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LEFSCHETZ THEOREMS ON QUASI-PROJECTIVE VARIETIES

BY

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RÉSUMÉ. — Nous démontrons un théorème du type de Lefschetz sur les sections hyperplanes pour des variétés quasi projectives. En particulier nous donnons une démonstration d'un énoncé de Deligne généralisant celui des théorèmes du type de Lefschetz à des morphismes finis dans un espace projectif. On démontre ces théorèmes en stratifiant et en utilisant la théorie de Morse sur les variétés à coins.

ABSTRACT. — We prove a theorem of Lefschetz type on hyperplane sections for quasi-projective varieties. In particular we give the proof of a statement of Deligne which generalizes the statement of theorems of Lefschetz type to finite morphisms into a projective space. The proofs use stratifications and Morse theory on manifolds with corners.

Introduction

In [H-L] we have shown that a theorem stated by O. ZARISKI in [Z] is actually a theorem of Lefschetz type for the complement of a projective hypersurface. This type of result has been extended to the case of the complement of a codimension $\geqslant 2$ projective variety by D. CHENIOT in [C 2] and to the case one considers the complement of a hypersurface in a space with singularities by M. KATO and LÊ D. T. in [L].

A strong interest in this type of problem arose from Fulton's viewpoint on connectivity problems (*see* e. g. [F] or [F-L]) and from Deligne's standpoint to solve the problem of the Abelianness of the complement of

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a projective plane curve with nodes in [D]. In his lecture, P. Deligne states a conjecture which gives a nice generalization of the result of [H-L]. In this paper we solve this conjecture in its strongest version, i. e. with a handlebody decomposition statement. Actually in [G-M] M. GORESKY and R. MACPHERSON indicated another proof of this conjecture with a homotopy statement and a large extension of it, as it was proposed by DELIGNE in [D]. However here we have a different treatment using an adapted Morse theory as it is shown in the Appendix.

Some results of this type have been already obtained by H. A. HAMM in [H 1] but in a weaker form than the ones we prove here.

The first paragraph gives the main techniques used further on.

One may view this paper as the first step to stronger results we have obtained in the local case (compare to [H-L]) and which will be published later.

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1. Generalization of the Lefschetz theorem on hypersurface sections

In this paragraph we consider a hyperplane section of a quasi-projective variety assuming that the whole variety or at least the complement of the hyperplane section is non-singular (Theorems 1.1.1 and 1.1.3). The results about the homotopy type have also been stated in [H 1], p. 552 (for a more general case), but here we consider the spaces up to homeomorphism or diffeomorphism, too, and provide a detailed proof.

1 HYPOTHESIS AND STATEMENT OF THE THEOREMS

Let X be a projective subvariety of $\mathbb{P}^N$. Let Z be an algebraic subspace of X and L be a complex hyperplane of $\mathbb{P}^N$ *We assume that* $X-(Z\cup L)$ *is non singular.*

We identify $\mathbb{P}^N-L$ with $\mathbb{C}^N$ and we denote the coordinates by $z_1, \ldots, z_N$. Let $V_R(L)$ be the complement in $\mathbb{P}^N$ of the open ball

$$\{\textstyle\sum_{i=1}^N |z_i|^2 < R\} \qquad \text{of} \quad \mathbb{P}^N-L.$$

We denote the boundary of $V_R(L)$ by $\partial V_R(L)$.

1.1.1. THEOREM. — *For all $R>0$ except a finite number of them, the space $X-Z$ is homeomorphic (and even diffeomorphic, if $X-Z$ is non singular) to the interior of a space obtained from $V_R(L)\cap(X-Z)$ by attaching handles of indexes at least equal to $\dim_{\mathbf{C}} X$. Furthermore $X-Z$ has the homotopy type of a topological space obtained from $V_R(L)\cap(X-Z)$ by attaching cells of dimension at least* $\dim_{\mathbf{C}} X$.

