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BERNARD HOST
JEAN-FRANÇOIS MÉLA
FRANÇOIS PARREAU

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NON SINGULAR TRANSFORMATIONS AND SPECTRAL ANALYSIS OF MEASURES

BY

BERNARD HOST, JEAN-FRANÇOIS MÉLA
AND FRANÇOIS PARREAU (*)

RÉSUMÉ. — Ce travail approfondit les interactions qui existent entre l'analyse harmonique des mesures et l'étude spectrale des systèmes dynamiques non singuliers. Il est centré sur l'étude de sous-groupes remarquables du cercle, groupes de valeurs propres, groupes de quasi-invariance des mesures..., dont les exemples les plus naturels sont définis par des conditions diophantiennes. La conjonction des points de vue permet d'obtenir nombre de résultats nouveaux dans les deux théories, y compris dans des problèmes classique d'analyse de Fourier.

ABSTRACT. — This work explores in depth the interactions existing between harmonic analysis of measures and spectral theory of non-singular dynamical systems. It focuses on the study of some classes of remarkable subgroups of the circle : eigenvalue groups, groups of quasi-invariance of measures..., the most natural examples of which are defined by diophantine conditions. The conjonction of these points of view leads to many new results in both theories, including some classical problems in Fourier analysis.

1. Introduction

The spectral study of non-singular transformations reveals a deep interplay between ergodic theory and harmonic analysis. The aim of this work is to display some aspects of these connections, mainly those involving the *eigenvalue group* $e(T)$ of a non-singular transformation T, and the *group of quasi-invariance* $H(\mu)$ of a positive finite Borel measure μ on $\mathbb{T}$, that is the group of all $t \in \mathbb{T}$ such that μ is equivalent to its translate by t.

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B. HOST, J.-F. MÉLA, F. PARREAU, Université Paris Nord, C.S.P., Dépt. de Mathématiques, Laboratoire d'Analyse et Applications, URA 742 du CNRS, avenue J.B. Clément, 93430 Villetaneuse, France.

The work finds its unity as much in the methods as in the results, as the reader will realize, we hope.

For sake of simplicity, we will focus on the case of the circle group, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is maybe the most interesting one. Most of our results can be easily extended to any second countable locally compact group and we will mention it only occasionally. We use the classical notations and results of Ergodic Theory and Fourier Analysis. The measure spaces we consider are all standard, that is : up to an isomorphism, they are polish spaces with a finite or σ-finite positive Borel measure. In the case of $\mathbb{T}$, we also allow complex Borel measures.

Along this paper, we will introduce some remarkable Borel subgroups of $\mathbb{T}$ which all belong to the class of so-called *saturated subgroups* of $\mathbb{T}$.

The notion of saturated subgroup had already been considered in a more general setting and under an other name, in the chapter 8 of [18]. The point was there to exhibit some examples of non locally compact group topologies for which we have an extension of the Bochner theorem. The property for a subgroup to be saturated is initially a Fourier Analysis property, related to the notions of *Dirichlet measure* and *weak Dirichlet set* ([18], [26]). C. MOORE and K. SCHMIDT ([32], [41]) already noticed that these notions appear naturally when studying the eigenvalue groups of non-singular dynamical systems.

Since then, there have been some developments and it seems useful to give a clear, new and more complete exposition of these topics. This is achieved in section 2, where are also discussed the connections with the classical theory of the absolute convergence of trigonometric series. We introduce a group which plays a key role at the crossroad of harmonic analysis and non-singular dynamics. Given any positive finite Borel measure on $\mathbb{T}$, we denote by $\overline{\mathbb{Z}}_1(\mu)$ the group of all measurable functions (of unit modulus) which are limits of group characters $e^{2\pi i n x}$, $n \in \mathbb{Z}$, in the $L^1(\mu)$ or $L^2(\mu)$ topology. We also characterize saturated subgroups in terms of these groups.

In the same section, we define a class of subgroups which provides typical examples throughout the paper : with any sequence of positive integers n_j and any sequence of nonnegative real numbers a_j such that $\sum a_j = +\infty$, we associate the group of all $t \in \mathbb{T}$ such that

$$\sum_{0}^{+\infty} a_j \left| e^{2\pi i n_j t} - 1 \right|^2 < +\infty.$$

Such a group will be called an H_2 *group.*

In section 3, we study the eigenvalue groups of ergodic non-singular transformations or flows, which have been paid attention by several authors ([14], [19], [3]) and which are also the T-sets of the Connes-Krieger theory of Von-Neumann algebras ([9]). These groups may be uncountable but may not be any subgroup of the circle; they are σ-compact and admit an intrinsic polish topology stronger than the circle topology. C. MOORE and K. SCHMIDT ([32], [41]) first noticed that eigenvalue groups are weak Dirichlet sets, except when equal to $\mathbb{T}$.

In fact it is proved in [30] that they enjoy the stronger property to be saturated. However, this proof is a little sketchy and the properties of saturated subgroups are simply stated, referring to [18] where the terminology and the context are different. The main theorem of [30] is proved here with full details and complements. Moreover, having in mind some general applications in Harmonic Analysis, we get rid of the unnecessarily restrictive assumption of ergodicity (restricting then the definitions to eigenfunctions of constant modulus).

We ask wether the conjunction of the saturation property and of the topological properties of eigenvalue groups characterizes this class of subgroups of $\mathbb{T}$. We prove that any H_2 group can be realized as the eigenvalue group of some ergodic non-singular transformation (3.5). Besides, any σ-compact saturated subgroup of the circle is close to being an H_2 group (2.3); it might even be that every eigenvalue group is an H_2 group.

The main proof in section 3 involves the construction of factors which play the same role as discrete factors for measure preserving transformations. Denoting by S the transformation of $\overline{\mathbb{Z}}_1(\tau)$ corresponding to the multiplication by $e^{2\pi i x}$, any non-singular system $(\overline{\mathbb{Z}}_1(\tau), \nu, S)$ is isomorphic to a non-singular compact group rotation (THEOREM 3.2). Given a non-singular dynamical system (X, μ, T), we associate such a factor with every positive Borel measure τ carried by $e(T)$.

In section 4, we consider some classical non-singular systems defined as Kakutani towers over the 2-odometer, which appear in [10], [23] and are also studied in [33], [19], [3]. Their eigenvalue groups turn out to be H_2 groups. We will prove two new results for these systems. First, under some growth condition for the height function, we show that the tower is isomorphic to a non-singular compact group rotation : we construct on the eigenvalue group a probability measure which is familiar to harmonic analysts — a so called "generalized Riesz product" — and then we prove that the factor given by the technique of section 3 is in fact isomorphic to the initial system.

On the other hand, in full generality, we compute the maximal spectral types of these systems which turn out to be classical Riesz products, and it is worth noting that any Riesz product with nonnegative coefficients may be interpreted as a maximal spectral type in this way.

Both results point out the key role of Riesz products in these topics and provide a link with the last sections of the paper, where we are mainly concerned with groups of quasi-invariance of measures on the circle.

Indeed, it is known that for any non-singular transformation T with maximal spectral type σ the eigenvalue group $e(T)$ is contained in the quasi-invariance group $H(\sigma)$. Whether the equality holds in general is an open problem. In the case of the tower over the 2-odometer and the Riesz product, we are able to prove it under a mild condition on the height function. This yields a condition for the equivalence of a Riesz product with its translates which improves the classical results (G. BROWN and W. MORAN [6], J. PEYRIÈRE [37]). Besides, when the tower is isomorphic to a non-singular compact group rotation, its spectral type is ergodic under the action of some countable group of translations. This provides a proof of the ergodicity of a class of Riesz products by a method quite different from PARREAU'S [34].

Apart from these results, the section 5 contains a general study of the groups $H(\mu)$ for an arbitrary positive measure μ on $\mathbb{T}$. Till recently, very little was known about these groups, except that $\mu(H(\mu)) = 0$ for any continuous singular measure μ ([29]; see also [12]). J. AARONSON and M. NADKARNI [3] show that $H(\mu)$ is the eigenvalue group of some non-singular transformation and thus is a saturated subgroup, under the assumption of ergodicity which is not quite natural from the Harmonic Analysis point of view. We prove the result without restriction, as a particular case of a more general theorem which deals with cocycle extension (THEOREM 5.4) : the "maximal group" for a cocycle associated with a group of translations on $\mathbb{T}$ is an eigenvalue group. This contains some results by H. HELSON and K. MERRILL ([16], [17]).

The property for a measure on $\mathbb{T}$ to be "ergodic under translations" may be restated in a non-classical way : it is possible to drop any requirement of quasi-invariance (see [7], [12]) and moreover any reference to a specified group of translations [34]. In this context the natural object to attach to any positive finite Borel measure μ on $\mathbb{T}$ is the set $A(\mu)$ of all $t \in \mathbb{T}$ such that μ and its translate by t are not mutually singular. These topics are discussed in section 5.5, where we also prove a significant property of the sets $A(\mu)$ for an arbitrary singular measure μ : any measure carried by $A(\mu)$, although it need not be a

Dirichlet measure, is concentrated on a countable union of weak Dirichlet sets.

The end of the paper is dedicated to some applications to the Fourier-Gelfand theory of the convolution algebra $M(\mathbb{T})$ of finite complex Borel measures on $\mathbb{T}$, in relation with the "Wiener Pitt phenomenon" (the range of the Fourier transform of a measure is generally not dense in its Gelfand spectrum). The eigenfunctions and eigenvalues of translations are closely related to the so-called "generalized characters" and the techniques of sections 2 and 3 involving the groups $\overline{\mathbb{Z}}_1(\tau)$ provide multiplicative linear functionals on the measure algebra $M(\mathbb{T})$ which are limits of continuous characters. These properties are used in [35] for constructing non-trivial examples of measures whose spectrum in $M(\mathbb{T})$ is the closure of the range of their Fourier transform.

2. Saturated subgroups

For sake of simplicity we will write the definitions and properties in the case of the circle group. But all we are going to say is meaningful for locally compact abelian groups and can be straightforwardly extended to the general case.

Let us denote $M(\mathbb{T})$ the convolution algebra of complex finite Borel measures on $\mathbb{T}$ which is a Banach algebra under the norm $\|\mu\| = \int \mathrm{d}|\mu|$. In the sequel a measure, without other specification, will mean an element of $M(\mathbb{T})$. The Fourier transform of a measure $\mu \in M(\mathbb{T})$ is defined by

$$\widehat{\mu}(n) = \int \mathrm{e}^{2\pi i n t}\,\mathrm{d}\mu(t), \qquad (n \in \mathbb{Z}).$$

We note $\widetilde{\mu}$ the measure defined by $\widetilde{\mu}(E) = \bar{\mu}(-E)$, for every Borel set E.

2.1. Definition and general properties.

THEOREM. — *For a Borel subgroup H of the circle group $\mathbb{T}$, the following properties are equivalent :*

(1) *For any measure $\mu \in M(\mathbb{T})$,*

$$|\mu(H)| \leq \sup\big|\widehat{\mu}(n)\big|.$$

(2) *For every compact $K \subset H$, every compact L disjoint from H, and every $\varepsilon > 0$, one can find a positive-definite continuous function ϕ with $\phi(0) = 1$ such that*

$$\big|1 - \phi(t)\big| \leq \varepsilon, \quad (t \in K), \qquad \big|\phi(t)\big| \leq \varepsilon, \quad (t \in L).$$

(3) *For any positive measure $\mu \in M(\mathbb{T})$, 1_H is in the closed convex hull in $L^1(\mu)$ of the exponentials $\mathrm{e}^{2\pi i n t}$.*

DEFINITION. — A Borel subgroup of the circle which shares these equivalent properties is called a *saturated subgroup.*

In the general case of a locally compact abelian group the theorem remains valid, replacing the $e^{2\pi int}$ by continuous group characters, and we define a saturated subgroup similarly.

Proof of the Theorem. — Let us recall that the positive-definite continuous functions on $\mathbb{T}$ are the sums of absolutely convergent Fourier series with nonnegative coefficients. The condition $\phi(0) = 1$ in (2) means that the sum of the Fourier coefficients of ϕ is 1, *i.e.* that ϕ belongs to the closed convex hull of the exponentials under the uniform norm.

(1) implies (2). Let us consider the space of real continuous functions on $K \cup L$ with the uniform norm. It is enough to prove that in this space 1_K is in the closed convex hull of the functions $\cos 2\pi nt$, $n \in \mathbb{N}$. If not, by the Hahn-Banach theorem, we could find a real measure μ supported by $K \cup L$ and a constant a such that

$$\operatorname{Re}\widehat{\mu}(n) = \int \cos(2\pi nt)\,\mathrm{d}\mu(t) < a, \qquad (n \in \mathbb{N}),$$
$$\mu(H) = \int 1_K(t)\,\mathrm{d}\mu(t) > a.$$

Let then ν be the symmetric measure $|a|\delta + \frac{1}{2}(\mu + \widetilde{\mu})$, where δ denotes the Dirac mass at 0. We have $\widehat{\nu}(n) = |a| + \operatorname{Re}\widehat{\mu}(n) > 0$ for every n and $\nu(H) = |a| + \mu(H) > \sup|\widehat{\nu}(n)|$, whence a contradiction.

(2) implies (3). Given any positive measure $\mu \in M(\mathbb{T})$ and any $\varepsilon > 0$, one can find a compact set $K \subset H$ and a compact set L disjoint from H, such that $\mu(K \cup L) > \|\mu\| - \varepsilon$. So, as the function ϕ given by (2) is a uniform limit of convex combinations of the exponentials, (3) is a direct consequence of (2).

(3) implies (1). Let $\mu \in M(\mathbb{T})$. The property (3) for $|\mu|$ implies immediately

$$|\mu(H)| = \Big|\int 1_H(t)\,\mathrm{d}\mu(t)\Big| \le \sup\Big|\int e^{2\pi int}\,\mathrm{d}\mu(t)\Big| = \sup\big|\widehat{\mu}(n)\big|.$$

COROLLARY 1. — *The properties of the theorem are also equivalent to :*

(4) *There exists some constant $C > 0$ such that, for any measure $\mu \in M(\mathbb{T})$,*

$$\big|\mu(H)\big| \le C\sup\big|\widehat{\mu}(n)\big|.$$

Proof. — Since (1) is obviously stronger than (4), we have to show that (4) implies (1). If (1) does not hold, one can find a measure μ such that $|\mu(H)| \geq 1$ and $\sup |\widehat{\mu}(n)| \leq 1 - \varepsilon$ for some $\varepsilon > 0$. Let us consider the measure $\nu = \mu * \widetilde{\mu}$. We have

$$\nu(H) = \int \mu(H + x)\, \mathrm{d}\bar{\mu}(x).$$

Now, there exist at most countably many disjoint classes $H + x$ with $\mu(H + x) \neq 0$; say $\mu(H + x) = 0$ if x does not belong to one of the classes $H + x_n$, $n \geq 1$, so that

$$\begin{aligned} \nu(H) &= \sum_{n\geq 1} \int_{H+x_n} \mu(H + x)\, \mathrm{d}\bar{\mu}(x) \\ &= \sum_{n\geq 1} \mu(H + x_n)\, \bar{\mu}(H + x_n) \\ &= \sum_{n\geq 1} \left|\mu(H + x_n)\right|^2 \geq \left|\mu(H)\right|^2 \geq 1. \end{aligned} \tag{2.1.1}$$

On the other hand, $\widehat{\nu}(n) = |\widehat{\mu}(n)|^2$ and $\sup |\widehat{\nu}(n)| \leq (1 - \varepsilon)^2$. By iterating this argument we will contradict (4).

REMARK 1. — For any Borel subgroup H of the circle, 1_H is a positive-definite Borel function. The property (3) of the theorem asserts that H is saturated if and only if, given any positive measure μ, 1_H is the limit in $L^1(\mu)$ (or μ-almost everywhere) of a sequence of continuous positive-definite functions.

Note that $\mathbb{T}$ itself is saturated. For a proper subgroup, the properties of the theorem can be strengthened.

COROLLARY 2. — *A proper Borel subgroup H of $\mathbb{T}$ is saturated if and only if*

(5) *for any measure* $\mu \in M(\mathbb{T})$, $\left|\mu(H)\right| \leq \limsup \left|\widehat{\mu}(n)\right|$.

It is clearly enough to prove that (1) implies (5). Since H is a proper subgroup, it has zero Lebesgue measure (otherwise $H = H - H$ has a non-empty interior and $H = \mathbb{T}$ follows). Let $\mu \in M(\mathbb{T})$. Given any positive integer N, we can find an absolutely continuous measure ν with $\widehat{\nu}(n) = \widehat{\mu}(n)$ for $-N \leq n \leq N$ and $\widehat{\nu}(n) = 0$ if $|n| \geq N$. Then $\nu(H) = 0$ and

$$\left|\mu(H)\right| = \left|(\mu - \nu)(H)\right| \leq \sup_{|n|>N} \left|\widehat{\mu}(n)\right|.$$

REMARK 2. — In the general case of a locally compact abelian group the assumption that H is a proper subgroup must then be replaced by the assumption that H is not an open subgroup or, equivalently, that it has zero Haar measure.

EXAMPLE . — *The group generated by a Kronecker set is a proper saturated subgroup.*

A compact subset E of the circle is a Kronecker set if every continuous function of unit modulus on E is a limit of group characters in the uniform topology on E. A Kronecker set is rationally independent and generates a proper subgroup. (See [26] for more details and examples.)

Proof. — For the group generated by a Kronecker set, the property (2) of the THEOREM 2.1 is proved in [18], p. 141.

2.2. Another characterization of saturated subgroups. The group $\overline{\mathbb{Z}}_1(\mu)$.

We will give another equivalent formulation of the property for a subgroup to be saturated, which will justify this terminology and will be used in the next section.

Given any positive measure μ, we denote $\overline{\mathbb{Z}}(\mu)$ the closure and $\widetilde{Z}(\mu)$ the closed convex hull of the exponentials $\mathrm{e}^{2\pi i n t}$ in the weak $*$-topology of $L^\infty(\mu)$. With this topology the pointwise multiplication is separately continuous and both sets are compact semi-topological semigroups. $\widetilde{Z}(\mu)$ is also the closed convex hull of the exponentials in the $L^1(\mu)$ topology.

$\overline{\mathbb{Z}}_1(\mu)$ denotes the closure of the exponentials in the $L^1(\mu)$ topology. It is contained in the multiplicative group of all functions in $L^\infty(\mu)$ with unit modulus, and it may be viewed at as the completion of $\mathbb{Z}$ under the invariant metric $d(m,n) = \int |\mathrm{e}^{2\pi i n t} - \mathrm{e}^{2\pi i m t}|\,\mathrm{d}\mu(t)$. Here, the $L^1(\mu)$ topology is equivalent to the $L^2(\mu)$ topology, since

$$\int |1 - \mathrm{e}^{2\pi i n t}|^2\,\mathrm{d}\mu(t) = 2\int (1 - \cos 2\pi n t)\,\mathrm{d}\mu(t).$$

It follows moreover that, for any sequence $n_k \in \mathbb{Z}$, $\mathrm{e}^{2\pi i n_k t}$ converges to 1 in the $L^1(\mu)$ topology if and only if it converges to 1 in the $L^\infty(\mu)$ weak $*$-topology and this is also equivalent to saying that $\lim_{k\to+\infty} \widehat{\mu}(n_k) = \|\mu\|$.

