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A NOTE ON FUNCTIONAL EQUATIONS OF THE P-ADIC POLYLOGARITHMS

BY

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RÉSUMÉ. — La fonction polylogarithme $\mathrm{Li}_n(z)$ d'ordre n est définie sur le disque ouvert de rayon 1 par la série $\sum_{k=1}^{\infty} z^k/k^n$. Cette fonction se prolonge analytiquement en une fonction multiforme sur tout plan complexe. La même série $\sum_{k=1}^{\infty} z^k/k^n$ définit une fonction analytique p-adique sur le disque unité ouvert dans $\mathbb{C}_p$ (la complétion de la clôture algébrique de $\mathbb{Q}_p$). Les analogues globales p-adiques des fonctions $\mathrm{Li}_n(z)$ sont construites dans le cadre d'une analyse p-adique rigide. Ce sont des polylogarithmes p-adiques. Dans ce papier nous trouvons les conditions suffisantes et nécessaires pour une existence d'une équation fonctionnelle de polylogarithmes en termes des applications induites par des fonctions rationnelles sur les groupes fondamentaux étales de la droite projective moins un nombre fini de points.

ABSTRACT. — The n-th order polylogarithm $L_n(z)$ is defined by the series $\sum_{k=1}^{\infty} z^k/k^n$ on the open unit disc. This function has multivalued analytic prolongation to $\mathbb{C}\setminus\{0,1\}$. The same series $\sum_{k=1}^{\infty} z^k/k^n$ defines an analytic p-adic function on the open unit disc in $\mathbb{C}_p$ (a completion of an algebraic closure of $\mathbb{Q}$ at some place above p). The global p-adic analogues of the functions $\mathrm{Li}_n(z)$ are constructed in the frame of rigid analysis. These functions are p-adic polylogarithms. In this paper we give sufficient and necessary conditions to have a functional equation of polylogarithms in terms of maps induced by rational functions on etale fundamental groups of projective lines minus finite numbers of points.

Dedicated to the memory of Prof. J. Frank Adams

0. Introduction

Let n be an integer. The series $\sum_{k=1}^{\infty} z^k/k^n$ converges on the open unit disc around 0 in the field of complex numbers $\mathbb{C}$. Hence it determines an analytic function on this disc. This function can be extended by an

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analytic continuation to a multivalued analytic function on $\mathbb{C} \setminus \{0,1\}$. We denote this function by $\mathrm{Li}_n(z)$ and we call it the *n-th order polylogarithm.*

The functions $\mathrm{Li}_n(z)$ are special cases of Chen iterated integrals (see [Ch]). We recall their definition. Let $\omega_1, \dots, \omega_n$ be one-forms on a smooth manifold M and let $\gamma : [0,1] \to M$ be a smooth path from x to z. Let $\gamma^t : [0,1] \to M$ be a restriction of γ. We define by a recursive formula

$$\int_\gamma \omega_1, \dots, \omega_n := \int_\gamma \Big(\int_{\gamma^t} \omega_1 \Big) \omega_2, \dots, \omega_n.$$

If x is fixed and $\omega_1, \dots, \omega_n$ are closed one-forms on M such that all possible products $\omega_{i_1} \wedge \cdots \wedge \omega_{i_k}$ vanish, then $F(z) = \int_\gamma \omega_1, \dots, \omega_n$ is an analytic multivalued function on M. We shall write also $\int_{x,\gamma}^z \omega_1, \dots, \omega_n$ or $\int_x^z \omega_1, \dots, \omega_n$ to denote the multivalued function $F(z)$.

It is clear that $\mathrm{Li}_n(z) = \displaystyle\int_0^z \frac{\mathrm{d}z}{1-z}, \frac{\mathrm{d}z}{z}, \dots, \frac{\mathrm{d}z}{z}$.

Let p be a finite prime of $\mathbb{Q}$ and let $\mathbb{C}_p$ denote a completion of an algebraic closure of $\mathbb{Q}$ at some place above p. Then the series

$$\ell_{n,p}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$$

determines an analytic function on the open unit disc around 0 in $\mathbb{C}_p$. However one cannot use analytic continuation to extend this function because the open unit disc is the maximal analytic domain for it.

The global p-adic analogs of $\mathrm{Li}_n(z)$ are constructed in the framework of rigid analysis. Our basic reference is the paper of COLEMAN (see [C]). We briefly sketch the necessary results from [C], asking the reader to consult [C] for any details.

To define iterated integrals in the p-adic realm we consider the following system of differential equations

$$(*) \qquad f_1' = \frac{1}{z-a_1}, \quad f_2' = \frac{f_1}{z-a_2}, \quad \dots \quad , \quad f_n' = \frac{f_{n-1}}{z-a_n}.$$

Let $a \in \mathbb{C}_p \setminus \{a_1, \dots, a_n\}$. We pose the following initial conditions

$$(**) \qquad f_1(a) = 0, \quad f_2(a) = 0, \quad \dots \quad , \quad f_n(a) = 0.$$

We set $D = \mathbb{C}_p \setminus \{a_1, \dots, a_n\}$. The following result is the direct consequence of [C] (Theorem 4.3, Lemma 5.2 and the whole section V in [C]).

THEOREM A. — *Let us choose a locally analytic homomorphism*

$$\log : \mathbb{C}_p^* \to \mathbb{C}_p.$$

There exists a logarithmic F-cristal $M(D)$ on $D = \mathbb{C}_p \setminus \{a_1, \ldots, a_n\}$ such that the system of differential equations (*) *has a unique solution $f_1(z), \ldots, f_n(z)$ in $M(D)$ which satisfies the initial conditions* (**).

It follows from the theory presented in [C] that the functions $f_k(z)$ are locally analytic. The function $f_n(z)$ we shall denote by

$$\int_a^z \frac{\mathrm{d}z}{z-a_1}, \frac{\mathrm{d}z}{z-a_2}, \ldots, \frac{\mathrm{d}z}{z-a_n}$$

and we shall call it an *iterated integral* in the *p*-adic realm.

The *p*-adic polylogarithms are defined in the section VI of [C]. We recall here their definition. Let $D = \mathbb{C}_p \setminus \{0, 1\}$. We consider the following system of differential equations

$$(*_1) \qquad \ell_1' = \frac{1}{z-1}, \quad \ell_2' = \frac{\ell_1}{z}, \quad \cdots \quad , \quad \ell_n' = \frac{\ell_{n-1}}{z}.$$

We pose the following initial conditions

$$(**_2) \qquad \lim_{z \to 0} \ell_k(z) = 0.$$

THEOREM A′. — *Let us choose a locally analytic homomorphism* $\log : \mathbb{C}_p^* \to \mathbb{C}_p$. *Then there exists a logarithmic F-crystal $M(D)$ on $D = \mathbb{C}_p \setminus \{0, 1\}$ such that the system of differential equations* $(*_1)$ *has a unique solution $\ell_1(z), \ldots, \ell_n(z)$ in $M(D)$ which satisfies the initial condition* $(**_2)$. *The function $\ell_k(z)$ extends to a locally analytic function on $\mathbb{C}_p \setminus \{1\}$ such that $\ell_k(0) = 0$.*

We shall denote $\ell_k(z)$ by $\mathrm{Li}_k(z)$ and we shall call it the *k-th p-adic polylogarithm.* The function $\mathrm{Li}_k(z)$ is analytic at 0 and has the convergent Taylor expansion $\sum_{n=1}^{\infty} \frac{z^n}{n^k}$ at 0. In fact we shall use only the fact that functions $\mathrm{Li}_n(z)$ and $\int_a^z \frac{\mathrm{d}z}{z-a_1}, \ldots, \frac{\mathrm{d}z}{z-a_n}$ exist in the *p*-adic realm, that they are locally analytic, that their Taylor power series at some points "coincide" with the Taylor power series of the corresponding complex functions and that the logarithmic *F*-crystal, where live p-adic iterated integrals satisfies a uniqueness principal.

The complex polylogarithms $\mathrm{Li}_n(z)$ have a lot of remarkable properties. For example, for small n, they have functional equations which generalize the functional equation

$$\log xy = \log x + \log y$$

satisfied by the logarithm. The dilogarithm

$$\mathrm{Li}_2(z) = \int_0^z \frac{-\log(1-z)}{z}\,\mathrm{d}z$$

satisfies the functional equation

$$\begin{aligned}\mathrm{Li}_2\Big(\frac{x}{1-x}\cdot\frac{y}{1-y}\Big) &= \mathrm{Li}_2\Big(\frac{y}{1-x}\Big) + \mathrm{Li}_2\Big(\frac{x}{1-y}\Big)\\ &\quad - \mathrm{Li}_2(x) - \mathrm{Li}_2(y) - \log(1-x)\log(1-y)\end{aligned}$$

(see [A]). In LEWIN's book one can find more examples (see [L1]). The basic reference for p-adic polylogarithms is the paper of COLEMAN (see [C]). For more general review of various aspects of polylogarithms and iterated integrals one can consult [Ca].

In this paper we give some sufficient and necessary conditions to have functional equations of polylogarithms. We discuss complex polylogarithms and p-adic polylogarithms as well. One of the main results is the following theorem.