1.1.2. *Remarks* — (i) If $X-Z$ is non-singular the space obtained from $V_R(L)\cap(X-Z)$ will be (as $V_R(L)\cap(X-Z)$ itself) a differentiable manifold with boundary. This observation clarifies the notion "interior" in this case and therefore also in general.

(ii) It may seem that the second statement — the "homotopy statement" — of Theorem 1.1.1 is a consequence of the first one, but one has to be careful if one requires (as we do) that the restriction of the homotopy equivalence to $V_R(L)\cap(X-Z)$ is the identity, since in general one cannot require the same property for the homeomorphism. But the homeomorphism (resp. diffeomorphism) can be chosen to be the identity in an arbitrary given subset of $V_R(L)\cap(X-Z)$ which is open in $X-Z$, as the proof will show, and this is already sufficient. Another possibility is to modify the proof of the existence of a homeomorphism slightly to obtain the homotopy statement.

(iii) In general $L\cap(X-Z)$ is not a deformation retract of $V_R(L)\cap(X-Z)$ even if R is large enough (as one sees easily when $X=\mathbb{P}^N$ and $Z=L$). One can prove that if L is "in general position" relatively to X and Z, $L\cap(X-Z)$ is a deformation retract of $V_R(L)\cap(X-Z)$ when R is large enough (even if $X-Z\cup L$ is singular): but if L is "in general position" and if we assume, as we do in Theorem 1.1.1, that $X-Z\cup L$ is non singular, this means we investigate the case when $X-Z$ is non singular (*cf.* Theorem 1.1.3 below).

(iv) The idea of considering $V_R(L)\cup(X-Z)$ when R is large enough is due to P. Deligne (*see* for instance [D] (§ 1)).

(v) The case $Z=\varnothing$ and $X-L$ non singular is essentially due to A. ANDREOTTI and T. FRANKEL in [A-F] (*see* [M 1], § 7).

(vi) Theorem 1.1.1 implies Theorem 2 of [H 1] when $X-Z\cup L$ is non singular. The proof of the homotopy statement of (1.1.1) was sketched in [*loc. cit.*].

(vii) If we consider the case when X is a finite union of algebraic subvarieties of $\mathbb{P}^N$, Theorem 1.1.1 holds considering the infimum of the complex dimensions of the components of X.

As we have announced in the remark 1.1.2 (iii) above, we moreover obtain:

1.1.3. THEOREM. — *Assume that $X-Z$ is non singular, there is an open dense set Ω of complex hyperplanes in $\mathbb{P}^N$ such that for any $L\in\Omega$, the following is true:*

(i) *There is a number $\bar{R}>0$ such that for every $R\geqslant\bar{R}$, $V_R(L)\cap(X-Z)$ is diffeomorphic to the unit disc bundle in the normal bundle of $L\cap(X-Z)$ in $X-Z$ (with respect to some Riemannian metric);*

(ii) *the space $X-Z$ has the homotopy type of a space obtained from $L\cap(X-Z)$ by attaching cells of indexes at least equal to* $\dim_{\mathbb{C}} X$.

1.2. PROOF OF THEOREM 1.1.1

We use essentially Morse theory in a way similar to A. ANDREOTTI and T. FRANKEL [A-F] but we use a different Morse function and we consider Morse theory on a space M modulo N, with $M-N$ being a manifold (*see* [*H* 2] (§ 1) for the homotopy version; for the handlebody version, *see* the Appendix below).

1.2.1. Let $f_1=\ldots=f_k=0$ be the algebraic equations of the affine set $Z-L$ in $\mathbb{C}^N$ (in the case $Z=\varnothing$ one may consider an "equation" $f_1=0$ where f_1 is a nonzero constant polynomial).

We call φ the restriction of the function

$$-\sum_{i=1}^{k}|f_i|^2(1+\sum_{j=1}^{N}|z_j|^2) \quad \text{to } X-Z\cup L.$$

We notice that, for any $R>0$ and for any $a>0$, the set

$$\{\varphi\leqslant -a\}\cap(X-(Z\cup \mathring{V}_R(L)))$$

is compact and

$$\{\varphi<0\}\cap(X-(Z\cup \mathring{V}_R(L)))=X-(Z\cup \mathring{V}_R(L)).$$

1.2.2. Using the corollary (2.8) of [M 2] for all $R>0$ except a finite number of them, the intersection $V_R(L)\cap(X-Z)$ is smooth.

