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DEGREES OF CURVES IN ABELIAN VARIETIES

BY

OLIVIER DEBARRE (*)

RÉSUMÉ. — Le degré d'une courbe C contenue dans une variété abélienne polarisée (X,λ) est l'entier $d = C\cdot\lambda$. Lorsque C est irréductible et engendre X, on obtient une minoration de d en fonction de n et du degré de la polarisation λ. Le plus petit degré possible est $d = n$ et n'est atteint que pour une courbe lisse dans sa jacobienne avec sa polarisation principale canonique (Ran, Collino). On étudie les cas $d = n+1$ et $d = n+2$. Lorsque X est simple, on montre de plus, en utilisant des résultats de Smyth sur la trace des entiers algébriques totalement positifs, que si $d \leq 1,7719\,n$, alors C est lisse et X est isomorphe à sa jacobienne. Nous obtenons aussi une borne supérieure pour le genre géométrique de C en fonction de son degré.

ABSTRACT. — The degree of a curve C in a polarized abelian variety (X,λ) is the integer $d = C\cdot\lambda$. When C is irreducible and generates X, we find a lower bound on d which depends on n and the degree of the polarization λ. The smallest possible degree is $d = n$ and is obtained only for a smooth curve in its Jacobian with its principal polarization (Ran, Collino). The cases $d = n+1$ and $d = n+2$ are studied. Moreover, when X is simple, it is shown, using results of Smyth on the trace of totally positive algebraic integers, that if $d \leq 1.7719\,n$, then C is smooth and X is isomorphic to its Jacobian. We also get an upper bound on the geometric genus of C in terms of its degree.

1. Introduction

Although curves in projective spaces have attracted a lot of attention for a long time, very little is known in comparison about curves in abelian varieties. We try in this article to partially fill this gap.

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Let (X, λ) be a principally polarized abelian variety of dimension n defined over an algebraically closed field k. The degree of a curve C contained in X is $d = C \cdot \lambda$.

The first question we are interested in is to find what numbers can be degrees of irreducibles curves C. When C generates X, we prove that $d \geq n(\lambda^n/n!)^{1/n} \geq n$. It is known (see [C], [R]) that $d = n$ if and only if C is smooth and X is isomorphic to its Jacobian JC with its canonical principal polarization. What about the next cases? We get partial characterizations for $d = n+1$ and $d = n+2$, and we show (example 6.11) that all degrees $\geq n+2$ actually occur when $\mathrm{char}(k) = 0$. However, it seems necessary to assume X simple to go further. We prove, using results of SMYTH [S], that *if C is an irreducible curve of degree $< 2n$ if $n \leq 7$, and $\leq 1.7719\,n$ if $n > 7$, on a simple principally polarized abelian variety X of dimension n, then C is smooth, has degree $(2n-m)$ for some divisor m of n, the abelian variety X is isomorphic to JC* (with a non-canonical principal polarization) *and C is canonically embedded in X.* We conjecture this result to hold for any n under the assumption that C has degree $< 2n$. This would be a consequence of our CONJECTURE 6.2, which holds for $n \leq 7$: *the trace of a totally positive algebraic integer σ of degree n is at least* $(2n-1)$ *and equality can hold only if σ has norm* 1. Smooth curves of genus n and degree $(2n-1)$ in their Jacobians have been constructed by MESTRE for any n in [Me].

The second question is the Castelnuovo problem : bound the geometric genus $p_g(C)$ of a curve C in a polarized abelian variety X of dimension n in terms of its degree d. We prove, using the original Castelnuovo bound for curves in projective spaces, the inequality $p_g(C) < (2d-1)^2/(2(n-1))$, which is far from being sharp (better bounds are obtained for small degrees). This in turn yields a lower bound in $\mathrm{O}(n^{3/2})$ on the degree of a curve in a *generic* principally polarized abelian variety of dimension n.

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2. Endomorphisms and polarizations of abelian varieties

Let X be an abelian variety of dimension n defined over an algebraically closed field k and let $\mathrm{End}(X)$ be its ring of endomorphisms. The degree $\deg(u)$ of an endomorphism u is defined to be 0 if u is not surjective, and the degree of u as a map otherwise. For any prime ℓ different from the characteristic of k, the Tate module $T_\ell(X)$ is a free $\mathbb{Z}_\ell$-module of rank $2n$ [Mu, p. 171] and the ℓ-adic representation $\rho_\ell : \mathrm{End}(X) \to \mathrm{End}(T_\ell(X))$

is injective. For any endomorphism u of X, the characteristic polynomial of $\rho_\ell(u)$ has coefficients in $\mathbb{Z}$ and is independent of ℓ. It is called the *characteristic polynomial* of u and is denoted by P_u. It satisfies

$$P_u(t) = \deg(t \operatorname{Id}_X - u)$$

for any integer t [Mu, thm 4, p. 180]. The opposite $\operatorname{Tr}(u)$ of the coefficient of t^{2n-1} is called the *trace* of u.

The *Néron-Severi group* of X is the group of algebraic equivalence classes of invertible sheaves on X. Any element μ of $NS(X)$ defines a morphism $\phi_\mu : X \to \operatorname{Pic}^0(X)$ [Mu, p. 60] whose scheme-theoretic kernel is denoted by $K(\mu)$. The Riemann-Roch theorem gives $\chi(X,\mu) = \mu^n/n\,!$, a number which will be called the *degree* of μ. One has $\deg \phi_\mu = (\deg \mu)^2$ [Mu, p. 150]. A *polarization* λ on X is the algebraic equivalence class of an ample invertible sheaf on X; it is said to be *separable* if its degree is prime to $\operatorname{char}(k)$. In that case, ϕ_λ is a separable isogeny and its kernel is isomorphic to a group $(\mathbb{Z}/\delta_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/\delta_n\mathbb{Z})^2$, where $\delta_1 \mid \cdots \mid \delta_n$ and $\delta_1 \cdots \delta_n = \deg(\lambda)$. We will say that λ is of type $(\delta_1|\cdots|\delta_n)$. We will need the following result.

