+k}] = f_i\,[e_i, e_{d_0+k}] + e_{d_0+k}(f_i)e_i(-1)^{|f_i|}.$$

From this, we get

$$L_D(\omega_{[e]}) = f_iL_{e_i}(\omega_{[e]}) + (-1)^{|f_i|}e_i(f_i)\omega_{[e]} \tag{4}$$

because, in the computation of $L_D(\omega_{[e]})$, the term $e_{d_0+k}(f_i)e_i(-1)^{|f_i|}$ is only to be taken into account if $d_0+k=i$.

- If e_i is even, then we have :

$$D\cdot e^j = \begin{cases} f_i(e_i\cdot e^j) & \text{if } j\neq i,\\ f_i(e_i\cdot e^i) + \sum_\ell e^\ell e_\ell(f_i). & \end{cases}$$

From this, we deduce that (4) holds whatever the grading of e_i might be. Then the relation (3) follows easily from (4) and (1).

b) The projective case. — Assume now that $\mathfrak{L}$ is a finitely generated projective A-module. Put $\mathrm{Spec}(A) = (X, O_X)$. We have :

$$X = \bigcup_{f\in\mathfrak{I}(\mathfrak{L})} D(f).$$

From the previous case, we know that for any f in $\mathfrak{I}(\mathfrak{L})$, $\widetilde{\mathrm{ber}(\mathfrak{L}^*_f)}$, the only non zero component of the complex of sheaves $\widetilde{\mathrm{Ber}(\mathfrak{L}^*_f)}$, has a natural right $\widetilde{\mathcal{V}(f)}$-module structure. The following remark shows that these structures can be glued to give a right $\widetilde{\mathcal{V}}$-module structure on $\widetilde{\mathrm{Ber}(\mathfrak{L}^*)}$: if we take another element f' in $\mathfrak{I}(\mathfrak{L})$, the restriction of the $\widetilde{\mathcal{V}}_{|D(f)}$ ($\simeq \widetilde{\mathcal{V}(f)}$) right module structure on $\widetilde{\mathrm{ber}(\mathfrak{L}^*)}_{|D(f)}$ ($\simeq \widetilde{\mathrm{ber}(\mathfrak{L}^*_f)}$) to $D(f)\cap D(f')$ is nothing but the $\widetilde{\mathcal{V}}_{|D(ff')}$ ($\simeq \widetilde{\mathcal{V}(ff')}$) right module structure on $\widetilde{\mathrm{ber}}(\mathfrak{L}^*)_{|D(ff')}$ ($\simeq \widetilde{\mathrm{ber}(\mathfrak{L}^*_{ff'})}$). Then, if we apply the functor global sections, we get a right $\mathcal{V}$-module structure on $\mathrm{Ber}(\mathfrak{L}^*)$. This finishes the proof of the PROPOSITION 4.0.1.

REMARKS :

1) Assume that $\mathfrak{L}$ is a free A-module of finite dimension. Let $(e_1, ..., e_n)$ be a basis of $\mathfrak{L}$ and let $\omega_{[e]}$ be the basis of $\mathrm{ber}(\mathfrak{L}^*)$ it determines. Let $D \in \mathfrak{L}$. We define the *divergence* $\mathrm{div}_{[e]}(D)$ of D in the basis $(e_1, ..., e_n)$, by :

$$D \cdot \omega_{[e]} = \mathrm{div}_{[e]}(D)\omega_{[e]}.$$

Let us consider the map :

$$\begin{array}{ccc} A \oplus \mathfrak{L} & \longrightarrow & A \oplus \mathfrak{L}, \\ a + D & \longmapsto & a - \mathrm{div}_{[e]}(D) - D. \end{array}$$

FEL'DMAN [F, p. 125] showed that this maps extends uniquely to an anti-involution σ of $\mathcal{V}$ which he called *antipodism*. The antipodism gives a correspondence between left and right $\mathcal{V}$-modules. Indeed, if M is a left $\mathcal{V}$-module, then we construct a right $\mathcal{V}$-module M^{right} as follows :

$$\forall v \in \mathcal{V},\ \forall m \in M, \quad m \cdot v = \sigma(v) \cdot m(-1)^{|v|\cdot|m|}.$$

So, in the free case, we have just proved that the right $\mathcal{V}$-module A^{right} and $\mathrm{ber}(\mathfrak{L}^*)$ are isomorphic. But the antipodism of Fel'dman can only be defined for finitely dimensional free A-modules whereas our procedure can be extended to finitely generated projective A-modules. The use of the Berezinian complex is also more canonical.

2) The PROPOSITION 4.0.1 generalizes the Bernstein's correspondence between left and right D-modules in the case of a manifold. This correspondence was extended by PENKOV [P] to the case of supermanifolds.

COROLLARY 4.0.2. — *Assume that $\mathfrak{L}$ is a k-A-Lie superalgebra which is a finitely generated projective A-module.*

(a) *If $M^\bullet$ is a complex of left $\mathcal{V}$-modules, then $M^\bullet \otimes_A \mathrm{Ber}(\mathfrak{L}^*)$ is a complex of right $\mathcal{V}$-modules. The right $\mathcal{V}$-module structure on the components of $M^\bullet \otimes_A \mathrm{Ber}(\mathfrak{L}^*)$ is given by the operations described below : let m be in M^n, ω in $\mathrm{Ber}(\mathfrak{L}^*)^p$, a in A and D in $\mathfrak{L}$. We put :*

$$(m \otimes \omega) \cdot a = (-1)^{|a|(|m|+|\omega|)} a \cdot m \otimes \omega = m \otimes (\omega \cdot a),$$

$$(m \otimes \omega) \cdot D = -(-1)^{|D|(|m|+|\omega|)} D \cdot m \otimes \omega + m \otimes (\omega \cdot D).$$

(b) *The functor $M^\bullet \mapsto M^\bullet \otimes_A \mathrm{Ber}(\mathfrak{L}^*)$ gives an equivalence between the category of complexes of left $\mathcal{V}$-modules and the category of complexes of right $\mathcal{V}$-modules. The converse functor is given by :*

$$N^\bullet \longmapsto \mathrm{Hom}_A(\mathrm{Ber}(\mathfrak{L}^*), N^\bullet).$$

Proof. — The proof of consist (a) in verifying the relations (**). For (b), see [Bo, p. 227].