Let us fix $R>0$, such that $V_R(L)\cap(X-Z)$ is smooth. We shall use Morse theory on $X-Z$ modulo the subspace $(X-Z)\cap V_R(L)$ in a similar way as in [H 2] (§ 1) (*see* also the Appendix) starting with the function φ which at first might not be a Morse function and, thus, which must be slightly changed to get a Morse function with Morse critical points of indexes $\geqslant \dim_{\mathbb{C}} X$ (compare with Theorem 2.7 of [M 3]).

1.2.3. Let us denote by $\mathscr{T}_\varepsilon$ the set $(\{\varphi \geqslant -\varepsilon\} \cup Z) - \mathring{V}_R(L)$. Using again the Corollary 2.8 of [M 2], one may prove that there is $\varepsilon_0 > 0$ such that any ε, $\varepsilon_0 \geqslant \varepsilon > 0$ is not a critical value of φ on $X - Z \cup V_R(L)$ and on $(X-Z) \cap \partial V_R(L)$. By building up a smooth vector field, one may show that $X - (\mathscr{T}_\varepsilon \cup \mathring{V}_R(L))$ is diffeomorphic to $X - (Z \cup \mathring{V}_R(L))$ in such a way that $(X - \mathscr{T}_\varepsilon) \cup (\mathring{V}_R(L) \cap (X - Z))$ is homeomorphic to $X - Z$ (and even diffeomorphic if $X - Z$ is non singular) (*cf.* the Appendix). Furthermore $(X - \mathring{\mathscr{T}}_\varepsilon) \cup (V_R(L) \cap (X - Z))$ is a deformation retract of $X - Z$.

1.2.4. Our theorem is proved if we show that the space $(X - \mathring{\mathscr{T}}_\varepsilon) \cup (V_R(L) \cap (X - Z))$ is obtained from $V_R(L) \cap (X - Z)$ by adding handles of indexes $\geqslant \dim_{\mathbb{C}} X$, together with the corresponding homotopy statement.

First we may approximate φ in the C^2-topology over the manifold with boundary and corners $(X - \mathring{\mathscr{T}}_\varepsilon) \cup (V_R(L) \cap (X - Z \cup L))$ by a function σ which has nondegenerate critical points with distinct values inside $X - \mathring{\mathscr{T}}_\varepsilon \cup \mathring{V}_R(L)$ and the restriction $\partial\sigma$ of which on $\partial V_R(L) \cap (X - Z)$ has non-degenerate critical points with distinct critical values different from the other ones. Moreover we ask that σ coincides with φ on

$$\partial \mathscr{T}_\varepsilon \cap (X - Z \cup \mathring{V}_R(L))$$

and at each point of $(X - \mathring{\mathscr{T}}_\varepsilon) \cup V_R(L) \cap (X - Z \cup L)$ *the Levi form of* σ *has at least the same index as the one of* φ (compare to [Mo], Theorem 8.7, p. 178 or Lemma 22.4 of [M 1]). Recall that if σ_0 is any extension of σ locally in $\mathbb{C}^N$ near to a point x then the Levi form L_x of σ at x is given for any $v \in T_x(X - Z \cup L)$ by

$$L_x(v, v) = \sum_{\nu,\ \mu=1}^{N} \partial^2 \sigma_0 / \partial z_\nu \partial \bar{z}_\mu (x)\, v_\nu \bar{v}_\mu.$$

Now we prove that the indexes of σ are at least $\dim_{\mathbb{C}} X$ and the ones of $\partial\sigma$ are at least $\dim_{\mathbb{C}} X - 1$, where the gradient points outwards, thus using the results of the Appendix and of [H 2] (§ 1), this will imply that $(X - \mathring{\mathscr{T}}_\varepsilon) \cup (V_R(L) \cap (X - Z))$ is obtained from $V_R(L) \cap (X - Z)$ by adding handles of indexes $\geqslant \dim_{\mathbb{C}} X$ and has the homotopy type of a space obtained from $V_R(L) \cap (X - Z)$ by attaching cells of dimensions $\geqslant \dim_{\mathbb{C}} X$.