THEOREM. — *Let K be the field of complex numbers or a p-adic completion of the algebraic closure of $\mathbb{Q}$ at some place above p. Let*

$$f_i : X = P^i(K) \setminus \{a_1, \dots, a_n\} \longrightarrow Y = P^1(K) \setminus \{0, 1, \infty\}$$

$(i = 1, \dots, N)$ be regular maps. Let $n_1, \dots, n_N$ be integers. There is a functional equation

$$\sum_{i=1}^{N} n_i \,\mathrm{Li}_n\big(f_i(z)\big) + \textit{ terms of lower degrees } = 0$$

if and only if $\sum_{i=1}^{N} n_i (f_i)_* = 0$ *in the group*

$$\begin{aligned}\mathrm{Hom}\Big(\Gamma^n\big(\pi_1(X,x)^{\ell}_{\mathrm{et}}\big)/\Gamma^{n+1}\big(\pi_1(X,x)^{\ell}_{\mathrm{et}}\big);&\\ \Gamma^n\big(\pi_1(Y,y)^{\ell}_{\mathrm{et}}\big)/(\Gamma^{n+1}\big(\pi_1(Y,y)^{\ell}_{\mathrm{et}}\big) + L_n)\Big)&\end{aligned}$$

where $\pi_1(X,x)^{\ell}_{\mathrm{et}}$ is the ℓ-profinite quotient of the etale fundamental group of X and where $(f_i)_*$ are maps induced by f_i on the etale fundamental groups. L_n is a closed subgroup of $\Gamma^n(\pi_1(Y,y)^{\ell}_{\mathrm{et}})$ defined in the following way. If K is the field of complex numbers $\mathbb{C}$, then L_n is topologically generated by all commutators in e_0 (loop around 0) and e_1 (loop around 1) which contain e_1 at least twice. If K is a non-Archimedean field $\mathbb{C}_p$ then any isomorphism $\mathbb{C}_p \approx \mathbb{C}$ induces an isomorphism

$$\pi_1\big(P^1(\mathbb{C}_p)\setminus\{0,1,\infty\},y\big)^{\ell}_{\mathrm{et}} \approx \pi_1\big(P^1(\mathbb{C})\setminus\{0,1,\infty\},y\big)^{\ell}_{\mathrm{et}}$$

and $L_n \subset \Gamma^n\big(\pi_1(P^1(\mathbb{C}_p)\setminus\{0,1,\infty\})^{\ell}_{\mathrm{et}}\big)$ is the image of

$$L_n \subset \Gamma^n\Big(\pi^n\big(P^1(\mathbb{C})\setminus\{0,1,\infty\},y\big)^{\ell}_{\mathrm{et}}\Big)$$

under this isomorphism.

We give also a sufficient and necessary condition to have a functional equation in terms of a differential Galois group of a certain system of differential equations.

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This paper grew out from preprints [W1] and [W2]. We would like to point the attention to our preprint [W3] which, we hope, will be a chapter of a book "Properties of polylogarithms" of various authors, where we discuss functional equations of complex polylogarithms.

In the present paper we concentrate mostly on a p-adic situation, though quite often to prove something about p-adic polylogarithms, we must show an analogous result about complex polylogarithms first.

We point also that in some aspects the p-adic situation is simpler than the complex situation. The reader can look in chapter 4 where the results are due to the absence of $2\pi i$ in a non-archimedean field $\mathbb{C}_p$.

Plan

1. Canonical unipotent connection on a projective line minus several points

If p is any prime of $\mathbb{Q}$ let $\mathbb{C}_p$ denote a completion of an algebraic closure of $\mathbb{Q}$ at some place above p. This definition includes also the case when $p = \infty$ and then $\mathbb{C}_p = \mathbb{C}$ is the field of complex numbers.

Let $X = P^1(\mathbb{C}_p) \setminus \{a_1, \ldots, a_{n+1}\}$. Observe that X is an affine algebraic variety over $\mathbb{C}_p$. Let $\Omega^*(X)$ be the algebraic De Rham complex of smooth, algebraic differential forms on X. Let $A^1(X)$ be a $\mathbb{C}_p$-subspace of $\Omega^1(X)$ generated by linear combinations with $\mathbb{C}_p$-coefficients of one forms $\dfrac{\mathrm{d}z}{z - a_i}$ for $i = 1, \ldots, n+1$. Observe that

$$A^1(X) = H^1_{DR}(X).$$

Let $H(X)$ be the dual of the $\mathbb{C}_p$-vector space $A^1(X)$. Let $\mathrm{Lie}(H(X))$ be a free Lie algebra over $\mathbb{C}_p$ on $H(X)$. Let

$$L(X) := \varprojlim_n \Big(\mathrm{Lie}\big(H(X)\big)/\Gamma^n \,\mathrm{Lie}\big(H(X)\big)\Big)$$

be the completion of $\mathrm{Lie}(H(X))$ with respect to the filtration given by the lower central series. We equipped $L(X)$ with a group law given by the Baker-Campbell-Hausdorff formula and a topology given by the inverse limit of finite dimensional $\mathbb{C}_p$-vector spaces with its natural p-adic topology if $p < \infty$ and the complex topology if $p = \infty$. We shall denote by $\pi(X)$ this topological group. Observe that each quotient $\pi(X)/\Gamma^n\pi(X)$ is an affine algebraic group, so $\pi(X)$ is an affine pro-algebraic group and $L(X)$ is its Lie algebra.

Definition. — The one form $\omega_X \in A^1(X) \otimes H(X)$ corresponds to $\mathrm{id}_{A^1(X)}$ under the natural isomorphism

$$A^1(X) \otimes \big(A^1(X)\big)^* \approx \mathrm{Hom}\big(A^1(X), A^1(X)\big).$$

We consider ω_X as an element of $A^1(X) \otimes L(X)$.

Let $T(H(X))$ be a tensor algebra over $\mathbb{C}_p$ on $H(X)$. Let I be an augmentation ideal of $T(H(X))$ and let

$$T[[H(X)]] := \varprojlim_n T\big(H(X)\big)/I^n$$

be the completed tensored algebra. Observe that $T(H(X))/I^n$ is a finite dimensional vector space over $\mathbb{C}_p$. Hence $T[[H(X)]]$ is equipped with the topology of an inverse limit of finite dimensional $\mathbb{C}_p$-vector spaces. Let $P(X)$ be a group of invertible elements in $T[[H(X)]]$ with leading term equal 1. From the discussion given above it follows that $P(X)$ is affine, pro-algebraic group over $\mathbb{C}_p$.

Remark. — $T[[H(X)]]$ is nothing else but an algebra of non-commutative formal power series over $\mathbb{C}_p$ on $H(X)$.

In $T(H(X))$ and $T[[H(X)]]$ we consider the Lie algebras of Lie elements (possibly of infinite length in a case of $T[[H(X)]]$). These Lie algebras are naturally isomorphic with $\mathrm{Lie}(H(X))$ and $L(X)$ respectively. After the identification of $L(X)$, which is the underlying set of $\pi(X)$ with the Lie elements (possibly of infinite length) in $T[[H(X)]]$ the exponential map

$$\exp : \pi(X) \longrightarrow P(X)$$

is defined by the standard formula

$$\exp(w) = 1 + \frac{w}{1!} + \frac{w^2}{2!} + \cdots$$

where we consider $w \in \pi(X)$ as a Lie element in $T[[H(X)]]$. The exponential map is a continuous monomorphism of topological groups, whose image is a closed subgroup of $P(X)$.

The inverse of exp is defined on the subgroup $\exp(\pi(X))$ of $P(X)$ and it is given by the formula

$$\log z = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 - \tfrac{1}{4}(z-1)^4 + \cdots$$

and homomorphisms exp and log are mutually inverse isomorphisms

$$\exp : \pi(X) \leftrightarrows \mathrm{im}(\exp) : \log .$$

Let $p(X)$ be a Lie algebra of $P(X)$. We identify $v \in H(X)$ with a tangent vector to $P(X)$ given by $[0,1] \ni t \mapsto 1 + t \cdot v \in P(V)$ if $\mathbb{C}_p = \mathbb{C}$

is the field of complex numbers and by the differentiation in the direction of v if $\mathbb{C}_p$ is arbitrary.

After this identification we shall consider ω_X as an element of $A^1(X) \otimes p(X)$ and provisionally we shall denote it by

$$\bar{\omega}(X) \in A^1(X) \otimes p(X).$$

LEMMA 1.2. — *The morphism* id $\times$ exp $: X \times \pi(X) \to X \times P(X)$ *maps* ω_X *into* $\bar{\omega}_X$.

Proof. — Let $v \in H(X)$. Then $\exp(tv)$ and $1 + tv$ define the same tangent vector. If $\mathbb{C}_p$ is non-archimedean one observes that exp tranforms the differentiation in the direction of v on $\pi(X)$ in the differentiation in the direction of v on $P(X)$.

It is clear that there is no need to distinguish between ω_X and $\bar{\omega}_X$, hence we shall denote both forms by ω_X.

Let $X = P^1(\mathbb{C}_p) \setminus \{x_1, \dots, x_{r+1}\}$ and let $Y = P^1(\mathbb{C}_p) \setminus \{y_1, \dots, y_{s+1}\}$. Let

$$f(z) = \alpha \prod_{i=1}^{n} (z - a_i)^{n_i} \Big/ \prod_{j=1}^{m} (z - b_j)^{m_j}$$

be a rational function. Let us assume that f restricts to a regular map $f : X \to Y$. Then f induces

$$f^* : A^1(Y) \to A^1(X) \quad \text{and} \quad f_* : H(X) \longrightarrow H(Y).$$

The map f_* induces the following five maps which we shall denote by the same letter f_*

$$f_* : \mathrm{Lie}\big(H(X)\big) \longrightarrow \mathrm{Lie}\big(H(Y)\big);$$

$$f_* : L(X) \longrightarrow L(Y), \quad f_* : \pi(X) \longrightarrow \pi(Y);$$

$$f_* : p(X) \longrightarrow p(Y), \quad f_* : P(X) \longrightarrow P(Y).$$

Hence $\pi(\ )$ and $P(\ )$ are functors on the category of pointed projective lines and regular maps. We shall denote by $G(\ )$ any of them. In this way we avoid formulations of separated statements for $\pi(\ )$ and for $P(\ )$.

