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A DISTORTION THEOREM FOR QUASICONFORMAL MAPPINGS

BY

MICHEL ZINSMEISTER (*)

RÉSUMÉ. — Un théorème de distorsion pour les applications quasiconformes. Nous généralisons aux applications quasiconformes dans la boule unité de \mathbb{R}^n un théorème de Pommerenke relatif aux transformations conformes du disque unité du plan.

ABSTRACT. — A distortion theorem for quasiconformal mappings. We extend to quasiconformal mappings in the unit ball of \mathbb{R}^n a theorem of Pommerenke concerning conformal mappings in the unit disk of the plane.

1. Introduction

(a) The purpose of this paper is to extend to quasi-conformal mappings the following distortion theorem for conformal mappings in the unit disk B^2 of \mathbb{R}^2 , a theorem due to Pommerenke. In this statement, $I(z)$ is, for $z \in B^2$, the interval of B^2 centered at $z/|z|$ of length $2\pi(1-|z|)$.

THÉORÈME [9]. — *There is a universal constant $C > 0$ such that if $f: B^2 \rightarrow \mathbb{R}^2$ is a conformal mapping, then, for every $z \in B^2$, there exists a non euclidean segment γ from z to $I(z)$ such that*

$$\text{Length}(f(\gamma)) \leq C \text{ distance}(f(z), \partial f(B^2)).$$

Before giving the precise results, we set some preliminary notations and results.

For $x \in \mathbb{R}^n$, $n \geq 2$, let $|x|$ be the euclidean norm of x . For $r > 0$,

$$B^n(x, r) = \{y \in \mathbb{R}^n; |y-x| < r\},$$
$$B^n = B^n(0, 1) \quad \text{and} \quad S^{n-1} = \partial B^n.$$

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If $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $d(x, E)$ is the distance from x to E . If $E \subset \mathbb{R}^n$ is measurable, we note $m(E)$ its Lebesgue measure; σ stands for Lebesgue measure on S^{n-1} and $\sigma_{n-1} = \sigma(S^{n-1})$. For a real $n \times n$ matrix A , let

$$|A| = \sup_{x \in S^{n-1}} |Ax|.$$

If $\Omega \subset \mathbb{R}^n$ is open and $f: \Omega \rightarrow \mathbb{R}^n$ is in the Sobolev space $W_{n, \text{loc}}^1(\Omega)$, $Df(x)$ will denote the Jacobian matrix of f , defined almost everywhere, and $J(x, f) = \det Df(x)$.

For $K \geq 1$, a continuous one-to-one mapping $f: \Omega \rightarrow \mathbb{R}^n$ is said to be K -quasiconformal if $f \in W_{n, \text{loc}}^1(\Omega)$ and if

$$|Df(x)|^n \leq KJ(x, f) \quad \text{a. e. on } \Omega.$$

For $x \in S^{n-1}$, we define the cone with vertex x as

$$\Gamma(x) = \{y \in B^n; |y-x| < 3(1-|y|)\}.$$

If $F: B^n \rightarrow \mathbb{R}^n$ is any function, the non-tangential maximal function of F is defined as

$$\forall x \in S^{n-1}, \quad F^*(x) = \sup_{y \in \Gamma(x)} |F(y)|.$$

If $z \in B^n$, we define the « cap » $S(z)$ as

$$S(z) = \{x \in S^{n-1}; z \in \Gamma(x)\} = S^{n-1} \cap B^n(z, 3(1-|z|)).$$

Let M be the group of Möbius self-maps of B^n . If $z \in B^n$, $z \neq 0$, we define

$$T_z(x) = \frac{(1-|z|^2)(x-z) - |x-z|^2 z}{|z|^2 |x - (z/|z|^2)|^2},$$

then $T_z \in M$ and $T_z(z) = 0$.

We will need the following elementary results, the proof of which we omit:

(1) $S(z) = S^{n-1}$ if $|z| < 1/2$.

(2) $T_z(S(z))$ always contains an hemisphere,

(3) If $x, y \in S(z)$,

$$(9(1-|z|))^{-1} |x-y| \leq |T_z(x) - T_z(y)| \leq 2(1-|z|)^{-1} |x-y|,$$

(4) $\forall z \in B^n, \quad B^n(0, 1/7) \subset T_z \left(B^n \left(z, \frac{1}{4}(1-|z|) \right) \right) \subset B^n(0, 1/2).$

(b) The main result will be the following.