5. Poincaré duality

5.1. Resolution of A as a left $\mathcal{V}$-module.

Consider the graded left $\mathcal{V}$-module

$$\mathcal{V} \underset{A}{\otimes} S(\Pi\mathfrak{L}) = \bigoplus_{n} \mathcal{V} \underset{A}{\otimes} S^n(\Pi\mathfrak{L})$$

where $\mathcal{V}$ acts by left multiplication. One can prove [R, p. 200] that there exists an endomorphism of degree -1 on this module such that : for all v in $\mathcal{V}$ and all μ_i in $\mathfrak{L}$, we have

$$\begin{aligned}\mathrm{d}(v \otimes \pi\mu_1 \cdots \pi\mu_p) = &\sum_{i=1}^{n} \varepsilon'(v,i) v\mu_i \otimes \pi\mu_1 ... \widehat{\pi\mu_i} \cdots \pi\mu_n \\ &+ \sum_{k<i} \varepsilon''(v,i,k) v \otimes \pi\mu_1 \cdots \pi\left[\mu_k, \mu_i\right] \cdots \widehat{\pi\mu_i} \cdots \pi\mu_n,\end{aligned}$$

where the notation $\hat{x}$ means that x is omitted and where :

$$\varepsilon'(v,i) = (-1)^{|v|}(-1)^{(|\mu_1|+\cdots+|\mu_{i-1}|+i-1)(|\mu_i|+1)},$$

$$\varepsilon''(v,i,k) = (-1)^{|v|+|\mu_1|+\cdots+|\mu_{i-1}|+i-1}(-1)^{|\mu_i|(\mu_{k+1}|+\cdots+|\mu_{i-1}|+i-k+1)},$$

Moreover, we define $\mathrm{d}_0 : \mathcal{V} \underset{A}{\otimes} S^0(\Pi\mathfrak{L}) \to A$ by :

$$\forall v \in \mathcal{V}, \quad \mathrm{d}_0(v \otimes 1) = v \cdot 1.$$

THEOREM 5.1.1. — *Let $\mathfrak{L}$ be an A-projective k-A Lie superalgebra. The complex $P^\bullet$ defined by*

$$\forall n \in \mathbb{Z}, \;\; P^{-n} = \mathcal{V} \otimes_A S^n(\Pi\mathfrak{L})$$

and the differential above is a $\mathcal{V}$-projective resolution of A.

Proof. — See [R, p. 202]. □

Let S be a multiplicative system of $A_{\bar{0}}$. If we localize the resolution $P^\bullet$ with respect to S, we get a $\mathcal{V}(S)$-projective resolution of A_S.

5.2. Statement.

As in [Bo, p. 94], using the fact that the category ${}_{\mathcal{V}}\mathcal{M}$ has enough projective objects, one constructs a functor denoted $\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}^\bullet(-,-)$ from $\mathcal{D}^-({}_{\mathcal{V}}\mathcal{M})^\circ \times \mathcal{D}({}_{\mathcal{V}}\mathcal{M})$ to $\mathcal{D}({}_k\mathcal{M})$. Similarly, using the fact that the categories ${}_{\mathcal{V}}\mathcal{M}$ and $\mathcal{M}_{\mathcal{V}}$ have enough projective objects, one can construct a functor denoted $\overset{L}{\underset{\mathcal{V}}{\otimes}}$ from $\mathcal{D}^-(\mathcal{M}_{\mathcal{V}}) \times \mathcal{D}^-({}_{\mathcal{V}}\mathcal{M})$ to $\mathcal{D}({}_k\mathcal{M})$. Lastly, we will write $\Omega^\bullet$ for the Berezinian complex of $\mathfrak{L}^*$.

THEOREM 5.2.1. — *Let $\mathfrak{L}$ be a k-A Lie superalgebra which is finitely generated and projective as an A-module. Let $X^\bullet$ be an object of $\mathcal{D}^-({}_{\mathcal{V}}\mathcal{M})$. We consider A as an element of $\mathcal{D}^-({}_{\mathcal{V}}\mathcal{M})$ in the natural way. There exists a morphism $\Psi(X^\bullet)$ (functorial in $X^\bullet$) from $(X^\bullet \underset{A}{\otimes} \Omega^\bullet) \overset{L}{\underset{\mathcal{V}}{\otimes}} A$ to $\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}}(A, X^\bullet)$. If $X^\bullet$ is quasi-isomorphic to a bounded complex consisting of projective modules, then $\Psi(X^\bullet)$ is an isomorphism.*

The last assertion can not be extended to any complex $X^\bullet$ in $\mathcal{D}^-({}_{\mathcal{V}}\mathcal{M})$. One can construct easily a counterexample by taking A to be k, σ to be 0, $\mathfrak{L}$ to be the completely odd Lie superalgebra Πk and $X^\bullet$ to be the trivial module k (see [C]). Nevertheless, we will see in the proof that, in the non graded case, $\Psi(X^\bullet)$ is an isomorphism without hypothesis on $X^\bullet$.

COROLLARY 5.2.2. — *We keep the same assumptions as in the theorem. Assume moreover that $\mathfrak{L}$ has a rank $d_0 + \epsilon d_1$. Let M be a left $\mathcal{V}(A, \mathfrak{L})$-module. Let i be in $\mathbb{Z}$. There exists a superspace morphism $\Phi^i(M)$ from $\operatorname{Tor}^{\mathcal{V}}_{d_0-i}(M \otimes \operatorname{ber}(\mathfrak{L}^*), A)$ to $\operatorname{Ext}^i_{\mathcal{V}}(A, M)$. If M has a finite projective resolution, then $\Phi^i(M)$ is an isomorphism and hence*

$$\operatorname{Tor}^{\mathcal{V}}_{d_0-i}\big(M \otimes_A \operatorname{ber}(\mathfrak{L}^*), A\big) \simeq \operatorname{Ext}^i_{\mathcal{V}}(A, M), \quad \forall i \in \mathbb{Z}.$$

From now on, we assume that $\mathfrak{L}$ is a finitely generated projective A-module. We will prove the THEOREM 5.2.1 in several steps. Let us first make some preliminary remarks.

5.3. Preliminary remarks.

Let M be a left $\mathcal{V}$-module. Using the resolution of A, $P^\bullet$, previously described, one can see that $\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}}(A, M)$ is isomorphic (in the derived category) to the complex

$$K(M) = \Big(\bigoplus_n \operatorname{Hom}_A\big(S^n(\Pi\mathfrak{L}), M\big), \delta_M\Big),$$

where δ_M is given by : for ϕ in $\mathrm{Hom}_A(S^n(\Pi\mathfrak{L}), M)$ and $(\mu_1, ..., \mu_n)$ in $\mathfrak{L}$,

$$\langle \pi\mu_1 \cdots \pi\mu_n, \delta_M(\phi)\rangle = \sum_{i=1}^{n} \varepsilon'(i)\mu_i\langle \pi\mu_1 \cdots \widehat{\pi\mu_i} \cdots \pi\mu_n, \phi\rangle$$
$$- \sum_{k<i} \varepsilon''(i,k)\big\langle \pi\mu_1 \cdots \pi[\mu_k, \mu_i] \cdots \widehat{\pi\mu_i} \cdots \pi\mu_n, \phi\big\rangle.$$

with :

$$\varepsilon'(i) = (-1)^{|\mu_i|+\ldots+|\mu_n|+n-i+1}(-1)^{|\mu_i|(|\mu_1|+\ldots|\mu_{i-1}|+i-1)},$$
$$\varepsilon''(v,i,k) = (-1)^{|\mu_i|+\ldots|\mu_n|+n-i+1}(-1)^{(|\mu_{k+1}|+\ldots+|\mu_{i-1}|+i-k-1)|\mu_i|}.$$

As $\mathfrak{L}$ is a finitely generated projective A-module and we are in characteristic 0, the A-modules $S^p_A(\Pi\mathfrak{L}^*)$ and $(S^p_A(\Pi\mathfrak{L}))^*$ (here « $*$ » means dual with respect to A) are isomorphic. Hence $\bigoplus_n \mathrm{Hom}_A(S^n(\Pi\mathfrak{L}), M)$ can be identified with $S_A(\Pi\mathfrak{L}^*) \underset{A}{\otimes} M$.