1.2.5. First consider a critical point x of σ inside $X-\mathring{\mathcal{T}}_\varepsilon \cup \mathring{V}_R(L)$. The Hessian H_x of σ at x is given by: for any $v \in T_x(X-Z \cup L)$,

$$H_x(v, v)=2\,\mathcal{R}e\, L_x(v, v)+2\,\mathcal{R}e\, Q_x(v, v),$$

where L_x is the Levi form of σ at x, $\mathcal{R}e\, L_x$ is the real part of L_x, Q_x is the complex quadratic form on $T_x(X-Z \cup L)$ defined by:

$$Q_x(w, w)=\sum_{\nu,\ \mu=1}^{n} \partial^2 \sigma/\partial\theta_\nu\, \partial\theta_\mu(x)\, w_\nu\, w_\mu,$$

for any $w \in T_x(X-Z \cup L)$, where $(\theta_\nu)_{1 \leqslant \nu \leqslant n}$ are complex local coordinates of X at x.

But, as we saw above, we have chosen σ in such a way that σ has its index at x at least equal to the index of the Levi form Λ_x of φ at x. This latter is negative definite, because, for any $v \in T_x X$:

$$\Lambda_x(v, v)=-\Big(\sum_{i=1}^{k} |df_i(v)|^2+\sum_{i=1}^{k}\sum_{\mu=1}^{N} |d(f_i z_\mu)(v)|^2\Big)$$

and, having $d(f_i z_\mu)=z_\mu\, df_i+f_i\, dz_\mu$, $\Lambda_x(v, v)=0$ if and only if $v=0$, as all the f_i cannot vanish together at $x \in X-Z \cup L$.

Reasoning as in [A-F], we obtain that the index of H_x at x is at least $\dim_{\mathbb{C}} X$. Thus, for $\eta>0$ small enough $\{\sigma \leqslant \sigma(x)+\eta\} \cup (V_R(L) \cap (X-Z))$ is obtained from $\{\sigma \leqslant \sigma(x)-\eta\} \cup (V_R(L) \cap (X-Z))$ by adding a handle of index $\geqslant \dim_{\mathbb{C}} X$ (*see* Appendix).

1.2.6. It remains to consider critical point of $\partial\sigma$ on $\partial V_R(L) \cap X-Z$. Let ψ be the restriction of $\sum_{i=1}^{N} |z_i|^2$ to $X-Z \cup L$. At such a critical point of $\partial\sigma$, one has: $d_x \sigma=\lambda\, d_x \psi$ — as linear forms on $T_x X$. But according to the Appendix the only critical points which add handles "modulo $V_R(L) \cap (X-Z)$" are precisely the critical points of $\partial\sigma$ where:

$$d_x \sigma=\lambda\, d_x \psi \qquad \text{with} \quad \lambda>0.$$

Moreover the Hessian H'_x at x of the restriction $\partial\sigma$ of σ to $\partial V_R(L) \cap (X-Z)$ is the restriction of the Hessian of $\sigma-\lambda\psi$ to the tangent space $T_x(\partial V_R(L) \cap X-Z)$ (compare to Lemme 4.1.9 of [H-L]).