THEOREM 1. — For $K \geq 1$ there exists a constant $C(K, n) > 0$ such that if $f: B^n \rightarrow \mathbb{R}^n$ is K -quasiconformal, then, for any $z \in B^n$, there exists a non-euclidean segment γ joining z to $S(z)$ such that

$$(5) \quad \text{Length}(f(\gamma)) \leq C(K, n) d(f(z), \partial f(B^n)).$$

Adapting an idea of B. DAVIS and J. LEWIS [2], we will show in a moment that Theorem 1 is a corollary of the following, of independent interest:

THEOREM 2. — For $f: B^n \rightarrow \mathbb{R}^n$ k -quasiconformal, define, for $x \in S^{n-1}$, $L_f(x) = \text{Length}[f([0, x])]$ where $[0, x]$ is the radius $\{tx; 0 \leq t \leq 1\}$. Then there exist $C(n, K) > 0$ and $p(n, K) > 0$ such that

$$(6) \quad \left(\int_{S^{n-1}} L_f(x)^p d\sigma(x) \right)^{1/p} \leq C(K, n) d(f(0), \partial f(B^n)).$$

Assuming Theorem 2 is true, let us prove Theorem 1. So let $f: B^n \rightarrow \mathbb{R}^n$ be a K -quasiconformal mapping and $z \in B^n$; put $g = f \circ T_z^{-1}$: Applying Theorem 2 to g , we see that for every $M > 0$,

$$(7) \quad \sigma(\{x \in S^{n-1}; L_g(x) > M d(f(z), \partial f(B^n))\}) \leq \left(\frac{C(K, n)}{M} \right)^p.$$

Now choose M large enough so that $(C/M)^p \leq (\sigma_{n-1})/4$. By (2) there exists then $x \in T^*(S(z))$ such that

$$L_g(x) \leq M d(f(z), \partial f(B^n)),$$

and this proves (5) with $\gamma = T_z^{-1}([0, x])$.

The main tools in proving Theorem 2 will be Theorem 3, due to P. JONES, which we discuss in Part 2, and an estimate for the nontangential maximal function f^* , which is proved in part 3.

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2. Let f be as in Theorem 2. Performing a preliminary translation, we may assume that f does not vanish in B^n and that

$$|f(0)| = d(f(0), \partial f(B^n)).$$

We will say that a function $g : B^n \rightarrow \mathbb{R}^n - \{0\}$ satisfies Harnack property if there exists a constant $C(g) > 0$, called the Harnack constant of g , such that:

$$(8) \quad \forall x \in B^n, \quad \forall y, z \in B^n \left(x, \frac{1}{4}(1-|x|) \right), \quad |g(y)| \leq C(g) |g(z)|.$$

LEMMA 1. — *If $f : B^n \rightarrow \mathbb{R}^n - \{0\}$ is K -quasiconformal, then f satisfies Harnack property with a constant depending only on K and n .*

Proof. — By the special distortion theorem for quasiconformal mappings [4], there exists $C(K, n) > 0$ such that

$$\forall x \in B^n, \quad \forall y, z \in B^n \left(x, \frac{1}{4}(1-|x|) \right), \quad \frac{|f(y) - f(z)|}{d(f(z), \partial f(B^n))} \leq C(K, n),$$

and Lemma 1 follows, for $d(f(z), \partial f(B^n)) \leq |f(z)|$ since f does not vanish in B^n .

Now let $f : B^n \rightarrow \mathbb{R}^n - \{0\}$ be K -quasiconformal. For almost every $x \in S^{n-1}$, we may write

$$L_f(x) \leq \int_0^1 |Df(tx)| dt \leq V_f(x) + 2^{n-1} f^*(x) H_f(x),$$

where

$$V_f(x) = \int_0^{1/2} |Df(tx)| dt$$

and

$$H_f(x) = \int_0^1 \frac{|Df(tx)|}{|f(tx)|} t^{n-1} dt.$$

LEMMA 2:

$$V_f \in L^1(S^{n-1}) \quad \text{with} \quad \|V_f\|_1 \leq C(K, n) |f(0)|.$$

Proof:

$$\|V_f\|_1 = \int_{B^n(0, 1/2)} \frac{|Df(y)|}{|y|^{n-1}} dm(y):$$

By Gehring's inequality [5], there exists $p = p(K, n) > n$ and $C(K, n) > 0$ such that