Let us recall that $\mathfrak{L}$ acts on $S_A(\Pi\mathfrak{L})$ (respectively on $\mathcal{V}$) by the adjoint action which is defined as follows : an element D of $\mathfrak{L}$ acts as a derivation determined by for all $a \in A$, for all $\Delta \in \mathfrak{L}$,

$$D \cdot a = D(a),$$
$$D \cdot \pi\Delta = (-1)^{|D|}\pi[D, \Delta].$$

Let M be a $\mathcal{V}$-module. Then $\bigoplus_A \mathrm{Hom}_A(S^n_A(\Pi\mathfrak{L}), M)$ is endowed with a $\mathfrak{L}$-module structure (preserving the $\mathbb{N}$ components) denoted τ. As in the Lie algebra case, one proves that τ induces an action on the cohomology modules $H^n(K(M))$ which turns out to be trivial.

If M is equal to $\mathcal{V}$ with left multiplication, then $K(\mathcal{V})$ is endowed with a right $\mathcal{V}$-module structure denoted by R and defined by right multiplication. Write $\bar{R}$ for the action induced by R on the cohomology modules $H^n(K(\mathcal{V}))$. Let ρ be the adjoint action of $\mathfrak{L}$ on $K(\mathcal{V})$. As $\bar{\tau}$ is trivial, we have for all $[\varphi] \in H^n(K(\mathcal{V}))$, for all $D \in \mathfrak{L}$,

$$[\varphi] \cdot \bar{R}(D) = \big[-(-1)^{|D|\cdot|\varphi|}\rho(D) \cdot \varphi\big].$$

Let M be a right $\mathcal{V}$-module. In the derived category $\mathcal{D}({}_k\mathcal{M})$, the complex $M \overset{L}{\underset{\mathcal{V}}{\otimes}} A$ is isomorphic to $B(M) = \big(M \underset{A}{\otimes} S^\bullet(\Pi\mathfrak{L}), \partial_M\big)$ where ∂_M is given by : if m belongs to M and $(\mu_1, \ldots, \mu_n)$ are elements of $\mathfrak{L}$, then :

$$\partial_M(m \otimes \pi\mu_1 \cdots \pi\mu_n)$$
$$= \sum_{i=1}^{n} \varepsilon'(m,i)m \cdot \mu_i \otimes \pi\mu_1 \cdots \widehat{\pi\mu_i} \cdots \pi\mu_n$$
$$+ \sum_{k<i} \varepsilon''(m,i,k)m \otimes \pi\mu_1 \cdots \pi[\mu_k, \mu_i] \cdots \widehat{\pi\mu_i} \cdots \pi\mu_n.$$

with

$$\varepsilon'(m,i) = (-1)^{|m|}(-1)^{(|\mu_i|+1)(|\mu_1|+\cdots+|\mu_{i-1}|+i-1)},$$
$$\varepsilon''(m,i,k) = (-1)^{|m|}(-1)^{|\mu_1|+\cdots+|\mu_{i-1}|+i-1}(-1)^{|\mu_i|(|\mu_{k+1}|+\cdots+|\mu_{i-1}|+i-k-1)}.$$

5.4. Study of the complex $K(\mathcal{V})$.

We are going to prove the following proposition.

PROPOSITION 5.4.1. — *Let $\mathfrak{L}$ be a k-A-Lie superalgebra which is finitely generated and projective as an A-module. For all n in $\mathbb{Z}$, the right $\mathcal{V}$-modules $H^n(K(\mathcal{V}))$ and Ω^n are isomorphic.*

Proof. — Assume first that $\mathfrak{L}$ is a free A-module. The complex $K(\mathcal{V})$ is filtered by the complexes $(F_N K(\mathcal{V}))$ where

$$\big(F_N K(\mathcal{V})\big)^q = \Big\{ \sum_{m-q\le N} v_m \otimes \omega^q \mid v_m \in \mathcal{V}_m,\ \omega^q \in S^q(\Pi\mathfrak{L}^*)\Big\}.$$

This filtration is preserved by the differential $\delta_{\mathcal{V}}$. Hence, we are led to introduce the quotient complex

$$W = \Big(\bigoplus_N W_N, \bar{\mathrm{d}}\Big),$$

where

$$W_N^q = \big(F_N K(\mathcal{V})\big)^q / \big(F_{N-1}K(\mathcal{V})\big)^q$$

and $\bar{\mathrm{d}}$ is the differential induced by $\delta_{\mathcal{V}}$. A small computation [C] shows W coincides with $J(\mathfrak{L}^*)$. A diagram chasing argument proves that $H^i(K(\mathcal{V}))$ is $\{0\}$ when $i \neq d_0$ and that the right A-module $H^{d_0}(K(\mathcal{V}))$ is free of dimension 1 or ϵ. A basis of $H^{d_0}(K(\mathcal{V}))$ is the class of

$$\omega_0 = e_{d_0+1}\cdots e_n \otimes \tilde{\lambda}_1 \cdots \tilde{\lambda}_{d_0}.$$

Let us prove that $H^{d_0}(K(\mathcal{V}))$ is canonically isomorphic to $\mathrm{ber}(\mathfrak{L}^*)$ as a right $\mathcal{V}$-module. The superspace $H^{d_0}(F_{d_1-d_0}K(\mathcal{V}))$ is endowed with a right $\mathcal{V}$-module because we know that, on the cohomology level, right multiplication by an element of $\mathfrak{L}$ coincides with the adjoint action. Moreover, $H^{d_0}(F_{d_1-d_0}K(\mathcal{V}))$ and $H^{d_0}(K(\mathcal{V}))$ are isomorphic as right $\mathcal{V}$-modules. Let us prove that $H^{d_0}(F_{d_1-d_0}K(\mathcal{V}))$ is canonically isomorphic to $\mathrm{ber}(\mathfrak{L}^*)$ as right $\mathcal{V}$-modules. If ω is a cycle of $S^{d_0}(\Pi\mathfrak{L}^*)\underset{A}{\otimes}\mathcal{V}_{d_1}$, we will write $\bar{\omega}$ for the projection of ω on $S^{d_0}(\Pi\mathfrak{L}^*)\underset{A}{\otimes} S(\mathfrak{L})$. We will denote by $[\omega]$ (resp. $[\overline{\omega}]$) the projection of ω (resp. $\overline{\omega}$) on $H^{d_0}(F_{d_1-d_0}K(\mathcal{V}))$ (resp. $\mathrm{ber}(\mathfrak{L}^*)$). The following map