THEOREM 2.1. (KEMPF, MUMFORD, RAMANUJAN). — *Let X be an abelian variety of dimension n, and let λ and μ be two elements of $NS(X)$. Assume that λ is a polarization. Then :*

(i) *The roots of the polynomial $P(t) = (t\lambda - \mu)^n$ are all real.*

(ii) *If μ is a polarization, the roots of P are all positive.*

(iii) *If P has no negative roots and r positive roots, there exist a polarized abelian variety (X', μ') of dimension r and a surjective morphism $f : X \to X'$ with connected kernel such that $\mu = f^*\mu'$.*

Proof. — The first point is part of [MK, thm 2, p. 98]. The second point follows from the same theorem and the fact that if M is an ample line bundle on X with class μ, one has $H^i(X, M) = 0$ for $i > 0$ [Mu, § 16]. For the last point, the same theorem from [MK] yields that the neutral component K of the group $K(\mu)$ has dimension $(n-r)$. The restriction of M to K is algebraically equivalent to 0 [*loc.cit.*, lemma 1, p. 95] hence, since the restriction $\operatorname{Pic}^0(X) \to \operatorname{Pic}^0(K)$ is surjective, there exists a line bundle N on X algebraically equivalent to 0 such that the restriction of $M \otimes N$ to K is trivial. It follows from theorem 1, p. 95 of *loc.cit.* that $M \otimes N$ is the pull-back of a line bundle on $X' = X/K$. □

2.2. — Suppose now that θ is a principal polarization on X, i.e. a polarization of degree 1. It defines a morphism of $\mathbb{Z}$-modules

$$\beta_\theta : NS(X) \longrightarrow \operatorname{End}(X)$$

by the formula $\beta_\theta(\mu) = \phi_\theta^{-1} \circ \phi_\mu$. Its image consists of all endomorphisms invariant under the *Rosati involution*, which sends an endomorphism u to $\phi_\theta^{-1} \circ \operatorname{Pic}^0(u) \circ \phi_\theta$ [Mu, (3) p. 190]. Moreover, one has, for any integer t :

$$\left(\frac{(t\theta - \mu)^n}{n!}\right)^2 = \deg(t\phi_\theta - \phi_\mu) = \deg\bigl(t \operatorname{Id}_X - \beta_\theta(\mu)\bigr) = P_{\beta_\theta(\mu)}(t).$$

2.3. — Let (X, λ) be a polarized abelian variety. For $0 < r \le n$, we set

$$\lambda^r_{\min} = \frac{\lambda^r}{r!\,\delta_1 \cdots \delta_r}.$$

If $k = \mathbb{C}$, the class of $\lambda^r_{\min}$ is minimal (i.e. non-divisible) in $H^{2r}(X, \mathbb{Z})$. If k is any algebraically closed field, and if ℓ is a prime number different from the characteristic of k, the group $H^1_{\text{ét}}(X, \mathbb{Z}_\ell)$ is a free $\mathbb{Z}_\ell$-module of rank n [Mi, thm 15.1] and the algebras $H^*_{\text{ét}}(X, \mathbb{Z}_\ell)$ with its cup-product structure and $\bigwedge^* H^1_{\text{ét}}(X, \mathbb{Z}_\ell)$ with its wedge-product structure, are isomorphic [Mi, rem. 15.4]. In particular, the class $[\lambda]_\ell$ in $H^2_{\text{ét}}(X, \mathbb{Z}_\ell)$ of the polarization λ can be viewed as an alternating form on a free $\mathbb{Z}_\ell$-module, and as such has elementary divisors. If λ is *separable*, (X, λ) lifts in characteristic 0 to a polarized abelian variety of the same type $(\delta_1 | \cdots | \delta_n)$. The elementary divisors of $[\lambda]_\ell$ are then the maximal powers of ℓ that divide $\delta_1, \ldots, \delta_n$. Since intersection corresponds to cup-product in étale cohomology, the class of $\lambda^r_{\min}$ is in $H^{2r}_{\text{ét}}(X, \mathbb{Z}_\ell)$ and is not divisible by ℓ.

Throughout this article, all schemes we consider will be defined over an algebraically closed field k. We will denote numerical equivalence by $\sim$. If C is a smooth curve, JC will be its Jacobian and θ_C its canonical principal polarization.

3. Curves and endomorphisms

We summarize here some results from [Ma] and [Mo]. Let C be a curve on a polarized abelian variety (X, λ) and let D be an effective divisor that represents λ. Morikawa proves that the following diagram, where d is the degree of C and S is the sum morphism, defines an endomorphism $\alpha(C, \lambda)$ of X which is independent on the choice of D :

$$\alpha(C,\lambda) : X \dashrightarrow C^{(d)} \xrightarrow{\ S\ } X \xrightarrow{\text{translation}} X$$

$$x \longmapsto (D + x) \cap C.$$

3.1. — Let N be the normalization of C. The morphism $\iota : N \to X$ factorizes through a morphism $p : JN \to X$. Set $q = \iota^* \circ \phi_\lambda : X \to JN$; Matsusaka proves that $\alpha(C, \lambda) = p \circ q$ [Ma, lemma 3].

3.2. — He also proves [*loc.cit.*, thm 2] that $\alpha(C,\lambda) = \alpha(C',\lambda)$ if and only if $C \sim C'$. Since $\alpha(\lambda^{n-1},\lambda) = (\lambda^n/n)\,\mathrm{Id}_X$, it follows that :

$$\alpha(C,\lambda) = m\ \mathrm{Id}_X \iff C \sim \frac{m}{(n-1)!\ \deg\lambda}\,\lambda^{n-1}.$$

If moreover λ is separable of type $(\delta_1|\cdots|\delta_n)$ and if ℓ is a prime distinct from $\mathrm{char}(k)$, the discussion of 2.3 yields that there exists a class ϵ in $H^2_{\text{ét}}(X,\mathbb{Z}_\ell)$ such that $\lambda^{n-1}\cdot\epsilon$ is $(n-1)!\,\delta_1\cdots\delta_{n-1}$ times a generator of $H^{2n}_{\text{ét}}(X,\mathbb{Z}_\ell)$. It follows that $c = m/\delta_n$ must be in $\mathbb{Z}_\ell$. But δ_n is prime to $\mathrm{char}(k)$, hence c is an integer and $C \sim c\lambda^{n-1}_{\min}$.