LEMMA. — *Let μ and ν be positive measures on $\mathbb{T}$. The following properties are equivalent :*

(1) *For any sequence $n_k \in \mathbb{Z}$,*

$$\lim_{k\to+\infty} \widehat{\mu}(n_k) = \|\mu\| \quad \textit{implies} \quad \lim_{k\to+\infty} \widehat{\nu}(n_k) = \|\nu\|.$$

(2) *For any sequence* $n_k \in \mathbb{Z}$,

$$\lim_{k\to+\infty} \int |e^{2\pi i n_k t} - 1| \, d\mu(t) = 0 \quad \textit{implies} \quad \lim_{k\to+\infty} \int |e^{2\pi i n_k t} - 1| \, d\nu(t) = 0.$$

(3) *Every element of* $\overline{\mathbb{Z}}(\mu+\nu)$ *which is* 1 μ*-almost-everywhere, is identically* 1.

(4) *There exists a continuous group homomorphism* $\overline{\mathbb{Z}}_1(\mu) \to \overline{\mathbb{Z}}_1(\nu)$ *which maps the function* $e^{2\pi i t}$ *of* $\overline{\mathbb{Z}}_1(\mu)$ *to the function* $e^{2\pi i t}$ *of* $\overline{\mathbb{Z}}_1(\nu)$.

(5) *Every element of* $\widetilde{Z}(\mu+\nu)$ *which is* 1 μ*-almost-everywhere, is identically* 1.

Proof. — The equivalence of (1), (2), (3) and (4) follows immediately from the remarks above and (5) implies obviously (3). It remains to prove that (3) implies (5) : every element ψ of $\widetilde{Z}(\mu+\nu)$ can be written as a barycenter of elements in $\overline{\mathbb{Z}}(\mu+\nu)$, say $\psi = \int \chi \, d\sigma(\chi)$ with some probability measure σ on $\overline{\mathbb{Z}}(\mu+\nu)$. Now if $\psi = 1$ μ-a.e., necessarily for σ-almost every χ we must have $\chi = 1$ μ-a.e., whence by (3) $\chi = 1$, and finally $\psi = 1$.

DEFINITION. — Let μ and ν be positive measures on $\mathbb{T}$. If the equivalent properties (1) to (5) of the theorem above hold, we will say that ν *sticks* to μ. More generally, we will say that a measure ν *sticks* to a measure μ if $|\nu|$ sticks to $|\mu|$.

REMARK 1. — *If μ is a positive measure such that* $\limsup |\widehat{\mu}(n)| < \|\mu\|$ *(in particular if the Fourier transform of μ vanishes at infinity), then every measure sticks to μ. Moreover* $\overline{\mathbb{Z}}_1(\mu) \simeq \mathbb{Z}$.

Proof. — Suppose that $\lim_{k\to+\infty} |\widehat{\mu}(n_k)| = \|\mu\|$; then necessarily the sequence n_k stays bounded. If $n_k = n$ for infinitely many k, $e^{\pi i n t}$ is equal to a constant μ-a.e., whence $|\widehat{\mu}(mn)| = \|\mu\|$ for all m and necessarily $n = 0$. So $n_k = 0$ for k large enough and, for an arbitrary positive measure ν, $\widehat{\nu}(n_k) = \|\nu\|$ for k large enough. This also shows that the topology of $\overline{\mathbb{Z}}_1(\mu)$ induces the discrete topology on $\mathbb{Z}$ and $\overline{\mathbb{Z}}_1(\mu) \simeq \mathbb{Z}$ follows by density.

The next theorem explains the choice of the terminology "saturated".

THEOREM. — *A Borel subgroup H of $\mathbb{T}$ is saturated if and only if every measure which sticks to a measure concentrated on H, is itself concentrated on H.*

Proof. — We shall prove the equivalence with the property (3) of the THEOREM 2.1, which can be stated : for any positive measure $\mu \in M(\mathbb{T})$, $1_H \in \widetilde{\mathbb{Z}}(\mu)$.

Assume that H is a saturated subgroup of $\mathbb{T}$ and let μ and ν be two positive measures such that μ is concentrated on H and ν sticks to μ. Since 1_H belongs to $\widetilde{\mathbb{Z}}(\mu+\nu)$, the property (5) of the lemma yields $1_H = 1$ ν-a.e. and therefore ν is concentrated on H.

Conversely, let μ be any positive measure on $\mathbb{T}$. The set of all non-negative elements of $\widetilde{\mathbb{Z}}(\mu)$ which are $\geq 1_H$ μ-a.e. is compact in the weak $*$-topology and thus admits a minimal element h for the natural order relation on μ-measurable functions (see [18], p. 21). Since $\widetilde{\mathbb{Z}}(\mu)$ is a multiplicative semi-group, h is a one-zero function. We claim that the measure $\nu = h\mu$ sticks to the measure $\mu_H = 1_H\mu$. Let ϕ be any element of $\widetilde{\mathbb{Z}}(\nu)$ which is equal to 1 μ_H-a.e.; by compacity ϕ may be extended to an element ϕ' of $\widetilde{\mathbb{Z}}(\mu)$ which is still equal to 1 μ_H-a.e. Then $\psi = \frac{1}{2}(1+\mathrm{Re}(\phi'))$ is a positive element of $\widetilde{\mathbb{Z}}(\mu)$ with $\psi = 1_H$ μ-a.e. and thus $\psi \geq h$, so that $\psi = 1$ ν-a.e. and it follows $\phi' = \phi = 1$ ν-a.e. By the property (5) of the lemma, ν sticks to μ_H. Finally, assuming the condition in the theorem, ν is concentrated on H; this proves $h = 1_H$ μ-a.e. and therefore $1_H \in \widetilde{\mathbb{Z}}(\mu)$.

EXAMPLE . — *Any countable subgroup H of $\mathbb{T}$ is saturated.*

Proof. — We denote $\widehat{H}$ the dual group of the discrete group H. Endowed with the dual topology, *i.e.* the pointwise convergence on H, $\widehat{H}$ is compact metrizable and contains Z as a dense subgroup. Let μ and ν be two positive measures such that μ is concentrated on H and ν sticks to μ. If a sequence $e^{2\pi i n_k t}$ converges pointwise on H, it converges in $\overline{\mathbb{Z}}_1(\mu)$ and, by the property (4) of the lemma, it converges in $\overline{\mathbb{Z}}_1(\nu)$; then $\widehat{\nu}(n_k)$ converges. Therefore, the Fourier transform $\widehat{\nu}$ may be extended to a continuous function on $\widehat{H}$. It follows from the Bochner theorem that ν is concentrated on H.

REMARK 2. — More generally, given a Borel subgroup H, we can define a group topology on $\mathbb{Z}$ such that a sequence n_k converges to 0 if and only if for every positive measure μ carried by H, the sequence $\widehat{\mu}(n_k)$ converges to $\|\mu\|$; by the theorem, H is saturated when any positive measure whose Fourier transform is continuous in this topology is concentrated on H (according to the terminology of [18], chap. 8, $M(H)$ is then a Bochner subalgebra of $M(\mathbb{T})$).

2.3. Saturated subgroups and $\mathbf{H_2}$ groups.

The most important example of saturated subgroups will arise from the next theorem which is nothing but a more precise version of the property (2) of the THEOREM 2.1.

THEOREM. — *Let H be a σ-compact subgroup of $\mathbb{T}$. The following properties are equivalent :*

(1) *H is saturated.*

(2) *For any compact L disjoint from H, one can find a sequence ϕ_j of real-valued positive-definite continuous functions with $\phi_j(0) = 1$ such that the sum*

$$\sum_{0}^{+\infty}(1 - \phi_j(t))$$

is finite for all $t \in H$ and infinite for all $t \in L$.

(3) *For any compact L disjoint from H, one can find a sequence a_j of nonnegative real numbers and a sequence n_j of positive integers (not necessarily distinct) such that the sum*

$$\sum_{0}^{+\infty} a_j(1 - \cos 2\pi n_j t)$$

is finite for all $t \in H$ and infinite for all $t \in L$.

Proof.

(1) implies (2). H can be written as a union of an increasing sequence of compact sets $K_j, j \geq 0$. Then, for $j \geq 0$, let ψ_j be given by the property (2) of the THEOREM 2.1 with $K = K_j$ and $\varepsilon = 2^{-j}$, and let $\phi_j = \mathrm{Re}(\psi_j)$.

(2) implies (3). For each $j \geq 1$, ϕ_j is the sum of an absolutely convergent series $\sum_{n=0}^{+\infty} a_{j,n} \cos 2\pi nt$, where $a_{j,n} \geq 0$ for all n and $\sum_{n=0}^{+\infty} a_{j,n} = 1$. Then

$$\sum_{j=0}^{+\infty}(1 - \phi_j(t)) = \sum_{j=0}^{+\infty}\sum_{n=1}^{+\infty} a_{j,n}(1 - \cos 2\pi nt).$$

(3) implies (1). We may assume $a_j \leq 1$ for all j (by splitting each term when needed). We shall show that the property (3) of the THEOREM 2.1 holds. Given any positive measure μ and any $\varepsilon > 0$, one can find a compact L disjoint from H and such that $\mu(H \cup L) \geq (1 - \varepsilon)\|\mu\|$. For every $k \geq 1$ we define

$$\psi_k(t) = \prod_{j=0}^{+\infty}\Big(1 - \frac{a_j}{k} + \frac{a_j}{k}\cos 2\pi n_j t\Big)\cdot$$

ψ_k is the pointwise limit of a sequence of positive-definite continuous functions taking the value 1 at 0 and therefore belongs to $\widetilde{\mathbb{Z}}(\mu)$. Now ψ_k

converges to the function ψ given by

$$\psi(t)=\begin{cases}1 & \text{if } \sum_0^{+\infty} a_j(1-\cos 2\pi n_j t)<+\infty,\\ 0 & \text{otherwise,}\end{cases}$$

so that $\psi(t)=1_H(t)$ for $t\in H\cup L$. We have

$$\psi\in\widetilde{\mathbb{Z}}(\mu) \quad\text{and}\quad \int|\psi-1_H|\,\mathrm{d}\mu<\varepsilon\|\mu\|.$$

DEFINITION. — Let $\alpha>0$. Given a sequence n_j of positive integers and a sequence $a_j\ge 0$, the set of all $t\in\mathbb{T}$ such that

$$\sum_{j=0}^{+\infty} a_j\left|1-\mathrm{e}^{2\pi \mathrm{i} n_j t}\right|^{\alpha}<+\infty \tag{2.3.1}$$

is a group, as can easily be checked. A subgroup of $\mathbb{T}$ which can be described in this way, for some sequence a_n, will be called an *H_α group.*

Notice that we do not suppose the integers n_j pairwise distinct and thus that we do not restrict these classes of groups if we moreover require that the sequence a_j be bounded.

For every $C>0$ the set of $t\in\mathbb{T}$ such that $\sum_{j=0}^{+\infty} a_j|1-\mathrm{e}^{2\pi \mathrm{i} n_j t}|^{\alpha}\le C$ is compact, so that H_α groups are σ-compact. When $\alpha=2$, (2.3.1) can be written

$$\sum_0^{+\infty} a_j(1-\cos 2\pi n_j t)<+\infty \tag{2.3.2}$$

and the next corollary is nothing but a reformulation of the theorem above.

COROLLARY.

(1) *Every H_2 group is a saturated σ-compact subgroup of $\mathbb{T}$.*

(2) *Let H be a σ-compact saturated subgroup of $\mathbb{T}$. For every compact L disjoint from H, there exists an H_2 group containing H and disjoint from L.*

EXAMPLES :

1) *Let ϕ_j be a sequence of real-valued positive-definite functions, with $\phi_j(0)=1$. The set of all $t\in\mathbb{T}$ such that*

$$\sum_0^{+\infty}\bigl(1-\phi_j(t)\bigr)<+\infty \tag{2.3.3}$$

is an H_2 group and so is a saturated subgroup.

Proof. — We have shown in the proof of the theorem that (2.3.3) may be written as (2.3.2).

2) *If* $0 < \alpha \leq 2$, *any* H_α *group is an* H_2 *group and so is a saturated subgroup.*

Proof. — For $0 < \alpha \leq 2$, it is well known that there exists a positive-definite continuous function ψ such that $1 - \psi(t) \simeq |t|^\alpha$ as $t \to 0$ (take for ψ the characteristic function of the stable law of order α). Then, we get an equivalent series when , for every $j \geq 0$, we replace $a_j|1 - e^{\pi i n_j t}|^\alpha$ by $1 - \phi_j(t)$ where $\phi_j(t) = 1 - a_j + a_j\psi(n_j t)$ (we may again assume $a_j \leq 1$ for every j).

We shall show in section 2.4 that the H_α groups are not saturated in general for $\alpha > 2$.

3) *If* $\sum_{j=0}^{+\infty} a_j = +\infty$, *the* H_α *group defined by* (2.3.1) *is a proper subgroup for any* $\alpha > 0$.

Proof. — Let us denote H this group and let $s_k = \sum_{j=0}^{k} a_j$ for $k \geq 0$. Since $s_k \to +\infty$, we have

$$\frac{1}{s_k}\sum_{j=0}^{k} a_j\big|1 - e^{2\pi i n_j t}\big|^\alpha \to 0, \qquad (t \in H).$$

Now, as we have averages of uniformly bounded functions, the convergence towards zero does not depend on the exponent α and, with $\alpha = 2$, we obtain

$$\frac{1}{s_k}\sum_{j=0}^{k} a_j \cos 2\pi n_j t \to 1, \qquad (t \in H),$$

and it follows by integrating that H has 0 Lebesgue measure. In fact, for any positive measure μ concentrated on H, we have

$$\frac{1}{s_k}\sum_{j=0}^{k} a_j \operatorname{Re}\widehat{\mu}(n_j) \to \|\mu\|$$

and therefore $\limsup|\widehat{\mu}(n)| = \|\mu\|$.

4) For an arbitrary choice of the sequences a_j and n_j, the correspon-ding H_2 group may pretty well be trivial. The most interesting case is when the sequence n_j satisfies some lacunarity condition. In particular, *the* H_2 *group is uncountable if*

$$\sum_{0}^{+\infty} a_j\Big(\frac{n_j}{n_{j+1}}\Big)^2 < +\infty. \tag{2.3.4}$$

This is shown in [34] by exhibiting a continuous measure concentrated on the group. We shall also give this construction in section 4.2. The hypothesis (2.3.4) is the best possible lacunarity condition : if $n_{j+1} = k_j n_j + 1$, where k_j is a sequence of positive integers such that $\sum a_j/k_j^2 = +\infty$, with the mild hypothesis that $a_j \sim a_{j+1}$, one proves in [34] that the corresponding H_2 group is trivial.

2.4. Non saturated $\mathbf{H}_\alpha$ groups.

THEOREM. — *Let, for $j \geq 0$, $n_{j+1} = m_j n_j$, where the m_j are positive integers ≥ 2 and $\sum_0^{+\infty} 1/m_j^2 < +\infty$. Neither the H_α groups defined by*

$$\sum_1^{+\infty} \left|1 - \mathrm{e}^{2\pi \mathrm{i} n_j t}\right|^\alpha < +\infty$$

for $\alpha > 2$, nor the group

$$H_\infty = \left\{t \in \mathbb{T}\ ;\ \mathrm{e}^{\pi \mathrm{i} n_j t} \to 1\right\}$$

are saturated.

Proof. — We choose positive integers $p_j < m_j$ such that $p_j/m_j \to 0$,

$$\sum_0^{+\infty} \frac{1}{p_j^2} = +\infty \quad \text{and} \quad \sum_0^{+\infty} \frac{1}{p_j^\alpha} < +\infty \quad \text{for all} \quad \alpha > 2.$$

For $j \geq 0$, let q_j be the integral part of $m_j/2p_j$ and $r_j = q_j/n_{j+1}$.

We consider independent random variables X_j which take the values 0 with probability $\frac{1}{2}$ and $\pm r_j$ with probability $\frac{1}{4}$, and independent random variables Y_j which take the values 0 with probability $1 - 1/(2p_j^2)$ and $\pm p_j r_j$ with probability $1/(4p_j^2)$. The distributions μ and ν of the convergent sums $X = \sum_1^{+\infty} X_j$ and $Y = \sum_1^{+\infty} Y_j$ are the infinite convolution products

$$\mu = \mathop{*}_0^{+\infty} \left(\tfrac{1}{2}\delta + \tfrac{1}{4}\delta_{r_j} + \tfrac{1}{4}\delta_{-r_j}\right),$$
$$\nu = \mathop{*}_0^{+\infty} \left[\left(1 - \frac{1}{2p_j^2}\right)\delta + \frac{1}{4p_j^2}\delta_{p_j r_j} + \frac{1}{4p_j^2}\delta_{-p_j r_j}\right].$$

The following lemma will prove clearly that none of the H_α groups with $\alpha > 2$ is saturated.

LEMMA. — (*With the notations above.*)

(a) *For every $\alpha > 2$, μ is concentrated on H_α.*

(b) $\nu(H_\infty) = 0$.

(c) *ν sticks to μ.*

Proof.

(a) Let $\alpha > 2$. The convergence of $\sum_1^{+\infty} |1 - e^{2\pi i n_j t}|^\alpha$ μ-a.e. is equivalent to the almost sure convergence of $\sum_0^{+\infty} |1 - e^{2\pi i n_j X}|^\alpha$. Let $j \geq 1$. For $k < j$, $n_j r_k = n_j q_k / n_{k+1}$ is an integer; as $q_k < m_k = n_{k+1}/n_k$ for all k, we also have $\sum_{j+1}^{+\infty} q_k/n_{k+1} \leq 1/n_{j+1}$, whence $n_j \sum_{j+1}^{+\infty} r_k \leq 1/m_j$. Thus, modulo 1, $n_j X$ is equal to $n_j X_j$ up to a term $\leq 1/m_j$. As $\sum_0^{+\infty} 1/m_j^2 < +\infty$, we only have to check $\sum_0^{+\infty} |n_j X_j|^\alpha < +\infty$ a.s. This follows from

$$\sum_0^{+\infty} E\big(|n_j X_j|^\alpha\big) = \tfrac{1}{2} \sum_0^{+\infty} (n_j r_j)^\alpha \leq \frac{1}{2^{1+\alpha}} \sum_0^{+\infty} \frac{1}{p_j^\alpha} < +\infty.$$

(b) Similarly, $n_j Y$ converges to 0 modulo 1 if and only if $n_j Y_j$ converges to 0. By construction, the sum of the probabilities

$$P\big(|n_j Y_j| = n_j p_j r_j\big) = \frac{1}{2p_j^2}$$

is infinite, and $n_j p_j r_j$ converges to $\frac{1}{2}$. As the variables $n_j Y_j$ are independent, we conclude by the Borel Cantelli lemma that almost surely $n_j Y_j$ and thus $1 - e^{2\pi i n_j Y}$ do not converge to 0. This proves $\nu(H_\infty) = 0$.

(c) For every $n \in \mathbb{Z}$, we have

$$\widehat{\mu}(n) = \prod_0^{+\infty} \tfrac{1}{2}(1 + \cos 2\pi n r_j),$$

$$\widehat{\nu}(n) = \prod_0^{+\infty} \Big(1 - \frac{1}{2p_j^2} + \frac{1}{2p_j^2} \cos 2\pi n p_j r_j\Big).$$

Since $1 - \cos t \geq 1/p^2(1 - \cos pt)$ for every $t \in \mathbb{R}$ and every positive integer p, we have $0 \leq \widehat{\mu}(n) \leq \widehat{\nu}(n)$ for every n. It is obvious from there that ν sticks to μ.