LEMMA 1.3. — *Let* X, Y *and* $f : X \to Y$ *be as above. Let*

$$f \times f_* : X \times G(X) \to Y \times G(Y)$$

be induced by f. Then we have

$$(\mathrm{id}\otimes f_*)\omega_X = (f^*\otimes \mathrm{id})\omega_Y,$$

where $f^* : A^1(Y)\to A^1(X)$.

Proof. — The form ω_X (resp. the form ω_Y) corresponds to $\mathrm{id}_{A_1(X)}$ (resp. to $\mathrm{id}_{A^1(Y)}$). The lemma follows immediately if we observe that $\mathrm{id}_{A^1(X)}\circ f^* = f^*\circ \mathrm{id}_{A^1(Y)}$.

2. Horizontal sections of the canonical connection

Let $X = P^1(\mathbb{C}_p)\setminus\{a_1,\dots,a_{n+1}\}$. Let us consider a principal $P(X)$-bundle

$$X\times P(X)\longrightarrow X$$

equipped with the integrable connection given by ω_X.

Let us choose a base of $A^1(X)$ given by one-forms

$$\omega_1 = \mathcal{T}_1(z)\,\mathrm{d}z,\quad \dots\quad ,\quad \omega_n = \mathcal{T}_n(z)\,\mathrm{d}z.$$

Let $X_1,\dots,X_n$ be a dual base of $H(X)$. Then $P(X)$ is a multiplicative group of non-commutative, formal power series with constant terms equal 1 in non-commutative variables $X_1,\dots,X_n$.

Let p be a finite prime. Let us choose a locally analytic homomorphism $\log : \mathbb{C}_p^*\to\mathbb{C}_p$. Then it follows from section 0 (THEOREM A) that there is a logarithmic F-crystal $M(X)$ on X such that iterated integrals $\int_x^z \omega_{i_1},\dots,\omega_{i_m}$ ($x\in X$, $i_1,\dots,i_m\in\{1,2,\dots,n\}$) exist in $M(X)$.

PROPOSITION 2.1. — *Let p be any prime of $\mathbb{Q}$. Let*

$$X = P^1(\mathbb{C}_p)\setminus\{a_1,\dots,a_{n+1}\}$$

and let $x\in X$. Let $\omega_1,\dots,\omega_n$ and $X_1,\dots,X_n$ be as above.

(i) *Let p be a finite prime. Then the map*

$$W\ni z\mapsto \Big(z, 1+\sum\Big\{(-1)^k\int_x^z \omega_{i_1},\dots,\omega_{i_k}\Big\}X_{i_k}\cdot\dots\cdot X_{i_1}\Big)\in X\times P(X)$$

(the summation is over all non-commutative monomials in $X_1,\dots,X_n$) is a horizontal section of a principal $P(X)$-bundle $X\times P(X)\to X$ equipped with an integrable connection given by ω_X. We shall denote this map shortly by

$$X\ni z\longmapsto \big(z,\lambda_X(z;x)\big)\in X\times P(X).$$

(ii) *Let $\mathbb{C}_p$ be the field of complex numbers $\mathbb{C}$. Let γ be a path in X from x to z. Then the map*

$$X \ni z \mapsto \Big(z, 1 + \sum \Big\{(-1)^k \int_{x,\gamma}^{z} \omega_{i_1}, \dots, \omega_{i_k}\Big\} X_{i_k} \cdots X_{i_1}\Big) \in X \times P(X)$$

is a horizontal section of a principal $P(X)$-bundle $X \times P(X) \to X$ equipped with an integrable connection given by ω_X. We shall denote this map shortly by

$$X \ni z \longmapsto \big(z, \lambda_X(z; x, \gamma)\big) \in X \times P(X)$$

or by

$$X \ni z \longmapsto \big(z, \lambda_X(z; x)\big) \in X \times P(X).$$

(iii) *The initial condition $\lambda_X(x;x) = 1$ determines $\lambda_X(z;x)$ (and $\lambda_X(z;x,\gamma)$ if $\mathbb{C}_p = \mathbb{C}$) uniquely.*

Proof. — The system of differential equations for the coefficient $f_k(z)$ at $X_{i_k} \cdots X_{i_2} \cdot X_{i_1}$ of the horizontal section is the following

$$(*) \qquad \mathrm{d}f_1 = -\omega_{i_1}, \quad \mathrm{d}f_2 = -f_1\,\omega_{i_2}, \quad \dots \quad , \quad \mathrm{d}f_k = -f_{k-1}\,\omega_{i_k}$$

with the initial condition

$$f_1(x) = 0, \quad f_2(x) = 0, \quad \dots \quad , \quad f_k(x) = 0.$$

If $\mathbb{C}_p = \mathbb{C}$ the solution of the system (*) is given by the iterated integrals $(-1)^\ell \int_{x,\gamma}^z \omega_{i_1}, \dots, \omega_{i_\ell}$ for $\ell = 1, \dots, k$ where γ is a path from x to z. If p is finite then the functions $(-1)^\ell \int_x^z \omega_{i_1}, \dots, \omega_{i_\ell}$, $\ell = 1, \dots, k$, exist in the logarithmic F-crystal $M(X)$ and satisfy the system (*).

The uniqueness principal is valid for analytic functions on a connected open set in the complex situation and for functions in $M(X)$ in the p-adic situation (see [C], Theorem 5.7). This implies (iii).

We shall denote by

$$X \ni z \longmapsto \big(z, \ell_X(z;x)\big) \in X \times \pi(X)$$

a horizontal section of a principal $\pi(X)$-bundle $X \times \pi(X) \to X$ equipped with the connection form ω_X which satisfies the initial condition $\ell_X(x;x) = 0$. If $\mathbb{C}_p = \mathbb{C}$ we shall also write $\ell_X(z;x,\gamma)$ instead of $\ell_X(z;x)$ to indicate the dependence on a path γ. It follows from LEMMA 1.2 that

$$\exp\big(\ell_X(z;x)\big) = \lambda_X(z;x).$$

Hence we have

$$\ell_X(z;x) = \log\big(\lambda_X(z;x)\big).$$

This implies that $\ell_X(z;x)$ exists (in $M(X)$ if p is finite) and it is uniquely determined by the initial conditions.

COROLLARY 2.2.

Let $X = P^1(\mathbb{C}_p)\setminus\{a_1,\dots,a_{n+1}\}$ and let $Y = P^1(\mathbb{C}_p)\setminus\{b_1,\dots,b_{m+1}\}$. Let $f : X \to Y$ be a regular map. The map $f\times f_* : X\times G(X) \to Y\times G(Y)$ maps horizontal sections of the bundle $X\times G(X)\to X$ equipped with the connection ω_X into horizontal sections of the bundle $Y\times G(Y)\to Y$ equipped with the connection ω_Y i.e. we have

$$f_*\big(\ell_X(z;x)\big) = \ell_Y\big(f(z);f(x)\big) \quad \text{if} \quad G(\ ) = \pi(\ ) \tag{2.2.1}$$

and

$$f_*\big(\lambda_X(z;x)\big) = \lambda_Y\big(f(z);f(x)\big) \quad \text{if} \quad G(\ ) = P(\ ). \tag{2.2.2}$$

Proof. — The corollary is an immediate consequence of LEMMA 1.3 and PROPOSITION 2.1.

3. Functional equations

Let X be a projective line $P^1(\mathbb{C}_p)$ minus a finite number of points. We recall from section 1 that $G(X)$ is an affine, pro-algebraic group. Let $\mathrm{Alg}(G(X))$ be an algebra of polynomial $\mathbb{C}_p$-valued functions on $G(X)$.

Let $X = P^1(\mathbb{C}_p)\setminus\{a_1,\dots,a_{n+1}\}$ and $Y = P^1(\mathbb{C}_p)\setminus\{b_1,\dots,b_{m+1}\}$. Let $f : X\to Y$ be a regular map. Let $x,z\in X$. Our principal tool to derive functional equations are equalities from COROLLARY 2.2

$$f_*\big(\ell_X(z;x)\big) = \ell_Y\big(f(z);f(x)\big) \tag{2.2.1}$$

and

$$f_*\big(\lambda_X(z;x)\big) = \lambda_Y\big(f(z);f(x)\big). \tag{2.2.1}$$

THEOREM 3.1. — *Let $f_1,\dots,f_N$ be regular functions. Let $\mathcal{T}_1,\dots,\mathcal{T}_N$ belong to $\mathrm{Alg}(G(Y))$ and let $p(t_1,\dots,t_n)$ be a polynomial in variables $t_1,\dots,t_n$.*

(i) *Let $G(\ ) = \pi(\ )$. There is a functional equation*

$$p\Big\{\mathcal{T}_1\Big(\ell_Y\big(f_1(z),f_1(x)\big)\Big),\dots,\mathcal{T}_n\Big(\ell_Y\big(f_N(z),f_N(x)\big)\Big)\Big\} = 0 \tag{1}$$

if and only if

$$p\big(\mathcal{T}_1\circ f_{1*},\dots,\mathcal{T}_N\circ f_{N*}\big) = 0. \tag{2}$$

(ii) *Let $G(\ ) = P(\ )$. If $p\big(\mathcal{T}_1\circ f_{1*},\dots,\mathcal{T}_N\circ f_{N*}\big) = 0$ then*

$$p\Big\{\mathcal{T}_1\Big(\lambda_Y\big(f_1(z);f_1(x)\big)\Big),\dots,\mathcal{T}_n\Big(\lambda_Y\big(f_N(z),f_N(x)\big)\Big)\Big\} = 0.$$

Proof. — Let us assume that we have the identity (2). The identity (2.2.1) implies that

$$\mathcal{T}_i\big(f_{i*}\big(\ell_X(z;x)\big)\big) = \mathcal{T}_i\big(\ell_Y\big(f_i(z);f_i(x)\big)\big).$$

Substituting $\mathcal{T}_i(f_{i*}(\ell_X(z;x)))$ by $\mathcal{T}_i(\ell_Y(f_i(z);f_i(x)))$ in the formula (2) we get the functional equation (1). The same arguments show also part (ii).