Let θ_N be the canonical principal polarization on JN. One has :

$$\text{(3.3)}\quad \phi_{q^*\theta_N} = \mathrm{Pic}^0(q)\circ\phi_{\theta_N}\circ q = \phi_\lambda\circ p\circ\phi_{\theta_N}^{-1}\circ\iota^*\circ\phi_\lambda = \phi_\lambda\circ\alpha(C,\lambda).$$

Similarly :

$$\phi_{p^*\lambda} = \mathrm{Pic}^0(p)\circ\phi_\lambda\circ p = \phi_{\theta_N}\circ q\circ\phi_\lambda^{-1}\circ\phi_\lambda\circ p = \phi_{\theta_N}\circ q\circ p.$$

Note that, if g is the genus of N, one has :

$$C\cdot\lambda = N\cdot p^*\lambda = \frac{\theta_N^{g-1}}{(g-1)!}\cdot p^*\lambda.$$

In particular, $-2(C\cdot\lambda)$ is the coefficient of t^{2g-1} in the polynomial :

$$\deg(t\,\theta_N - p^*\lambda)^2 = \deg(t\,\phi_{\theta_N} - \phi_{p^*\lambda}) = \deg(t\ \mathrm{Id}_{JN} - q\circ p).$$

Since $\mathrm{Tr}(q\circ p) = \mathrm{Tr}(p\circ q) = \mathrm{Tr}(\alpha(C,\lambda))$, the following equality, originally proved by MATSUSAKA [Ma, cor., p. 8], holds :

$$\text{(3.4)}\qquad \mathrm{Tr}\big(\alpha(C,\lambda)\big) = 2(C\cdot\lambda).$$

3.5.—If the Néron-Severi group of X has rank 1 (this holds for a generic principally polarized X by [M, thm 6.5], hence for a generic X with any polarization by [Mu, cor. 1, p. 234]), and ample generator ℓ', we can write $q^*\theta_N = r\ell'$ and $\ell = s\ell'$ with r and s integers. We get $r\phi_{\ell'} = s\phi_{\ell'}\circ\alpha(C,\ell)$ hence $\alpha(C,\ell)(sx) = rx$ for all x in X. By taking degrees, one sees that s divides r and $\alpha(C,\ell) = (r/s)\,\mathrm{Id}_X$. By (3.2), any curve C is numerically equivalent to a rational multiple of λ^{n-1} and its degree is a multiple of n. If ℓ' is separable of type $(\delta_1|\cdots|\delta_n)$, any curve C is numerically equivalent to an integral multiple of $\lambda^{n-1}_{\min}$ and its degree is a multiple of $n\delta_n$.

LEMMA 3.6. — *Let C be an irreducible curve that generates a polarized abelian variety (X,λ) of dimension n. Then, the polynomial $P_{\alpha(C,\lambda)}$ is the square of a polynomial whose roots are all real and positive.*

Proof. — Let $\alpha = \alpha(C,\lambda)$. By (3.3), one has $\phi_{q^*\theta_N} = \phi_\lambda \circ \alpha$, hence, for any integer t :

$$\begin{aligned} P_\alpha(t)\ \deg\phi_\lambda &= \deg(t\,\mathrm{Id}_X - \alpha)\ \deg\phi_\lambda \\ &= \deg(t\phi_\lambda - \phi_\lambda\circ\alpha) = \deg(t\phi_\lambda - \phi_{q^*\theta_N}) \\ &= \deg(\phi_{t\lambda - q^*\theta_N}) = \Big[\frac{1}{n\,!}\,(t\lambda - q^*\theta_N)^n\Big]^2. \end{aligned}$$

The lemma then follows from THEOREM 2.1. ∎

We end this section with a proof of Matsusaka's celebrated criterion.

THEOREM 3.7. (MATSUSAKA). — *Let C be an irreducible curve in a polarized abelian variety (X,λ) of dimension n. Assume that $\alpha(C,\lambda) = \mathrm{Id}_X$. Then C is smooth and (X,λ) is isomorphic to (JC,θ_C).*

Proof. — Let N be the normalization of C. The morphism $\alpha(C,\lambda)$ is the identity and factors as :

$$X \dashrightarrow N^{(n)} \longrightarrow W_n(N) \longrightarrow JN \longrightarrow X.$$

It follows that $\dim JN = g(N) \geq n$. Moreover, the image of X in JN has dimension n, hence is the entire $W_n(N)$, which is therefore an abelian variety. This is possible only if $g(N) \leq n$. Hence N has genus n. It follows that the morphism $q : X \to JN$ is an isogeny, which is in fact an isomorphism since $p\circ q = \alpha(C,\lambda) = \mathrm{Id}_X$. By (3.3), the polarizations $q^*\theta_N$ and λ are equal, hence q induces an isomorphism of the polarizations. ∎

4. Degrees of curves

Let C be an irreducible curve that generates a polarized abelian variety (X,λ) of dimension n. We want to study its degree $d = C\cdot\lambda$. First, by description (3.1), the dimension of the image of $\alpha(C,\lambda)$ is the dimension of the abelian subvariety $\langle C\rangle$ generated by C. This and the definition of $\alpha(C,\lambda)$ imply :

$$C\cdot\lambda \geq n.$$

It was proved by RAN [R] for $k = \mathbb{C}$ and by COLLINO [C] in general, that if $C\cdot\lambda = n$, the minimal value, then C is smooth and (X,λ) is isomorphic to its Jacobian (JC,θ_C). This suggests that there should be a better lower bound on the degree that involves the type of the polarization λ. The following proposition provides such a bound.

PROPOSITION 4.1. — *Let C be an irreducible curve that generates a polarized abelian variety (X,λ) of dimension n. Then :*

$$C\cdot\lambda \geq n\,(\deg\lambda)^{\frac{1}{n}}.$$

If λ is separable, there is equality if and only if C is smooth and (X,λ) is isomorphic to $(JC,\delta\theta_C)$, for some integer δ prime to $\mathrm{char}(k)$.

Recall that by (3.5), the degree of any curve on a *generic* polarized abelian variety (X,λ) is a multiple of n. When λ is separable of type $(\delta_1|\cdots|\delta_n)$, this degree is even a multiple of $n\delta_n$.