$$\begin{array}{rcl} I : H^{d_0}\big(F_{d_1-d_0}K(\mathcal{V})\big) & \longrightarrow & \mathrm{ber}(\mathfrak{L}^*), \\ {[\omega]} & \longmapsto & [\overline{\omega}] \end{array}$$

is well defined because we restrict ourselves to elements of $F_{d_1-d_0}K(\mathcal{V})$. Let ω be a non-zero cycle in $S^{d_0}(\Pi\mathfrak{L}^*)\underset{A}{\otimes}\mathcal{V}_{d_1}$ and let a be in A. The following remark

$$\omega \cdot R(a) - (-1)^{|a|\cdot|\omega|}a \cdot \omega \in S^{d_0}(\Pi\mathfrak{L}^*)\underset{A}{\otimes}\mathcal{V}_{d_1-1}$$

proves that $[\omega \cdot R(a)] = (-1)^{|a|\cdot|\omega|}a[\omega]$. Hence I is a morphism of right A-modules. As I sends the basis $[\omega_0]$ of the right A-module $H^{d_0}(K(\mathcal{V}))$ to the basis $[\overline{\omega_0}]$ of the right A module $\mathrm{ber}(\mathfrak{L}^*)$, it is an isomorphism. It is easy to check that actually I is an isomorphism of right $\mathcal{V}$-modules.

In the case where $\mathfrak{L}$ is only projective, we use as previously a localization and gluing procedure to conclude.

5.5. Description of the morphism $\Psi(X^\bullet)$.

5.5.1. Case where $\mathfrak{L}$ has a rank.

We will denote by $d_0 + \epsilon d_1$ the rank of $\mathfrak{L}$.

1) *Preliminary remark.* Using the isomorphism between $\mathrm{ber}(\mathfrak{L}^*)$ and $H^{d_0}(K(\mathcal{V}))$, we can express any element of $\mathrm{ber}(\mathfrak{L}^*)$ as a cohomology class $[\sum_\alpha v_\alpha \otimes \omega_\alpha]$, where v_α is in $\mathcal{V}$ and ω_α is in $S^{d_0}(\Pi\mathfrak{L}^*)$. The map

$$\begin{array}{rcl} \Phi^{d_0}(M) : H^{d_0}\big((M\otimes\Omega^\bullet)\underset{\mathcal{V}}{\otimes}P^\bullet\big) & \longrightarrow & H^{d_0}\big(K(M)\big), \\ \Big[m\otimes\big[\sum_\alpha \omega_\alpha\otimes v_\alpha\big]\otimes 1\Big] & \longmapsto & \Big[\sum_\alpha \omega_\alpha \otimes v_\alpha\cdot m(-1)^{(|\omega_\alpha|+|v_\alpha|)|m|}\Big] \end{array}$$

is well defined. Indeed, if $\sum_\alpha v_\alpha\otimes\omega_\alpha$ is a cycle (resp. a boundary), it is clear that

$$\sum_\alpha \omega_\alpha \otimes v_\alpha \cdot m(-1)^{(|\omega_\alpha|+|v_\alpha|)|m|}$$

is a cycle (resp. a boundary). Lastly, if $m\otimes[\sum_\alpha \omega_\alpha\otimes v_\alpha]\otimes 1$ is a boundary, then it is of the type

$$\begin{aligned}
&m \otimes \Big[\sum_\alpha \omega_\alpha \otimes v_\alpha\Big] \otimes 1 \\
&= \partial_{M\otimes\Omega^\bullet}\Big(\sum_i m_i \otimes \Big[\sum_\alpha \omega_\alpha \otimes v_\alpha\Big] \otimes \pi D_i\Big) \\
&= \sum_i -D_i m_i \otimes \Big[\sum_\alpha \omega_\alpha \otimes v_\alpha\Big] \otimes 1 (-1)^{(|m_i|+|\omega_\alpha|+|v_\alpha|)(|D_i|+1)} \\
&\qquad + \sum_i m_i \otimes \Big[\sum_\alpha \omega_\alpha \otimes v_\alpha D_i\Big] \otimes 1 (-1)^{|m_i|+|\omega_\alpha|+|v_\alpha|}
\end{aligned}$$

(where the m_i are elements of M and the D_i are elements of $\mathfrak{L}$) and it is easy to check that its image by $\Phi^{d_0}(M)$ is 0.

Assume that $\mathfrak{L}$ is free. If ω is a basis of the A-module $\mathrm{ber}(\mathfrak{L}^*)$, then $\mathcal{V} \otimes \Omega^\bullet$ is a free right $\mathcal{V}$-module of basis $1 \otimes \omega$. Hence, if M is a free $\mathcal{V}$-module, it is obvious that $\Phi^{d_0}(M)$ is an isomorphism. One generalizes easily this assertion first to the case where $\mathfrak{L}$ has only a rank and then to the case where M is a projective $\mathcal{V}$-module.

2) *Construction of* $\Psi(X^\bullet)$. Let $X^\bullet$ be a complex of left $\mathcal{V}$-modules bounded below. Let $Q^\bullet \to X^\bullet \to 0$ be a projective resolution of $X^\bullet$. We can assume without loss of generality that $Q^\bullet$ is of the form

$$\cdots \to Q^s \to Q^{s+1} \to \cdots \to Q^0 \to \{0\}.$$

The total complex associated to the double complex

$$\big[C^{n,s} = \mathrm{Hom}_{\mathcal{V}}(P^{-n}, Q^s)_{(n,s)\in\mathbb{Z}}\big]$$

(with the natural differentials) is a subcomplex of the complex

$$\mathrm{Hom}^\bullet(P^\bullet, Q^\bullet).$$

In the case where $Q^\bullet$ (or $P^\bullet$ for the non graded case) is bounded, it is $\mathrm{Hom}^\bullet(P^\bullet, Q^\bullet)$. In the case where $\mathfrak{L}$ has a rank, this complex is such that its lines have cohomology in only one degree, namely the d_0 degree. It is known [Bo, p. 283] that such a double complex is quasi-isomorphic to a double complex with non zero coefficients in only one column. Let us recall how one gets this result. Consider the truncated double complex $\tau_{\le d_0} C^{\bullet\bullet}$ defined by

$$\big(\tau_{\le d_0} C^{\bullet\bullet}\big)^{n,s} = \begin{cases} C^{n,s} & \text{if } s < d_0, \\ \ker(\delta^n_{d_0}) & \text{if } s = d_0, \\ \{0\} & \text{if } s > d_0. \end{cases}$$

There is a natural morphism from $\tau_{\leq d_0} C^{\bullet\bullet}$ to $C^{\bullet\bullet}$ inducing a morphism on the associated total complex

$$\operatorname{Tot}\big(\tau_{\leq d_0} C^{\bullet\bullet}\big) \longrightarrow \operatorname{Tot}(C^{\bullet\bullet}).$$