Let us assume that we have a functional equation (1). Let $\mathbb{C}_p = \mathbb{C}$ be the field of complex numbers. Observe that the subset

$$\{\ell_X(x;x,\gamma) \in \pi(X) \mid \gamma \in \pi_1(X,x)\}$$

of $\pi(x)$ is Zariski dense in $\pi(X)/\Gamma^2\pi(X)$. Hence this subset is Zariski dense in $\pi(X)/\Gamma^k\pi(X)$ for any k. The vanishing of a regular function $p(\mathcal{T}_1 \circ f_{1*},\dots,\mathcal{T}_N \circ f_{N*})$ on a Zariski dense subset implies that this regular function is the zero function.

Now we shall assume that p is finite. Let us choose an isomorphism of fields $\alpha : \mathbb{C}_p \approx \mathbb{C}$. If

$$q(t_1,\dots,t_n) = \sum a_{i_1,\dots,i_n}(t_1)^{i_1}(t_2)^{i_2}\cdots(t_n)^{i_n} \in \mathbb{C}_p[[t_1,\dots,t_n]]$$

then we set

$$q^\alpha(t_1,\dots,t_n) := \sum \alpha(a_{i_1,\dots,i_n})(t_1)^{i_1}(t_2)^{i_2}\cdots(t_n)^{i_n} \in \mathbb{C}\big[[t_1,\dots,t_n]\big].$$

If $X = P^1(\mathbb{C}_p) \setminus \{a_1,\dots,a_{n+1}\}$ then we set

$$X^\alpha := P^1(\mathbb{C}) \setminus \{\alpha(a_1),\dots,\alpha(a_{n+1})\}.$$

Let us identify $\Big(\dfrac{\mathrm{d}z}{z-a_i}\Big)^*$ with $\Big(\dfrac{\mathrm{d}z}{z-\alpha(a_i)}\Big)^*$. After this identification, if $\mathcal{T} \in \mathrm{Alg}(\pi(X))$ then $\mathcal{T}^\alpha \in \mathrm{Alg}(\pi(X^\alpha))$.

Let $q_i(z)$ be a Taylor series of $\mathcal{T}_i(\ell_Y(f_i(z);f_i(x)))$ at $x \in \mathbb{C}_p$. Then if follows from (1) that $p(q_1(z),\dots,q_N(z)) = 0$ and consequently also $p^\alpha(q_1^\alpha(z),\dots,q_N^\alpha(z)) = 0$. The power series $q_i^\alpha(z)$ is a Taylor power series of $\mathcal{T}_i^\alpha(\ell_Y\alpha(f_i^\alpha(z),f_i^\alpha(\alpha(x))))$ at $\alpha(x) \in \mathbb{C}$. Hence locally, in a neighbourhood of $\alpha(x)$ we have a functional equation

$$p^\alpha\Big\{\mathcal{T}_1^\alpha\Big(\ell_Y\alpha\big(f_1^\alpha(z);f_1^\alpha(\alpha(x))\big)\Big),\dots,\mathcal{T}_N^\alpha\Big(\ell_Y\alpha\big(f_N^\alpha(z);f_N^\alpha(\alpha(x))\big)\Big)\Big\} = 0.$$

By the principle of analytic continuation we have

$$p^{\alpha}\Big\{\mathcal{T}_1^{\alpha}\Big(\ell_Y\alpha\big(f_1^{\alpha}(z);f_1^{\alpha}(\alpha(x)),f_1^{\alpha}(\gamma)\big)\Big),\dots,\Big\}=0$$

for any smooth path γ from $\alpha(x)$ to z. Hence we have

$$p^{\alpha}\{\mathcal{T}_1^{\alpha}\circ f_{1*}^{\alpha},\dots,\mathcal{T}_N^{\alpha}\circ f_{N*}^{\alpha}\}=0$$

by the result proved above for the field of complex numbers. This implies

$$p\{\mathcal{T}_1\circ f_{1*},\dots,\mathcal{T}_N\circ f_{N*}\}=0.$$

We recall that $\operatorname{Lie} H(Y)$ is a free Lie algebra on

$$Y_1=(\omega_1)^*,\quad \dots\quad ,\quad Y_m=(\omega_m)^*$$

where $\omega_1,\dots,\omega_m$ is a base of $A^1(Y)$. We fixed a base $\mathcal{B}_Y$ of $\operatorname{Lie} H(Y)$ given by basic Lie elements corresponding to the ordering $Y_1,\dots, Y_m$ (see [MKS], Theorem 5.8). Let $v\in\mathcal{B}_Y$ and let $v^*\in\operatorname{Hom}\,(\operatorname{Lie} H(Y),\mathbb{C})$ be a linear functional on $\operatorname{Lie} H(Y)$ dual to v with respect to the base $\mathcal{B}_Y$. We consider the linear functional $v^*\in\operatorname{Lie} H(Y)$ as an element of $\operatorname{Alg}(\pi(Y))$. We set

$$\mathcal{L}_{v,\mathcal{B}_Y}(z;x):=v^*\big(\ell_Y(z;x)\big).$$

If the choice of the base $\mathcal{B}_Y$ is clear we shall omit the subscript $\mathcal{B}_Y$ and we shall write $\mathcal{L}_v(z;x)$ instead of $\mathcal{L}_{v,\mathcal{B}_Y}(z;x)$.

The following results are immediate corollaries of THEOREM 3.1.

COROLLARY 3.2. — *Let $f_1,\dots,f_N$ be regular functions, let $n_1,\dots,n_N$ be integers and let $v_1,\dots,v_N$ be homogeneous of degree n and let they belong to the base $\mathcal{B}_Y$ of* $\operatorname{Lie}(H(Y))$. *There is a functional equation*

$$\sum_{i=1}^{N} n_i\mathcal{L}_{v_i}\big(f_i(z);f_i(x)\big)=0$$

if an only if

$$\sum_{i=1}^{N} n_i\big(v_i^*\circ(f_i)_*\big)=0$$

in $\operatorname{Hom}\big(\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X);\mathbb{C}_p\big)$, *where*

$$(f_i)_*:\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X)\longrightarrow\Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y)$$

is induced by f_i.

COROLLARY 3.3. — *Let $\mathcal{B}_X$ be a base of* $\mathrm{Lie}\, H(X)$ *given by basic Lie elements. The functions $\{\mathcal{L}_v(z;x_o) \mid v \in \mathcal{B}_X\}$ are algebraically independent on X.*

Now we shall concentrate on polylogarithms. Let

$$Y = P^1(\mathbb{C}_p) \setminus \{0\,;1;\infty\}.$$

Let $\mathcal{B}_Y$ be a base of $\mathrm{Lie}\, H(Y)$ given by basic Lie elements corresponding to the ordering $e_0 = \left(\frac{\mathrm{d}z}{z}\right)^*$ and $e_1 = \left(\frac{\mathrm{d}z}{z-1}\right)^*$. Let us set $e_2 := [e_1, e_0]$ and $e_{n+1} := [e_n, e_0]$. Let e_n^* denote the linear functional on $\mathrm{Lie}\, H(Y)$ dual to e_n with respect to the base $\mathcal{B}_Y$. We shall consider e_n^* as an element of $\mathrm{Alg}(\pi(Y))$.

We recall that $\mathcal{L}_{e_n}(z;x) = e_n^*(\ell_Y(z;x))$. To simplify notation we set

$$\mathcal{L}_n(z;x) := \mathcal{L}_{e_n}(z;x).$$

Let $\mathcal{T}_n : P(Y) \hookrightarrow \mathbb{C}_p[[e_o, e_1]]^* \to \mathbb{C}_p$ associate to an element of $P(Y)$ its coefficient at $e_0^n \cdot e_1$. We set

$$\mathrm{Li}_n(z;x) := (-1)^{n-1}\mathcal{T}_{n-1}\big(\lambda_Y(z;x)\big).$$

It is an easy observation that

$$\mathrm{Li}_n(z;x) = \int_x^z \frac{-\mathrm{d}z}{z-1}, \frac{\mathrm{d}z}{z}, \dots, \frac{\mathrm{d}z}{z},$$

where $\mathrm{d}z/z$ appears $(n-1)$ times.

We shall express the function $\mathcal{L}_n(z;x)$ by functions $\mathrm{Li}_k(z;x)$. Let

$$\lambda = \exp(ae_0) + \sum_{n=0}^{\infty} b_{n+1}\, e_0^n\, e_1 \in P(Y).$$

We recall that

$$\log t = (t-1) - \tfrac{1}{2}(t-1)^2 + \tfrac{1}{3}(t-1)^3 - \tfrac{1}{4}(t-1)^4 + \cdots.$$

Let c_{n+1} be a coefficient at $e_0^n \cdot e_1$ in the power series $\log\lambda$. We have

$$\begin{aligned} c_{n+1} = b_{n+1} &- \tfrac{1}{2}\Big(\sum_{1\le i\le n} \frac{a^i}{i\,!}\, b_{n+1-i}\Big) + \tfrac{1}{3}\Big(\sum_{i=j=1}^{i+j\le n} \frac{a^{i+j}}{i\,!\,j\,!}\, b_{n+1-i-j}\Big) \\ &- \tfrac{1}{4}\Big(\sum_{i=j=k=1}^{i+j+k\le n} \frac{a^{i+j+k}}{i\,!\,j\,!\,k\,!}\, b_{n+1-i-j-k}\Big) + \cdots. \end{aligned}$$

We shall compute the coefficient at $a^k b_{n+1-k}$. Observe that this coefficient is equal to a coefficient at z^k in a power series

$$\varphi(z) = -\tfrac{1}{2}(\mathrm{e}^z - 1) + \tfrac{1}{3}(\mathrm{e}^z - 1)^2 - \tfrac{1}{4}(\mathrm{e}^z - 1)^3 + \cdots.$$

We have

$$(\mathrm{e}^z - 1) + \varphi(z)(\mathrm{e}^z - 1) = \log \mathrm{e}^z = z.$$

Hence

$$\varphi(z) = \frac{z}{\mathrm{e}^z - 1} - 1 = \sum_{n=1}^{\infty} \frac{B_i}{i\,!}\, z^i$$

where B_i are Bernoulli numbers ($B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{12}$, $B_3 = 0, \ldots$). The immediate consequence of this discussion is the following lemma.