Proof of the proposition. — We know by LEMMA 3.6 that $P_{\alpha(C,\lambda)}$ is the square of a polynomial Q whose roots $\beta_1,\ldots,\beta_n$ are real and positive. We have :

$$\begin{aligned}C\cdot\lambda &= \tfrac{1}{2}\operatorname{Tr}\alpha(C,\lambda) = \tfrac{1}{2}(2\beta_1+\cdots+2\beta_n)\\ &\geq n\,(\beta_1\cdots\beta_n)^{\frac{1}{n}} = n\,Q(0)^{\frac{1}{n}} = n\,P_{\alpha(C,\lambda)}(0)^{\frac{1}{2n}}\\ &= n\,\big(\deg\alpha(C,\lambda)\big)^{\frac{1}{2n}} \geq n(\deg\phi_\lambda)^{\frac{1}{2n}} = n\,(\deg\lambda)^{\frac{1}{n}}.\end{aligned}$$

This proves the inequality in the proposition. If there is equality, $\beta_1,\ldots,\beta_n$ must be all equal to the same number m, which must be an integer since $P_{\alpha(C,\lambda)}$ has integral coefficients. It follows from the proof of LEMMA 3.6 that :

$$\left[\frac{1}{n!}\,(t\lambda-q^*\theta_N)^n\right]^2 = P_\alpha(t)\deg\phi_\lambda = (t-m)^{2n}\deg\phi_\lambda.$$

Thus THEOREM 2.1 (iii) yields $m\lambda = q^*\theta_N$. It follows from (3.3) that $\alpha(C,\lambda)=m\,\mathrm{Id}_X$.

If λ is separable of type $(\delta_1|\cdots|\delta_n)$, by (3.2), the number $c=m/\delta_n$ is an integer and C is numerically equivalent to $c\lambda_{\min}^{n-1}$. We get :

$$c\,n\,\delta_n = C\cdot\lambda = n\,(\deg\lambda)^{\frac{1}{n}} = n\,(\delta_1\cdots\delta_n)^{\frac{1}{n}} \leq n\,\delta_n.$$

This implies $c=1$ and $\delta_1=\cdots=\delta_n=\delta$. But then λ is δ times a principal polarization θ [Mu, thm 3, p. 231] and $C\sim\theta_{\min}^{n-1}$. The conclusion now follows from Matsusaka's criterion 3.7. ∎

COROLLARY 4.2 (RAN, COLLINO). — *Let C be an irreducible curve that generates a polarized abelian variety (X, λ) of dimension n. Assume that $C \cdot \lambda = n$. Then C is smooth and (X, λ) is isomorphic to its Jacobian (JC, θ_C).*

Proof. — Although the converse of the proposition was proved only for λ separable, we still get from its proof that $\alpha(C, \lambda)$ is the identity of X and we may then apply Matsusaka's criterion 2.7. This is the same proof as Collino's. □

More generally, if $C \cdot \lambda = \dim\langle C \rangle$, the same reasoning can be applied on $\langle C \rangle$ with the induced polarization to prove that C is smooth and that (X, λ) is isomorphic to the product of (JC, θ_C) with a polarized abelian variety.

COROLLARY 4.3. — *Let X be an abelian variety with a separable polarization λ of type $(\delta_1 | \cdots | \delta_n)$. Let C be an irreducible curve in X and let m be the dimension of the abelian subvariety that it generates. Then :*

$$C \cdot \lambda \geq m\,(\delta_1 \cdots \delta_m)^{\frac{1}{m}}.$$

Proof. — Apply the proposition on the abelian subvariety Y generated by C. All there is to show is that the degree $Y \cdot \lambda^m / m\,!$ of the restriction λ' of λ to Y is at least $\delta_1 \cdots \delta_m$. We will prove that it is actually *divisible* by $\delta_1 \cdots \delta_m$. When $k = \mathbb{C}$, this follows from the fact that the class $\lambda_{\min}^m$ is integral. The following argument for the general case was kindly communicated to the author by KEMPF. Let ι be the inclusion of Y in X. Then $\phi_{\lambda'} = \mathrm{Pic}^0(\iota) \circ \phi_\lambda \circ \iota$, hence $\deg(\lambda')^2$, which is the order of the kernel of $\phi_{\lambda'}$, is a multiple of the order of its subgroup $K(\lambda) \cap Y$, hence a fortiori a multiple of the order of its (r, r) part K'. In other words, since $K' \simeq (\mathbb{Z}/\delta'_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/\delta'_m\mathbb{Z})^2$ for some integers $\delta'_1 \mid \delta'_2 \mid \cdots \mid \delta'_m$ prime to $\mathrm{char}(k)$, it is enough to show that $\delta'_1 \delta'_2 \cdots \delta'_m$ is a multiple of $\delta_1 \delta_2 \cdots \delta_m$.

Let ℓ be a prime number distinct from $\mathrm{char}(k)$ and let $\mathbb{F}_\ell$ be the field with ℓ elements. For any integer s, let X_s be the kernel of the multiplication by ℓ^s on X. Then X_s / X_{s-1} is a $\mathbb{F}_\ell$-vector space of dimension $2n$ of which Y_s / Y_{s-1} is a subspace of dimension $2m$. Since $K(\lambda)$ is isomorphic to $(\mathbb{Z}/\delta_1\mathbb{Z})^2 \times \cdots \times (\mathbb{Z}/\delta_n\mathbb{Z})^2$, the rank over $\mathbb{F}_\ell$ of $(K(\lambda) \cap X_s)/(K(\lambda) \cap X_{s-1})$ is twice the cardinality of the set $\{i \in \{1, \ldots, n\}\,;\ \ell^s \mid \delta_i\}$. The dimension formula yields :

$$\mathrm{rank}\big(K(\lambda) \cap Y_s\big)/\big(K(\lambda) \cap Y_{s-1}\big) \geq 2\,\mathrm{Card}\{i\,;\ \ell^s \mid \delta_i\} - 2n + 2m.$$

But the rank of $(K(\lambda) \cap Y_s)/(K(\lambda) \cap Y_{s-1}) = (K' \cap X_s)/(K' \cap X_{s-1})$ is also twice the cardinality of $\{i \in \{1,\dots,m\}\ ;\ \ell^s \mid \delta'_i\}$. It follows that :

$$\mathrm{Card}\{i \in \{1,\dots,n\}\ ;\ \ell^s \nmid \delta_i\} \geq \mathrm{Card}\{i \in \{1,\dots,m\}\ ;\ \ell^s \nmid \delta'_i\}.$$

This implies what we need. □

COROLLARY 4.4. — *Let C be an irreducible curve that generates a principally polarized abelian variety (X,θ) of dimension n. Assume that C is invariant by translation by an element ϵ of X of order m. Then $C \cdot \theta \geq n\,m^{1-1/n}$.*

Proof. — Let H be the subgroup scheme generated by ϵ. The abelian variety $X' = X/H$ has a polarization λ of degree m^{n-1} whose pull-back on X is $m\theta$ [Mu, cor., p. 231]. If C' is the image of C in X', the proposition yields $C \cdot \theta = C' \cdot \lambda \geq n\,m^{1-1/n}$. □

Note that in the situation of COROLLARY 4.4, if (X,θ) is a *generic* principally polarized abelian variety of dimension n, and m is prime to $\mathrm{char}(k)$, then mn divides $C \cdot \theta$. With the notation of the proof above, this follows from the fact that any curve on X' is numerically equivalent to an integral multiple of $\lambda_{\min}^{n-1}$ (see (3.5)).