$$\begin{aligned} (9) \quad & \left(\int_{B^n(0, 1/2)} |Df(y)|^p dm(y) \right)^{1/p} \\ & \leq C(K, n) \left(\int_{B^n(0, 1/2)} J(x, f) dm(x) \right)^{1/n} \\ & = C(K, n) m[f(B^n(0, 1/2))]^{1/n} \leq C(K, n) |f(0)|, \end{aligned}$$

the last inequality being a consequence of Lemma 1. We now apply Hölder's inequality to the expression of $\|V_f\|_1$, to obtain

$$\begin{aligned} \|V_f\|_1 & \leq \left(\int_{B^n(0, 1/2)} |Df(y)|^p dm(y) \right)^{1/p} \\ & \quad \times \left(\int_{B^n(0, 1/2)} |y|^{-(n-1)p/(p-1)} dm(y) \right)^{(p-1)/p} \\ & \leq C(K, n) |f(0)|, \end{aligned}$$

by (9) and the fact that $(n-1)p/(p-1) < n$.

An estimate for H_f is given by the following theorem, due to P. Jones:

THEOREM 3 [8]:

$$H_f \in L^1(S^{n-1}) \quad \text{with} \quad \|H_f\|_1 \leq C(K, n).$$

As P. Jones has observed, Theorem 3 implies that

$$\begin{aligned} \sup_{T \in \mathcal{M}} \|H_{f \circ T}\|_1 & \leq C(K, n) \iff \\ (10) \quad & \sup_{T \in \mathcal{M}} \int_{B^n} \frac{|Df(x)|}{|f(x)|} |DT(x)|^{n-1} dm(x) \leq C(K, n), \end{aligned}$$

and (10) exactly says that $|Df|/|f| dm$ is a Carleson measure in B^n [3]. Since $|\nabla|f|| \leq |Df|$, the same is true for $|\nabla u| dm$, where $u = \log|f|$. Writing

$$\|u\|_* = \sup_{T \in \mathcal{M}} \int_{B^n} |\nabla u(x)| |DT(x)|^{n-1} dm(x),$$

we can now invoke the following theorem, due to Varopoulos:

THEOREM 4 [10]. — *Let $u \in W_{1 \text{ loc}}^1(B^n)$ be a real-valued function having radial limit $\tilde{u}(x)$ a. e. on S^{n-1} . If $|\nabla u| dm$ is a Carleson measure in B^n , then $\tilde{u} \in BMO(S^{n-1})$ with $\|\tilde{u}\|_{BMO} \leq C(n) \|u\|_*$.*

(Varopoulos proves Theorem 4 for $n=2$ only, but his argument is easily seen to extend to the general case.)

From Theorems 3 and 4, it follows that if $f: B^n \rightarrow \mathbb{R}^n - \{0\}$ is K -quasi-conformal, then

$$\|\text{Log} |f(rx)|\|_{BMO(S^{n-1})} \leq C(K, n),$$

for $0 \leq r \leq 1$. If we now apply the John and Nirenberg inequality [7] and Lemma 1, we get

LEMMA 3. — *If $f: B^n \rightarrow \mathbb{R}^n - \{0\}$ is K -quasiconformal there exist $C(K, n) > 0$ and $p = p(K, n) > 0$ such that*

$$(11) \quad \sup_{0 < r \leq 1} \int_{S^{n-1}} |f(rx)|^p d\sigma(x) \leq C(K, n) |f(0)|^p.$$

In the case $n=2$ and f conformal, (11) implies that f is in the Hardy space $H^p(B^2)$, so that $f^* \in L^p(S^{n-1})$, and Theorem 2 follows in this particular case. In the general case, we need an extra argument, provided by the next section.

3. Non-tangential maximal function

PROPOSITION 1. — *Let $f: B^n \rightarrow \mathbb{R}^n - \{0\} \in W_{1 \text{ loc}}^1(B^n)$, and $u = \text{Log} |f|$. If $|\nabla u| dm$ is a Carleson measure and if f satisfies Harnack property (8) then, for every $p > 0$,*

$$\int_{S^{n-1}} f^*(x)^p d\sigma(x) \leq C_p \int_{S^{n-1}} |f(x)|^p d\sigma(x),$$

where C_p depends only on $n, p, \|u\|_*$ and $C(f)$.

We first notice that this statement makes sense, since $|f(x)|$ has radial limits a. e. on S^{n-1} , for $|\nabla u| dm$ is a Carleson measure.