2.5. Dirichlet subgroups and N sets.

We have shown that, if H is a proper saturated subgroup of $\mathbb{T}$, every positive measure μ concentrated on H satisfies

$$\limsup \big|\widehat{\mu}(n)\big| = \|\mu\|. \tag{2.5.1}$$

In particular, H has measure zero for the Lebesgue measure and, more generally, for every measure whose Fourier transform tends to 0 at infinity.

DEFINITION 1. — Any positive measure $\mu \in M(\mathbb{T})$ enjoying the property (2.5.1) is called a *Dirichlet measure.* More generally, a complex measure $\mu \in M(\mathbb{T})$ is called a *Dirichlet measure* if $|\mu|$ is one.

This definition is introduced in [18] where one can find a thorough study of Dirichlet measures. The property (2.5.1) is equivalent to the fact that 1 is the limit, in the $L^1(\mu)$ topology, of a sequence of group characters $e^{2\pi i n_j t}$, where n_j tends to infinity (see [18], p. 34). Here again, the $L^1(\mu)$ topology can be replaced by the $L^2(\mu)$ topology, or by the weak $*$-topology of $L^\infty(\mu)$, or by the topology of convergence in measure.

DEFINITION 2.

(a) A compact set $E \subset \mathbb{T}$ is a *Dirichlet set* if 1 is the limit, in the uniform topology on E, of a sequence of group characters $e^{2\pi i n_k t}$, where n_k tends to infinity.

(b) A Borel set $E \subset \mathbb{T}$ is a *weak Dirichlet set* if, for every measure μ carried by E, 1 is the limit, in the $L^1(\mu)$ topology, of a sequence of group characters $e^{2\pi i n_k t}$, where n_k tends to infinity; or, equivalently, if every measure carried by E is a Dirichlet measure.

(c) A Borel subgroup of T is a *Dirichlet subgroup* if it is a weak Dirichlet set.

EXAMPLES :

1) *Every saturated proper subgroup is a Dirichlet subgroup.*

2) *Every discrete measure is a Dirichlet measure.*

When μ is supported by a finite number of points, this property is a nothing but a classical diophantine approximation theorem of Dirichlet, which justifies our terminology (see [20] from which these ideas originate).

We will see later several examples of continuous Dirichlet measures. It is useful to remark that a Dirichlet measure is always singular because, for every positive measure, we have $\limsup |\widehat{\mu}(n)| \leq \|\mu_s\|$, where μ_s is the singular part of μ.

3) *If* $\sum_{j=0}^{+\infty} a_j = +\infty$, *the* H_α *group defined by* (2.3.1) *is a Dirichlet subgroup for any* $\alpha > 0$.

That is nothing but what we actually proved in example 3, section 2.3.

4) *There exists a Dirichlet subgroup H such that every point mass measure sticking to a measure concentrated on H is itself concentrated on H, but which is not saturated.*

It is enough to consider the group H_∞ defined in section 2.4. The proof is obvious.

The terminology of Dirichlet set and of weak Dirichlet set is used by many authors (see [26]). The interest of the last notion is enhanced by the fact that, for a compact set, being a weak Dirichlet set is equivalent to being a N set.

DEFINITION 3. — A set $E \subset \mathbb{T}$ is a *N-set* if there exists a trigonometric series $\sum_{n=0}^{+\infty} a_n \cos(2\pi nt - \alpha_n)$, with $a_n \geq 0$ and $\sum a_n = +\infty$, which is absolutely convergent on E. It is well known (see [4], [5]) that we obtain an equivalent definition when considering only series of sines $\sum a_n \sin 2\pi nt$.

The problem of absolute convergence and the description of N sets has long been an important topic in the classical theory of trigonometric series, studied by the best Fourier analysts (see [4], [5]). We would like now to show its connections with our subject. It is worth noticing that the next theorem contains many classical results on N sets.

THEOREM. — *For a Borel subset E of T, the following properties are equivalent :*

(1) *E is a N set;*

(2) *E is contained in an H_1 proper subgroup;*

(3) *E is contained in an H_2 proper subgroup;*

(4) *E is contained in a proper σ-compact saturated subgroup;*

(5) *E is contained in a σ-compact weak Dirichlet set (or subgroup).*

Proof.

(1) implies (2). This is obvious from $|\sin 2\pi nt| = \frac{1}{2}|1 - e^{4\pi int}|$.

(2) implies (3). Every H_1 group is also an H_2 group (see 2.3, example 2).

(3) implies (4). Every H_2 subgroup of T is saturated (COROLLARY 2.2).

(4) implies (5). Every proper saturated subgroup is a weak Dirichlet set.

(5) implies (1). We may assume that E is a weak Dirichlet set. Let K be a compact contained in E, and N be an integer ≥ 1. In the space of continuous functions on K with the uniform topology, 0 belongs to the closed convex hull of the functions $|\sin 2\pi nt|$, $n \geq N$. Indeed, if not, by the Hahn-Banach theorem, we would have a real measure μ concentrated on K such that

$$\int |\sin 2\pi nt| \, d\mu(t) \geq C > 0, \qquad (n \geq N),$$

and, as $|\sin 2\pi nt| \leq |1 - e^{2\pi int}|$, this would contradict the property for E of being a weak Dirichlet set. Therefore, for every $\varepsilon > 0$ and for every $N \geq 1$, one can find numbers $a_n \geq 0$ ($N \leq n < N'$), such that

$$\sum_{N \leq n < N'} a_n = 1 \quad \text{and} \quad \sum_{N \leq n < N'} a_n |\sin 2\pi nt| \leq \varepsilon \qquad (t \in K).$$

Now E is σ-compact and can be written as a union of an increasing sequence of compact sets K_j, $j \geq 1$. Using the previous argument with $K = K_j$ and $\varepsilon = 2^{-j}$, $j \geq 1$, we build a sequence N_j and blocks of coefficients a_n, $N_j \leq n < N_{j+1}$, so that $\sum_{n=1}^{+\infty} a_n = +\infty$ and $\sum_{n=1}^{+\infty} a_n |\sin 2\pi nt| < +\infty$ for every $t \in E$.

3. Eigenvalue groups of non-singular dynamical systems

3.1. The eigenvalue groups are saturated.

Let μ be a finite or σ-finite positive measure on a standard Borel space (X, B) and let T be a non-singular automorphism of the measure space $(X, B, \mu) = (X, \mu)$. A complex number λ is said to be an L^∞ *eigenvalue* of T (or of the dynamical system (X, μ, T)) if there exists a non-zero function $\phi_\lambda \in L^\infty(\mu)$ such that $\phi_\lambda(Tx) = \lambda\phi_\lambda(x)$ μ-a.e.; ϕ_λ will be called an L^∞ *eigenfunction* corresponding to the eigenvalue λ.

If T is ergodic, every such ϕ_λ has constant modulus μ-a.e. and every eigenvalue has modulus 1. Later on, we will have to deal with non-ergodic transformations. In that case, the L^∞ eigenfunctions have no longer constant modulus in general. Most of what we are going to say will stay valid if we restrict ourselves to considering only those eigenfunctions which have constant modulus. The corresponding eigenvalues are, of course, of modulus 1.

Definition. — For any non-singular transformation T of a standard measure space (X, μ), we denote by $E(T)$ the set of all L^∞ eigenfunctions of unit modulus, and by $e(T)$ the set of all $t \in \mathbb{T}$ such that $\mathrm{e}^{2\pi i t}$ is an eigenvalue corresponding to an eigenfunction in $E(T)$.

$e(T)$ is a subgroup of the circle group. $E(T)$ is a subgroup of the multiplicative group of L^∞ functions of unit modulus, which will be denoted $U(\mu)$.

When the measure μ is finite, we shall give $U(\mu)$ the $L^1(\mu)$ topology. It is equivalent to the $L^2(\mu)$ topology and, in fact, the convergence of φ_j towards φ in $U(\mu)$ is equivalent to the weak $*$-convergence in $L^\infty(\mu)$. This follows from

$$\int |\varphi - \varphi_j|^2 \,\mathrm{d}\mu = 2\Big(\|\mu\| - \mathrm{Re}\int \varphi_j \overline{\varphi} \,\mathrm{d}\mu\Big)$$

and this implies that we do not change the topology when we replace μ by an equivalent finite measure. If μ is infinite, we define the topology on $U(\mu)$ by the above metric for any equivalent finite measure.

Then $U(\mu)$ is a polish group; the mapping $\phi \mapsto (\phi \circ T)/\phi$ from $U(\mu)$ into itself is a continuous group homomorphism which maps $E(T)$ into the group of constant functions and whose kernel is the group $E_0(T)$ of all invariant functions of unit modulus. Thus $E_0(T)$ and $E(T)$ are closed and are also polish groups. Endowed with the quotient topology of $E(T)/E_0(T)$, $e(T)$ is a polish group, continuously imbedded in $\mathbb{T}$. We shall need a few basic properties of polish spaces or polish groups that can be found in [24], [36], or [8], chapter 9.

(3.1.1) *Let f be a continuous one-to-one map from a polish space X to a metric space Y. Then $f(X)$ is a Borel subset of Y and f is a Borel isomorphism of X with $f(X)$.*

(3.1.2) *A group homomorphism between polish groups is continuous whenever it is Borel. Then it has a Borel cross section (not necessarily a group homomorphism).*

We deduce from (3.1.1) and (3.1.2) :

LEMMA. — *There exists a unique polish topology on $e(T)$ under which the imbedding of $e(T)$ in $\mathbb{T}$ is continuous, and this topology has the same Borel structure as the Borel structure inherited from $\mathbb{T}$. Moreover, there exists a Borel map $t \mapsto \varphi_t$ from $e(T)$ to $E(T)$ such that, for all $t \in e(T)$, $e^{2\pi i t}$ is the eigenvalue corresponding to φ_t.*

We state here the main result of this section, which will be proved in section 3.2, using a remarkable property of the measures concentrated on $e(T)$.

THEOREM. — *Let T be a conservative non-singular automorphism of a standard measure space (X, μ). Then $e(T)$ is a proper saturated subgroup (and so a Dirichlet subgroup) of the circle group.*

In the ergodic case, this result was given by J.F. MÉLA [30] and the property that $e(T)$ is a Dirichlet subgroup was already obtained by K. SCHMIDT [41]. In fact, the only property we know which is specific to the ergodic case is the following :

PROPOSITION. — *If T is ergodic, $e(T)$ is σ-compact in the topology of $\mathbb{T}$.*

Proof. — We consider here the closed unit ball $B(\mu)$ of $L^\infty(\mu)$ endowed with the weak $*$-topology and the set $E'(T)$ of all eigenfunctions of modulus ≤ 1. $B(\mu)$ is compact and metrizable, and $E'(T)$ is contained in $B(\mu) \setminus \{0\}$ which is σ-compact. Now, for any $f \in L^1(\mu)$, the mapping $\varphi \mapsto \int \varphi(Tx) f(x)\, \mathrm{d}\mu(x) = \int \varphi(x) f(T^{-1}x)\, \mathrm{d}T\mu(x)$ is continuous in the weak $*$-topology and it follows easily that $E'(T)$ is closed in $B(\mu) \setminus \{0\}$, hence σ-compact, and that the mapping which assigns to any $\varphi \in E'(T)$ the corresponding eigenvalue is continuous. So, the set of all eigenvalues is a σ-compact subset of $\mathbb{T}$ (and this set is $e(T)$ when T is ergodic).

REMARK. — We do not know if every polish group continuously embedded in $\mathbb{T}$ as a σ-compact saturated subgroup is the eigenvalue group of some non singular ergodic transformation. That is the case for the H_2 groups, as we shall see (section 3.5).

3.2. Measures concentrated on the eigenvalue group. Discrete factors.

In the case of measure-preserving transformations, the discrete factors are the compact group rotations $(\widehat{D}, \lambda, S)$ where D is any subgroup of the eigenvalue group, $\widehat{D}$ the dual group of D, λ the Haar measure of $\widehat{D}$ and S the rotation $x \mapsto x+1$ of $\widehat{D}$ (D being a subgroup of $\mathbb{T}$, there is a natural homomorphism of $\mathbb{Z}$ onto a dense subgroup of $\widehat{D}$). Such factors can be defined for non-singular transformations as well, except that the measure λ may be any S-quasi-invariant measure. A further extension will be to associate a factor with every measure concentrated on the eigenvalue group, the previous construction being the case when we choose a discrete measure. We shall first use this construction in order to prove the THEOREM 3.1 and then show that all the factors we obtain this way turn out to be compact group rotations.

We shall use the following standard result (see [42], p. 65) :

(3.2.1) *Let (X, μ) and (Y, ν) be standard measure spaces. If $x \mapsto \psi_x$ is a Borel map from X to $U(\nu)$, there exists a Borel function $\varphi(x, y)$ on $X \times Y$, such that, for μ-almost every x, $\varphi(x, y) = \psi_x(y)$ ν-a.e(y).*

Conversely, if $\varphi(x, y)$ is a unit modulus Borel function on $X \times Y$, the map $y \mapsto \varphi(\cdot, y)$ is a Borel map from Y to $U(\mu)$. Therefore, for each Borel map $x \mapsto \psi_x$ from X to $U(\nu)$, we have a Borel map $y \mapsto \varphi_y$ from Y to $U(\mu)$ with $\psi_x(y) = \varphi_y(x)$ $\mu \otimes \nu$-a.e.

LEMMA 1. — *A positive measure $\tau \in M(\mathbb{T})$ is concentrated on $e(T)$ if and only if there exists a Borel map $x \mapsto \psi_x$ from X to $U(\tau)$ such that $\psi_{Tx}(t) = \mathrm{e}^{2\pi\mathrm{i}t}\psi_x(t)$ τ-a.e(t) for μ-almost every x.*

Proof. — According to the previous remark, it is equivalent to have a Borel map $x \mapsto \psi_x$ from X to $U(\tau)$ such that, for μ-almost every x,

$$\psi_{Tx}(t) = \mathrm{e}^{2\pi\mathrm{i}t}\psi_x(t), \qquad \tau\text{- a.e}(t),$$

or a Borel map $t \mapsto \varphi_t$ from $\mathbb{T}$ to $U(\mu)$ with, for τ-almost every t,

$$\varphi_t(Tx) = \mathrm{e}^{2\pi\mathrm{i}t}\varphi_t(x), \qquad \mu\text{-a.e}(x),$$

and the latter property implies clearly that τ is concentrated on $e(T)$. Conversely, when τ is concentrated on $e(T)$, the existence of such a map is given by the LEMMA 3.1.

For any positive measure $\tau \in M(\mathbb{T})$, we are led to consider the multiplication by $e^{2\pi it}$ on $U(\tau)$ as a continuous transformation S of $U(\tau)$:

$$S\varphi = e^{2\pi it}\varphi, \qquad (\varphi \in U(\tau)).$$

S is also a transformation of the subgroup $\overline{\mathbb{Z}}_1(\tau)$ (the closure of the exponentials $e^{2\pi int}$, $n \in \mathbb{Z}$, in $U(\tau)$).

LEMMA 2. — *For any positive measure $\tau \in M(\mathbb{T})$, there exists a Borel map from $U(\tau)$ onto $\overline{\mathbb{Z}}_1(t)$ which commutes with S.*

Proof. — Since $\overline{\mathbb{Z}}_1(t)$ is a closed subgroup of $U(\tau)$, the quotient group $U(\tau)/\overline{\mathbb{Z}}_1(\tau)$ is a polish group and admits a Borel cross section, according to (3.1.2). In other words, one can find a Borel map $\psi \mapsto \psi'$ from $U(\tau)$ into itself which is constant on every class modulo $\overline{\mathbb{Z}}_1(\tau)$ and such that $\psi \equiv \psi'$ mod $\overline{\mathbb{Z}}_1(\tau)$. Then we have $(S\psi)' = \psi'$. It is clear that $\chi(\psi) = \psi\psi'^{-1}$ belongs to $\overline{\mathbb{Z}}_1(\tau)$ and $\chi(S\psi) = S\chi(\psi)$.

COROLLARY. — *A positive measure $\tau \in M(\mathbb{T})$ is concentrated on $e(T)$ if and only if there exists a Borel map $x \mapsto \chi_x$ from X to $\overline{\mathbb{Z}}_1(\tau)$ such that $\chi_{Tx}(t) = e^{2\pi it}\chi_x(t)$ τ-a.e(t) for μ-almost every x.*

This is immediate from the LEMMAS 1 and 2.

Proof of the theorem 3.1. — Let τ a positive measure concentrated on $e(T)$, and let ω be another positive measure sticking to τ. By the LEMMA 2.2, we have a continuous group homomorphism $\chi \mapsto \widetilde{\chi}$ from $\overline{\mathbb{Z}}_1(\tau)$ to $\overline{\mathbb{Z}}_1(\omega)$ which maps the function $e^{2\pi it}$ of $\overline{\mathbb{Z}}_1(\tau)$ to the function $e^{2\pi it}$ of $\overline{\mathbb{Z}}_1(\omega)$, and thus commutes with the transformations S of $\overline{\mathbb{Z}}_1(\tau)$ and $\overline{\mathbb{Z}}_1(\omega)$.

By the corollary above, there exists a Borel map $x \mapsto \chi_x$ from X to $\overline{\mathbb{Z}}_1(\tau)$ such that, for μ-almost every x, $\chi_{Tx} = S\chi_x$ and consequently $\widetilde{\chi}_{Tx} = S\widetilde{\chi}_x$. Using the corollary in the other way, we conclude that the measure ω is carried by $e(T)$. This proves that $e(T)$ is saturated (THEOREM 2.2).

It remains to prove that T is non conservative when $e(T) = \mathbb{T}$. Then, we may apply the corollary with τ being the Lebesgue measure. Since $\overline{\mathbb{Z}}_1(\tau) \simeq \mathbb{Z}$ (see the remark 1 in section 2.2), $\chi_x(t) = e^{2\pi in(x)t}$ where $n(x)$ is a Borel function from X to $\mathbb{Z}$ such that $n(Tx) = n(x) + 1$, μ-a.e. Thus T is non conservative.

REMARK. — One can also use the corollary to obtain a more precise version of the main theorem in [11]. Let T be a non-singular ergodic transformation and let S be a finite measure preserving ergodic transformation with reduced maximal spectral type σ. The cartesian product $S \times T$ is ergodic if and only if $\sigma(e(T)) = 0$. We omit the proof.

According to the corollary, for each positive measure τ concentrated on $e(T)$, we have a factor $(\overline{\mathbb{Z}}_1(\tau),\nu,S)$ of the system (X,μ,T) (taking for ν the image measure of μ by the map $x \mapsto \chi_x$). In section 4, we shall give an example where, for some choice of τ, we can construct explicitly a one-to-one map $x \mapsto \chi_x$. Then we will conclude that the system is isomorphic to a compact group rotation by using the next simple result, interesting by itself.

THEOREM. — *Any non-singular dynamical system $(U(\tau),\nu,S)$ (or $(\overline{\mathbb{Z}}_1(\tau),\nu,S)$) is isomorphic to a compact group rotation.*

Proof. — It is enough to show that the σ-algebra $\mathcal{A}$ of $U(\tau)$ spanned by the Borel eigenfunctions of modulus 1 is the whole Borel σ-algebra (see [3], theorem 1.2.). Note that $\mathcal{A}$ contains the ν-null Borel sets, since eigenfunctions are only determined almost everywhere.