5. Bounds on the genus

We keep the same setting : C is an irreducible curve that generates a polarized abelian variety (X,λ), its normalization is N, and its degree is $d = C \cdot \lambda$. The composition :

$$X \dashrightarrow N^{(d)} \longrightarrow W_d(N) \longrightarrow JN$$

is a morphism with finite kernel (since $\alpha(C,\lambda)$ is an isogeny), hence $W_d(N)$ contains an abelian variety of dimension n. We can apply the ideas of [AH] to get a bound of Castelnuovo type on the genus of N. Note that if C does not generate X, the same bound holds with n replaced by the dimension of $\langle C\rangle$.

THEOREM 5.1. — *Let C be an irreducible curve that generates a separably polarized abelian variety (X,λ) of dimension $n > 1$. Let N be the normalization of C and let $d = C \cdot \lambda$. Then :*

$$g(N) < \frac{(2d-1)^2}{2(n-1)}.$$

The inequality in the second part of lemma 8 in [AH] would improve this bound when $\mathrm{char}(k) = 0$, but its proof is incorrect.

Proof. — Let A be the image of X in $W_d(N)$ and let A_2 be the image of $A \times A$ in $W_{2d}(N)$ under the addition map. We want to show that the morphism associated with a generic point of A_2 is generically injective on N. The linear systems corresponding to points of A_2 are of the form $|\mathcal{O}_N(2D_x)|$, where x varies in X, where D is an effective divisor that represents λ and $D_x = D + x$. It is therefore enough to show that the restriction to $C-x$ of the morphism ϕ_{2D} associated with $|2D|$ is generically injective for x generic. If not, for x generic in X and a generic in $C - x$, there exists b in $C - x$ with $a \neq b$ and $\phi_{2D}(a) = \phi_{2D}(b)$. The same holds for a generic in X and x generic in $C - a$. Since ϕ_{2D} is finite [Mu, p. 60], b does not depend on x, hence $C - a = C - b$. Since C generates X, this implies that $\epsilon = a - b$ is torsion, hence does not depend on a. Letting a vary, we see that any divisor in $|2D|$ is invariant by translation by ϵ. The argument in [Mu, p. 164], yields a contradiction.

It follows that the image of the morphism $N \to \mathbf{P}^r$ that corresponds to a generic point in A_2 is a curve of degree a divisor d' of $2d$, with normalization N. Moreover, one has $r \geq n$ [AH, lemma 1]. Castelnuovo's bound ([AH, lemma 1] and [B] when $\mathrm{char}(k) > 0$) then gives :

$$g(N) \leq m(d'-1) - \tfrac{1}{2}m(m+1)(r-1),$$

where $m = \big[(d'-1)/(r-1)\big]$. Hence :

$$\begin{aligned} g(N) &\leq m\big(d'-1-\tfrac{1}{2}(m+1)(r-1)\big) \\ &< \frac{d'-1}{r-1}\big(d'-1-\tfrac{1}{2}(d'-1)\big) \\ &\leq \frac{(d'-1)^2}{2(n-1)} \leq \frac{(2d-1)^2}{2(n-1)}. \end{aligned}$$

This finishes the proof of the theorem. ☐

In particular, in a principally polarized abelian variety (X, λ) of dimension n, any *smooth* curve numerically equivalent to $c\,\theta_{\min}^{n-1}$ has genus $< (2cn-1)^2/(2(n-1))$. For curves in *generic* principally polarized abelian varieties of dimension n, I conjecture the stronger inequality

$$g(C) \leq cn + (c-1)^2.$$

The theorem also gives a lower bound on the degree of any curve in a *generic* complex polarized abelian variety of dimension n, whose only merit is to go to infinity faster than n.

COROLLARY 5.2. — *Let C be a curve in a generic complex polarized abelian variety (X, λ) of dimension n and let c be the integer such that C is numerically equivalent to $c\lambda_{\min}^{n-1}$. Then :*

$$c > \sqrt{\frac{n}{8}} - \frac{1}{4}.$$

Proof. — We may assume that λ is a principal polarization and that $n > 12$. Let N be the normalization of C. Corollary 5.5 in [AP] yields $g(N) > 1 + \frac{1}{4}n(n+1)$, which, combined with the proposition, gives what we want. □

We can get better bounds on the genus when d/n is small.

PROPOSITION 5.3. — *Let C be an irreducible curve that generates a complex polarized abelian variety (X, λ) of dimension n. Let N be the normalization of C and let $d = C \cdot \lambda$. Then :*

(i) *If $d < 2n$, then $g(N) \leq d$.*

(ii) *If $d = 2n$, then $g(N) < \frac{3}{2}d = 3n$.*

(iii) *If $d \leq 3n$, then $g(N) \leq 4d$.*

(iv) *If $d \leq 4n$, then $g(N) \leq 6d$.*

Proof. — We keep the notation of the proof of THEOREM 5.1. In particular, $W_d(N)$ contains an abelian variety A of dimension n. If $2n > d$, it follows from proposition 3.3 of [DF] that $g(N) \leq d$. Recall that we proved earlier that the morphisms that correspond to generic points in A_2 are *birational* onto their image. It follows from corollary 3.6 of *loc.cit.* that $g(N) < \frac{3}{2}d$ when $d = 2n$. This proves (ii). We will do (iv) only, (iii) being analogous. First, we may assume that the embedding of A in $W_d(N)$ satisfies the minimality assumptions made in [A1]. Let A_k be the image of $A \times \cdots \times A$ in $W_{kd}(N)$ under the addition map and let r_k be the maximum integer such that A_k is contained in $W_{kd}^{r_k}(N)$. If $g(N) > 6d$, we get, as in the proof of proposition 3.8 of [DF], the inequalities $r_6 \geq 8n + 2$ and $n \leq 6d - 3r_6$. It follows that $d \geq \frac{1}{6}(n + 3r_6) \geq \frac{1}{6}(25n + 6) > 4n$. This proves (iv). □

The inequality (ii) should be compared with the inequality

$$g(C) \leq 2n + 1$$

proved by WELTERS in [W] when $\mathrm{char}(k) = 0$ for any irreducible curve C numerically equivalent to $2\theta_{\min}^{n-1}$ on a principally polarized abelian variety (X, θ) of dimension n (so that $C \cdot \theta = 2n$). Equality is obtained only with the Prym construction.