LEMMA 3.4. — *We have*

$$c_{n+1} = b_{n+1} + \sum_{k=1}^{n} \frac{B_k}{k\,!}\, a^k b_{n+1-k}.$$

This Lemma implies the following result.

COROLLARY 3.5. — *We have*

$$\mathcal{L}_{n+1}(z;x) = \mathrm{Li}_{n+1}(z;x) + \sum_{k=1}^{n} \frac{B_k}{k\,!} \Big(\int_x^z \frac{\mathrm{d}z}{z}\Big)^k \mathrm{Li}_{n+1-k}(z;x).$$

(Observe that $\mathcal{L}_2(z) := \mathrm{Li}_2(z) + \frac{1}{2}\log z \log(1-z)$ is the Rogers function (see [R]).)

THEOREM 3.6.
Let $X = P^1(\mathbb{C}_p) \setminus \{a_1, \ldots, a_{n+1}\}$ and let $Y = P^1(\mathbb{C}_p) \setminus \{0, 1, \infty\}$. Let $f_1, \ldots, f_N : X \to Y$ be regular functions and let $n_1, \ldots, n_N$ be integers. There is a functional equation

$$\sum_{i=1}^{N} n_i \mathcal{L}_n\big(f_i(z); f_i(x)\big) = 0 \tag{0}$$

if and only if one of the following equivalent conditions is satisfied :

(1) $\sum_{i=1}^{N} n_i e_n^* \circ (f_i)_* = 0$ *in the group*

$$\mathrm{Hom}\big(\Gamma^n \pi(X) / \Gamma^{n+1} \pi(X); \mathbb{C}_p\big)\ ;$$

(2) $\sum_{i=1}^{N} n_i(f_i)_* = 0$ *in the group*

$$\operatorname{Hom}\big(\Gamma^n\pi(X)\backslash_{\Gamma^{n+1}\pi(X)};\Gamma^n\pi(Y)/(\Gamma^{n+1}\pi(Y)+L_n)\big),$$

where L_n *is a* $\mathbb{C}_p$*-vector subspace of* $\Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y)$ *generated by all commutators in* e_0 *and* e_1 *which contain* e_1 *at least twice;*

(3) $\sum_{i=1}^{N} n_i(f_i)_* = 0$ *in the group*

$$\operatorname{Hom}\Big(\Gamma^n\big(\pi_1(X,x)^\ell_{\text{et}}\big)/\Gamma^{n+1}\big(\pi_1(X,x)^\ell_{\text{et}}\big);\\ \Gamma^n\big(\pi_1(Y,y)^\ell_{\text{et}}\big)/(\Gamma^{n+1}\big(\pi_1(Y,y)^\ell_{\text{et}}\big)+L_n)\Big)$$

where $\pi_1(X,x)^\ell_{\text{et}}$ *is the* ℓ*-profinite quotient of the etale fundamental group of* X *and where* $(f_i)_*$ *are maps induced by* f_i *on etale fundamental groups.* L_n *is a closed subgroup of* $\Gamma^n(\pi_1(Y,y)^\ell_{\text{et}})$ *defined in the following way. If* $\mathbb{C}_p$ *is the field of complex numbers* $\mathbb{C}$ *then* L_n *is topologically generated by all commutators in* e_0 *(loop around* 0*) and* e_1 *(loop around* 1*) which contain* e_1 *at least twice. If* $\mathbb{C}_p$ *is a non-archimedean field then any isomorphism* $\mathbb{C}_p \approx \mathbb{C}$ *induces an isomorphism*

$$\pi_1\big(P^1(\mathbb{C}_p)\setminus\{0,1,\infty\},y\big)^\ell_{\text{et}} \approx \big(P^1(\mathbb{C})\setminus\{0,1,\infty\},y\big)^\ell_{\text{et}}$$

and $L_n \subset \Gamma^n\pi_1(P^1(\mathbb{C}_p)\setminus\{0,1,\infty\},y)^\ell_{\text{et}}$ *is the image of*

$$L_n \subset \Gamma^n\Big(\pi_1\big(P^1(\mathbb{C})\setminus\{0,1,\infty\},y\big)^\ell_{\text{et}}\Big)$$

under this isomorphism.

Proof. — It follows immediately from COROLLARY 3.2 that (0) and (1) are equivalent. Observe that $L_n = \ker e_n^*$. This implies that (1) and (2) are equivalent. Observe that the map induced by f_i on quotient groups $\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X)$ "coincides" with the map induced by f_i on $\Gamma^n(\pi_1(X,x)^\ell_{\text{et}})/\Gamma^{n+1}(\pi_1(X,x)^\ell_{\text{et}})$. This implies that conditions (2) and (3) are equivalent.

Definition 3.7. — Let n be a natural number. We note $\operatorname{ldt}(n)$ a polynomial in variables $Li_k(g_i(z))$, where $k < n$ and $g_i(z)$ are rational functions.

COROLLARY 3.8. — *There is a functional equation*

$$\sum_{i=1}^{N} n_i\big(\mathrm{Li}_n(f_i(z)) - \mathrm{Li}_n(f_i(x))\big) + \mathrm{ldt}(n) = 0 \tag{1}$$

if and only if

$$\sum_{i=1}^{N} n_i(f_i)_* = 0 \tag{2}$$

in the group $\mathrm{Hom}(\Gamma^n\pi(X)/\Gamma^{n+1}\pi(X); \Gamma^n\pi(Y)/(\Gamma^{n+1}\pi(Y) + L_n))$.

Proof. — Let us assume that $\sum_{i=1}^{N} n_i(f_i)_* = 0$. THEOREM 3.6 implies that $\sum_{i=1}^{N} n_i\mathcal{L}_n(f_i(z); f_i(x)) = 0$. It is a trivial observation that $\mathrm{Li}_n(z;x) = \mathrm{Li}_n(z) - \mathrm{Li}_n(x) + \mathrm{ldt}(n)$. Hence COROLLARY 3.5 implies that

$$\sum_{i=1}^{N} n_i\left[\mathrm{Li}_n\big(f_i(z)\big) - \mathrm{Li}_n\big(f_i(x)\big)\right] + \mathrm{ldt}(n) = 0.$$

Let us assume that (1) holds. Let $\mathbb{C}_p = \mathbb{C}$ be the field of complex numbers. Calculating the monodromy of the function $\mathrm{Li}_n(z)$ on elements of $\Gamma^n\pi_1(Y,y)$ we get a linear function $\hat{e}_n$ from $\Gamma^n\pi_1(Y,y)/\Gamma^{n+1}\pi_1(Y,y)$ to $(2\pi i)^n \cdot \mathbb{Z}$ which after the identification of $e_0 \in \pi(Y)$ (resp. $e_1 \in \pi(Y)$) with a loop around 0 (resp. loop around 1) coincides with

$$(2\pi i)^n \cdot e_n^* : \Gamma^n\pi(Y)/\Gamma^{n+1}\pi(Y) \to \mathbb{C}.$$

Calculating the monodromy of $\mathrm{Li}_n(f_i(z))$ on $\Gamma^n\pi_1(X,x)$ we get a linear function

$$\hat{e}_n \circ (f_i)_* : \Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x) \longrightarrow (2\pi i)^n \cdot \mathbb{Z}$$

where $(f_i)_* : \pi_1(X,x) \to \pi_1(Y,y)$ is the map induced by f_i. The functional equation (1) implies that we have $\sum_{i=1}^{N} n_i\,\hat{e}_n \circ (f_i)_* = 0$ in $\mathrm{Hom}(\Gamma^n\pi_1(X,x)/\Gamma^{n+1}\pi_1(X,x); (2\pi i)^n\mathbb{Z})$. This condition is of course equivalent to (2).

Let $\mathbb{C}_p$ be a non-archimedean field. We rewrite the equation (1) in the form

$$\sum_{i=1}^{N} n_i\,\mathrm{Li}_n\big(f_i(z); f_i(x)\big) + \mathrm{ldt}(n) = 0$$

where $\mathrm{ldt}(n)$ is a polynomial in $L_k(g_j(z); g_j(x))$ (here $k < n$ and $g_j(z)$ are rational functions) and constants. We replace the functions $\mathrm{Li}_n(f_i(z); f_i(x))$ and $\mathrm{Li}_k(g_j(z); g_j(x))$ by their Taylor power series at x. We choose an isomorphism $\alpha : \mathbb{C}_p \approx \mathbb{C}$ and we interpret the Taylor power series over $\mathbb{C}_p$ as Taylor power series of complex functions $\mathrm{Li}_k(\ ;\ )$. Hence we get a functional equation of complex polylogarithms

$$\sum_{i=1}^{N} n_i \,\mathrm{Li}_n\big(f_i^\alpha(z); f_i^\alpha(x)\big) + \mathrm{ldt}(n) = 0$$

and this situation we have already considered.

Now we shall show that the ideal of polynomial relations between functions $\mathrm{Li}_n(f_i(z))$, where $f_i(z)$ are rational functions is generated by linear relations from THEOREM 3.6.

Let $f_1(z), \dots, f_N(z)$ be rational functions. Let $p(x_1, \dots, x_N, t_1, \dots, t_R)$ be a polynomial whose degree with respect to $x_1, \dots, x_N$ is strictly smaller than k. We set

$$\mathrm{LDT}_k(n) := p\Big(\mathrm{Li}_n\big(f_i(z)\big), \dots, \mathrm{Li}_n\big(f_N(z)\big), T_1, \dots, T_R\Big)$$

where $T_1 = \mathrm{ldt}(n), \dots, T_R = \mathrm{ldt}(n)$.