By (3.2.1), there exists a Borel map $t \mapsto \varphi_t$ from $\mathbb{T}$ to $U(\nu)$ with $\varphi_t(\psi) = \psi(t)$ $\nu \otimes \tau$-a.e. Then, given any $\psi_0 \in U(\tau)$,

$$\int \big|\psi(t) - \psi_0(t)\big| \,\mathrm{d}\tau(t) = \int \big|\varphi_t(\psi) - \psi_0(t)\big| \,\mathrm{d}\tau(t), \qquad \nu\text{-a.e.}(\psi).$$

For τ-almost every t, $\varphi_t(S\psi) = \mathrm{e}^{2\pi \mathrm{i} t}\varphi_t(\psi)$ holds ν-a.e.(ψ), so that φ_t is an eigenfunction of modulus 1 and thus belongs to $L^1(U(\tau),\mathcal{A},\nu)$. It follows that the integral above, as a function of ψ, belongs to $L^1(U(\tau),\mathcal{A},\nu)$ and therefore is $\mathcal{A}$-measurable.

We conclude that $\mathcal{A}$ contains the balls of $U(\tau)$ and so coincides with the Borel σ-algebra.

3.3. Group actions.

Let us consider a locally compact second countable abelian group G acting non singularly on a standard measure space (X,μ). A continuous character γ of G is called an L^∞ *eigenvalue* if there exists a non-zero function $\phi \in L^\infty(\mu)$ such that

$$\phi(T_g x) = \gamma(g)\,\phi(x), \qquad \mu\text{-a.e.},$$

for every $g \in G$. We denote $e(G)$ the set of all L^∞ eigenvalues corresponding to constant modulus eigenfunctions (all L^∞ eigenvalues in the ergodic case). $e(G)$ is a subgroup of the dual group $\widehat{G}$ which enjoys the same general properties as $e(T)$ above. The THEOREM 3.1 can be extended to group actions. We will restrict ourselves the ergodic case.

Let us recall that the action is said to be of *type I* if the measure μ is carried by some G-orbit.

THEOREM. — *Let G be a locally compact second countable abelian group, acting non-singularly and ergodically on a standard measure space (X, μ), and assume the action is not of type I. Then $e(G)$ is a proper saturated subgroup (and so a Dirichlet subgroup) of the dual group $\widehat{G}$.*

Proof. — The LEMMAS 3.2 and the THEOREM 3.1 can be readily extended. In particular when $e(G) = \widehat{G}$, one can find a Borel map $x \mapsto \chi_x$ from X to G, such that $\chi_{T_g x} = g\chi_x$ μ-a.e., for every $g \in G$. Thus X can be written (modulo a null-set) as the disjoint union of $T_g B$, $g \in G$, for some Borel set B, and it follows easily by ergodicity that μ is carried by some G-orbit.

3.4. Cocycles and T-sets.

Given a non-singular dynamical system (X, μ, T) and a locally compact second countable abelian group G (noted additively), a $\mathbb{Z}$-*cocycle* with values in G is a Borel map h from $\mathbb{Z} \times X$ to G, satisfying

$$h(m+n, x) = h(m, T^n x) + h(n, x), \qquad \mu\text{-a.e.}$$

Since h is completely defined by the Borel function $h(x) = h(1, x)$, one commonly speaks of "the cocycle $h(x)$". The cocycle is a *coboundary* if there exists a Borel function $\varphi(x)$ from X to G such that

$$h(x) = \varphi(Tx) - \varphi(x), \qquad \mu\text{-a.e.}$$

For any $\gamma \in \widehat{G}$, $\gamma \circ h$ is a cocycle with values in the circle group. If $\gamma \circ h$ is a coboundary for every $\gamma \in \widehat{G}$, then h itself is a coboundary : when $G = \mathbb{R}$ and $h = \mathrm{Log}(\mathrm{d}T\mu/\mathrm{d}\mu)$ this is a theorem of HAMACHI, OKA and OSIKAWA ([13], [14]). In [32], MOORE and SCHMIDT extend this result, showing that if h is not a coboundary, the group of $\gamma \in \widehat{G}$ such that $\gamma \circ h$ is a coboundary cannot be "too thick" : namely it must be a Dirichlet subgroup of $\widehat{G}$. One can improve these results as a direct consequence of the previous theorem.

THEOREM. — *Let (X, μ, T) be an ergodic non-singular dynamical system and let $h(x)$ be a cocycle with values in a locally compact second countable abelian group G. If h is not a coboundary, the set of all $\gamma \in \widehat{G}$ such that $\gamma \circ h$ is a coboundary, is a proper saturated subgroup (and so a Dirichlet subgroup) of $\widehat{G}$.*

Sketch of the proof. — As in [13], [14], we can associate with the cocycle h a non-singular ergodic flow S_g, $g \in G$, whose eigenvalue group is exactly the subgroup of $\gamma \in \widehat{G}$ such that $\gamma \circ h$ is a coboundary. We skip the details, referring to [14]. Actually, the proof in [14] is given for the Radon-Nikodym cocycle, but can easily be extended to the case of a general cocycle. So the result is a mere consequence of the THEOREM 3.3.

DEFINITION. — The *T-set* of a non-singular transformation T is the set of real numbers t such that $\exp(it\,\mathrm{Log}(\mathrm{d}T\mu/\mathrm{d}\mu))$ is a coboundary. It can be identified with the T-set, as defined by A. CONNES [9], of the Krieger-Von Neumann factor constructed from T (see [13]).

COROLLARY 1. — *The T-set of any ergodic non-singular dynamical system (X,μ,T) is a saturated subgroup of $\mathbb{R}$. It is a proper saturated subgroup (and so a Dirichlet subgroup) if and only if μ is not equivalent to a σ-finite T-invariant measure.*

Proof. — The additive cocycle $\mathrm{Log}(\mathrm{d}T\mu/\mathrm{d}\mu)$ is a coboundary if and only if there exists a σ-finite T-invariant measure equivalent to μ (see [13]).

COROLLARY 2. — *Let $(T_t)_{t\in\mathbb{R}}$ be an ergodic non-singular flow, not of type I. The set*

$$\{0\} \cup \{t \in \mathbb{R}\,;\ T_{1/t} \text{ is not ergodic}\}$$

is a proper saturated subgroup (and so a Dirichlet subgroup) of $\mathbb{R}$.

Proof. — The fact that we have a Dirichlet subgroup of $\mathbb{R}$ is already proved in [27]. Going back to [27], we can check that this group can be written as a countable increasing union of groups of the type considered in the theorem above (and which are saturated). Now it is an easy exercise to show that a countable increasing union of saturated subgroups is a saturated subgroup.

3.5. The $\mathbf{H_2}$ groups are eigenvalue groups.

Let H be an H_2 group, defined by the convergence of the series

$$\sum_{0}^{+\infty} a_j(1-\cos 2\pi n_j t) < +\infty.$$

H will be endowed with the translation invariant metric

$$d(t,t') = \|t-t'\| + |t-t'|,$$

where

$$\|t\|^2 = \sum_{0}^{+\infty} a_j(1-\cos 2\pi n_j t).$$

It is elementary that this metric is complete and separable. So, H is a polish group with a topology stronger than the circle topology. Moreover, as we noticed in section 2.3, it is σ-compact in the last topology.

THEOREM. — *Any H_2 group is the eigenvalue group of some ergodic non-singular compact group rotation.*

Proof. — Let H be as above. We shall assume, without loss of generality, $0 \leq a_j < 1$ for every $j \geq 0$. We also assume that H is infinite, in order to avoid trivialities. Let D be a countable dense subgroup of H in the polish topology and let $\widehat{D}$ be the dual group of D, which is a compact metrizable group containing $\mathbb{Z}$ as a dense subgroup. The group rotation T on $\widehat{D}$ will be defined by $Tx = x + 1$.

By duality, any element t of D defines a character γ_t of $\widehat{D}$, such that $\gamma_t(n) = \mathrm{e}^{2\pi i n t}$ for every $n \in \mathbb{Z}$, and the Fourier transform at t of a finite measure μ on $\widehat{D}$ is defined as $\widehat{\mu}(\tau) = \int \gamma_t \,\mathrm{d}\mu$.

Let $\nu = \mathop{*}\limits_{0}^{+\infty} \nu_j$ be the infinite convolution product of the probability measures on $\widehat{D}$

$$\nu_j = (1 - a_j)\delta + \tfrac{1}{2} a_j \,\delta_{n_j} + \tfrac{1}{2} a_j \,\delta_{-n_j}, \qquad (j \geq 0).$$

The weak $*$-convergence is guaranteed by the convergence at any $t \in D$ of the infinite product $\prod_1^{+\infty} \widehat{\nu}_j(t)$, that is

$$\prod_{0}^{+\infty} \big((1 - a_j) + a_j \cos 2\pi n_j t\big). \tag{3.5.1}$$

ν can be viewed at as the distribution of the sum $X = \sum_0^{+\infty} X_j$ in $\widehat{D}$ of a series of independent random variables X_j which take the values 0 with probability $(1 - a_j)$ and $\pm n_j$ with probability $\frac{1}{2} a_j$. In fact the convergence of (3.5.1) implies the almost sure convergence of $\prod \mathrm{e}^{2\pi i t X_j}$ for every $t \in D$ and thus the convergence a.s. of $\sum X_j$ in $\widehat{D}$ (see [36]).

We build from ν the T-quasi-invariant probability measure

$$\mu = \tfrac{1}{3} \sum_{-\infty}^{+\infty} 2^{-|k|}\, T^k \nu.$$

The ergodicity of the dynamical system $(\widehat{D}, \mu, T)$ is then an immediate consequence of the zero-one law for the random variables X_j (see [7] or [12] for the details). We claim that its eigenvalue group is exactly the group H.

Let ϕ be an non-zero eigenfunction corresponding to some eigenvalue $\mathrm{e}^{2\pi i t}$. Let us denote

$$E_j = E\Big[\phi\Big(\sum_{k=j}^{+\infty} X_k\Big)\Big], \qquad (j \geq 0).$$

For j large enough, since ϕ is non zero, the conditional expectation $E(\phi(X)\,;\ X_0,\dots,X_{j-1})$ cannot vanish identically, and we can find integers $m_0,\dots,m_{j-1}$ such that

$$0 \neq E\Big[\phi\Big(\sum_{k<j} m_k + \sum_{k=j}^{+\infty} X_k\Big)\Big] = E\Big[\exp\Big(2\pi it \sum_{k<j} m_k\Big)\phi\Big(\sum_{k=j}^{+\infty} X_k\Big)\Big]$$
$$= \exp\Big[2\pi it \sum_{k<j} m_k\Big] E_j.$$

This proves $E_j \neq 0$ for j large enough. We also have the induction relation

$$\begin{aligned} E_j &= \big((1-a_j) + \tfrac{1}{2}a_j\,\mathrm{e}^{2\pi\mathrm{i}n_jt} + \tfrac{1}{2}a_j\,\mathrm{e}^{-2\pi\mathrm{i}n_jt}\big)E_{j+1} \\ &= \big((1-a_j) + a_j\cos 2\pi n_jt\big)E_{j+1} \end{aligned}$$

and thus, for every j,

$$E_0 = \Big[\prod_{k<j} \big((1-a_k) + a_k\cos 2\pi n_kt\big)\Big]E_j.$$

Therefore $E_0 \neq 0$ and, since $|E_j| \leq 1$ for all j, it follows that the infinite product (3.5.1) converges and so $t \in H$.

Conversely, we prove now that for every $t \in H$, $\mathrm{e}^{2\pi\mathrm{i}t}$ is an eigenvalue. Let us first consider a sequence t_k in D which converges to 0 in the polish group topology of H. Then, the infinite product (3.5.1) converges to 1, that is $\widehat{\nu}(t_k)$ tends to 1 and

$$\int |1-\gamma_{t_k}|^2\,\mathrm{d}\nu = 2\big(1-\mathrm{Re}(\widehat{\nu}(t_k)\big) \longrightarrow 1.$$

Moreover, as t_k converges to 0 in the circle topology, for every $n \in \mathbb{Z}$, $\mathrm{e}^{2\pi\mathrm{i}nt_k}$ converges to 1 and, as $\gamma_{t_k}(x+n) = \mathrm{e}^{2\pi\mathrm{i}nt_k}\gamma_{t_k}(x)$,

$$\int \big|1-\gamma_{t_k}\big|^2\,\mathrm{d}(T^n\nu) = \int \big|1-\mathrm{e}^{2\pi\mathrm{i}nt_k}\gamma_{t_k}\big|^2\,\mathrm{d}\nu \longrightarrow 1.$$

This proves that γ_{t_k} converges to 1 in $L^2(\mu)$.

Now, any $t \in H$ is the limit in the polish topology of a sequence t_k of elements of D. Then the corresponding sequence of characters γ_{t_k} converges in $L^2(\mu)$ to a measurable function χ_t of unit modulus. Since $\mathrm{e}^{2\pi\mathrm{i}t_k}$ converges to $\mathrm{e}^{2\pi\mathrm{i}t}$, $\chi_t(x+1) = \mathrm{e}^{2\pi\mathrm{i}t}\chi_t(x)$ μ-a.e. follows and we conclude that $\mathrm{e}^{2\pi\mathrm{i}t}$ is an eigenvalue.

4. Towers over the 2-odometer

We will consider now classical examples of non-singular dynamical systems (X, μ, T) given by a Kakutani tower over the 2-odometer (see [1], [3], [19], [33]). We will prove two new results for these systems, both involving the measures called *Riesz products.* Under some condition on the height function, we will show that the corresponding tower is isomorphic to a non-singular compact group rotation, by means of the technique introduced in section 3.2 : we will build a special Riesz product τ on $e(T)$, and then an explicit isomorphism from X to $\overline{\mathbb{Z}}_1(\tau)$. Besides, we will compute in full generality the maximal spectral types of these systems, which turn out to be classical Riesz products.

We begin with a brief description of the systems and the definition of Riesz products.

4.1. Definition.

Let $\Omega = \{0,1\}^N$, identified to the group of dyadic integers, let S be the transformation of Ω defined by $S\omega = \omega + 1$, and, given any two sequences $p_j > 0$ and $q_j > 0$ such that $p_j + q_j = 1$, let ν be the product measure

$$\bigotimes_{j=0}^{+\infty}(p_j\delta_0 + q_j\delta_1).$$

Then ν is S-quasi-invariant and ergodic. Moreover ν is non-atomic if and only if

$$\sum \min(p_j, q_j) = +\infty.$$

From now on we will assume that this condition holds.

Next, let $h(\omega)$ be a measurable positive integer-valued function on Ω. Let us consider the subspace $X \subset \Omega \times \mathbb{N}$, of elements (ω, n) such that $1 \leq n \leq h(\omega)$, and let T be the measurable transformation of X defined by

$$T(\omega, n) = \begin{cases} (\omega, n+1) & \text{if } 1 \leq n < h(\omega), \\ (\omega+1, 1) & \text{if } n = h(\omega). \end{cases}$$

We identify Ω with $\Omega \times \{1\}$ and extend the measure ν to a σ-finite measure μ on X by setting

$$\int_X f(\omega, n)\,\mathrm{d}\mu(\omega, n) = \int_\Omega \Big[\sum_{n=1}^{h(\omega)} f(\omega, n)\Big]\,\mathrm{d}\nu(\omega)$$

for every positive Borel function $f(\omega, n)$ on X. Then T is invertible, and μ is non-atomic, T-quasi-invariant and ergodic. The ergodic, conservative, non-singular dynamical system (X, μ, T) is called a "Kakutani tower over Ω" (Ω is called the *basis* of the tower and $h(\omega)$ the *height function*).

Now, we choose the height function as follows. Let be given a sequence n_j of positive integers such that

$$n_k > \sum_{j<k} n_j, \qquad (k \geq 0). \tag{4.1.1}$$

For $\omega \in \Omega$, let us denote $k(\omega)$ the smallest index k such that $\omega_k = 0$ and

$$h(\omega) = n_{k(\omega)} - \sum_{j<k(\omega)} n_j.$$

Strictly speaking, $h(\omega)$ is not defined for $\omega = (1, 1, 1, \ldots)$ and we shall have to except all the ω with only finitely many coordinates equal to 0, or to 1 (*i.e.* the integers), but this does not matter since the measure ν has no atom. Since the value of $(\omega+1)_j - \omega_j$ is -1 for $j < k(\omega)$, 1 for $j = k(\omega)$ and 0 for $j > k(\omega)$, we have

$$h(\omega) = \sum_{j \geq 0} n_j \big((\omega + 1)_j - \omega_j \big). \tag{4.1.2}$$

From the definition of T we get that, for any $\omega \in \Omega$ and $m \in \mathbb{N}$,

$$\omega + m = T^{h(\omega)+h(\omega+1)+\cdots+h(\omega+m)}\omega = T^{\Sigma n_j ((\omega + m)_j - \omega_j)}\omega. \tag{4.1.3}$$

REMARK. — When n_{j+1} is a multiple of n_j, say $n_{j+1} = m_j n_j$ for every $j \geq 1$, X can be embedded in the adding machine

$$\prod_{j \geq 0} \{0, \ldots, m_j - 1\}$$

in such a way that T matches the mapping $x \mapsto x + 1$ of this group : this is immediate from (4.1.2) and (4.1.3). So, in this case, the system is isomorphic to a non-singular compact group rotation.

It is known that the group $e(T)$ of L^∞ eigenvalues of the dynamical system (X, μ, T) is an H_2 group ([3], theorem 2.2) :

THEOREM. — *The group of L^∞ eigenvalues of the dynamical system (X,μ,T) is the H_2 group of all $t \in \mathbb{T}$ such that*

$$\sum_0^{+\infty} p_j\, q_j \left|1 - \mathrm{e}^{2\pi\mathrm{i} n_j t}\right|^2 < +\infty.$$

4.2. Riesz products.

A generalized Riesz product is a probability measure defined as an infinite product

$$\prod_0^{+\infty} P_j(n_j\, t)$$

where the P_j are nonnegative trigonometric polynomials

$$\sum_{|k|\leq m_j} c_j(k)\, \mathrm{e}^{2\pi\mathrm{i} kt}$$

of integral 1 and the convergence is the weak $*$-convergence of measures. In order to guarantee the convergence, we have to make some lacunarity assumption. A sufficient condition is

$$n_k > 2 \sum_{j<k} m_j\, n_j, \qquad (j \geq 1). \tag{4.2.1}$$

This implies that, in the expansion of the finite products, all the frequencies are distinct (we say that the polynomials P_j are dissociate) and from this fact it is elementary to compute their Fourier coefficients and to prove the convergence (*cf.* [18], p. 176). The weak $*$-limit is a probability measure τ whose Fourier transform is explicitly known : namely $\widehat{\tau}(n) = 0$ except if n is a finite sum $\sum k_j\, n_j$, with $k_j \in \mathbb{Z}$ and $|k_j| \leq m_j$, in which case

$$\widehat{\tau}\Big(\sum k_j n_j\Big) = \prod c_j(-k_j) = \prod \widehat{\tau}(k_j\, n_j). \tag{4.2.2}$$

In particular $\widehat{\tau}(n_j) = c_j(-1)$.