6. Curves of low degrees

Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n. We keep the same notation : N is the normalization of C and $q : X \longrightarrow JN$ is the induced morphism. From (2.2), we get that the square of the monic polynomial $Q_C(T) = (T\theta - q^*\theta_N)^n/n!$ has integral coefficients (and is the characteristic polynomial of $\alpha(C, \theta)$). It follows that Q_C itself has integral coefficients, and we get from THEOREM 2.1 and (3.4) :

(i) The roots of Q_C are all real and positive.

(ii) The sum of the roots of Q_C is the degree $d = C \cdot \theta$.

(iii) The product of the roots of Q_C is the degree of the polarization $q^*\theta_N$.

SMYTH obtained in [S] a lower bound on the trace of a totally real algebraic integer in terms of its degree. His results can be partially summarized as follows.

THEOREM 6.1. (SMYTH). — *Let σ be a totally positive algebraic integer of degree m. Then* $\mathrm{Tr}(\sigma) > 1.7719\, m$, *unless σ belongs to an explicit finite set, in which case* $\mathrm{Tr}(\sigma) = 2m - 1$ *and* $\mathrm{Nm}(\sigma) = 1$.

It is tempting to conjecture :

CONJECTURE 6.2 (conjecture C_m).—*Let σ be a totally positive algebraic integer of degree m. Then we have* $\mathrm{Tr}(\sigma) \geq 2m - 1$. *If there is equality, then* $\mathrm{Nm}(\sigma) = 1$.

6.3. — The inequality in the conjecture follows from Smyth's theorem for $m \leq 8$ (and holds also for $m = 9$ according to further calculations). Smyth also worked out a list of all totally positive algebraic integers σ for which $\mathrm{Tr}(\sigma) - \deg(\sigma) \leq 6$. It follows from this list that *the full conjecture holds for $m \leq 7$.*

There are infinitely many examples for which the conjectural bound is obtained : if M is an odd prime, the algebraic integer $4\cos^2(\pi/2M)$ is totally positive, has degree $\frac{1}{2}(M-1)$, trace $(M-2)$ and norm 1.

PROPOSITION 6.4. — *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n and let Q_C be the polynomial defined above. Then, if $|Q_C(0)| = 1$, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded.*

Proof. — By fact (iii) above, the polarization $q^*\theta_N$ is principal. The proposition then follows from the next lemma. $\square$

LEMMA 6.5. — *Let (JN, θ_N) be the Jacobian of a smooth curve, let X be a non-zero abelian variety and let $q : X \to JN$ be a morphism. Assume that $q^*\theta_N$ is a principal polarization. Then q is an isomorphism.*

Proof. — Since $q^*\theta_N$ is a principal polarization, q is a closed immersion. By Mumford's proof of Poincaré's complete reducibility theorem [Mu, p. 173], there exist another abelian subvariety Y of JN and an isogeny $f : X \times Y \to JN$ such that $f^*\theta_N$ is the product of the induced polarizations on each factor. As in *loc.cit.*, for any k–scheme S, the set $(X \cap Y)(S)$ is contained in $K(q^*\theta_N)(S)$, which is trivial. Hence f is an isomorphism of polarized varieties. But a Jacobian with its canonical principal polarization cannot be a product, hence Y is 0 and q is an isomorphism. □

We now give a result on curves on *simple* abelian varieties. The part that depends on the validity of CONJECTURE 6.2 holds in particular for $n \leq 7$.

THEOREM 6.6. — *Let C be an irreducible curve in a simple principally polarized abelian variety (X, θ) of dimension n. Assume that either $C \cdot \theta \leq 1.7719\, n$, or that conjecture C_m holds for all divisors m of n and $C \cdot \theta < 2n$. Then, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded.*

Proof. — Since X is simple, the polynomial $P_{\alpha(C,\theta)}$, hence also its «square root» Q_C, are powers of an irreducible polynomial R of degree some divisor m of n. If the degree of C, which is equal to the sum of the roots of Q_C, is $\leq 1.7719\, n$, the sum of the roots of R is also $\leq 1.7719\, m$. It follows from THEOREM 6.1 that $|R(0)| = 1$. On the other hand, if $C\cdot\theta < 2n$, the sum of the roots of R is also $< 2m$, hence, since C_m is supposed to hold, we also have $|R(0)| = 1$. The theorem then follows in both cases from PROPOSITION 6.4. □

It follows from the proof of the theorem that C has degree $2n - m$ for some divisor m of n. In particular, for n *prime*, either C has degree n and θ is the canonical principal polarization, or it has degree $2n - 1$.

If one wants curves of degree between n and $2n$ in a simple abelian variety X, and if one believes in CONJECTURE 6.2, X needs to be a *Jacobian with real multiplications* (in the sense that the ring $\mathrm{End}(X) \otimes \mathbb{Q}$ contains a totally real number field different from $\mathbb{Q}$). Examples have been constructed in [Me] (see also [TTV]). More precisely, for any integer $M \geq 4$, MESTRE constructs an explicit 2–dimensional family of complex hyperelliptic Jacobians JC of dimension $[\frac{1}{2} M]$ whose endomorphism rings contain a subring isomorphic to $\mathbb{Z}[T]/G_M(T)$, where :

$$G_M(T) = \prod_{0<k\leq[M/2]} \Big(T - 4\cos^2 \frac{k\pi}{M}\Big),$$

whose elements are invariant under the Rosati involution. By (2.2), they correspond to polarizations on JC. Take M *odd* and set $n = \dim(JC) = \frac{1}{2}(M-1)$. Then, the endomorphism of X that corresponds to T gives rise to a *principal* polarization on JC, with respect to which the degree of C, canonically embedded, is $2n-1$. Therefore, for any $n \geq 2$, we have examples of *complex principally polarized abelian varieties of dimension n that contain curves of degree* $2n-1$. They are simple if $2n+1$ is prime. For $n = 2$, these examples are Humbert surfaces, which contain curves of degree 3 (see [vG, p. 221]).

If the assumption X simple is dropped, much less can be said. If Q is a monic polynomial with integral coefficients whose roots are all real, we will say that a curve C has *real multiplications by* Q if there is an endomorphism of JC whose characteristic polynomial (see §2) is Q^2. If $k = \mathbb{C}$, this is the same as asking that the characteristic polynomial of the endomorphism acting on the space of first-order differentials of C be Q.