If $Q^\bullet$ is bounded, a spectral sequence argument shows that this morphism is a quasi-isomorphism. It is also true if $Q^\bullet$ is unbounded but the proof is slightly more subtle. We don't reproduce it here because we don't really need it. Now, we can truncate $\tau_{\leq d_0} C^{\bullet\bullet}$ on the right. We get $\tau_{\geq d_0}(\tau_{\leq d_0} C^{\bullet\bullet})$ given by

$$\big(\tau_{\geq d_0}(\tau_{\leq d_0} C^{\bullet\bullet})\big)^{n,s} = \begin{cases} \{0\} & \text{if } s \neq d_0, \\ H^{d_0}\big(\operatorname{Hom}(P^\bullet, Q^n)\big) & \text{if } s = n_0. \end{cases}$$

There is a natural morphism from $\tau_{\leq d_0} C^{\bullet\bullet}$ to $\tau_{\geq d_0}\tau_{\leq d_0} C^{\bullet\bullet}$ which induces a quasi-isomorphism on the associated total complex as a spectral sequence argument shows it. So we have the following picture :

$$\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0} C^{\bullet\bullet}) \xleftarrow{\sim} \operatorname{Tot}(\tau_{\leq d_0} C^{\bullet\bullet}) \longrightarrow \operatorname{Tot}(C^{\bullet\bullet}).$$

Similarly, the total complex associated to the double complex $(G^{n,s})_{(n,s\in\mathbb{Z})}$ defined by

$$G^{n,s} = \big(Q^s \underset{A}{\otimes} \operatorname{ber}(\mathfrak{L}^*)\big) \underset{\mathcal{V}}{\otimes} P^{n-d_0}$$

is the complex $(Q^\bullet \underset{A}{\otimes} \Omega^\bullet) \underset{\mathcal{V}}{\otimes} P^\bullet$. As the double complex $G^{\bullet\bullet}$ is such that its lines have cohomology in only the degree d_0, we can do the same reasoning as before. We have the following quasi-isomorphisms

$$\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0} G^{\bullet\bullet}) \xleftarrow{\sim} \operatorname{Tot}(\tau_{\leq d_0} G^{\bullet\bullet}) \xrightarrow{\sim} \operatorname{Tot}(G^{\bullet\bullet}),$$

the bicomplex $\tau_{\geq d_0}\tau_{\leq d_0} G^{\bullet\bullet}$ given by :

$$\tau_{\geq d_0}\tau_{\leq d_0} G^{n,s} = \begin{cases} 0 & \text{if } s \neq d_0, \\ H^{d_0}((Q^n \underset{A}{\otimes} \Omega^\bullet) \underset{\mathcal{V}}{\otimes} P^\bullet) & \text{if } s = d_0. \end{cases}$$

We know that we have an isomorphism $\Phi(Q^\bullet)$ from $\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0} G^{\bullet\bullet})$ to

$\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0}C^{\bullet\bullet})$ described by the diagram below :

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & H^{d_0}(\mathcal{Q}'_r) \arrow[r, leftrightarrow, "\Phi^{d_0}(Q_r)"] \arrow[d] & H^{d_0}(\mathcal{Q}''_r) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{d_0}(\mathcal{Q}'_{r+1}) \arrow[r, leftrightarrow, "\Phi^{d_0}(Q_{r+1})"] \arrow[d] & H^{d_0}(\mathcal{Q}''_{r+1}) \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \\
0 \arrow[r] & H^{d_0}(\mathcal{Q}'_0) \arrow[r, leftrightarrow, "\Phi^{d_0}(Q_0)"] \arrow[d] & H^{d_0}(\mathcal{Q}''_0) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}$$

with $\mathcal{Q}'_i = (Q_i \otimes_A \Omega^\bullet) \overset{L}{\underset{\mathcal{V}}{\otimes}} A$ and $\mathcal{Q}''_i = \mathrm{R}_{II}\mathrm{R}^-_I \operatorname{Hom}_{\mathcal{V}}(A, Q_i)$. The following diagram

$$\begin{tikzcd}
 & & (X^\bullet \underset{A}{\otimes} \Omega^\bullet) \overset{L}{\underset{\mathcal{V}}{\otimes}} A \\
\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0}G^{\bullet\bullet}) \arrow[d, leftrightarrow, "\Phi^{d_0}(Q^\bullet)"] & \operatorname{Tot}\tau_{\leq d_0}G^{\bullet\bullet} \arrow[l, "\sim"'] \arrow[r, "\sim"] & \operatorname{Tot}G^{\bullet\bullet} \arrow[u, "\wr"'] \\
\operatorname{Tot}(\tau_{\geq d_0}\tau_{\leq d_0}C^{\bullet\bullet}) & \operatorname{Tot}\tau_{\leq d_0}C^{\bullet\bullet} \arrow[l, "\sim"'] \arrow[r] & \operatorname{Tot}C^{\bullet\bullet} \arrow[d] \\
 & & \mathrm{R}_{II}\mathrm{R}^-_I \operatorname{Hom}_{\mathcal{V}}(A, Q^\bullet) \arrow[d, "\wr"] \\
 & & \mathrm{R}_{II}\mathrm{R}^-_I \operatorname{Hom}_{\mathcal{V}}(A, X^\bullet)
\end{tikzcd}$$

defines a morphism $\Psi(X^\bullet)$ from $((X^\bullet \underset{A}{\otimes} \Omega^\bullet) \overset{L}{\underset{\mathcal{V}}{\otimes}} A)$ to $\mathrm{R}_{II}\mathrm{R}^-_I \operatorname{Hom}_{\mathcal{V}}(A, X^\bullet)$. One checks easily that $\Psi(X^\bullet)$ does not depend on the projective resolution chosen for $X^\bullet$. From its construction, it is obvious that, if $X^\bullet$ admits a finite projective resolution (or if $P^\bullet$ is bounded as in the non graded case), then $\Psi(X^\bullet)$ is an isomorphism.

5.5.2. General case. — Assume now that $\mathfrak{L}$ is only a finitely generated projective A-module. We make a partition of $\operatorname{Spec}(A)$ into open subsets $(U_i)_{i\in I}$ over which $\mathfrak{L}^*$ has a given rank. The previous case allows us to construct $\Psi(X^\bullet)$ on each U_i. Then, we finish the proof of the THEOREM 5.2.1 by applying a gluing procedure.

6. Examples of computations

From now on we always assume that $\mathfrak{L}$ is a k-A-Lie superalgebra which is finitely generated and projective as an A-module.