We saw above that the Levi form of σ at any point $x \in X-Z \cup L$ is negative definite as the Levi form of φ. Besides of it the Levi form of ψ is positive definite. The fact that at the critical points x which are involved we have $\lambda>0$ implies that the Levi form of $\sigma-\lambda\psi$ is negative definite on $T_x(X)$. Using an argument as in [A-F], or in the proof of Lemma 5.6 of [M 2], the Hessian of $\partial\sigma$ at a critical point x on $\partial V_R(L)\cap(X-Z)$ has an index at least equal to $\dim_{\mathbf{C}} X-1$ (this follows by restriction to the unique complex subspace of dimension $\dim_{\mathbf{C}} X-1$ contained in $T_x(\partial V_R(L)\cap(X-Z)))$. But as it is shown in the Appendix, a critical point x of $\partial\sigma$ of index k gives a handle of index $k+1$, i.e. for $\eta>0$ small enough $\{\sigma\leqslant\sigma(x)+\eta\}\cup(V_R(L)\cap X-Z)$ is obtained from $\{\sigma\leqslant\sigma(x)-\eta\}\cup(V_R(L)\cap X-Z))$ by adding a handle of index $k+1$; the corresponding homotopy statement is true by the results of [H 2] (§ 1). This ends the proof of Theorem 1.1.1.

1.3. Proof of Theorem 1.1.3

1.3.1. Let $F_1=\ldots=F_k=0$ be the homogeneous equations of Z in $\mathbb{P}^N$. Let $d_1,\ldots,d_k$ be the degrees of $F_1,\ldots,F_k$ and let d be the least common multiple of the d_i $(i=1,\ldots,k)$. We consider the real algebraic function:

$$\frac{\sum_{i=1}^{k}|F_i|^{2d/d_i}}{\sum_{i=0}^{N}|Z_i|^{2d}},$$

where Z_i are the homogeneous coordinates of $\mathbb{P}^N$. This real algebraic function induces a real algebraic function on $\mathbb{P}^N$. Let us call τ its restriction to X.

We notice that $\tau=0$ is the set Z. Moreover using again the Corollary 2.8 of [M 2] and an argument similar to the one of Theorem 3.1 of [M 1], one obtains that, for any $\varepsilon>0$ small enough, the sets $\{\tau>\varepsilon\}$ are diffeomorphic to $X-Z$.

1.3.2. Let us stratify X by $\mathcal{S}=(X_i)_{i\in I}$ (*see* the Definitions 1.2.1 and 1.2.2 in [L-T 2]) such that:

(1) each X_i is a real semi-algebraic subset of X;

(2) $\mathcal{S}$ is a Whitney stratification;

(3) Z is a union of strata;

(4) $\mathcal{S}$ satisfies the Thom condition for τ (*cf.* 1.4.4 of [L-T 2]).

The existence of such a stratification is obtained from [W] for (1), (2), (3) (*see* [Hi] (§ 3) too) and there is a refinement of a stratification having properties (1), (2) and (3) which satisfies (4) because of [Hi] (§ 5), Corollary 1).

Notice that the number $|I|$ of strata in $\mathcal{S}$ is finite.

1.3.3. Let Ω be the set of complex projective hyperplanes of $\mathbb{P}^N$ transverse to all the strata X_i of $\mathcal{S}$: then Ω is the complement of some real semi-algebraic set strictly contained in the complex projective space $\check{\mathbb{P}}^N$ of projective hyperplanes of $\mathbb{P}^N$ and is therefore an open dense subset of $\check{\mathbb{P}}^N$.

1.3.4. Let $L \in \Omega$. Let $x_n \in \mathbb{P}^N - L$, $n \in \mathbb{N}$, be a sequence of points tending to $x \in L$. Assume that the limit $\lim_{n \to \infty} T_{x_n}(\partial V_{R_n}(L))$, where $R_n = \|x_n\|^2$ in $\mathbb{C}^N = \mathbb{P}^N - L$, exists and equals T. Then one notices that $T \supset T_x(L)$. As L is transverse to all the strata in $\mathcal{S}$ and $\mathcal{S}$ satisfies the Thom condition for τ, there is $\varepsilon_0 > 0$ such that for any $R > 0$ large enough $\partial V_R(L)$ is transverse to any real hypersurface $\{\tau = \varepsilon\}$ in X, for $\varepsilon_0 \geqslant \varepsilon > 0$ (compare to [H-L] (Lemma 2.1.4)). Constructing an adequate vector field, one shows that $(X-Z) \cap L$ is a deformation retract of $(X-Z) \cap V_R(L)$.