We recall that I is a homogeneous ideal in $\mathbb{C}_p[x_1, \dots, x_m]$ if the following two conditions holds :

(i) for any two homogeneous elements in I of the same degree, their sum is in I;

(ii) for any homogeneous element a in I and any homogeneous element c in $\mathbb{C}_p[x_1, \dots, x_m]$, the element $c \cdot a$ is in I.

THEOREM 3.9. — *Let $f_1(z), \dots, f_N(z)$ be rational functions. Let*

$$\begin{aligned} I_n(f_1, \dots, f_N) &\\ = \Big\{ &p(x_1, \dots, x_N) \in \mathbb{C}_p[x_1, \dots, x_N] \\ &\mid p(x_1, \dots, x_N) \textit{ homogeneous of degree } k > 0, \\ &\quad p\big(\mathrm{Li}_n\big(f_1(z)\big), \dots, \mathrm{Li}_n\big(f_N(z)\big)\big) + \mathrm{LDT}_k(n) = 0 \Big\}. \end{aligned}$$

Then $I_n(f_1, \dots, f_N)$ is a homogeneous ideal generated by a finite number of linear forms in $x_1, \dots, x_N$.

Proof. — It is clear that $I_n(f_1,\dots,f_N)$ is a homogeneous ideal. Let $X = P^1(\mathbb{C}_p)\setminus\{a_1,\dots,a_t\}$ be such that the maps $f_i : X \to P^1(\mathbb{C}_p)\setminus\{0,1,\infty\}$ are regular for $i=1,\dots,N$. It follows from THEOREM 3.1 and the fact that $p(x_1,\dots,x_N)$ is homogeneous that

$$p\big(\mathrm{Li}_n\big(f_1(z)\big),\dots,\mathrm{Li}_n\big(f_N(z)\big)\big) + \mathrm{LDT}_k(n) = 0$$

if and only if $p(e_n^*\circ(f_1)_*,\dots,e_n^*\circ(f_N)_*) = 0$ in $\mathrm{Alg}(\pi(X))$. This is equivalent to the condition $p(e_n^*\circ(f_1)_*,\dots,e_n^*\circ(f_N)_*) = 0$ in $S(V^*)$ where $V = \Gamma^n\pi(X)/\Gamma^{n+1}\pi(X)$ and $S(V^*)$ is the symmetric algebra over $\mathbb{C}_p$ on the vector space $V^* = \mathrm{Hom}(V,\mathbb{C}_p)$.

Then $I_n(f_1,\dots,f_N)$ is the maximal homogeneous ideal contained in $\ker(\mathbb{C}_p[x_1,\dots,x_N] \xrightarrow{\pi} S(V^*))$, where $\pi(x_i) = e_n^*\circ(f_i)_*$. The map π is induced by a linear map $\bigoplus_{i=1}^N \mathbb{C}_p x_i \to V^*$. Hence the ideal $I_n(f_1,\dots,f_N)$ is generated by one-forms.

4. Sometimes it is easier without $2\pi i$

In this section p is a finite prime, so we are working in a p-adic realm. Let

$$P_{n+1}(z) = \sum_{i=0}^{n} \alpha_i(\log z)^i\,\mathrm{Li}_{n+1-i}(z)$$

$(\mathrm{Li}_1(z) = -\log(1-z))$. Let $V_{n+1}\subset \mathbb{C}_p^{n+1}$ be given by

$$V_{n+1} = \Big\{(a_0,\dots,a_n)\in\mathbb{C}_p^{n+1} \mid \sum_{i=0}^{n}\frac{a_k}{(n+1-k)\,!} = 0\Big\}.$$

LEMMA 4.1. — *Let*

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(P_{n+1}(z)\big) - \alpha_n\Big[\frac{(\log z)^{n-1}\log(1-z)}{z} + \frac{(\log z)^n}{1-z}\Big] = \sum_{i=0}^{n-1}\beta_i\,\frac{(\log z)^i\,\mathrm{Li}_{n-i}(z)}{z}.$$

If $(\alpha_0,\dots,\alpha_n)\in V_{n+1}$ *then* $(\beta_0,\dots,\beta_{n-1})\in V_n$.

Proof. — We have

$$\begin{aligned}\beta_k &= a_k + (k+1)a_{k+1} \qquad \text{if}\quad k<n-1,\\ \beta_{n-1} &= a_{n-1} + (n+1)a_n.\end{aligned}$$

Hence
$$\sum_{i=0}^{n-1} \frac{\beta_i}{(n-i)\,!} = \sum_{i=0}^{n-1} \frac{a_i + (i+1)a_{i+1}}{(n-i)\,!} + a_n$$
$$= \frac{a_0}{n!} + \sum_{i=1}^{n} \frac{(n+1)}{(n+1-i)\,!} a_i = 0.$$

The p-adic k-th polylogarithm satisfies the functional equation

$$(*) \qquad \mathrm{Li}_k\Big(\frac{1}{z}\Big) = (-1)^{k+1}\,\mathrm{Li}_k(z) + (-1)^{k+1}\frac{(\log z)^k}{k\,!}$$

(see [C], Proposition 6.4).

LEMMA 4.2. — *Let $P_{n+1}(z)$ be such that $(\alpha_0,\ldots,\alpha_n) \in V_{n+1}$. Then* $P_{n+1}\Big(\dfrac{1}{z}\Big) + (-1)^{n+1} P_{n+1}(z) = 0.$

Proof. — This follows immediately from (*).

Following [C], we set

$$\lim_{z\to a}{}' f(z) = \lim_{\substack{z\to a\\ z\in K}} f(z)$$

if all the limits on the right side exist and coincide, for an arbitrary finitely ramified extension K of $\mathbb{Q}_p$ such that coordinates of a are in K.

LEMMA 4.3. — *Let $P_{n+1}(z)$ be such that $(\alpha_0,\ldots,\alpha_n) \in V_{n+1}$. Then* $\lim_{z\to 0}{}' P_{n+1}(z) = 0$ *and* $\lim_{z\to\infty}{}' P_{n+1}(z) = 0$.

Proof. — The fact that $\log z$ is bounded on any finitely ramified extension of $\mathbb{Q}_p$ and $\mathrm{Li}_k(0) = 0$ implies that the first limit vanishes. It follows from LEMMA 4.2 that the second limit vanishes.

Now we give some examples of functions $P_{n+1}(z)$ such that the sequence of coefficients $(\alpha_0,\ldots,\alpha_n) \in V_{n+1}$.

Example 1. — Let

$$\alpha_i = \frac{(-1)^i}{i\,!} \quad \text{if} \quad i < n \quad \text{and} \quad \alpha_n = \frac{(-1)^n}{n\,!} + \frac{(-1)^{n+1}}{(n+1)\,!}.$$

We must show that the expression $\sum_{i=0}^{n+1} (-1)^i / i\,!\,(n+1-i)\,!$ vanishes. Observe that this expression is a coefficient at z^{n+1} in the power series $e^{-z}\cdot e^{z} = 1$, hence

$$\sum_{i=0}^{n+1} \frac{(-1)^i}{i\,!\,(n+1-i)\,!} = 0.$$

The function

$$\sum_{i=0}^{n}\frac{(-1)^i}{i\,!}(\log z)^i\,\mathrm{Li}_{n-i}(z)+\frac{(-1)^{n+1}}{(n+1)\,!}(\log z)^n\,\mathrm{Li}_1(z)$$

appeared in [L2] while its single valued analogue in [W4].

Example 2. — Let $\alpha_i=\dfrac{\beta_i}{i\,!}$ where β_i are Bernoulli numbers $\beta_0=1$, $\beta_1=-\frac{1}{2}$, $\beta_2=\frac{1}{6}$, $\beta_3=0,\ldots$. Observe that $\sum_{i=0}^{n}\beta_i/i\,!\,(n+1-i)\,!$ is a coefficient at z^{n+1} in the power series $\dfrac{z}{\mathrm{e}^z-1}\cdot(\mathrm{e}^z-1)$, hence

$$\sum_{i=0}^{n}\frac{\beta_i}{i\,!\,(n+1-i)\,!}=0.$$

The function $\displaystyle\sum_{i=0}^{n}\frac{\beta_i}{i\,!}(\log z)^i\,\mathrm{Li}_{n+1-i}(z)$ appeared in [D2] and in a non explicit way in [W2] as a solution of a system of differential equations defining horizontal sections.

Example 3. — Let

$$\alpha_0=1,\quad \alpha_i=0\quad\text{for}\quad 0<i<n\quad\text{and}\quad \alpha_n=-\frac{1}{(n+1)\,!}\cdot$$

The corresponding function is $\mathrm{Li}_{n+1}(z)+\dfrac{1}{(n+1)\,!}(\log z)^n\log(1-z)$.

Observe that $\dim V_{n+1}=n$. Hence for $n=1$ there is only one function (up to a multiplication by a constant) such that its sequence of coefficients belongs to V_2. This is the Rogers function $\mathrm{Li}_2(z)+\frac{1}{2}\log(z)\cdot\log(1-z)$.

We hope that a function $P_{n+1}(z)$ such that its sequence of coefficients $(\alpha_0,\ldots,\alpha_n)\in V_{n+1}$, has all functional equations without lower degree terms. We give a partial result in this direction. We shall imitate D. Zagier (see [Z1]).