• Let us denote by σ the Lie superalgebra and A-module morphism from $\mathfrak{L}$ to $\operatorname{Der}_k(A)$. The restriction of σ to $\mathfrak{L}_{\bar 0}$ produces a Lie algebra and $A_{\bar 0}$-module morphism for which we construct $\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})$. Assume that $A_{\bar 1}$ is a projective $A_{\bar 0}$-module, then it is not hard to prove that $\mathcal{V}(A,\mathfrak{L})$ is a projective $\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})$-module. We have :

$$\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})}(A,\mathcal{V}) \simeq \Big(\mathcal{V} \underset{A_{\bar 0}}{\otimes} \operatorname{Ber}(\mathfrak{L}_{\bar 0}^*)\Big) \underset{\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})}{\otimes} A.$$

Hence, if the rank $d_0+\epsilon d_1$ of $\mathfrak{L}$ exists, we can compute $\operatorname{Ext}^i_{\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})}(A,\mathcal{V})$:

$$\operatorname{Ext}^i_{\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})}(A,\mathcal{V}) \begin{cases} = \{0\} & \text{if } i \neq d_0, \\ \simeq \Big(\mathcal{V}\underset{A_{\bar 0}}{\otimes} \bigwedge^{d_0}(\mathfrak{L}_{\bar 0}^*)\Big) \underset{\mathcal{V}(A_{\bar 0},\mathfrak{L}_{\bar 0})}{\otimes} A & \text{if } i = d_0. \end{cases}$$

• Assume that we are in the non graded case then we can apply our theorem to the $\mathcal{V}$-module A. We get the following isomorphism :

$$\mathrm{R}_{II}\mathrm{R}_I^- \operatorname{Hom}_{\mathcal{V}}(A,A) \simeq \operatorname{Ber}(\mathfrak{L}^*) \overset{L}{\underset{\mathcal{V}}{\otimes}} A.$$

If $\mathfrak{L}$ has a rank d, we get :

$$\operatorname{Ext}^i_{\mathcal{V}}(A,A) \simeq \operatorname{Tor}^{\mathcal{V}}_{d-i}\big(\operatorname{ber}(\mathfrak{L}^*),A\big).$$

• Let N be a n-dimensional smooth Poisson manifold admitting a global volume form ω annihilated by Hamiltonian vector fields. Put $A = C^\infty(N)$ and denote by $\{\ \}$ the Poisson bracket on A. We have already seen that $\Omega^1(N)$ was endowed with a natural $\mathbb{R}$-A Lie algebra structure and that $\Omega^1(N)$ was a finitely generated projective A-module.

The volume form ω is a basis of the free A-module $\bigwedge^n(\Omega^1(N))$. We denote by σ its dual basis. We can compute the action of $\Omega^1(N)$ on $\bigwedge^n(\Omega^1(N)^*)$. We have :

$$(g\sigma)\cdot \mathrm{d}f = \{g,f\}\sigma,$$

using the fact that ω is annihilated by Hamiltonian vector fields. On another hand, we know from [Hu2]) that A has a natural right $\mathcal{V}(A,\Omega^1(N))$-module structure defined by the two following operations : for all f,g and h in A,

$$f\cdot g = fg,\quad f\cdot g\mathrm{d}h = \{fg,h\}.$$

The algebra A, endowed with this right $\mathcal{V}(A,\Omega^1(N))$-module structure, will be denoted A_r. So it is easy to see that the map

$$\begin{array}{ccc}\bigwedge^n(\Omega^1(N)^*) & \longrightarrow & A_r,\\ f\sigma & \longmapsto & f\end{array}$$

is a right $\mathcal{V}(A,\Omega^1(N))$-module morphism. Let $H_*^{\mathrm{can}}(N)$ be the canonical homology of the Poisson manifold N (see [Br], [Kos]). We know [Hu2] that

$$\mathrm{Tor}^i_{\mathcal{V}(A,\Omega^1(N))}(A_r,A)\simeq H_i^{\mathrm{can}}(N).$$

Combining the two previous remarks, our theorem gives :

$$H_i^{\mathrm{can}}(N)\simeq \mathrm{Ext}^{n-i}_{\mathcal{V}(A,\Omega^1(N))}(A,A).$$

Assume now that N is symplectic. Then, the $\mathbb{R}$-A Lie algebra $\Omega^1(N)$ is isomorphic to the $\mathbb{R}$-A Lie algebra of smooth vectors. Hence, the right hand side is isomorphic to the de Rham cohomology of N and we recover a BRYLINSKI's result [Br, p. 101]

$$H_i^{\mathrm{can}}(N)\simeq H^{n-i}_{DR}(N),$$

where $H^*_{DR}(N)$ denotes the de Rham cohomology of N.

• As $S_A(\mathfrak{L})$ is the associated graded superalgebra of $\mathcal{V}(A,\mathfrak{L})$, it is naturally endowed with a Poisson bracket [Br, p. 106] characterized by : for all (D,Δ) in $\mathfrak{L}$ and for all a in A, we have

$$\forall D\in\mathfrak{L},\forall\Delta\in\mathfrak{L},\quad \begin{cases}\{D,\Delta\}=[D,\Delta]\\ \{D,a\}=D(a).\end{cases}$$

Take for A the algebra of regular functions on an affine smooth variety N and put $\mathfrak{L} = \mathrm{Der}(A)$. Then $S = S_A(\mathrm{Der}(A))$ is the algebra of regular functions on $T^*(N)$ and the Poisson bracket is induced by the symplectic structure on $T^*(N)$. Let $H_i^{\mathrm{can}}(S)$ be the canonical homology of the Poisson algebra S (see [Br, p. 106]). The previous example shows that, if n is the dimension of N, we have :

$$H_i^{\mathrm{can}}(S) \simeq \mathrm{Tor}_i^{\mathcal{V}(S,D_S)}\Big(\bigwedge^{2n}(D_S^*), S\Big).$$

Hence, if we apply our theorem to the $\mathcal{V}(S, D_S)$-module S, we get :

$$H_i^{\mathrm{can}}(S) \simeq H_{DR}^{2n-i}(T^*N) \simeq H_{DR}^{2n-i}(N).$$

We recover a result of BRYLINSKI [Br, thm 3.3.1].

7. Duality properties for induced representations of Lie superalgebras

In this section, we will keep using the correspondence introduced in the COROLLARY 4.0.2. We are going to prove the following result :

THEOREM 7.0.1. — *Let $\mathfrak{g}$ be a k-Lie superalgebra. Let $\mathfrak{h}$ and $\mathfrak{k}$ be two finite dimensional subsuperalgebras of $\mathfrak{g}$. Set* $\dim \mathfrak{h}_{\bar{0}} = h_0$ *and* $\dim \mathfrak{k}_{\bar{0}} = s_0$. *Let V (respectively W) be a finite dimensional $\mathfrak{h}$-module (respectively $\mathfrak{k}$-module). Then, for all n in $\mathbb{Z}$, we have :*

$$\begin{aligned}&\mathrm{Ext}_{U(\mathfrak{g})}^{n-s_0}\big(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(W)\big)\\ &\qquad\simeq \mathrm{Ext}_{U(\mathfrak{g})}^{n-h_0}\Big(\mathrm{IND}_{\mathfrak{k}}^{\mathfrak{g}}\big(W^* \otimes \mathrm{ber}(\mathfrak{k}^*)\big), \mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}\big(V^* \otimes \mathrm{ber}(\mathfrak{h}^*)\big)\Big),\end{aligned}$$

where in the left hand side (resp. right hand side) the Ext *is taken over left (resp. right) $U(\mathfrak{g})$-modules.*

It is easy to see that this result is wrong if V or W is not finite dimensional. The situation $\mathfrak{k} = \{0\}$, $W = \{0\}$, $V = U(\mathfrak{h})$ provides a counterexample.