PROPOSITION 6.7. — *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension n. Then, if $C \cdot \theta = n+1$, the curve C is smooth, X is isomorphic to its Jacobian and C is canonically embedded. Moreover, the curve C has real multiplications by $(T-1)^{n-2}(T^2-3T+1)$.*

Proof. — By THEOREM 6.1 and Smyth's list in [S], the polynomial Q_C can only be $(T-1)^{n-1}(T-2)$ or $(T-1)^{n-2}(T^2-3T+1)$. By PROPOSITION 6.4, we need only exclude the first case. By THEOREM 2.1, there exist a polarized elliptic curve (X', λ') and a morphism $f' : X \to X'$ such that $f'^*\lambda' = q^*\theta_N - \theta$. Similarly, there exist an $(n-1)$-dimensional polarized abelian variety (X'', λ'') and a morphism $f'' : X \to X''$ such that $f''^*\lambda'' = 2\theta - q^*\theta_N$. The isogeny $(f', f'') : (X, \theta) \to (X', \lambda') \times (X'', \lambda'')$ is a morphism of polarized abelian varieties. Since θ is principal, it is an isomorphism and λ' and λ'' are both principal polarizations. Then, $(X, q^*\theta_N)$ is isomorphic to $(X', 2\lambda') \times (X'', \lambda'')$. In particular, the pull-back of θ_N by $X'' \to JN$ is a principal polarization. By LEMMA 6.5, this cannot occur. ∎

In the next case where $\deg(C) = n+2$, the same techniques give partial results.

PROPOSITION 6.8. — *Let C be an irreducible curve that generates a principally polarized abelian variety (X, θ) of dimension $n > 2$. Assume that* $\mathrm{char}(k) \neq 2, 3$. *Then, if $C \cdot \theta = n+2$, one of the following possibilities occurs :*

(i) *The curve C is smooth of genus n, X is isomorphic to its Jacobian and C is canonically embedded. Moreover, the curve C has real multiplications by $(T-1)^{n-2}(T^2-4T+1)^2$, $(T-1)^{n-3}(T^3-5T^2+6T-1)$ or $(T-1)^{n-4}(T^2-3T+1)^2$.*

(ii) *The curve C is smooth of genus n and bielliptic, i.e. there exists a morphism of degree 2 from C onto an elliptic curve E. The abelian variety X is the quotient of JC by an element of order 3 that comes from E.*

(iii) *The normalization N of C has genus n and real multiplications by $(T-1)^{n-2}(T^2-4T+2)$ or $(T-1)^{n-3}(T-2)(T^2-3T+1)$. There is an isogeny $JN \to X$ of degree 2, and either C is smooth, or it has one node and N is hyperelliptic.*

(iv) *The curve C is smooth and bielliptic of genus $n+1$, and has real multiplications by*

$$T(T-1)^{n-2}(T^2-4T+2) \quad \text{or} \quad T(T-1)^{n-3}(T-2)(T^2-3T+1).$$

The abelian variety X is the « Prym variety » associated with the bi-elliptic structure.

REMARKS 6.9.1

1) MESTRE's construction for $M = 7$ gives examples of curves of degree 5 in principally polarized abelian varieties of dimension 3, which fit into case (i) of the PROPOSITION. Example 6.11 below shows that case (ii) does occur. These are the only examples I know of.

2) In general, if a curve C has real multiplications by a polynomial $(T-a)^m Q(T)$, where a is an integer and $Q(a) \neq 0$, then there is a morphism from JC onto an abelian variety of dimension m (this follows for example from [MK, thm 2, p. 98]).

Proof. — By THEOREM 6.1 and Smyth's list in [S], the polynomial Q_C can only be one of the following :

$$(T-1)^{n-2}(T-2)^2, \qquad (T-1)^{n-2}(T^2-4T+1)^2,$$
$$(T-1)^{n-3}(T^3-5T^2+6T-1), \qquad (T-1)^{n-4}(T^2-3T+1)^2,$$
$$(T-1)^{n-1}(T-3), \qquad (T-1)^{n-2}(T^2-4T+2),$$
$$(T-1)^{n-3}(T-2)(T^2-3T+1).$$

The first polynomial is excluded as in PROPOSITION 6.7 (use $n > 2$). If the constant term is ± 1, the same proof as above yields that we are in case (i).

If $Q_C(T) = (T-1)^{n-1}(T-3)$, as in the proof of PROPOSITION 6.7, there exist a polarized elliptic curve (X', λ') and a morphism $f' : X \to X'$ with connected kernel X'' such that $f'^*\lambda' = q^*\theta_N - \theta$ or equivalently $q^*\theta_N = \theta + (\deg \lambda')[X'']$. The identity

$$\frac{1}{n!}(T\theta - q^*\theta_N)^n = (T-1)^{n-1}(T-3)$$

yields $(\deg \lambda')(\deg \theta_{|X''}) = 2$. If $\deg \lambda' = 2$, one gets a contradiction as in the proof of PROPOSITION 6.7. If λ' is principal, one has $\deg\big((q^*\theta_N)_{|X''}\big) = 2$. We use the following result.

LEMMA 6.10. — *Let (JN, θ_N) be the Jacobian of a smooth curve, let X be a non-zero abelian variety and let $r : X \to JN$ be a morphism with finite kernel. Assume that* $\deg(r^*\theta_N)$ *is* $\leq \dim(X)$ *and prime to* $\mathrm{char}(k)$. *Then* $g(N) < \dim(X) + \deg(r^*\theta_N)$.