It is proved in [18] that such a measure τ is continuous, in a more general setting (see theorem 5, p 184). In the case $m_j = 1$ for every j, we have the standard Riesz product

$$\prod_0^{+\infty} \left(1 + c_j\, \mathrm{e}^{2\pi\mathrm{i} n_j t} + \bar{c}_j\, \mathrm{e}^{-2\pi\mathrm{i} n_j t}\right) = \prod_0^{+\infty} \left(1 + 2\,\mathrm{Re}(c_j\, \mathrm{e}^{2\pi\mathrm{i} n_j t})\right).$$

The following construction was originally given by one of the authors in a study of the ergodic properties of Riesz products [34].

LEMMA. — *Let a_j be a sequence of nonnegative real numbers and n_j a sequence of positive integers such that*

$$\sum_0^{+\infty} a_j\Big(\frac{n_j}{n_{j+1}}\Big)^2 < +\infty.$$

Then there exists a symmetric generalized Riesz product τ concentrated on the H_2 group $\sum_0^{+\infty} a_j(1-\cos 2\pi n_j t) < +\infty$ and this group is uncountable.

Proof. — We will choose for m_j the integral part of $(n_{j+1}-n_j)/(2n_j)$, so that (4.2.1) clearly holds, and

$$P_j(t) = \frac{2}{m_j+2}\Big|\sum_1^{m_j+1} \sin\frac{k\pi}{m_j+2}\,\mathrm{e}^{2\pi ikt}\Big|^2.$$

A simple computation gives that

$$c_j(0) = 1 \quad\text{and}\quad c_j(1) = c_j(-1) = \cos\Big(\frac{\pi}{m_j+2}\Big)\cdot$$

(Actually this is the best possible choice for a positive trigonometric polynomial with frequencies in $[-m_j, m_j]$ if we want that $|1-c_j(1)|$ be minimal.) By (4.2.2), $\widehat{\tau}(n_j) = c_j(-1)$ whenever $m_j > 0$, *i.e.* $n_{j+1} \geq 3n_j$, and we have $1-\widehat{\tau}(n_j) = \mathrm{O}((n_j/n_{j+1})^2)$, whence

$$\sum_0^{+\infty} a_j\big(1-\widehat{\tau}(n_j)\big) < +\infty \tag{4.2.3}$$

and

$$\sum_0^{+\infty}\int a_j(1-\cos 2\pi n_j t)\,\mathrm{d}\tau(t) < +\infty.$$

It follows that $\sum_0^{+\infty} a_j(1-\cos 2\pi n_j t) < +\infty$ τ-a.e. and thus τ is concentrated on the H_2 group. Since τ is continuous, this group cannot be countable.

4.3. Isomorphism with a compact group rotation.

THEOREM. — (*With the notations of* 4.1.) *Assume*

$$\sum_0^{+\infty} p_j\, q_j\Big(\frac{n_j}{n_{j+1}}\Big)^2 < +\infty.$$

Then the dynamical system (X, μ, T) is isomorphic to a non-singular compact group rotation.

Proof. — We keep the notations of section 4.2 and use the LEMMA 4.2 with $a_j = p_j q_j$. Let τ be the generalized Riesz product constructed there. τ is concentrated on $e(T)$ (THEOREM 4.1) and, according to the COROLLARY 3.2, there exists a Borel map $x \mapsto \chi_x$ from X to $\overline{\mathbb{Z}}_1(\tau)$ such that, for μ-almost every x,

$$\chi_{Tx} = e^{2\pi i t}\chi_x. \tag{4.3.1}$$

Here we are going to build explicitly such a mapping on a Borel subset of full measure in X and show that it is one-to-one. So, by the THEOREM 3.2, it will provide an isomorphism of the given dynamical system with an ergodic non-singular group rotation.

For every $j \geq 0$, let α_j be 0 if $p_j \geq \frac{1}{2}$ and 1 if $p_j < \frac{1}{2}$. For every $x = (\omega, n)$ in X, let $\varepsilon_j = \varepsilon_j(x) = \omega_j - \alpha_j$.

We observe from the lacunarity condition, that n_{j+1}/n_j cannot be bounded. Indeed, we constantly assume that the sum

$$\sum \min(p_j, q_j) \leq \sum 2p_j\, q_j$$

is infinite (4.1) and, given any positive constant C, the sum $\sum p_j\, q_j$ restricted to the set of all j with $n_{j+1}/n_j < C$ is finite . As the probability to have $\varepsilon_j = \omega_j - \alpha_j \neq 0$ is $\min(p_j, q_j)$, it also follows that for almost every x there exist only a finite number of j with $n_{j+1}/n_j < C$ and $\varepsilon_j \neq 0$.

We shall first restrict the definition of χ_x to the set X_0 of all $x = (\omega, n)$ satisfying this property. Then, the integers m_j in the definition of the Riesz product τ being chosen as in the LEMMA 4.2, we have $m_j = 0$ only if $n_{j+1}/n_j < 3$, and there exist only finitely many j with $\varepsilon_j \neq 0$ and $m_j = 0$, that is $|\varepsilon_j| > m_j$.

We define formally, for $x \in X_0$ and $t \in \mathbb{T}$,

$$\chi_x(t) = e^{2\pi i n t} \prod_{0}^{+\infty} e^{2\pi i \varepsilon_j n_j t} \tag{4.3.2}$$

and we have to show that this infinite product converges in $\overline{\mathbb{Z}}_1(\tau)$ for μ-almost every x, *i.e.* for ν-almost every ω. The limit will satisfy the relation (4.3.1), obviously if $n < h(\omega)$, and by the formula (4.1.2) when $n = h(\omega)$.

For a given ω, for $0 \leq k < k'$, we have

$$\int \Big|1 - \prod_{k}^{k'} e^{2\pi i \varepsilon_j n_j t}\Big|^2 \mathrm{d}\tau = 2 - 2\,\mathrm{Re}\Big[\widehat{\tau}\Big(\sum_{k}^{k'} \varepsilon_j\, n_j\Big)\Big].$$

$\widehat{\tau}$ is real and $\widehat{\tau}(\sum_k^{k'} \varepsilon_j n_j) = \prod_k^{k'} \widehat{\tau}(\varepsilon_j n_j)$ by (4.2.2) if $|\varepsilon_j| \leq m_j$ for $k \leq j \leq k'$. Since $x \in X_0$, this condition holds for k large enough; we have then

$$\int \Big|1 - \prod_k^{k'} \mathrm{e}^{2\pi \mathrm{i}\varepsilon_j n_j t}\Big|^2 \mathrm{d}\tau = 2 - 2\prod_k^{k'} \widehat{\tau}(\varepsilon_j n_j)$$

and the infinite product (4.3.2) converges in $L^2(\tau)$ if and only if the series $\sum_0^{+\infty}(1 - \widehat{\tau}(\varepsilon_j n_j))$ converges.

Now, $\sum_0^{+\infty}(1 - \widehat{\tau}(\varepsilon_j n_j))$ converges for ν-almost every ω if the series $\sum_0^{+\infty} E[1 - \widehat{\tau}(\varepsilon_j n_j)]$ converges. According to the choice we made for α_j, we have

$$E\big[1 - \widehat{\tau}(\varepsilon_j n_j)\big] = q_j\big(1 - \widehat{\tau}(n_j)\big) \qquad \text{if} \quad p_j \geq \tfrac{1}{2},$$
$$E\big[1 - \widehat{\tau}(\varepsilon_j n_j)\big] = p_j\big(1 - \widehat{\tau}(-n_j)\big) \quad \text{if} \quad p_j < \tfrac{1}{2},$$

whence

$$\sum_0^{+\infty} E\big[1 - \widehat{\tau}(\varepsilon_j n_j)\big] \leq 2\sum_0^{+\infty} p_j\, q_j\big(1 - \widehat{\tau}(n_j)\big),$$

and this series converges according to (4.2.3). This proves that the set X_1 in X_0 where the infinite product (4.3.2) converges in $L^2(\tau)$ has full measure.

It remains to show that the mapping $x \mapsto \chi_x$, from X_1 into $\overline{\mathbb{Z}}_1(\tau)$, is one-to one. Let $x = (\omega, n)$ and $x' = (\omega', n')$ be two elements of X_1 with $\chi_{x'} = \chi_x$ and let

$$r_k = n' - n + \sum_0^k (\omega'_j - \omega_j) n_j, \qquad (k \geq 0).$$

From the definition of χ_x and $\chi_{x'}$, $\mathrm{e}^{2\pi \mathrm{i} r_k t}$ converges to 1 in $\overline{\mathbb{Z}}_1(\tau)$ and thus $\widehat{\tau}(r_k) \to 1$ as $k \to +\infty$. Since $x, x' \in X_0$, we may choose an integer k such that $\widehat{\tau}(r_k) \neq 0$ and which moreover satisfies

$$n_{k+1} > 4n_k + 2|n' - n|, \tag{4.3.3}$$

$$\omega'_j = \omega_j = \alpha_j \ \text{ for all } j > k \text{ such that } m_j = 0. \tag{4.3.4}$$

As $\widehat{\tau}(r_k) \neq 0$, r_k may be written as a finite sum $r_k = \sum \beta_j n_j$ where $|\beta_j| \leq m_j$ for all j. (4.3.3) implies $|r_k| < \frac{1}{2} n_{k+1}$ and, by (4.2.1), $\sum_{j<k'} |\beta_j| n_j < \frac{1}{2} n_{k'}$ for every $k' \geq 0$. It follows that β_j must be 0 for all $j > k$. Thus, for all $k' \geq k$, $r_{k'}$ is the sum $\sum_0^k \beta_j n_j + \sum_{k+1}^{k'} (\omega'_j - \omega_j) n_j$, where each coefficient has absolute value $\leq m_j$, so that

$$\widehat{\tau}(r_{k'}) = \prod_0^k \widehat{\tau}(\beta_j n_j) \prod_{k+1}^{k'} \widehat{\tau}\big((\omega'_j - \omega_j) n_j\big).$$

Since the product converges to 1 as $k' \to +\infty$, all the coefficients β_j ($j \leq k$) and $\omega'_j - \omega_j$ ($j > k$) must be zero. In other words $\omega' - \omega$ is the integer $\sum_0^k (\omega'_j - \omega_j) 2^j$ and

$$n - n' = \sum_0^k (\omega'_j - \omega_j) n_j.$$

Therefore, by the formula (4.1.3), $T^{n-n'}\omega = \omega'$ and finally $x = T^n \omega = T^{n'} \omega' = x'$.

REMARK. — In fact, there is a great latitude in the construction of χ_x. Alternately, we could choose the α_j at random, independently, and independently from the ω_j, with probabilities p_j for $\alpha_j = 0$ and q_j for $\alpha_j = 1$. See also [18] for a complete description of $\overline{\mathbb{Z}}_1(\tau)$.

4.4. Maximal spectral type.

The basic notions and results of spectral theory that we shall use can be found in [38]. By the maximal spectral type of the system (X, μ, T), we mean the maximal spectral type of the unitary operator $U = U_T$ of $L^2(\mu)$ associated with the transformation T, defined by

$$Uf = \Big(\frac{\mathrm{d}T^{-1}\mu}{\mathrm{d}\mu}\Big)^{1/2} f \circ T, \qquad \big(f \in L^2(\mu)\big).$$

In the case of the tower over the 2-odometer, it turns out that the maximal spectral type can be computed explicitly in terms of the sequences p_j, q_j, and n_j. The next theorem is new, as far as we know (see [25], [38] for similar results when $h(\omega) = 0$). We shall have a Riesz product

$$\prod (1 + b_j \cos 2\pi n_j t)$$

where the dissociation condition (4.2.1) (with $m_j = 1$) is not satisfied. Indeed we only assume the condition (4.1.1) : $n_k > \sum_{j<k} n_j$. With this weaker condition, it is still possible to prove the weak $*$-convergence of the product, but we lose the formula (4.2.2) for the Fourier coefficients and the fact that the Riesz product must be continuous (however in our case the convergence will follow from the proof).

THEOREM. — *(With the above notations.) The spectral multiplicity of the system (X, μ, T) is 1 and its maximal spectral type, up to a discrete measure, is the Riesz product*

$$\prod_{j=0}^{+\infty}\big(1 + 2(p_j\, q_j)^{1/2} \cos 2\pi n_j t\big). \tag{4.4.1}$$

Proof. — Remind that we identify Ω with $\Omega \times \{1\}$ and the measure ν on Ω with the restriction of μ to $\Omega\times\{1\}$. For every $k \geq 0$, given $\varepsilon_0, \dots, \varepsilon_{k-1}$ with values 0 or 1, we denote $[\varepsilon_0, \dots, \varepsilon_{k-1}]$ the set of all $\omega = (\omega_j) \in \Omega$ such that $\omega_0 = \varepsilon_0, \dots, \omega_{k-1} = \varepsilon_{k-1}$; such a set will be called a *k-cylinder* in Ω. Let us denote Ω_k the k-cylinder $[0, \dots, 0]$ $(\Omega_0 = \Omega)$, f_k the normalized function

$$f_k = \nu(\Omega_k)^{-1/2} 1_{\Omega_k} \quad \text{and} \quad \sigma_k = \sigma_{f_k}.$$

Given any $k \geq 0$ and any choice of $\varepsilon_0, \dots, \varepsilon_{k-1}$ with values 0 or 1, let us consider $m = \sum_0^{k-1} \varepsilon_j\, 2^j$ and $n = \sum_0^{k-1} \varepsilon_j\, n_j$. We have, for every $\omega \in \Omega_k$, $(\omega + m)_j = \varepsilon_j$ for $1 \leq j < k$, $(\omega + m)_j = \omega_j$ for $j \geq k$ and thus, by (4.1.3), $\omega + m = T^n\omega$. So, $[\varepsilon_0, \dots, \varepsilon_{k-1}]$ is the image of Ω_k under T^n and the Radon-Nikodym derivative of $T^n\mu$ with respect to μ is constant on $[\varepsilon_0, \dots, \varepsilon_{k-1}]$. It follows that $U^{-n} f_k$ is the normalized characteristic function of $[\varepsilon_0, \dots, \varepsilon_{k-1}]$.

This shows that the subspace spanned by the $U^{-n} f_k$ contains all the functions in $L^2(\mu)$ which vanish out of Ω, and it is clear that these functions span $L^2(\mu)$ under U. Moreover, as Ω_k is the union of Ω_{k+1} and of the $(k+1)$-cylinder $[0, ..., 0, 1] = T^{n_k}\Omega_{k+1}$, f_k is a combination of f_{k+1} and $U^{-n_k} f_{k+1}$. So, f_k belongs to the span of the iterates of f_{k+1}, and $L^2(\mu)$ is spanned by a non-decreasing sequence of cyclic subspaces. This proves that the spectral multiplicity is 1. Also, the maximal spectral type is determined by the σ_k.

More precisely, as $\nu(\Omega_{k+1}) = p_k \nu(\Omega_k)$ and $\nu(T^{n_k}\Omega_{k+1}) = q_k \nu(\Omega_k)$, we have

$$f_k = p_k^{1/2} f_{k+1} + q_k^{1/2} U^{-n_k} f_{k+1}, \qquad (k \geq 0). \tag{4.4.2}$$

For every $k \geq 0$, let N_k denotes the set of all sums $\sum_0^{k-1} \varepsilon_j\, n_j$ with $\varepsilon_j = 0$ or 1 $(1 \leq j < k)$. Iterating (4.4.2), we get the expansion

$$f_0 = \Big[\prod_0^{k-1} (p_j^{1/2} I + q_j^{1/2} U^{-n_j})\Big] \cdot f_k = \sum_{n \in N_k} c_k(n)\, U^{-n} f_k$$

(the functions $c_k(n)\,U^{-n}f_k$, $n \in N_k$, are merely the characteristic functions of the k-cylinders).

The spectral measure σ_0 is given by

$$\begin{aligned}(4.4.3)\qquad \widehat{\sigma}_0(n) &= \langle U^n f_0, f_0\rangle \\ &= \sum_{r,s\in N_k} c_k(r)\, c_k(s)\, \langle U^{n-r} f_k, U^{-s} f_k\rangle, \qquad (n \in \mathbb{Z}).\end{aligned}$$

Substituting $\widehat{\sigma}_k(n - r + s)$ for $\langle U^{n-r} f_k, U^{-s} f_k\rangle$ yields the relation $\sigma_0 = P_k \sigma_k$, where

$$P_k(t) = \Big|\sum_{n\in N_k} c_k(n)\mathrm{e}^{-2\pi int}\Big|^2 = \prod_0^{k-1}\big(1 + 2(p_j\, q_j)^{1/2}\cos 2\pi n_j t\big).$$

Since a trigonometric polynomial has only a finite number of zeros, σ and σ_k are equivalent up to finitely many point masses and the continuous part of the maximal spectral type is the continuous part of σ_0. We claim that σ_0 is the Riesz product (4.4.1), that is the weak $*$-limit of the P_k. We have to prove $\widehat{\sigma}_0(n) = \lim_{k\to+\infty}\widehat{P}_k(n)$ for all $n \in \mathbb{Z}$.

Let $n \in \mathbb{Z}$ be given. We may identify the expansion (4.4.3) with the decomposition

$$\begin{aligned}\widehat{\sigma}_0(n) &= \int\Big(\frac{\mathrm{d}T^{-n}\mu}{\mathrm{d}\mu}\Big)^{1/2} 1_{T^{-n}\Omega}\, 1_\Omega\, \mathrm{d}\mu \\ &= \sum_{r,s\in N_k}\int\Big(\frac{\mathrm{d}T^{-n}\mu}{\mathrm{d}\mu}\Big)^{1/2} 1_{T^{r-n}\Omega_k}\, 1_{T^s\Omega_k}\, \mathrm{d}\mu.\end{aligned}$$

Let $\omega \in \Omega$ with $T^n\omega \in \Omega$. Necessarily $T^n\omega = \omega + m$ for some integer m and then $n = \sum n_j[(\omega+m)_j - \omega_j]$ by (4.1.3). When k is large enough, $\omega_j = (\omega+m)_j$ for all $j \geq k$ and then, if r and s are the elements of N_k such that $\omega \in T^s\Omega_k$ and $T^n\omega \in T^r\Omega_k$, we find $r = s + n$. This proves that $\widehat{\sigma}_0(n)$ is the limit as k tends to infinity of the integral restricted to the union of the $T^{r-n}\Omega_k \cap T^s\Omega_k$ with $r, s \in N_k$ and $r = s + n$. Therefore,

$$\widehat{\sigma}_0(n) = \lim_{k\to+\infty}\sum_{s\in N_k\cap N_{k-n}} c_k(s+n)\, c_k(s) = \lim_{k\to+\infty}\widehat{P}_k(n).$$

Remark. — The maximal spectral type has point masses if and only if the dynamical system is of type II_1, that is to say if there exists a T-invariant finite measure μ' equivalent to μ (indeed, a point mass in the maximal spectral type arises if and only if there exists some eigenfunction ϕ in $L^2(\mu)$ for U, and then $|\phi|^2\mu$ is a finite invariant measure equivalent to μ).

It is immediate that μ is equivalent to a (possibly infinite) T-invariant σ-finite measure μ' if and only if ν is equivalent to the equidistributed product measure ν' on Ω. Using Kakutani's criterion (see [21], [6]), this happens if and only if $\sum(\frac{1}{2} - p_j)^2 < +\infty$. Then,

$$\|\mu'\| = \int_\Omega h(\omega)\,\mathrm{d}\nu'(\omega) = \sum_0^{+\infty} 2^{-(k+1)}\Big(n_k - \sum_{j<k} n_j\Big)$$

and this sum is finite if and only if $n_k = \mathrm{O}(2^k)$.