Generalizing a result of G. ZUCKERMAN [B-C], A. GYOJA [G] proved a part of this theorem (namely the case where $\mathfrak{h} = \mathfrak{k}$ and $n = h_0 = s_0$) under the assumptions that $\mathfrak{g}$ is split semisimple and $\mathfrak{h}$ is a parabolic subalgebra of $\mathfrak{g}$. D.H. COLLINGWOOD and B. SHELTON proved also a duality property of this type but in a slightly different context [C-S].

Moreover, the THEOREM 7.0.1 allows us to recover the following result of M. DUFLO [D2]. Assume that λ is a character of $\mathfrak{h}$. Then, for the adjoint representation of $\mathfrak{h}$ on $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(k_\lambda)$ and $\mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}(k_\lambda^* \otimes \bigwedge^{\dim \mathfrak{h}}(\mathfrak{h}^*))$ (interpreted as quotients of $U(\mathfrak{g})$), the following duality holds :

$$\forall i \in \mathbb{Z}, \quad H^i\big(\mathfrak{h}, \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(\lambda)\big) \simeq H^i\Big(\mathfrak{h}, \mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}\big(k_\lambda^* \otimes \bigwedge^{\dim h}(\mathfrak{h}^*)\big)\Big).$$

As Duflo's proof, our proof will rely on the LEMMA 7.0.2.

LEMMA 7.0.2. — *Assume that E is a $\mathfrak{h} \times \mathfrak{k}$-bimodule which is free as a left $\mathfrak{h}$-module and as a right $\mathfrak{k}$-module, then we have the following isomorphisms* :

$$H^n\big(\mathfrak{h} \times \mathfrak{k}, E \otimes \mathrm{ber}(\mathfrak{k})\big) \simeq H^{n-h_0}\Big(\mathfrak{k}, \frac{E \otimes \mathrm{ber}(\mathfrak{k}) \otimes \mathrm{ber}(\mathfrak{h}^*)}{(E \otimes \mathrm{ber}(\mathfrak{k}) \otimes \mathrm{ber}(\mathfrak{h}^*))\mathfrak{h}}\Big)$$
$$\simeq H^{n-s_0}\Big(\mathfrak{h}, \frac{E}{E\mathfrak{k}}\Big).$$

Proof of the lemma 7.0.2. — This follows from the Hochschild-Serre spectral sequence applied to the ideals $\mathfrak{h} \times \{0\}$ and $\{0\} \times \mathfrak{k}$ (see [Fu, p. 40]) and from the THEOREM 5.2.1. $\square$

LEMMA 7.0.3. — *Let $\mathfrak{a}$ be a finite dimensional Lie superalgebra, we have* :

$$\mathrm{Ext}^i_{U(\mathfrak{a})}(V, W) \simeq H^i\big(\mathfrak{a}, \mathrm{Hom}_k(V, W)\big).$$

If $W = U(\mathfrak{a})$ (or more generally an $\mathfrak{a} \times \mathfrak{a}$ bimodule), then $\mathrm{Ext}^i_{U(\mathfrak{a})}(V, U(\mathfrak{a}))$ is endowed with a natural right $\mathfrak{a}$-module structure provided by right multiplication. If we also endow $\mathrm{Hom}_k(V, U(\mathfrak{a}))$ with the right $\mathfrak{a}$-module structure given by right multiplication, then the isomorphism above is a right $U(\mathfrak{a})$-module isomorphism.

Proof of lemma 7.0.3. — See [Kn, p. 185]. $\square$

LEMMA 7.0.4. — *Let $\mathfrak{a}$ be a finite dimensional Lie superalgebra. Let U be a left $\mathfrak{a}$-module and let V be a right $\mathfrak{a}$-module. We have the following isomorphism*

$$\mathrm{Ext}^i_{U(\mathfrak{a})}\big(U, V \otimes \mathrm{ber}(\mathfrak{a})\big) \simeq \mathrm{Ext}^i_{U(\mathfrak{a})}\big(U \otimes \mathrm{ber}(\mathfrak{a}^*), V\big),$$

where, in the left hand side (resp. right hand side), the Ext *is taken over left modules (resp. right modules).*

Proof of the theorem 7.0.1. — We put $E = V^* \otimes U(\mathfrak{g}) \otimes W$. We endow E with a $\mathfrak{h} \times \mathfrak{k}$-bimodule structure by the following operations : for all $H \in \mathfrak{h}$, $K \in \mathfrak{k}$, $u \in U(\mathfrak{g})$, $w \in W$, $v^* \in V^*$, we have

$$H \cdot (v^* \otimes u \otimes w) = H \cdot v^* \otimes u \otimes v + (-1)^{|H| \cdot |v^*|} v^* \otimes Hu \otimes w,$$

$$(v^* \otimes u \otimes w) \cdot K = v^* \otimes uK \otimes w(-1)^{|K| \cdot |w|} - v^* \otimes u \otimes K \cdot w(-1)^{|K| \cdot |w|}.$$

It is clear that E is free as a left $U(\mathfrak{h})$-module and as a right $U(\mathfrak{k})$-module (the reader can refer to the LEMMA 7.0.6 a bit further). We now apply the LEMMA 7.0.2 to the $\mathfrak{h} \times \mathfrak{k}$-bimodule E. On one hand, we get :

$$H^n\big(\mathfrak{h} \times \mathfrak{k}, E \otimes \mathrm{ber}(\mathfrak{k})\big) \simeq H^{n-h_0}\left(\mathfrak{k}, \frac{E \otimes \mathrm{ber}(\mathfrak{k}) \otimes \mathrm{ber}(\mathfrak{h}^*)}{(E \otimes \mathrm{ber}(\mathfrak{k}) \otimes \mathrm{ber}(\mathfrak{h}^*))\,\mathfrak{h}}\right)$$

$$\simeq H^{n-h_0}\Big(\mathfrak{k}, \mathrm{IND}_{\mathfrak{k}}^{\mathfrak{g}}\big(V^* \otimes \mathrm{ber}(\mathfrak{h}^*)\big) \otimes W \otimes \mathrm{ber}(\mathfrak{k})\Big).$$

Then, by using successively the LEMMA 7.0.3 and 7.0.4, we have

$$H^n\big(\mathfrak{h} \times \mathfrak{k}, E \otimes \mathrm{ber}(\mathfrak{k})\big) \simeq \mathrm{Ext}_{U(\mathfrak{k})}^{n-h_0}\Big(W^*, \mathrm{IND}_{\mathfrak{k}}^{\mathfrak{g}}\big(V^* \otimes \mathrm{ber}(\mathfrak{h}^*)\big) \otimes \mathrm{ber}(\mathfrak{k})\Big)$$

$$\simeq \mathrm{Ext}_{U(\mathfrak{k})}^{n-h_0}\Big(W^* \otimes \mathrm{ber}(\mathfrak{k}^*), \mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}\big(V^* \otimes \mathrm{ber}(\mathfrak{h}^*)\big)\Big).$$