Let $\mathcal{A}_{\mathrm{loc}}(\mathbb{C}_p)$ be a ring of functions which are locally analytic on some $\mathbb{C}_p\setminus$ several points. Let $\mathrm{Sym}^k(\mathbb{C}_p(z)^*)$ be the k-th symmetric power of the multiplicative group $\mathbb{C}_p(z)^*$. Let us set

$$L_{n+1}\bigl(\mathbb{C}_p(z)^*\bigr):=\mathrm{Sym}^{n-1}\bigl(\mathbb{C}_p(z)^*\bigr)\otimes\bigl(C_p(z)^*\wedge\mathbb{C}_p(z)^*\bigr)\otimes\mathbb{Q}/R$$

where R is generated by expressions of the form

$$(**)\qquad f_1 \odot \cdots \odot f_{n-2} \odot a \otimes b \otimes c + f_1 \odot \cdots \odot f_{n-2} \odot b \otimes c \otimes a + f_1 \odot \cdots \odot f_{n-2} \odot c \otimes a \otimes b$$

and

$$(***)\qquad c_1 \odot \cdots \odot c_{n-1} \otimes c_n \otimes c_{n+1}$$

where $c_i \in \mathbb{C}_p^*$ for $i = 1, \ldots, n+1$. Let $K_{n+1} : L_{n+1}(\mathbb{C}_p(z)^*) \to \mathcal{A}_{\mathrm{loc}}(\mathbb{C}_p)$ be given by

$$K_{n+1}\big((f_1 \odot f_2 \odot \ldots \odot f_{n-1}) \otimes f_n \wedge f_{n+1} \otimes \alpha\big) = \alpha A(f_1) \cdots A(f_{n-1}) \cdot \Big(A(f_{n+1}) \cdot B(f_n) - A(f_n) \cdot B(f_{n+1})\Big)$$

where $A(f) = \log f$ and $B(g) = g'/g$.

Let $\mathcal{B}(\mathbb{C}_p(z)^*)$ be a free abelian group on the set $\mathbb{C}_p(z)^*$. We shall denote by $[f]$ the generator corresponding to $f \in \mathbb{C}_p(z)^*$. Let

$$b_{n+1} : \mathcal{B}\big(\mathbb{C}_p(z)^*\big) \longrightarrow L_{n+1}\big(\mathbb{C}_p(z)^*\big)$$

be a homomorphism given by

$$b_{n+1}\big([f]\big) = f \odot \cdots \odot f \otimes f \otimes 1 - f.$$

PROPOSITION 4.4. — *Let*

$$P_{n+1}(z) = \sum_{i=0}^{n} \alpha_i (\log z)^i \operatorname{Li}_{n+1-i}(z)$$

be such that $(\alpha_0, \ldots, \alpha_n) \in V_{n+1}$. *If* $f = \sum_{k=1}^{m} n_k [f_k] \in \ker b_{n+1}$, *then we have*

$$\sum_{k=1}^{m} n_k \Big(P_{n+1}\big(f_k(z)\big) - P_{n+1}\big(f_k(z_0)\big)\Big) = 0$$

where z_0 *is a fixed element of* $\mathbb{C}_p$.

Proof. — Let $n = 1$. The space V_2 is one-dimensional generated generated by $(1, -\frac{1}{2})$. Then for $P_2(z) = \operatorname{Li}_2(z) + \frac{1}{2}\log z \log(1-z)$ we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z} \sum_{k=1}^{m} n_k \Big(P_2\big(f_k(z)\big)\Big) &= -\tfrac{1}{2} \sum_{k=1}^{m} n_k \left\{ \frac{f_k'(z)}{f_k(z)} \log\big(1 - f_k(z)\big) - \frac{\big(1 - f_k(z)\big)'}{1 - f_k(z)} \log\big(f_k(z)\big) \right\} \\ &= -\tfrac{1}{2} K_2\big(b_2(f)\big) = 0.\end{aligned}$$

Hence $\sum_{k=1}^{m} n_k(P_2(f_k(z)))$ is constant. Assume that the theorem holds for n. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}z}\sum_{k=1}^{m} n_k P_{n+1}\big(f_k(z)\big) &= \sum_{k=1}^{m} n_k\,\frac{f_k'(z)}{f_k(z)}\cdot \mathcal{Q}_n\big(f_k(z)\big)\\ &\quad + \alpha_n\sum_{k=1}^{m} n_k\Big\{\frac{f_k'(z)}{f_k(z)}\big((\log f_k(z))^{n-1}\log(1-f_k(z))\big)\\ &\qquad\qquad - \frac{\big(1-f_k(z)\big)'}{1-f_k(z)}\big(\log f_k(z)\big)^n\Big\}\end{aligned}$$

where $\mathcal{Q}_n(z)=\sum_{i=0}^{n-1}\beta_i(\log z)^i\,\mathrm{Li}_{n-i}(z)$ and

$$\beta_i=\alpha_i+(i+1)\alpha_{i+1}\quad\text{if}\quad i<n-1,\qquad \beta_{n-1}=\alpha_{n-1}+(n+1)\alpha_n.$$

The second summand is equal to $\alpha_n K_{n+1}((b_{n+1}(f))=0$. Let $v_{z-a}(f(z))$ be a valuation of a rational function $f(z)$ at $z-a$. The first summand $\sum n_k f_k'(z)/f_k(z)\cdot\mathcal{Q}_n(f_k(z))$ is equal to

$$\sum_{a\in\mathbb{C}_p\cup\{\infty\}}\frac{1}{z-a}\sum_{k=1}^{m} n_k v_{z-a}\big(f_k(z)\big)\mathcal{Q}_n\big(f_k(z)\big).$$

For any $a\in\mathbb{C}_p$ the element

$$\sum_{k=1}^{m} n_k v_{z-a}\big(f_k(z)\big)\big[f_k(z)\big]\in\mathcal{B}\big(\mathbb{C}_p(z)^*\big)$$

belongs to $\ker b_n$. By LEMMA 4.1 and the inductive assumption the expression $\sum_{k=1}^{m} n_k v_{z-a}(f_k(z))\mathcal{Q}_n(f_k(z))$ is constant. It follows from LEMMA 4.3 that this constant is zero. Hence $(\mathrm{d}/\mathrm{d}z)\sum_{k=1}^{m} n_k P_{n+1}(f_k(z))$ vanishes and the function $\sum_{k=1}^{m} n_k P_{n+1}(f_k(z))$ is constant.

COROLLARY 4.5 (Conjectured by L. LEWIN (see [L2], pp. 7–8)). — *Let $f=\sum_{i=1}^{N} n_i[f_i(z)]\in\mathcal{B}(\mathbb{C}_p(z)^*)$ belong to* $\ker b_{n+1}$. *The lower degree terms of the functional equation of* $\mathrm{Li}_{n+1}(z)$ *for f involve only constants and logarithms.*

Proof. — The sequence $(1,0,\dots,0,-1/(n+1)\,!)$ belongs to V_{n+1}. Let $P_{n+1}(z)=\mathrm{Li}_{n+1}(z)+\dfrac{1}{(n+1)\,!}\log z\log(1-z)$. It follows from PROPOSITION 4.4 that

$$\sum_{i=1}^{N} n_i\Big(P_{n+1}\big(f_i(z)\big)-P_{n+1}\big(f_i(x)\big)\Big)=0.$$

This implies the corollary.

5. Differential Galois groups and functional equations

Let

$$X'=AX \tag{*}$$

be a linear system of differential equations on $P^1(\mathbb{C}_p)$ where $X(z):=(X_1(z),\dots,X_n(z))$ and $A(z)=(A_{ij}(z))_{i,j=1,\dots,n}$. Assume that the elements of the matrix A are in $K=\mathbb{C}_p(z-a)$. Assume also that the functions $A_{ij}(z)$ for $i,j=1,\dots,n$ have no poles at $a\in\mathbb{C}_p$. Then there exists n-solutions $Y_1,\dots,Y_n$ of (*) in $\mathbb{C}_p[[z-a]]$ linearly independent over $\mathbb{C}_p$. The subfield $F=\mathbb{C}_p(z-a)(Y_1,\dots,Y_n)$ of the field of fractions of $\mathbb{C}_p[[z-a]]$ is preserved by the derivation $\partial=\dfrac{\mathrm{d}}{\mathrm{d}(z-a)}$. The differential Galois group of F/K is the group $\mathrm{Aut}_\partial(F/K)$ of automorphism of F which commute with ∂ and fix K (see [An]).

Our fundamental example is the following system of differential equations

$$\begin{cases} \mathcal{T}_0'=0, \quad \Psi=\dfrac{\mathcal{T}_0}{z},\\ \mathcal{T}_1'=\dfrac{\mathcal{T}_0}{1-z}, \quad \mathcal{T}_2'=\dfrac{\mathcal{T}_1}{z}, \quad \dots \quad , \quad \mathcal{T}_n'=\dfrac{\mathcal{T}_{n-1}}{z}\end{cases} \tag{**}$$

with initial conditions $\mathcal{T}_0=1$, $\Psi(a)=0$ and $\mathcal{T}_k(a)=0$ for $k>0$. Its differential Galois group $\mathcal{G}_n$ is given by the following automorphisms of $F=\mathbb{C}_p(z-a)(\Psi,\mathcal{T}_1,\dots,\mathcal{T}_n)$:

$$\Psi\longmapsto\Psi+\alpha$$

$\theta\big(\alpha,\beta_1,\dots,\beta_n\big)$:

$$\mathcal{T}_k\longmapsto\mathcal{T}_k+\sum_{i=1}^{k}\frac{\beta_i}{(k-i)\,!}\,\Psi^{k-i}$$

for $k = 1, \dots, n$, where $\alpha, \beta_1, \dots, \beta_n$ are independent parameters. Observe that

$$\begin{aligned}\theta(\alpha,\beta_1,\dots,\beta_n)\circ\theta(\alpha',\beta'_1,\dots,\beta'_n) &\\ = \theta\Big(\alpha+\alpha',\beta_1+\beta'_1,\beta_2+\beta'_2+\beta_1\frac{\alpha'^1}{1!},&\\ \beta_3+\beta'_3+\beta_2\frac{\alpha'}{1!}+\beta_1\frac{\alpha'^2}{2!},\dots\Big).&\end{aligned}$$