In all other cases, the maximal spectral type is the Riesz product σ_0 and $L^2(\mu)$ is spanned by the iterates of $f_0 = 1_\Omega$.

5. Quasi-invariance and ergodicity under translations

5.1. The group of quasi-invariance of a measure.

As before we restrict the discussion to the case of $\mathbb{T}$ although everything stays valid for any metrizable locally compact abelian group.

DEFINITION. — The group of quasi-invariance of a positive measure $\mu \in M(\mathbb{T})$ is the group

$$H(\mu) = \{t \in \mathbb{T}\ ;\ \delta_t * \mu \sim \mu\}$$

In section 5.2 we shall give the example of the standard Riesz products (defined in section 4.2) $\rho = \prod_0^{+\infty}(1 + 2\,\mathrm{Re}(c_j\,\mathrm{e}^{2\pi \mathrm{i} n_j x}))$, where the positive integers n_j satisfy the dissociation condition $n_k > 2\sum_{j<k} n_j$ $(k \geq 1)$. Then $H(\rho)$ is the H_2-group $\{t \in \mathbb{T}\ ;\ \sum_0^{+\infty} |c_j|^2|1 - \mathrm{e}^{2\pi \mathrm{i} n_j t}|^2 < +\infty\}$.

The main property of the group of quasi-invariance of a measure, which will be obtained in section 5.4, is the following :

For any positive measure $\mu \in M(\mathbb{T})$, $H(\mu)$ is a saturated subgroup of $\mathbb{T}$. It is a proper subgroup whenever μ is singular.

This result has already been obtained by J. AARONSON and M. NADKARNI [3] with the assumption that μ is $H(\mu)$-ergodic. The methods we will use, although developed independently, are not very different from theirs. But we get rid of any ergodicity condition and we will prove moreover a general theorem on the extension of cocycles, which provides a new class of saturated subgroups of the circle containing the groups $H(\mu)$

for every measure $\mu \in M(\mathbb{T})$. The problem of extension of a multiplicative cocycle has been paid attention by many authors, specially for our concern by HELSON and MERRILL [16], [17].

The proof, which makes clear the spectral theoretic nature of this problem, will allow us, in section 7, to deduce some new and rather surprising results in the Fourier-Gelfand theory of the convolution algebra $M(\mathbb{T})$.

In the present section we will be interested in finding the general properties of $H(\mu)$, without any restriction on the measure μ. The assumption of ergodicity is not relevant as far as we are concerned with the harmonic analysis properties of μ. On the other hand, it is interesting to study the measures which are ergodic under translation subgroups. But then, as we shall see in section 5.5 and in order to include classical examples, we will be led to drop any *a priori* assumption of quasi-invariance and to consider measures μ for which the group $H(\mu)$ can pretty well be trivial. The significant object in this context is the set

$$A(\mu) = \{t \in \mathbb{T}\,;\ \delta_t * \mu \not\perp \mu\}.$$

We shall establish for $A(\mu)$ properties close to those of $H(\mu)$.

For every positive measure $\mu \in M(\mathbb{T})$, we denote $L(\mu)$ the (closed) subspace of $M(\mathbb{T})$ of all measures ν such that $|\nu| \ll \mu$. For $h \in H(\mu)$, we denote T_h the non-singular translation by h on $(\mathbb{T}, \mu)$ and the corresponding operator on $L(\mu)$. We define the strong topology of $H(\mu)$ as the strong topology of operators on $L(\mu)$: a sequence h_n tends to h in this topology if and only if $T_{h_n}\nu$ tends to $T_h\nu$ in norm for every $\nu \in L(\mu)$. As we can take for ν a measure with an arbitrary small support, h_n must converge to h in $\mathbb{T}$. So, the strong topology of $H(\mu)$ is stronger than the usual topology; moreover, it is clearly polish. According to (3.1.1) and (3.1.2), this implies :

(5.1.1) *$H(\mu)$ is Borel and the topology of the action on $L(\mu)$ is the unique polish group topology on $H(\mu)$, compatible with its Borel structure inherited from $\mathbb{T}$.*

From the uniqueness, the following characterization is immediate :

A sequence h_n converges to h in the polish topology of $H(\mu)$ if and only if h_n tends to h in the usual topology, and $T_{h_n}\mu$ tends to $T_h\mu$ in norm.

On the other hand, it is known that $A(\mu)$ is a Borel set and that it has zero Lebesgue measure when μ is purely singular (see [12], chap. 8.3). It follows that $H(\mu)$ is a proper subgroup whenever μ is singular, but we have a stronger result :

(5.1.2) *If μ is continuous and non-equivalent to the Lebesgue measure, $\mu(x+H(\mu))=0$ for all $x\in\mathbb{T}$.*

As pointed out in [29], this follows from a theorem of MACKEY ([28], p. 146) which states that, if a polish group H carries a non-zero H-quasi-invariant measure, then its topology must be locally compact. In the case of a subgroup of $\mathbb{T}$ then, either H is finite or countable, or $H=\mathbb{T}$. A different proof, avoiding Mackey's theorem, can be found in [31].

If H is an uncountable subgroup of $H(\mu)$ we say that μ is H-ergodic if $\mu(B)=0$ or $\mu(B)=\|\mu\|$ for every Borel set B which is H-invariant modulo null sets (*i.e.* such that, for every $h\in H$, T_hB is equal to B up to a μ-null set). We shall need the following remark :

(5.1.3) *If a positive measure μ is H-ergodic for some subgroup H of $H(\mu)$, it is D-ergodic for any countable subgroup D of H which is dense in H in the polish topology of $H(\mu)$.*

Indeed, let B be a D-invariant Borel set of $\mathbb{T}$. The space of all measures absolutely continuous with respect to μ and concentrated on B is closed and D-invariant. Referring to the definition of the polish topology of $H(\mu)$, B is also H-invariant modulo null sets.

5.2. Quasi-invariance and ergodicity of Riesz products.

All the statements in the next theorem have already been proved by means of harmonic analysis ([6], [37], [34]), but we want to point out the interplay between the behaviour of Riesz products with respect to translations and the results of section 4.

THEOREM

Let n_j $(j\geq 0)$ be a sequence of positive integers with $n_k>2\sum_{j<k}n_j$, $(k\geq 1)$, and let c_j $(j\geq 0)$ be any sequence of complex coefficients of modulus ≤ 1. Let ρ denote the Riesz product $\prod_0^{+\infty}(1+2\operatorname{Re}(c_j\,\mathrm{e}^{2\pi\mathrm{i}n_jx}))$.

(a) *$H(\rho)$ is the H_2 group*

$$H=\Big\{t\in\mathbb{T}\,;\ \sum_0^{+\infty}|c_j|^2\,|1-\mathrm{e}^{2\pi\mathrm{i}n_jt}|^2<+\infty\Big\}.$$

(b) *If we assume either that n_j divides n_{j+1} for every $j\geq 0$, or that*

$$\sum_0^{+\infty}|c_j|^2\Big(\frac{n_j}{n_{j+1}}\Big)^2<+\infty, \tag{5.2.1}$$

then ρ is H-ergodic.

Proof. — The translate of ρ by t is clearly the Riesz product

$$\prod_{0}^{+\infty}\left(1+2\operatorname{Re}(c_j\,\mathrm{e}^{-2\pi i n_j t}\,\mathrm{e}^{2\pi i n_j x})\right).$$

The criterion of Brown and Moran [6], and Peyrière [37] asserts that two Riesz products, constructed with the same sequence n_j (satisfying $n_k > 2\sum_{j<k} n_j$, $k \geq 1$), and coefficients c_j and c'_j respectively, are mutually singular if $\sum_0^{+\infty} |c_j - c'_j|^2 = +\infty$. Thus $\delta_t * \rho \perp \rho$ if $\sum_0^{+\infty} |c_j|^2 |1 - \mathrm{e}^{2\pi i n_j t}|^2 = +\infty$. In particular $H(\rho) \subset A(\rho) \subset H$.

The same criterion yields the equivalence of the Riesz products if $\sum |c_j - c'_j|^2 < +\infty$, under some extra condition which is satisfied in particular if $\limsup_{j\to+\infty} |c_j| < 1$ (see Kilmer and Saeki [22] for recent improvements). This proves $H \subset H(\rho)$ under some restriction. In [34], this restriction is dropped, and the ergodicity is proved under the condition (5.2.1). When n_j divides n_{j+1}, a previous result of Brown and Moran (see [12]) asserts that ρ is D-ergodic, where D is the group generated by the $1/n_j$ (obviously contained in H).

Now, in the case when the coefficients c_j are positive, we can give an alternative proof, except for the inclusion $H(\rho) \subset H$. According to the Theorem 4.4, the Riesz product ρ can be interpreted as the maximal spectral type of the tower over the 2-odometer defined in 4.1, with the same sequence n_j, when choosing p_j and q_j such that $2(p_j\,q_j)^{1/2} = c_j$. In 4.1 we have made the general assumption that $\sum \min(p_j, q_j) = +\infty$, equivalent to $\sum c_j^2 = +\infty$, but we may restrict to that case : otherwise ρ is equivalent to the Lebesgue measure (see [18], p. 127), H is equal to $\mathbb{T}$ and the Lebesgue measure is H-ergodic.

Under either assumption in (b) the tower is isomorphic to a non-singular ergodic compact group rotation, according to the remark 4.1 or the Theorem 4.3. So, the inclusion $H \subset H(\rho)$ and the ergodicity of ρ under the assumptions in (b) follow from the more general property :

(5.2.2) *For any non-singular dynamical system (X, μ, T), the maximal spectral type is quasi-invariant under the translations by the eigenvalue group $e(T)$; it is $e(T)$-ergodic when (X, μ, T) is isomorphic to an ergodic non-singular compact group rotation.*

This result is already in [3] ; we will prove a slightly different version of it in the next section.

REMARKS :

1) As mentioned in section 2.2 (example 4), we can get

$$H(\rho) = H = \{0\}$$

by releasing slightly the conditions in (b). Then, of course, ρ is not H-ergodic.

2) Actually, the THEOREMS 4.1 and 4.4, and thus the inclusion $H \subset H(\rho)$ and the property (b), hold under the weaker hypothesis

$$n_k > \sum_{j<k} n_j, \qquad (k \geq 1).$$

But, then, we do not know if the converse inclusion $H(\rho) \subset H$ is still valid. This leads to natural questions which can be asked either for the tower over the 2-odometer or, more generally, for an arbitrary non-singular ergodic dynamical system : whether the group of quasi-invariance of the maximal spectral type is exactly the eigenvalue group, and whether this spectral type is ergodic if and only if the system is isomorphic to a non-singular ergodic compact group rotation ?

5.3. Spectral type of a cocycle.

We introduce now some prerequisites for the discussion of cocycle extension. Let μ be a probability measure on $\mathbb{T}$. Let us consider the (algebraic) unitary representation of $H(\mu)$ in $L^2(\mu)$, defined by

$$U_t f(x) = \Big(\frac{\mathrm{d}T_{-t}\mu}{\mathrm{d}\mu}\Big)^{1/2}(x) f(T_t x), \qquad (t \in H(\mu)).$$

Let H be a possibly uncountable subgroup of $H(\mu)$, acting non-singularly by translation on $(\mathbb{T}, \mu)$ and let $a(t,x)$ be an (algebraic) H-cocycle with values in the circle group; we assume simply that, for every $t \in H$, $x \mapsto a(t,x)$ is a unit modulus Borel function and that

$$a(t+t', x) = a(t,x)\, a(t', T_t x) \ \ \mu\text{-a.e.}, \qquad (t, t' \in H). \tag{5.3.1}$$

With the cocycle $a(t,x)$ we associate the representation of H on $L^2(\mu)$ defined by

$$W_t f(x) = a(t,x)\, U_t f(x), \qquad \big(f \in L^2(\mu)\big).$$

Notice that operators so defined yield a representation of H if and only if $a(t,x)$ satisfies the cocycle equation (5.3.1).

Let us also denote V the unitary operator of multiplication by $e^{2\pi i x}$:

$$Vf(x) = e^{2\pi i x} f(x), \qquad (f \in L^2(\mu)).$$

We have the so called *Weyl commutation relation*, both for U_t and W_t :

$$U_t V = e^{2\pi i t} V U_t, \qquad (t \in H(\mu)), \tag{5.3.2}$$

$$W_t V = e^{2\pi i t} V W_t, \qquad (t \in H). \tag{5.3.3}$$

H will be given the strong topology defined by the representation W_t; it is metrizable and separable but not necessarily complete. Notice that the strong topology on $H(\mu)$ defined by the representation U_t is the same as the strong topology already defined on $H(\mu)$, by a straightforward computation or by the uniqueness property (5.1.1); it is also immediate that the strong topology on H is stronger than the topology given by the metric of $H(\mu)$ and than the usual topology of $\mathbb{T}$.

Let D be a countable dense subgroup of H. The compact metric dual group $\widehat{D}$ will be denoted additively. It contains the eigenvalue group $e(D)$ of the non-singular action of D on $(\mathbb{T}, \mu)$ (see section 3.3); as $D \subset \mathbb{T}$ there is a natural homomorphism of $\mathbb{Z}$ into $\widehat{D}$, whose image is clearly contained in $e(D)$ since every function $e^{2\pi i n x}$ ($n \in \mathbb{Z}$) is an eigenfunction. We still denote by n the image of $n \in \mathbb{Z}$ by this homomorphism, and by S the translation by 1 in $\widehat{D}$. For every $d \in D$, we denote γ_d the character of $\widehat{D}$ defined by duality, such that $\gamma_d(\alpha) = \alpha(d)$ for $\alpha \in \widehat{D}$; in particular, $\gamma_d(n) = e^{2\pi i n d}$ for $n \in \mathbb{Z}$.

Let us consider the spectral representation of $(W_d)_{d \in D}$. We recall that the spectral measure of any $f \in L^2(\mu)$ is the positive measure σ_f on $\widehat{D}$, defined by $\widehat{\sigma}_f(d) = \langle W_d f, f \rangle$, $d \in D$. There exists a measure $\sigma = \sigma_{f_0}$ (unique up to an equivalence) such that $\sigma_f \ll \sigma$ for all $f \in L^2(\mu)$; σ is called the *maximal spectral type* of $(W_d)_{d \in D}$.

By the spectral theory, we have an isomorphism $\psi \mapsto W_\psi$ of $L^\infty(\sigma)$ onto the subalgebra of operators on $L^2(\mu)$ spanned by the W_d in the strong (or weak) operator topology, which maps γ_d to W_d and satisfies $\langle W_\psi f, f \rangle = \int \psi \, d\sigma_f$ for all $\psi \in L^\infty(\sigma)$ and $f \in L^2(\mu)$. A bounded sequence ψ_n in $L^\infty(\sigma)$ converges to ψ in the $L^2(\sigma)$ topology if and only if W_{ψ_n} converges to $W\psi$ in the strong operator topology (see for example [38], chap. II).

In particular, from the definition of the topology on H and the assumption that D is dense in H, there is, for every $t \in H$, a (unique) function γ_t in $L^\infty(\sigma)$ with $W_{\gamma_t} = W_t$. Every γ_t has unit modulus and the mapping $t \mapsto \gamma_t$ is continuous from H to $L^\infty(\sigma)$ endowed with the $L^1(\sigma)$ topology.

LEMMA. — (*With the above notations.*)

(a) *The maximal spectral type* σ *of* $(W_d)_{d\in D}$ *is translation quasi-invariant under* $e(D)$.

(b) *If* μ *is* D*-ergodic,* σ *is* S*-ergodic.*

Proof. — Let $\alpha \in e(D)$ and let φ be a corresponding eigenfunction of unit modulus. Let V_φ be the unitary operator of multiplication by φ on $L^2(\mu)$. As for V, we have a Weyl commutation relation

$$W_d V_\varphi = \alpha(d)\, V_\varphi W_d, \qquad (d \in D), \tag{5.3.4}$$

whence, for every $f \in L^2(\mu)$,

$$\langle W_d V_\varphi f, V_\varphi f\rangle = \alpha(d)\,\langle V_\varphi W_d f, V_\varphi f\rangle = \alpha(d)\,\langle W_d f, f\rangle, \qquad (d \in D),$$

that is $\widehat{\sigma}_{V_\varphi f}(d) = \alpha(d)\,\widehat{\sigma}_f(d)$, $(d \in D)$, which implies

$$\sigma_{V_\varphi f} = \delta_\alpha * \sigma_f. \tag{5.3.5}$$

Since V_φ is an isomorphism, it follows $\delta_\alpha * \sigma \sim \sigma$. This proves (a).

In particular $\delta_1 * \sigma \sim \sigma$; in other words σ is S-quasi-invariant. Assume now that μ is D-ergodic and let us show that σ is S-ergodic. The eigenvalue $\alpha = 1$ corresponds to the eigenfunction $\varphi(x) = e^{2\pi i x}$, and then $V_\varphi = V$. Let B be a S-invariant Borel subset of $\widehat{D}$. By (5.3.5), the subspace $\mathcal{H}$ of all $f \in L^2(\mu)$ such that σ_f is concentrated on B is V-invariant. Therefore it consists of all functions of $L^2(\mu)$ which are null outside some Borel set E of $\mathbb{T}$. As a spectral subspace, $\mathcal{H}$ is also invariant under $(W_d)_{d\in D}$, and thus E must be D-invariant. By ergodicity we conclude that $\mathcal{H} = \{0\}$ or $\mathcal{H} = L^2(\mu)$; it follows that $\sigma(B) = 0$ or $\sigma(\widehat{D} \setminus B) = 0$. This proves (b).

REMARKS :

1) For each $d \in D$, we have $\gamma_d(Sx) = e^{2\pi i d}\gamma_d(x)$ for all $x \in \widehat{D}$. By density and continuity, it follows that, for every $t \in H$, the unit modulus function γ_t is an eigenfunction of $(\widehat{D}, \sigma, S)$ with eigenvalue $e^{2\pi i t}$. Therefore $H \subset e(S)$.

2) In the second part of the proof, if we do not assume the ergodicity we still obtain the following property, which will be useful in section 6.1 :

Any S*-invariant Borel set* B *of* $\widehat{D}$ *is also invariant modulo a* σ*-null set under the translation by any eigenvalue* $\alpha \in e(D)$.

Indeed, with the same notations, $\mathcal{H}$ is invariant under any multiplication operator; so, it is invariant under V_φ. It follows by (5.3.5) that B is α-invariant modulo a null set.

3) In the proof of (a), we need not suppose that D acts by translation on a compact group. We may consider any non-singular action and the property (5.2.2) used in the previous section is a particular case. Moreover, introducing cocycles in the example of the tower on the 2-odometer would allow to find any Riesz product as spectral type, and so to prove the THEOREM 5.1 without the restriction $c_j > 0$. We skip the proof.

5.4. Maximal groups for cocycles.

DEFINITION. — A subgroup H of $H(\mu)$ will be said to be *maximal for an H-cocycle $a(h,x)$* if it cannot be extended to an H'-cocycle for any subgroup H' of $H(\mu)$ strictly larger than H.