Lastly, using Shapiro's lemma [Kn, p. 286], we obtain

$$H^n\big(\mathfrak{h} \times \mathfrak{k}, E \otimes \mathrm{ber}(\mathfrak{k})\big)$$
$$\simeq \mathrm{Ext}_{U(\mathfrak{g})}^{n-h_0}\Big(\mathrm{IND}_{\mathfrak{k}}^{\mathfrak{g}}\big(W^* \otimes \mathrm{ber}(\mathfrak{k}^*)\big), \mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}\big(V^* \otimes \mathrm{ber}(\mathfrak{h}^*)\big)\Big).$$

On the other hand, by the same series of arguments, we get :

$$H^n\big(\mathfrak{h} \times \mathfrak{k}, E \otimes \mathrm{ber}(\mathfrak{k})\big) \simeq \mathrm{Ext}_{U(\mathfrak{g})}^{n-s_0}\big(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), \mathrm{Ind}_{\mathfrak{k}}^{\mathfrak{g}}(W)\big).$$

This finishes the proof of the THEOREM 7.0.1.

In the case where $\mathfrak{k} = \{0\}$ and $W = \{0\}$, the THEOREM 7.0.1 allows us to compute $\mathrm{Ext}_{U(\mathfrak{g})}^{n}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), U(\mathfrak{g}))$:

- If $n \neq h_0$, then $\mathrm{Ext}_{U(\mathfrak{g})}^{n}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), U(\mathfrak{g}))$ equals $\{0\}$.
- For $n = h_0$, we can improve the result.

THEOREM 7.0.5. — *Let $\mathfrak{g}$ be a Lie superalgebra and let $\mathfrak{h}$ be a finite dimensional subsuperalgebra of $\mathfrak{g}$. Let V be a finite dimensional left $U(\mathfrak{h})$-module. Put $h_0 = \dim \mathfrak{h}_{\bar{0}}$. The right $U(\mathfrak{g})$-modules $\mathrm{Ext}_{U(\mathfrak{g})}^{h_0}(\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V), U(\mathfrak{g}))$ and $\mathrm{IND}_{\mathfrak{h}}^{\mathfrak{g}}(V^* \otimes \mathrm{ber}(\mathfrak{h}^*))$ are isomorphic.*

This result was proved by BROWN and LEVASSEUR [B-L, p. 410] and KEMPF [K] in the case where $\mathfrak{g}$ is a finite dimensional semi-simple Lie algebra and $\mathrm{Ind}_{\mathfrak{h}}^{\mathfrak{g}}(V)$ is a Verma module.

To prove the THEOREM 7.0.5, we will need the following lemma.

LEMMA 7.0.6. — *Let us denote by* $(V^*\otimes U(\mathfrak{h}))_1$ *the superspace* $V^*\otimes U(\mathfrak{h})$ *endowed with the following* $\mathfrak{h}\times\mathfrak{h}$*-bimodule structure : for all* $f\in V^*$*, for all* $u\in U(\mathfrak{g})$*, for all* $H\in\mathfrak{h}$*,*

$$(f\otimes u)\cdot H = f\otimes uH,$$

$$H\cdot(f\otimes u) = H\cdot f\otimes u + (-1)^{|H|\cdot|f|} f\otimes Hu.$$

Let us denote by $(V^*\otimes U(\mathfrak{h}))_2$ *the superspace* $V^*\otimes U(\mathfrak{h})$ *endowed with the following* $\mathfrak{h}\times\mathfrak{h}$*-bimodule structure : for all* $f\in V^*$*, for all* $u\in U(\mathfrak{g})$*, for all* $H\in\mathfrak{h}$*,*

$$H\cdot(f\otimes u) = f\otimes Hu(-1)^{|H|\cdot|f|},$$

$$(f\otimes u)\cdot H = -H\cdot f\otimes u(-1)^{|H|(|f|+|u|)} + f\otimes uH.$$

Then the $\mathfrak{h}\times\mathfrak{h}$ *bimodules* $(V^*\otimes U(\mathfrak{h}))_1$ *and* $(V^*\otimes U(\mathfrak{h}))_2$ *are isomorphic.*

Proof of lemma 7.0.6. — The map (see [D1, p. 387])

$$\begin{array}{ccc} (V^*\otimes U(\mathfrak{h}))_2 & \longrightarrow & (V^*\otimes U(\mathfrak{h}))_1 \\ v^*\otimes u & \longmapsto & \displaystyle\sum_i u_i' v^*\otimes u_i''(-1)^{|u_i'|\cdot|v^*|}, \end{array}$$

where $\Delta(u)=\sum_i u_i'\otimes u_i''$, provides an isomorphism between the two structures.

Proof of the theorem 7.0.5. — Let us compute the right $U(\mathfrak{g})$-module $\mathrm{Ext}^{h_0}_{U(\mathfrak{g})}(Ind^{\mathfrak{g}}_{\mathfrak{h}}(V),U(\mathfrak{g}))$. One sees easily [B-L, p. 397] that we have the following isomorphism of right $U(\mathfrak{g})$-modules :

$$\mathrm{Ext}^{h_0}_{U(\mathfrak{g})}\big(\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{h}}(V),U(\mathfrak{g})\big)\simeq \mathrm{Ext}^{h_0}_{U(\mathfrak{h})}\big(V,U(\mathfrak{h})\big)\underset{U(\mathfrak{h})}{\otimes}U(\mathfrak{g}).$$

Hence, we have now to compute the right $U(\mathfrak{h})$-module $\mathrm{Ext}^{h_0}_{U(\mathfrak{h})}(V,U(\mathfrak{h}))$. Using successively the LEMMA 7.0.3 and the LEMMA 7.0.6, we see that we have the following isomorphism of right $U(\mathfrak{h})$-modules :

$$\begin{aligned}\mathrm{Ext}^{h_0}_{U(\mathfrak{h})}\big(V,U(\mathfrak{h})\big) &\simeq H^{h_0}\big(\mathfrak{h},(V^*\otimes U(\mathfrak{h}))_1\big)\\ &\simeq H^{h_0}\big(\mathfrak{h},(V^*\otimes U(\mathfrak{h}))_2\big)\\ &\simeq V^*\otimes H^{h_0}\big(\mathfrak{h},U(\mathfrak{h})\big).\end{aligned}$$

Then, the PROPOSITION 5.4.1 tells us that the right $U(\mathfrak{h})$-modules $\mathrm{Ext}^{h_0}_{U(\mathfrak{h})}(V,U(\mathfrak{h}))$ and $\mathrm{ber}(\mathfrak{h}^*)$ are isomorphic. This finishes the proof of the THEOREM 7.0.5.

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