Before going into the proof of Proposition 1, let us see why it implies Theorem 2. So let $f: B^n \rightarrow \mathbb{R}^n - \{0\}$ be K -quasiconformal with

$|f(0)| = \text{dist}(f(0), f(B^n))$. By the results in Part 2 we can apply Proposition 1 to $f(x)$ and, by Lemma 3, we get

$$\|f^*\|_p \leq C(K, n) |f(0)|,$$

for some p depending only on K and n . Recalling now that

$$L_f(x) \leq V_f(x) + 2^{n-1} f^*(x) H_f(x),$$

Lemma 2, Theorem 3 and (12) imply that $L_f \in L^{p/p+1}(S^{n-1})$ with

$$\|L_f\|_{p/p+1} \leq C(K, n) |f(0)|,$$

and Theorem 2 is proved.

To prove Proposition 1, we need 3 lemmas:

LEMMA 4. — *If f is as in Proposition 1 and $N > C(f)^2$,*

$$\sigma\left(\left\{x \in S^{n-1}; |f(x)| \leq \frac{|f(0)|}{N}\right\}\right) \leq \frac{7^n \|u\|_*}{\text{Log } N}.$$

Proof: Put

$$F_N = \left\{x \in S^{n-1}; |f(x)| < \frac{|f(0)|}{N}\right\}$$

and

$$G(x) = \int_0^1 |\nabla u(tx)| t^{n-1} dt.$$

If $x \in F_N$,

$$\begin{aligned} G(x) &\geq 7^{1-n} \int_{1/7}^1 |\nabla u(tx)| dt \\ &\geq 7^{1-n} \left| \int_{1/7}^1 |f(tx)|^{-1} \frac{\partial}{\partial t} (|f(tx)|) dt \right| \\ &= 7^{1-n} \left| \text{Log} \frac{|f(x)|}{|f(x/7)|} \right|. \end{aligned}$$

By Harnack property, $|f(0)| \leq C(f) |f(x/7)|$; so, if $x \in F_N$ and $N > C(f)^2$,

$$G(x) \geq 7^{1-n} \left| \text{Log} \frac{N}{C(f)} \right| \geq 7^{-n} \text{Log } N,$$

and from this it follows that

$$\sigma(F_N) \leq \frac{7^n \|G\|_1}{\text{Log } N} \leq \frac{7^n \|u\|_*}{\text{Log } N}.$$

LEMMA 5. — *There is an universal constant $\alpha > 0$ such that if f is as in Proposition 1 and $z \in B^n$,*

$$\sigma\left(\left\{x \in S(z); |f(x)| \leq \frac{|f(z)|}{N}\right\}\right) \leq \frac{\alpha 7^n \|u\|_*}{\text{Log } N} \sigma(S(z)).$$

Proof. — Define $g(x) = f \circ T_z^{-1}(x)$. By (4), if $x \in S^{n-1}$ then $|g(0)| \leq C(f)|g(x/7)|$; also, by definition $\|\text{Log } |g|\|_* = \|u\|_*$. By Lemma 4 we then have, for $N \geq C(f)^2$,

$$\sigma\left(\left\{x \in S^{n-1}; g(x) \leq \frac{|f(z)|}{N}\right\}\right) \leq \frac{7^n \|u\|_*}{\text{Log } N},$$

which implies

$$\sigma\left(T_z\left(\left\{y \in S(z); |f(y)| \leq \frac{|f(z)|}{N}\right\}\right)\right) \leq \frac{7^n \|u\|_*}{\text{Log } N},$$

and the result follows from (3).

LEMMA 6. — *There exist $0 < C(n) < 1$ and $N(n, \|u\|_*, C(f))$ such that the following inequality holds:*

$$(13) \quad \forall \lambda > 0, \quad \sigma\left(\left\{x \in S^{n-1}; f^*(x) > \lambda, |f(x)| \leq \frac{\lambda}{N}\right\}\right) \leq C(n) \sigma(\{x \in S^{n-1}; f^*(x) \geq \lambda\}).$$

Proof. — Let

$$\mathcal{E}(\lambda) = \{z \in B^n, |f(z)| > \lambda\}$$

and

$$\mathcal{U}(\lambda) = \{x \in S^{n-1}; f^*(x) > \lambda\}.$$

Then

$$\mathcal{U}(\lambda) = \bigcup_{z \in \mathcal{E}(\lambda)} S(z).$$

By Vitali covering lemma, there exists $\alpha(n) \in (0, 1)$ and a sequence $\{z_j\} \subset \mathcal{E}(\lambda)$ such that the $S(z_j)$'s are mutually disjoint and

$$\sum_{j \in \mathbf{N}} \sigma(S(z_j)) \geq \alpha(n) \sigma(\mathcal{Q}(\lambda)).$$