Given any subgroup H of $H(\mu)$ and any H-cocycle $a(h,x)$, H is contained in a maximal group for $a(h,x)$, by application of Zorn's lemma.

THEOREM. — *Let $\mu \in M(\mathbb{T})$ and let H be a subgroup of $H(\mu)$, acting non-singularly by translation on $(\mathbb{T},\mu)$. Let $a(h,x)$ be a cocycle defined on $H \times \mathbb{T}$. If H is maximal for a, H is the eigenvalue group $e(S)$ of some non-singular transformation S, and so is a saturated subgroup of $\mathbb{T}$. If μ is H-ergodic the conclusion holds with S ergodic.*

Proof. — We keep the notations of section 5.3. Let D be a countable subgroup of H which is dense in the strong topology. D is a fortiori dense in H in the topology of $H(\mu)$ and, when μ is H-ergodic, μ is still D-ergodic (5.1.3). By the LEMMA 5.3, we have a non-singular dynamical system $(\widehat{D},\sigma,S)$, which is ergodic when μ is D-ergodic. We already noticed that $H \subset e(S)$. So, it will be enough to prove that $e(S)$ is contained in H.

For $\psi \in L^\infty(\sigma)$, let $S\psi(x) = \psi(Sx) = \psi(x+1)$. For every $d \in D$, we have $S\gamma_d(x) = \gamma_d(1)\gamma_d(x) = \mathrm{e}^{2\pi \mathrm{i} d}\gamma_d(x)$. The Weyl commutation relation $W_d V = \mathrm{e}^{2\pi \mathrm{i} d} V W_d$ yields

$$W_\psi V = V W_{S\psi}$$

when $\psi = \gamma_d$ for some $d \in D$ and, by linearity and continuity, for every $\psi \in L^\infty(\sigma)$.

Let $t \in e(S)$ and let ψ be a corresponding eigenfunction of unit modulus; then $S\psi(x) = \mathrm{e}^{2\pi \mathrm{i} t}\psi(x)$ σ-a.e. and thus we have

$$W_\psi V = \mathrm{e}^{2\pi \mathrm{i} t} V W_\psi.$$

We shall conclude that t belongs to H and so prove the theorem by the following lemma :

LEMMA. — *(With the above notations.) Let $t \in \mathbb{T}$. t belongs to $H(\mu)$ and the cocycle a can be extended to the subgroup $H' = H + \mathbb{Z}t$ if and only if there exists a unitary operator W of $L^2(\mu)$ which commutes with W_h for all $h \in H$ and satisfies moreover*

$$WV = e^{2\pi i t} VW. \tag{5.4.1}$$

Proof. — If $t \in H(\mu)$ and if the cocycle $a(h, x)$ can be extended to H', the unitary operator $W = W_t$ defined with the extended cocycle satifies both properties. Let us prove the converse.

Let W be a unitary operator commuting with all the W_h $(h \in H)$ and satisfying (5.4.1). This relation expresses that the operators V and $e^{2\pi i t}V$ are unitarily equivalent. It follows that their maximal spectral types, which are respectively μ and $\delta_t * \mu$, are equivalent; in other words $t \in H(\mu)$. Moreover, W and U_t satisfy the same commutation relation with V and thus the operator WU_t^{-1} commutes with V; therefore it is the operator of multiplication by some measurable function of unit modulus $b(x)$:

$$Wf(x) = b(x)\, U_t f(x), \qquad \big(f \in L^2(\mu)\big).$$

For every $n \in \mathbb{Z}$ and every $h \in H$,

$$W^n f(x) = b_n(x)\, U_{nt} f(x), \qquad \big(f \in L^2(\mu)\big),$$

where $b_n(x) = b(x)\, b(x+t) \cdots b(x + (n-1)t)$ for $n > 0$, and

$$W^n W_h f(x) = b_n(x+h)\, a(h, x)\, U_{h+nt} f(x), \qquad \big(f \in L^2(\mu)\big). \tag{5.4.2}$$

Since W commutes with all the W_h, we have a representation of the group $H \times \mathbb{Z}t$. If $H \cap (\mathbb{Z}t) = \{0\}$, (5.4.2) yields a representation of $H' = H + \mathbb{Z}t$, and this is equivalent to the fact that

$$a'(h + nt, x) = b_n(x+h)\, a(h, x) \tag{5.4.3}$$

is an H'-cocycle; then, we have an extension of a to H'.

If $H \cap (\mathbb{Z}t) \neq 0$, let m be the least positive integer with $mt \in H$, and let $c(x) = b_m(x)/a(mt, x)$; we have

$$W^m f(x) = c(x)\, W_{mt} f(x), \qquad \big(f \in L^2(\mu)\big)$$

and thus the operator of multiplication by $c(x)$ commutes both with W and with all the W_h. $c(x)Wf(x) = W(cf)(x)$ μ-a.e. for all $f \in L^2(\mu)$ implies clearly $c(x+t) = c(x)$ μ-a.e. Similarly $c(x)$ is invariant under the translation by any $h \in H$; thus it is H'-invariant. We may write $c(x) = \mathrm{e}^{\mathrm{i}m\alpha(x)}$ where $\alpha(x)$ is some real-valued H'-invariant Borel function. Let then

$$W'f(x) = \mathrm{e}^{-\mathrm{i}\alpha(x)}Wf(x), \qquad \big(f \in L^2(\mu)\big).$$

Conversely, the multiplication by $\mathrm{e}^{\mathrm{i}\alpha(x)}$ commutes with all the W_h and with W; therefore W' still commutes with the W_h, and moreover we have $W'^m = W_{mt}$; it follows that $W'^n W_h$ depends only on $h + nt$. Substituting W' for W, we obtain a representation of H' and so (5.4.3) defines an extension of a to H'.

COROLLARY 1. — *For any positive measure $\mu \in M(\mathbb{T})$, $H(\mu)$ is a saturated subgroup of $\mathbb{T}$. If μ is continuous singular, $H(\mu)$ is a proper saturated subgroup of $\mathbb{T}$, and so it is a Dirichlet subgroup.*

Proof. — $H(\mu)$ is maximal for the cocycle $a(h,x) = 1$. This proves the first part of the corollary. The second part follows from (5.1.2).

We turn now to the problem of extension of a multiplicative cocycle when μ is the Lebesgue measure (*cf.* HELSON and MERRILL [16], [17]). Let us recall that a multiplicative cocycle $a(h,x)$ on $H \times \mathbb{T}$ is *trivial* if it is cohomologous to a cocycle which is constant in x for each h, that is if there exists some Borel function ϕ of unit modulus, such that $a(h,x) = c_h\phi(x+h)\bar{\phi}(x)$, μ-a.e., with c_h a constant of modulus 1.

The theorem in [17] can be rephrased by saying that any maximal group for a non-trivial cocycle is a Dirichlet subgroup (section 2.4, definition 2). This result can be strengthened as a consequence of the theorem.

COROLLARY 2. — *Let μ be the Lebesgue measure of $\mathbb{T}$, and let $a(h,x)$ be a non trivial cocycle. Then any maximal subgroup for $a(h,x)$ is a proper saturated subgroup of $\mathbb{T}$, and so is a Dirichlet subgroup.*

5.5. Ergodicity without quasi-invariance. The sets $\mathbf{A}(\mu)$.

We extend now the notion of ergodicity with respect to a group of translations in a way which is not quite classical in ergodic theory, but very natural in harmonic analysis (see [7], [12], [34]) :

DEFINITION 1. — If D is a countable subgroup of $\mathbb{T}$ we shall say that a finite positive Borel measure μ on $\mathbb{T}$ is *D-ergodic* if $\mu(B) = 0$ or $\mu(B) = \|\mu\|$ for every D-invariant Borel set B.

We do not request any quasi-invariance property for μ. As a matter of fact, the most natural examples of ergodic measures are the infinite convolution products of discrete measures (see [7]), and they are generally not quasi-invariant under any non-trivial translation (in other words $H(\mu)$ is trivial). In section 3.5 we constructed such a measure on a compact group, and we had to replace it by a combination of its translates in order to get a quasi-invariant ergodic measure.

We recall the notation, for any positive measure $\mu \in M(\mathbb{T})$,

$$A(\mu) = \{t \in \mathbb{T}\,;\ \delta_t * \mu \not\perp \mu\}.$$

When μ is $H(\mu)$-ergodic, then $A(\mu) = H(\mu)$; indeed, any translate of μ is still quasi-invariant and ergodic under the translations by elements of $H(\mu)$, and two such measures are either equivalent or mutually singular.

In the general case, $A(\mu)$ will be given a natural separable metric topology in a similar way as $H(\mu)$. Namely, with every $t \in A(\mu)$ we associate the operator T_t of $L(\mu)$ such that, for every $\nu \in L(\mu)$, $T_t\nu$ is the part of $\delta_t * \nu$ which is absolutely continuous with respect to μ in the Lebesgue decomposition; $A(\mu)$ will be endowed with the strong topology of the operators T_t on $L(\mu)$ (referred to, simply, as the strong topology). When restricted to $H(\mu)$, this topology is nothing but the intrinsic polish topology of $H(\mu)$.

The ergodicity property can be formulated without reference to any subgroup D :

PROPOSITION ([34]). — *For any positive measure $\mu \in M(\mathbb{T})$, the following properties are equivalent :*

(1) *For any positive measures $\mu' \ll \mu$ and $\mu'' \ll \mu$, there exists some $t \in \mathbb{T}$ such that $\mu' \not\perp \delta_t * \mu''$.*

(2) *There exists some countable subgroup D of $\mathbb{T}$ such that μ is D-ergodic.*

(3) *μ is D-ergodic for any countable subgroup D of $\mathbb{T}$ such that $D \cap A(\mu)$ is dense in $A(\mu)$ in the strong topology.*

Proof. — When $\mu' \perp \delta_d * \mu''$ for every d in some countable subgroup D, we can find a D-invariant Borel set B such that μ' is concentrated on B and $\mu''(B) = 0$; therefore (1) follows from (2).

Assume now that $D \cap A(\mu)$ is dense in $A(\mu)$ and let B and B' be two disjoint D-invariant Borel sets. $T_d(1_{B'}\mu)$ is concentrated on B' for all $d \in D$; by the density of $D \cap A(\mu)$ in $A(\mu)$ in the strong topology, $T_t(1_{B'}\mu)$ is still concentrated on B' for all $t \in A(\mu)$. It follows $1_B\mu \perp \delta_t * 1_{B'}\mu$ for every $t \in A(\mu)$, and thus for every $t \in \mathbb{T}$.

Therefore (1) implies (3) and the proposition follows.

DEFINITION 2. — We shall say that μ is *translation ergodic* (or simply *ergodic*) if the equivalent properties of the proposition hold.

The COROLLARY 1 in section 5.4 states that, when μ is any continuous singular positive measure in $M(\mathbb{T})$, $H(\mu)$ is a weak Dirichlet set. We shall prove now a weaker property of the same kind for $A(\mu)$.

Let D be a countable subgroup of $\mathbb{T}$ generated by a dense subset of $A(\mu)$ in the strong topology. Taking for ν a convex combination with positive coefficients of all the translates of μ by elements of D, we get a D-quasi-invariant measure with the following property :

LEMMA. — *(With the above notations.) Let B and B' be two disjoint D-invariant Borel sets. For every $t \in \mathbb{T}$, $1_B\nu \perp \delta_t * 1_{B'}\nu$.*

Proof. — By construction, if $1_B\nu \not\perp \delta_t * 1_{B'}\nu$ for some $t \in \mathbb{T}$, there is some $d \in D$ such that $1_B\mu \not\perp \delta_{t+d} * 1_{B'}\mu$. But, from the proof of the proposition, we have $1_B\mu \perp \delta_t * 1_{B'}\mu$ for every $t \in T$.

REMARK 1. — At first glance the definition of ν depends strongly on the choice of D. It turns out that, when μ is ergodic, ν is unique, up to an equivalence. If ν' is another measure with the properties : $\mu \ll \nu'$ and ν' is quasi-invariant and ergodic under the subgroup generated by $A(\mu)$, then, ν and ν' being not mutually singular, they are necessarily equivalent by ergodicity. We summarize this remark as follows :

Given any translation ergodic measure μ, there exists a unique measure ν, up to an equivalence, such that $\mu \ll \nu$, ν is $H(\nu)$-ergodic and $H(\nu)$ is the group generated by $A(\mu)$.

THEOREM. — *Let μ be any purely singular positive measure in $M(\mathbb{T})$.*

(a) *If μ is translation ergodic, then the group generated by $A(\mu)$ is a proper saturated subgroup.*

(b) *In the general case, every finite positive Borel measure on $A(\mu)$ is concentrated on a countable union of weak Dirichlet sets.*

Proof. — Let D and ν be defined as above. It is clear that ν is still purely singular.

(a) If μ is translation-ergodic, by the proposition μ is D-ergodic, and it follows immediately that ν is D-ergodic; it is a fortiori $H(\nu)$-ergodic. Obviously $A(\mu)$ is contained in $A(\nu) = H(\nu)$. Conversely, if $\delta_t * \nu \sim \nu$, there exists some $d \in D$ such that $\delta_{t+d} * \mu \not\perp \mu$, which implies that $t + d$ belongs to $A(\mu)$, whence t belongs to the group generated by $A(\mu)$; so, $H(\nu)$ is exactly the group generated by $A(\mu)$ and by the COROLLARY 1 (section 5.4), it is a proper saturated subgroup.

(b) In the general case, let $\mathcal{D}$ denote the σ-algebra of all D-invariant Borel sets. We introduce the ergodic decomposition of ν with respect to the action of D (see [40], p. 63). It may be written $\nu = \int \omega_s \, \mathrm{d}\nu(s)$, where the ω_s are D-quasi-invariant and D-ergodic probability measures, the map $s \mapsto \omega_s$ is weakly $\mathcal{D}$-measurable and

$$1_B \nu = \int_B \omega_s \, d\nu(s) \quad \text{for every } B \in \mathcal{D}. \tag{5.5.1}$$

As ν is purely singular, ω_s must be singular for ν-almost every s; we may assume that it is the case for all s. Let us also notice that

$$E = \{(s,t)\,;\ t \in H(\omega_s)\}$$

is a Borel set in $\mathbb{T}^2$. Indeed, as ω_s is D-ergodic, $H(\omega_s) = A(\omega_s)$ and $t \in A(\omega_s)$, *i.e.* $\delta_t * \omega_s \not\perp \omega_s$, is equivalent to $\|\delta_t * \omega_s - \omega_s\| < 2$; as the norm is semi-continuous in the weak $*$-topology of $M(\mathbb{T})$, this inequality defines a Borel set.

Let $t \in A(\mu)$ and let us write $\delta_t * \nu = \phi\nu + \nu'$, where ϕ is a Borel function and $\nu' \perp \nu$. Clearly $t \in A(\nu)$ and $\phi\nu > 0$. For any D-invariant Borel set B, we have $\phi 1_B \nu \le \delta_t * \nu$ and, applying the lemma with $B' = \mathbb{T} \setminus B$, $\phi 1_B \nu \perp \delta_t * 1_{B'} \nu$, whence

$$\phi 1_B \nu \le \delta_t * 1_B \nu.$$

Comparing with (5.5.1), we get

$$\int_B (\phi\omega_s) \, d\nu(s) \le \int_B \delta_t * \omega_s \, d\nu(s) \quad \text{for every} \quad B \in \mathcal{D}.$$

The mappings $s \mapsto \phi\omega_s$ and $s \mapsto \delta_t * \omega_s$ being weakly $\mathcal{D}$-measurable, it follows $\phi\omega_s \le \delta_t * \omega_s$ ν-a.e. When $\phi\omega_s > 0$, this implies $t \in A(\omega_s) = H(\omega_s)$; as $\phi\nu = \int \phi\omega_s \, d\nu(s) > 0$, we get

$$\nu(\{s\,;\ t \in H(\omega_s)\}) > 0 \quad \text{for each } t \in A(\mu).$$

Now, let τ be any positive finite Borel measure concentrated on $A(\mu)$; by Fubini's theorem, $\nu \otimes \tau(E) > 0$ and there exists some s with

$$\tau(\{t\,;\ (s,t) \in E\}) = \tau(H(\omega_s)) > 0.$$

Since ω_s is singular, $H(\omega_s)$ is a weak Dirichlet set (COROLLARY 1, section 5.4). Thus τ gives positive measure to some weak Dirichlet set. We may consider a countable union F of such sets with maximal τ-measure; then the non-negative measure $\tau' = \tau - 1_F\tau$ is concentrated on $A(\mu)$ and gives zero measure to every weak Dirichlet set. By the previous argument, $\tau' = 0$ and τ is concentrated on F.

REMARK 2. — It is classical that the union of two Dirichlet sets need not be a Dirichlet set [26] and actually $A(\mu)$ may be not weak Dirichlet. However, it is still a null set for every measure whose Fourier transform vanishes at infinity.

6. Applications to harmonic analysis

This section is devoted to the connections between non-singular dynamics and the Fourier-Gelfand theory for measures. The link is provided by the eigenvalues and eigenfunctions of translations which give rise to generalized characters in the sense of convolution measure algebras theory. We develop the techniques and results of the previous sections which involve the groups $\overline{\mathbb{Z}}_1(\tau)$ to show a deep property of the eigenvalues of the action of $H(\mu)$, namely that they are in the closure of the continuous group characters. This is a key property for solving subtle problems about spectra of measures around the Wiener-Pitt phenomenon.

6.1. H(μ)-eigenvalues.

Let μ be a positive measure in $M(\mathbb{T})$. We consider $H(\mu)$, or more generally a subgroup H of $H(\mu)$, acting non-singularly by translation on $(\mathbb{T},\mu)$. As in the case of locally compact group actions, we will say that a non identically zero function $\varphi \in L^\infty(\mu)$ is an *eigenfunction* and that a complex function α on H is the corresponding eigenvalue if, for every $t \in H$,

$$\varphi(T_t x) = a(t)\,\varphi(x), \qquad \mu\text{-a.e.} \tag{6.1.1}$$

Since $|\varphi(x)|$ and $|\varphi(T_t x)|$ have the same essential supremum, necessarily $|\alpha(t)| = 1$ for all $t \in H$ and it is immediate that α is a group character. On the other hand, for every $t \in H$, integrating (6.1.1) with respect to any measure $\nu \in L(\mu)$, we have

$$\int \varphi\,\mathrm{d}(T_t\nu) = \alpha(t)\int \varphi\,\mathrm{d}\nu.$$

The first integral is continuous with respect to the topology of $H(\mu)$ and, since $\varphi \neq 0$, one can choose ν such that $\int \varphi\,\mathrm{d}\nu \neq 0$. This shows that α is a continuous group character of H, when H is endowed with the strong topology of $H(\mu)$.

THEOREM. — *Let μ be any positive measure on $\mathbb{T}$, let H be a Borel subgroup of $H(\mu)$ and let α be an eigenvalue of the action of H on $(\mathbb{T},\mu)$. For every positive measure τ carried by H, α belongs to $\overline{\mathbb{Z}}_1(\tau)$.*

Proof. — Let φ be a non identically zero function in $L^\infty(\mu)$ which satisfies (6.1.1). The positive measure $\nu = |\varphi|\mu$ is H-quasi-invariant. We have $H \subset H(\nu)$ and φ is an eigenfunction for the action of H on $(\mathbb{T},\nu)$, with the same eigenvalue α, and moreover $\varphi(x) \neq 0$ ν-a.e. So, replacing μ by ν and φ by $\varphi/|\varphi|$, we may assume that φ has modulus 1.