Proof. — Let K be the kernel of r and let $\iota : X/K \to JN$ be the induced embedding. By Poincaré's complete reducibility theorem [Mu, p. 173], there exist an abelian subvariety X' of JN and an isogeny $f : X/K \times X' \to JN$ such that the pull-back $f^*\theta_N$ is the product of the induced polarizations. Note that $\deg(\iota^*\theta_N)$ divides $\deg(r^*\theta_N)$. In particular, under our assumptions, the polarization $\iota^*\theta_N$ is *separable* and has a non-empty base locus F, of dimension $\geq \dim(X) - \deg(r^*\theta_N)$. If Θ is a theta divisor for JN, it follows from the equation of $f^*\Theta$ given in [D, prop. 9.1], that $f(F \times X')$ is contained in Θ. Lemma 5.1 from [DF] (which is valid in any characteristic) then yields

$$\dim(F \times X') + \dim(X') \leq g(N) - 1,$$

from which the lemma follows. ∎

Since $\mathrm{char}(k) \neq 2$, it follows from the lemma applied to the inclusion $X'' \to JN$ that $g(N) = n$ hence that the morphism $q : X \to JN$ is an isogeny of degree 3. It is not difficult to see (using for example [D, § 9]) that since $\mathrm{char}(k) \neq 2, 3$, there is a commutative diagram of separable isogenies :

$$\begin{array}{ccccc} X'' \times X' & \xrightarrow{3:1} & X'' \times E & \xrightarrow{3:1} & X'' \times X' \\ {\scriptstyle 4:1}\downarrow & & {\scriptstyle 4:1}\downarrow & & {\scriptstyle 4:1}\downarrow \\ X & \xrightarrow{\quad q \quad} & JN & \xrightarrow{\quad p \quad} & X \end{array}$$

where E is the quotient of X' by a subgroup of order 3. The middle vertical arrow induces an injection of E into JN whose image has degree 2 with respect to θ_N. By duality, one gets a morphism $f : N \to E$ of degree 2. In this situation, one checks that since $n > 2$, for any two points x and y of N, one cannot have $x - y \equiv f^*e$, for $e \neq 0$ in E. Thus C, image of N in X by p, is smooth.

If $Q_C(T) = (T-1)^{n-2}(T^2-4T+2)$ or $(T-1)^{n-3}(T-2)(T^2-3T+1)$, the polarization $q^*\theta_N$ has degree 2. It follows from LEMMA 6.10 that :

- Either $g(N) = n$ and C is the image of N by an isogeny $p : JN \to X$ of degree 2. In particular, either C is smooth or N is hyperelliptic and C is obtained by identifying two Weierstrass points of N (so that, in a sense, C is bi-elliptic).
- Or $g(N) = n+1$ and q is a closed immersion. The proof of LEMMA 6.10 yields an elliptic curve X' in JN and an isogeny $f : X \times X' \to JN$ of degree 4. Moreover, $\deg(\theta_N)_{|X'} = 2$, hence the morphism $N \to X'$ obtained by duality has degree 2. One checks as above that C is smooth. The abelian variety X is the Prym variety associated with the bielliptic structure, i.e. is isomorphic to the quotient JN/X'. It remains to prove the statement about real multiplications. With the notation of (2.2), we calculate the characteristic polynomial of the endomorphism $\beta_{\theta_N}(p^*\theta)$ of JN. If t is any integer, one has :

$$\begin{aligned}\deg\big(t\ \mathrm{Id}_{JN} - \beta_{\theta_N}(p^*\theta)\big) &= \deg(t\,\theta_N - p^*\theta)^2 \\ &= \big(\tfrac14 \deg(t\,f^*\theta_N - f^*p^*\theta)\big)^2 \\ &= \big(\tfrac14 \deg(t\,(\theta_N)_{|X'})\deg(t\,q^*\theta_N - q^*p^*\theta)\big)^2 \\ &= \tfrac14 t^2 \deg\big(t\,\phi_{q^*\theta_N} - \phi_{q^*p^*\theta}\big)\ .\end{aligned}$$

Set $\alpha = \alpha(C,\theta)$. Using (3.1) and (3.3), we get :

$$\begin{aligned}\deg\big(t\ \mathrm{Id}_{JN} - \beta_{\theta_N}(p^*\theta)\big) &= \tfrac14 t^2 \deg\big(t\,\phi_\theta \circ \alpha - \phi_{\alpha^*\theta}\big) \\ &= \tfrac14 t^2 \deg\big(t\ \mathrm{Id}_{\mathrm{Pic}^0(X)} - \mathrm{Pic}^0(\alpha)\big)\deg(\phi_\theta \circ \alpha) \\ &= P_{\mathrm{Pic}^0(\alpha)}(t)\,t^2 = Q_C(t)^2\,t^2.\end{aligned}$$

It follows that N has real multiplications by $T\,Q_C(T)$. This finishes the proof of the proposition. □

EXAMPLE 6.11. — Case (ii) of the PROPOSITION does occur as a particular case of the following construction. Let C be a smooth curve of genus n with a morphism of degree r onto an elliptic curve E. Assume that r is prime to $\mathrm{char}(k)$ and that the induced morphism $E \to JC$ is a closed immersion. Let s be an integer prime to $\mathrm{char}(k)$ and congruent to 1 modulo r, and let $q : JC \to X$ be the quotient by a cyclic subgroup of order s of E. There exist an abelian variety Y of dimension $(n-1)$ with a polarization λ_Y of type $(1|\cdots|1|r)$ and an isogeny $f : E \times Y \to JC$ with kernel isomorphic to $(\mathbb{Z}/r\mathbb{Z})^2$, such that $f^*\theta_C = \mathrm{pr}_1^*(r\lambda_E) \otimes \mathrm{pr}_2^*\lambda_Y$, where λ_E is the polarization on E defined by a point. Because $s \equiv 1 \pmod r$, one checks that there exists a principal polarization θ on X such that $f^*q^*\theta = \mathrm{pr}_1^*(rs\lambda_E) \otimes \mathrm{pr}_2^*\lambda_Y$. I claim that *the degree of the curve $q(C)$ on X with respect to the principal polarization θ is $n+s-1$*. In fact, one has :

$$\frac{f^*\theta_C^{n-1}}{(n-1)!} \sim \frac{1}{(n-2)!} r\lambda_E \, (\mathrm{pr}_2^*\lambda_Y)^{n-2} + \frac{1}{(n-1)!} (\mathrm{pr}_2^*\lambda_Y)^{n-1}$$

hence

$$\begin{aligned}\frac{f^*\theta_C^{n-1}}{(n-1)!} f^*q^*\theta &= rs \deg \lambda_Y + r(n-1)\deg \lambda_Y \\ &= r^2(s+n-1).\end{aligned}$$

It follows that $C \cdot q^*\theta = n+s-1$, which proves the claim.

When $\mathrm{char}(k) = 0$, this construction yields examples of curves of degree $n+t$ in principally polarized abelian varieties of dimension n, for any $n \geq 2$ and $t \geq 2$.

P.S. — C. Smyth recently found a totally positive algebraic integer of degree 15, trace 28 and norm 1. This disproves conjecture C_{15}. His construction may give counterexamples to conjecture C_m for infinitely many values of m. He also has a conterexample to C_{64} with norm 2.

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