Now let

$$E_N = \left\{ x \in S^{n-1}; |f(x)| \leq \frac{\lambda}{N} \right\}.$$

Then

$$\sigma(E_N \cap \mathcal{Q}(\lambda)) \leq \sum_{j \in \mathbf{N}} \sigma(E_N \cap S(z_j)) + (1 - \alpha(n)) \sigma(\mathcal{Q}(\lambda)).$$

But

$$E_N \cap S(z_j) \subset \left\{ x \in S(z_j); |f(x)| \leq \frac{|f(z_j)|}{N} \right\}$$

and so

$$\sigma(E_N \cap S(z_j)) \leq \frac{\alpha 7^n \|u\|_*}{\text{Log } N} \sigma(S(z_j)) \quad \text{if } N \geq C(f)^2,$$

by Lemma 5. So, choosing N so large that

$$\frac{\alpha 7^n \|u\|_*}{\text{Log } N} < \frac{\alpha(n)}{2},$$

we get Lemma 6 with $C(n) = 1 - \alpha(n)/2$. We can now complete the proof of Proposition 1. For $\lambda > 0$, let

$$\chi(\lambda) = \sigma(\{x \in S^{n-1}; f^*(x) > \lambda\})$$

and

$$\theta(\lambda) = \sigma(\{x \in S^{n-1}; |f(x)| > \lambda\}).$$

By (13),

$$\chi(\lambda) \leq C(n) \chi(\lambda) + \theta(\lambda/N) \Leftrightarrow \chi(\lambda) \leq \frac{1}{1 - C(n)} \theta(\lambda/N).$$

Finally,

$$\|f^*\|_p^p = p \int_0^\infty \lambda^{p-1} \chi(\lambda) d\lambda \leq \frac{p}{1 - C(n)} \int_0^\infty \lambda^{p-1} \theta\left(\frac{\lambda}{N}\right) d\lambda = \frac{N^p}{1 - C(n)} \|f\|_p^p,$$

which proves Proposition 1.

Remark. — A mapping $f: B^n \rightarrow \mathbb{R}^n$ is said to be K -quasiregular if $f \in W_{n, \text{loc}}^1(B^n)$ and if

$$|Df(x)|^n \leq KJ(x, f) \quad \text{a. e. on } B^n.$$

The above methods imply the following:

PROPOSITION 2. — Let $f: B^n \rightarrow \mathbb{R}^n - \{0\}$ be a K -quasiregular mapping and $u = \text{Log} |f|$. If $|\nabla u| dm$ is a Carleson measure in B^n , there exists an exponent $p = p(n, K, \|u\|_*) > 0$ such that $L_f \in L^p(S^{n-1})$, where L_f has the same meaning as in Theorem 1.

Sketch of Proof. — Using the same notations as in 2, one may write

$$L_f(x) \leq V_f(x) + 2^{n-1} K f^*(x) \tilde{H}_f(x),$$

where

$$\tilde{H}_f(x) = \int_0^1 |\nabla u(tx)| t^{n-1} dt.$$

From results in [1], Lemma 2 is valid for V_f ; also, $\tilde{H}_f \in L^1$ by hypothesis. To prove Proposition 2, it suffices then to show that $f^* \in L^p$ for some $p > 0$. Using Proposition 1 and Theorem 4, this reduces to proving that f satisfies Harnack property. To do this, we use a recent result of IWANIEC and NOLDER [6]: They proved that u is actually in $W_{q, \text{loc}}^1(B^n)$ for some $q(K, n) > n$ and that

$$\begin{aligned} \forall x \in B^n, \quad & \left(\int_{B^n(x, 1/2(1-|x|))} |\nabla u(y)|^q dm(y) \right)^{1/q} \\ & \leq C(K, n)(1-|x|)^{n(1/q-1)} \\ & \quad \times \int_{B^n(x, 3/4(1-|x|))} |\nabla u(y)| dm(y), \end{aligned}$$

and the result follows from the Sobolev embedding theorem and the fact that $|\nabla u| dm$ is a Carleson measure.

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