The group $\mathcal{G}_n$ is a nilpotent, affine, algebraic group over $\mathbb{C}_p$. The nilpotence class of $\mathcal{G}_n$ is n, $\mathcal{G}_n^{ab} = \mathbb{C}_p \oplus \mathbb{C}_p$ and, for each $k \leq n$, $\Gamma^k\mathcal{G}_n/\Gamma^{k+1}\mathcal{G}_n \approx \mathbb{C}_p$ is generated by the class of some $\theta(0,\dots,0,\beta_k,\dots)$ with $\beta_k \neq 0$, $\alpha = 0$ and $\beta_i = 0$ for $i < k$. Observe that

$$\mathrm{Gal}_\partial\big(F/\mathbb{C}_p(z-a)(\Psi,\mathcal{T}_1,\dots,\mathcal{T}_k)\big) \approx \Gamma^{k+1}\mathcal{G}_n.$$

Let $f_1(z), \dots, f_m(z)$ be rational functions on $P^1(\mathbb{C}_p)$. We consider the following system of differential equations

$$(***)\qquad \begin{cases} \mathcal{T}' = 0, \quad \Psi'_i = \mathcal{T}\cdot\dfrac{f'_i}{f_i}, \\ \mathcal{T}'_{1,i} = \mathcal{T}\cdot\dfrac{f'_i}{1-f_i}, \quad \mathcal{T}'_{k,i} = \mathcal{T}_{k-1,i}\cdot\dfrac{f'_i}{f_i} \end{cases}$$

$k = 2, \dots, n$, $i = 1, \dots, m$ with initial conditions

$$\mathcal{T} = 1, \quad \Psi_i(a) = 0 \quad \text{and} \quad \mathcal{T}_{k,i}(a) = 0.$$

Let $\mathcal{G}$ be a differential Galois group of the system (***). Let

$$F_s = \mathbb{C}_p(z-a)(\Psi_i,\mathcal{T}_{k,i})_{\substack{i=1,\dots,m,\\ k=1,\dots,s.}}$$

Then $\mathrm{Gal}_\partial(F_n/\mathbb{C}_p(z-a)) = \mathcal{G}$ and $\mathrm{Gal}_\partial(F_n/F_s) \approx \Gamma^{s+1}\mathcal{G}$.

THEOREM 5.1. — *There is a functional equation*

$$(****)\qquad \sum_{i=1}^{m} n_i\mathcal{T}_{n,i}(z) + p(z) = 0$$

where $p(z) \in F_{n-1}$ *and* $\sum_{i=1}^{m} |n_i| > 0$ *if and only if* $\dim\Gamma^n\mathcal{G} < m$.

Proof. — The differential Galois group $\mathcal{G}$ is given by the following automorphisms of the field F_n :

$$\Psi_i \longmapsto \Psi_i + \alpha_i$$

$\theta\big((\alpha_i)_{i=1,\dots,n}, \beta_{k,i}\big)_{\substack{i=1,\dots,n \\ k=1,\dots,m}}$:

$$\mathcal{T}_{k,i} \longmapsto \mathcal{T}_{k,i} + \sum_{\ell=1}^{k} \frac{\beta_{\ell,i}}{(k-\ell)\,!}\Psi_i^{k-\ell}.$$

The parameters α_i, $\beta_{k,i}$ need not be longer independent hence we have $\dim \mathcal{G} \leq m(n+1)$. We shall denote by $\theta(\beta_{k_0 i_0})$ the element θ of $\mathrm{Aut}_\partial(F_n/\mathbb{C}_p(z-a))$ such that all $\alpha_i = 0$ and all $\beta_{ki} = 0$ but $\beta_{k_0 i_0 \neq 0}$.

Let us assume that we have a functional equation (****). Applying $\theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m})$ to (****), we get

$$\sum_{i=1}^{n} n_i\big(\mathcal{T}_{n,i}(z) + \beta_{n,i}\big) + p(z) = 0.$$

Hence $\sum_{i=1}^{n} n_i\beta_{n,i} = 0$. Observe that the group $\Gamma^n\mathcal{G}$ is generated by elements $\theta(\beta_{n,i})$ $i = 1,\dots,m$. This implies that $\dim \Gamma^n\mathcal{G}g < m$.

The group $\Gamma^n\mathcal{G}$ is an abelian group, quotient of $\mathbb{C}_p^m$, generated by the elements $\theta(\beta_{n,i})$ for $i = 1,\dots,m$. Hence every element $\theta \in \Gamma^n\mathcal{G}$ has the form

$$\theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m}).$$

Let us assume that $\dim \Gamma^n\mathcal{G} < m$. Then there is a non-trivial relation $\sum_{i=1}^{m} n_i\beta_{n,i} = 0$ for any $\theta = \theta(\beta_{n,1}) \circ \theta(\beta_{n,2}) \circ \cdots \circ \theta(\beta_{n,m}) \in \Gamma^n\mathcal{G}$. Observe that the function $\sum_{i=1}^{m} n_i\mathcal{T}_{n,i}(z)$ is fixed by $\Gamma^n\mathcal{G}$, because

$$\begin{aligned}\theta : \sum_{i=1}^{m} n_i\mathcal{T}_{n,i}(z) \longmapsto \sum_{i=1}^{m} n_i\big(\mathcal{T}_{n,i}(z) + \beta_{n,i}\big) \\ = \sum_{i=1}^{m} n_i\mathcal{T}_{n,i}(z) + \sum_{i=1}^{m} n_i\beta_{n,i}.\end{aligned}$$

The isomorphism $\mathrm{Aut}_\partial(F_n/F_{n-1}) \approx \Gamma^n\mathcal{G}$ implies that

$$\sum_{i=1}^{m} n_i\mathcal{T}_{n,i}(z) \in F_{n-1}.$$

Remark. — Solutions of (**) are

$$\mathcal{T}_0 = 1, \quad \Psi(z) = \int_a^z \frac{\mathrm{d}z}{z}, \quad \mathcal{T}_k(z) = \int_a^z \frac{\mathrm{d}z}{1-z}, \frac{\mathrm{d}z}{z}, \dots, \frac{\mathrm{d}z}{z}$$

and solutions of (***) are

$$\mathcal{T} = 1, \quad \Psi_i(z) = \int_{f_i(a)}^{f_i(z)} \frac{\mathrm{d}z}{z}, \quad \mathcal{T}_{k,i}(z) = \int_{f_i(a)}^{f_i(z)} \frac{\mathrm{d}z}{1-z}, \frac{\mathrm{d}z}{z}, \dots, \frac{\mathrm{d}z}{z}.$$

BIBLIOGRAPHIE

[A] ABEL (N.H.). — *Note sur la fonction* $\Psi(x) = x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots$, Oeuvres, Bd. II, pp. 189–193.

[An] ANDRE (Y.). — *Quatre descriptions des groupes de Galois différentiels*, Proceedings of the Séminaire d'algèbre de Paris, Lecture Notes in Math. 1296, Springer–Verlag.

[C] COLEMAN (R.F.). — *Dilogarithms, Regulators and p-adic L-functions*, Invent. Math., t. **69**, 1982, p. 171–208.

[Ca] CARTIER (P.). — *Jacobiennes généralisées, monodromie unipotente et intégrales itérées*, Séminaire Bourbaki, 1987–1988, 687, *Astérisque*, pp. 161–162, 1988.

[Ch] CHEN (K.T.). — *Algebra of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc., t. **156**, 1971, p. 359–379.

[D1] DELIGNE (P.). — *Le groupe fondamental de la droite projective moins trois points*, in Galois Groups over $\mathbb{Q}$, Springer–Verlag New York, 1989.

[D2] DELIGNE (P.). — *Interprétation motivique de la conjecture de Zagier reliant polylogarithmes et régulateurs.*

[L1] LEWIN (L.). — *Polylogarithms and Associated Functions.* — North Holland, New York-Oxford, 1981.

[L2] LEWIN (L.). — *The order-independence of the polylogarithmic ladder structure, implications for a new category of functional equations*, Aequationes Math., t. **30**, 1986, p. 1–20.

[MKS] MAGNUS (W.), KARRASS (A.) and SOLITAR (D.). — *Combinatorial Group Theory, Pure and Applied Mathematics*, volume XIII, Interscience Publishers, 1966, John Wiley and Sons, Inc..

[R] ROGERS (L.J.). — *On function sum theorems connected with the series* $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$, Proc. London Math. Soc.(2), t. **4**, 1907, p. 169–189.

[S] SANDHAM (H.F.). — *A Logarithmic Transcendent*, J. London Math. Soc., t. **24**, 1949, p. 83–91.

[W1] WOJTKOWIAK (Z.). — *A note on the functional equation of the dilogarithm*, preprint 1, CRM (Bellaterra), 1984.

[W2] WOJTKOWIAK (Z.). — *A note on functional equation of polylogarithms*, preprint 22, MPI, 1990.

[W3] WOJTKOWIAK (Z.). — *The basic structure of polylogarithmic functional equations*, preprint 88, MPI, 1990.

[W4] WOJTKOWIAK (Z.). — *A construction of analogs of the Bloch-Wigner function*, Math. Scand., t. **65**, 1989, p. 140–142.

[Z1] ZAGIER (D.). — *Polylogarithms, Dedekind zeta function and the algebraic K-theory of fields*, preprint 44, MPI, 1990.

[Z2] ZAGIER (D.). — *The remarkable dilogarithm*, J. Math. Phys. Sci., t. **22**, 1988, p. 131–145.

[Z3] ZAGIER (D.). — *The Bloch-Wigner-Ramakrishnan polylogarithm function*, Math. Ann., t. **28**, 1990, p. 613–624.