Let us come back to the construction of section 5.3 (with the cocycle $a(h,x)$ equal to 1) : D is a countable dense subgroup of H in the strong topology, $\widehat{D}$ its dual group and σ is the maximal spectral type of the representation $(U_d)_{d\in D}$. We keep the notations γ_d for the character of $\widehat{D}$ defined by $d \in D$ and γ_t $(t \in H)$ for the family of unit modulus functions in $L^\infty(\sigma)$ defined by continuity from the γ_d. We still denote by α the restriction of α to D; since φ has modulus 1, it is an element of $e(D)$ and the LEMMA 5.3 shows that σ is quasi-invariant under both the translation $Sx = x + 1$ and the translation by α in $\widehat{D}$. We have noticed (5.3, remark 1) that $H \subset e(S)$ and more precisely that for every $t \in H$

$$\gamma_t(Sx) = \mathrm{e}^{2\pi \mathrm{i} t}\gamma_t(x) \qquad \sigma\text{-a.e.} \tag{6.1.2}$$

Moreover, for every $d \in D$, $\gamma_d(x+\alpha) = \alpha(d)\gamma_d(x)$ σ-a.e.; since, by a previous remark, α is continuous in the topology of H, we still have for every $t \in H$

$$\gamma_t(x+\alpha) = \alpha(t)\,\gamma_t(x) \qquad \sigma\text{-a.e.} \tag{6.1.3}$$

On the other hand, given any positive measure τ carried by H, since $H \subset e(S)$, the COROLLARY 3.2 yields a mapping $\chi \mapsto \chi_x$ from $\widehat{D}$ to $\overline{\mathbb{Z}}_1(\tau)$ such that $\chi_{Sx}(t) = \mathrm{e}^{2\pi \mathrm{i} t}\chi_x(t)$ τ-a.e(t) for σ-almost all x; by the remark (3.2.1) we may write $\chi_x(t) = \chi(x,t)$ $\sigma \otimes \tau$-a.e., where χ is a unit modulus Borel function on $\widehat{D} \times \mathbb{T}$, and then

$$\chi(Sx,t) = \mathrm{e}^{2\pi \mathrm{i} t}\chi(x,t) \qquad \sigma \otimes \tau\text{-a.e.} \tag{6.1.4}$$

It is clear from (6.1.2) and (6.1.4) that, for τ-almost every t, the function $\chi(x,t)/\gamma_t(x)$ of $L^\infty(\sigma)$ is S-invariant. Now, according to the remark 2 in section 5.3, any S-invariant Borel function is also invariant under translation by α (modulo a σ-null function). Therefore,

$$\chi(x+\alpha,t)/\chi(x,t) = \gamma_t(x+\alpha)/\gamma_t(x) \quad \sigma \otimes \tau\text{-a.e.}$$

By (6.1.4) we have, for σ-almost every x, $\chi_{x+\alpha}\overline{\chi}_x = \alpha$. We conclude that α belongs to $\overline{\mathbb{Z}}_1(\tau)$.

6.2. A property of the measures carried by $\mathbf{H}(\boldsymbol{\mu})$.

For every saturated subgroup H, the property for a measure to be concentrated on H depends only on the behaviour at infinity of its Fourier transform (see remark 2, section 2.2). When μ is a positive measure and $H = H(\mu)$, is it possible to give an explicit criterion? The next theorem gives half of the answer. We ask whether its converse is true when μ is ergodic.

THEOREM. — *Let μ be a positive measure on $\mathbb{T}$. Let τ be any probability measure carried by $H(\mu)$. For every $\varepsilon > 0$, there exists a positive measure $\nu \ll \mu$ such that*

$$\widehat{\nu}(n) \neq 0 \implies \left|\widehat{\tau}(n) - 1\right| \leq \varepsilon.$$

Proof. — We keep the notations of the previous section. Let τ be any probability measure carried by $H(\mu) = e(S)$ and $\chi \mapsto \chi_x$ the Borel map from $\widehat{D}$ to $\overline{\mathbb{Z}}_1(\tau)$ given by the COROLLARY 3.2. Given $\varepsilon > 0$, for some $\chi_0 \in \overline{\mathbb{Z}}_1(\tau)$ the set B of $x \in \widehat{D}$ such that $\|\chi_x - \chi_0\|_{L^2(\tau)} \leq \frac{1}{2}\varepsilon$ has positive σ-measure. When $\sigma(B \cap S^n B) > 0$, we can find some $x \in B \cap S^n B$ for which we have

$$\left|\widehat{\tau}(n) - 1\right| \leq \left\|e^{2\pi i n t} - 1\right\|_{L^2(\tau)} \leq \left\|\chi_{S^n x} - \chi_x\right\|_{L^2(\tau)} \leq \varepsilon. \tag{6.2.1}$$

Now we choose some $f \in L^2(\mu)$ whose spectral measure σ_f is concentrated on B. Let $\nu = |f|^2\mu$. Its Fourier transform is

$$\widehat{\nu}(n) = \int e^{2\pi i n x} f(x)\,\bar{f}(x)\,d\mu(x) = \langle V^n f, f\rangle, \qquad (n \in \mathbb{Z}).$$

If $\widehat{\nu}(n) \neq 0$, the spectral measures of f and of $V^n f$ cannot be mutually singular. We know that $\sigma_{V^n f} = \delta_n * \sigma_f$ is carried by $S^n B$. Therefore, necessarily, $\sigma(B \cap T^n B) > 0$ and (6.2.1) holds. This proves the theorem.

6.3. The Fourier-Gelfand representation of the convolution algebra M(T).

The convolution algebra $M(\mathbb{T})$ of all finite complex Borel measures on $\mathbb{T}$, is a commutative Banach algebra with the total variation norm $\|\mu\| = \int d|\mu|$. We summarize here some basic facts of the Fourier-Gelfand theory of this algebra (we refer the reader to [18], chapter 4 for more details). Let us recall that, for any positive measure $\mu \in M(\mathbb{T})$, we denote by $L(\mu)$ the space of all measures $\nu \ll \mu$.

Let $\Delta = \Delta M(\mathbb{T})$ be the maximal ideal space of the algebra $M(\mathbb{T})$. An element χ of Δ is a non-zero complex homomorphism of $M(\mathbb{T})$; for $0 < \mu \in M(\mathbb{T})$, its restriction to the subspace $L(\mu)$ is given by the function χ_μ of $L^\infty(\mu)$ such that

$$\langle \chi, \nu\rangle = \int \chi_\mu \, d\nu, \qquad \big(\nu \in L(\mu)\big). \tag{6.3.1}$$

The family of functions χ_μ ($\mu \in M(\mathbb{T})$, $\mu > 0$) is called a *generalized character*, and satisfies the following properties (for every positive μ, $\nu \in M(\mathbb{T})$) :

(6.3.2) $$\|\chi_\mu\|_\infty \leq 1\,;$$

(6.3.3) $$\chi_\nu = \chi_\mu \quad |\nu|\text{-a.e.}, \qquad (0 < \nu \ll \mu)\,;$$

(6.3.4) $$\chi_{\mu*\nu}(x+y) = \chi_\mu(x)\,\chi_\nu(y), \quad \mu\otimes\nu\text{-a.e.}$$

Conversely, any non identically zero family of functions χ_μ ($\mu \in M(\mathbb{T})$, $\mu > 0$), enjoying the properties (6.3.2), (6.3.3) and (6.3.4), defines a non-zero complex homomorphism of $M(\mathbb{T})$ by the formula (6.3.1). For every $n \in \mathbb{Z}$, the mapping $\mu \mapsto \widehat{\mu}(n)$ is obviously a complex homomorphism corresponding to the family $\chi_\mu(x) = \mathrm{e}^{2\pi i n x}$. So we have a natural imbedding of $\mathbb{Z}$ in Δ, such that the Gelfand transform of a measure restricted to $\mathbb{Z}$ is nothing but its Fourier transform. The Gelfand topology of Δ, that is the weak $*$-topology of $M(\mathbb{T})'$, induces on $\mathbb{Z}$ the usual (discrete) topology. Δ is weak $*$-compact and contains the weak $*$-closure of $\mathbb{Z}$, that we will denote $\overline{\mathbb{Z}}$. The corresponding generalized characters χ_μ are those for which χ_μ belongs to $\overline{\mathbb{Z}}(\mu)$ for every $\mu \in M(\mathbb{T})$ (see section 2.2).

The following results are classical and can be found in [18].

(6.3.5) *Except for group characters, one does not have $|\chi_\mu| = 1$ for every $\mu \in M(\mathbb{T})$ and one cannot find a single Borel function ϕ such that $\chi_\mu = \phi$, μ-a.e. for every $\mu \in M(\mathbb{T})$.*

(6.3.6) *There exist generalized characters which are not weak $*$-limits of group characters (in other words, $\mathbb{Z}$ is not dense in the maximal ideal space $\Delta M(\mathbb{T})$).*

(6.3.6) expresses that the range of the Fourier transform of a measure $\mu \in M(\mathbb{T})$ may be not dense in the range of its Gelfand transform, which is nothing but its spectrum. It is well known that μ may be non-invertible in the convolution algebra $M(\mathbb{T})$ although $|\widehat{\mu}(n)| \geq c > 0$, $n \in \mathbb{Z}$ (the so-called Wiener-Pitt phenomenon).

In particular, the spectrum of any continuous symmetric probability measure on $\mathbb{T}$ whose convolution powers are mutually singular is the entire unit disc (see [18], chapter 4), although its Fourier transform is real-valued (an example is provided by the standard Riesz product, defined in 4.2, when c_j is a non-zero constant real sequence).

In the opposite direction, it is classical that $\widehat{\mu}(\mathbb{Z})$ is dense in the spectrum of μ in the following cases (see [12], [18]) :

(a) if μ is discrete;

(b) if μ is absolutely continuous;

(c) if $|\mu|^n$ is absolutely continuous for some $n \geq 1$ and, more generally, if μ belongs to the radical of $L^1(\mathbb{T})$ (*i.e.* the intersection of all the maximal ideals of $M(\mathbb{T})$ containing the absolutely continuous measures).

The problem to decide whether these cases were the the only ones (when μ is symmetric), stayed opened for long. The answer, given by PARREAU in [35], is negative. It is a nice application of the results that we will now draw from the properties of $H(\mu)$-eigenvalues and saturated subgroups.

6.4. Generalized characters and $\mathbf{H}(\mu)$-eigenvalues.

There is an obvious relationship between eigenfunctions and generalized characters. Let $\chi \in \Delta M(\mathbb{T})$ and let t be any element of $H(\mu)$. Since $\delta_t * \mu \sim \mu$, using (6.3.3) and (6.3.4) we obtain

$$\chi_\mu(x+t) = \chi_\mu(x)\,\chi_{\delta_t}(t) \quad \mu\text{-a.e.}$$

Defining $\alpha(t) = \chi_{\delta_t}(t)$, we have

$$\chi_\mu(x+t) = \chi_\mu(x)\,\alpha(t) \quad \mu\text{-a.e.}$$

This expresses that χ_μ is an eigenfunction for the action of $H(\mu)$ and that α is the corresponding eigenvalue. The following properties follow from the discussion in section 6.1.

COROLLARY. — *Let $0 < \mu \in M(\mathbb{T})$ and let $\chi \in \Delta M(\mathbb{T})$ such that χ_μ is not identically* 0.

(a) *There exists a Borel group character α of $H(\mu)$ (equivalently, a continuous group character in the polish topology of $H(\mu)$) such that, for every measure τ carried by $H(\mu)$, $\chi_\tau = \alpha$, τ-a.e.*

(b) *For every measure τ carried by $H(\mu)$, χ_τ is in $\overline{\mathbb{Z}}_1(\tau)$.*

Proof. — First notice that, for any measure τ carried by $H(\mu)$, $\tau * \mu \sim \mu$. Indeed, for every Borel set,

$$\tau * \mu(B) = \int \delta_t * \mu(B)\,\mathrm{d}\tau(t).$$

This formula shows that $\tau * \mu(B) = 0$ if and only if $\delta_t * \mu(B) = 0$ for τ-almost-every t, which in turn is equivalent to $\mu(B) = 0$. Using again (6.1.3) and (6.1.4) we obtain

$$\chi_\mu(x+t) = \chi_\mu(x)\,\chi_\tau(\tau) \quad \mu \otimes \tau\text{-a.e.}$$

By assumption $\chi_\mu \neq 0$. Comparing (6.3.6) and (6.3.7) we conclude that

$$\chi_\tau(t) = \alpha(t) \quad \tau\text{-a.e.}$$

α, being an eigenvalue, is a continuous group character of $H(\mu)$ and, according to the THEOREM 6.1, belongs to $\overline{\mathbb{Z}}_1(\tau)$. This proves both properties (a) and (b) of the corollary.

REMARKS :

1) For every $\chi \in \Delta$ such that $\chi_\mu \neq 0$, χ_μ is an eigenfunction for the action of $H(\mu)$. We don't know if any $H(\mu)$-eigenfunction is of the form χ_μ for some $\chi \in \Delta$.

2) If $\chi_\mu = 1$, $\chi_\tau = 1$ for every measure τ carried by $H(\mu)$; in particular τ sticks to μ (definition 2.2). Thus every saturated subgroup of $\mathbb{T}$ which carries μ contains necessarily $H(\mu)$. But this property is of no interest when the Fourier transform of μ tends to 0 at infinity, because then every measure sticks to μ (remark 2.2).

6.5. Saturated subgroups and generalized characters.

We restate here the separation properties of saturated subgroups discussed in section 2 in terms of generalized characters. Let us recall that, given $\omega \in M(\mathbb{T})$, the measure $\widetilde{\omega}$ is defined by $\widetilde{\omega}(E) = \overline{\omega}(-E)$ for every Borel set E; then $\widehat{\widetilde{\omega}}(n) = \overline{\widehat{\omega}}(n)$ for $n \in \mathbb{Z}$.

THEOREM. — *Let H be a saturated subgroup of $\mathbb{T}$. There exists some $\varphi \in \overline{\mathbb{Z}}$ with $\varphi_\tau = 1$ for any positive measure τ concentrated on H, and $\varphi_\omega = 0$ for any positive measure ω such that $\omega(H + x) = 0$ for all $x \in \mathbb{T}$.*

Proof. — By compactness, it is enough to prove that given finitely many positive measures $\tau_1, \dots, \tau_m$ concentrated on H and finitely many positive measures $\omega_1, \dots, \omega_n$ with $\omega_k(H + x) = 0$ $(1 \le k \le n)$ we can find a positive measure μ with $\tau_j \ll \mu$ $(1 \le j \le m)$, $\omega_k \ll \mu$ $(1 \le k \le n)$, and $\varphi \in \overline{\mathbb{Z}}(\mu)$ such that $\int \varphi \, \mathrm{d}\tau_j = \int \mathrm{d}\tau_j$ $(1 \le j \le m)$ and $\int \varphi \, \mathrm{d}\omega_k = 0$ $(1 \le k \le n)$. Let

$$\mu = \sum_{1 \le j \le m} \tau_j + \sum_{1 \le k \le n} \omega_k + \sum_{1 \le k \le n} \omega_k * \widetilde{\omega}_k.$$

Since H is saturated, 1_H, as an element of $L^\infty(\mu)$, belongs to the closed convex hull of $\overline{\mathbb{Z}}(\mu)$ (section 2.2), and it can be written as a barycenter of elements of $\overline{\mathbb{Z}}(\mu)$; say $1_H = \int \varphi \, \mathrm{d}\sigma(\varphi)$ with some probability measure σ on $\overline{\mathbb{Z}}(\mu)$. We have then, for $1 \le j \le m$

$$\int \mathrm{d}\tau_j = \int 1_H \, \mathrm{d}\tau_j = \int \Big(\int \varphi \, \mathrm{d}\tau_j \Big) \, \mathrm{d}\sigma(\varphi),$$

whence $\varphi = 1$ τ_j-a.e. for σ-almost every φ.

On the other hand, for $1 \leq k \leq n$,

$$\omega_k * \widetilde{\omega}_k(H) = \int \omega_k(H - x)\,\mathrm{d}\widetilde{\omega}_k(x) = 0\,;$$

since $\mathbb{Z}$ is dense in $\overline{\mathbb{Z}}(\mu)$, we have $\int \varphi\,\mathrm{d}(\omega_k * \widetilde{\omega}_k) = |\int \varphi\,\mathrm{d}\omega_k|^2$ for every $\varphi \in \overline{\mathbb{Z}}(\mu)$ and thus

$$0 = \int 1_H\,\mathrm{d}(\omega_k * \widetilde{\omega}_k) = \int \Big(\int \varphi\,\mathrm{d}(\omega_k * \widetilde{\omega}_k)\Big)\,\mathrm{d}\sigma(\varphi) = \int \Big|\int \varphi\,\mathrm{d}\omega_k\Big|^2\,\mathrm{d}\sigma(\varphi).$$

It follows $\int \varphi\,\mathrm{d}\omega_k = 0$ σ-a.e. So, σ-almost every φ satisfies the desired properties.

REMARKS :

1) What is significant in the previous theorem is to have $\varphi \in \overline{\mathbb{Z}}$. Indeed, given any Borel subgroup H, there always exists a $\varphi \in \Delta$ such that $\varphi_\tau = 1$ for any positive measure τ concentrated on a translate of H, and $\varphi_\omega = 0$ for any positive measure ω such that $\omega(H + x) = 0$ for all $x \in \mathbb{T}$ (see [18], chap. 8). The theorem is already stated in other words in [18], p. 248, by saying that, when $M(H)$ is a "Bochner algebra" (*i.e.* H is saturated), then the algebra spanned in $M(\mathbb{T})$ by all the measures carried by the translates of H is still a Bochner algebra. Let us note that, for a countable subgroup H, the latter algebra is nothing but the algebra of all discrete measures; then our result is simply equivalent to the well-known inequality

$$\|\widehat{\mu}_d\|_\infty \leq \|\widehat{\mu}\|_\infty$$

for every $\mu \in M(\mathbb{T})$, where μ_d denotes the discrete part of μ.

2) If μ is a continuous positive measure non-equivalent to the Lebesgue measure, $\mu(H(\mu) + x) = 0$ for all $x \in \mathbb{T}$ (5.1.3). According to the theorem, one can find some $\chi \in \overline{\mathbb{Z}}$ such that $\chi_\mu = 0$ and $\chi_\tau = 1$ for every measure τ carried by $H(\mu)$.

This theorem, under the form of this last remark, and the COROLLARY 6.4 are quoted in [35] (theorem 3.2) in order to prove the following result ([5], theorem 3.3) :

THEOREM. — *Let $\{\rho_t\}_{t \in (0,1)}$ be a weakly measurable family of continuous singular probability measures on $\mathbb{T}$ such that ρ_s is carried by $H(\rho_t)$ whenever $s < t$ and let $\mu = \int_0^1 \rho_t\,\mathrm{d}t$. Then all the convolution powers of μ are continuous and singular and, for every measure $\nu \ll \mu$, $\widehat{\nu}(\mathbb{Z})$ is dense in the spectrum of ν.*

Actually, in [35], these results are stated for any second countable locally-compact abelian group instead of $\mathbb{T}$; but the extension is straightforward; besides, explicit examples of families $\{\rho_t\}$ of measures on the circle group fulfilling these assumptions are given.

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