This gives us a proof of Theorem 1.1.3 (ii) using the statement of Theorem 1.1.1.

Constructing another adequate vector field, one shows that $(X - \{\tau \leqslant \varepsilon\}) \cap V_R(L)$ and $X \cap V_R(L) - Z$ are diffeomorphic. Since $(X - \{\tau < \varepsilon\}) \cap V_R(L)$ can be regarded as a closed tubular neighbourhood of $(X - \{\tau < \varepsilon\}) \cap L$ for $R \gg 0$ we get (1.1.3) (i).

(1.4) Some remarks

1.4.1. We can find a complex stratification $(X'_i)_{i \in I} = \mathcal{S}'$ of X such that:

(1) each X'_i is a complex locally closed non singular algebraic subset of X;

(2) $\mathcal{S}'$ is a Whitney stratification;

(3) Z is a union of strata.

The existence of such a stratification has been seen above and is due to Whitney (*see* [W]). However in this new situation our strata are complex locally closed algebraic subsets of X: this is allowed, as we have not the condition (4) of 1.3.2 which forced us to have semi-algebraic strata.

Now there is in $\check{\mathbb{P}}^N$ a Zariski open subset $\Omega' \neq \emptyset$ of complex projective hyperplanes transverse to all the strata of $\mathcal{S}'$; for instance Ω' is the complement in $\check{\mathbb{P}}^N$ of the projective algebraic subset which is the topological closure of the set of all hyperplanes tangent to some X'_i. In this case Ω' is connected.

Now we have :

1.4.2. For any L', $L'' \in \Omega'$, the spaces $L' \cap (X-Z)$ and $L'' \cap (X-Z)$ ate diffeomorphic. Furthermore there are suitable tubular neighbourhoods in $X-Z$ which are diffeomorphic.

The proof of this statement uses Thom-Mather first isotopy theorem (*cf.* [Ma] or *see* [T]), as L' and L'' are transverse to a Whitney stratification of X. We may proceed as in [Cl] where $X = \mathbb{P}^N$ and Z is a projective hypersurface of $\mathbb{P}^N$.

Notice that this statement is true even if $X-Z$ is singular with "homeomorphic" instead of "diffeomorphic".

1.4.3. Now let L be any projective complex hyperplane in $\mathbb{P}^N$. Using again the Corollary 2.8 of [M 2], one finds that for all $R>0$ except $R_1, \ldots, R_l$, $\partial V_R(L)$ intersects all be the strata of X transversally. Using again Thom-Mather first isotopy theorem, for any R, R', $R_i < R \leqslant R', < R_{i+1}$ ($i=0, \ldots, l$, with $R_0=0$ and $R_{l+1}=+\infty$) the spaces $V_R(L) \cap (X-Z)$ and $V_{R'}(L) \cap (X-Z)$ are diffeomorphic.

Notice again that this statement is true even if $X-Z$ is singular with "homeomorphic" instead of "diffeomorphic".

2. An answer to a conjecture of deligne

In [D], P. Deligne states a conjecture (conjecture 1.3). We want to show that our methods used in the first paragraph give a positive answer to Deligne's conjecture. In [G-M] (Theorem 4.1), M. Goresky and R. MacPherson indicate a proof of a more general conjecture due to P. Deligne, too (*see* [D], § 1), As our method is different from the one of M. Goresky and R. MacPherson and will be applied to obtain a global Lefschetz-Zariski theorem in the singular case, we give here an alternate proof of Deligne's conjecture which moreover gives a handlebody version of this conjecture, as Deligne already hoped.

2.1. Statement of the results

2.1.1. Let c be an integer and L be a projective subspace of $\mathbb{P}^N$ of codimension c and denote $n^* = n-c$. Let $Z_0 = \ldots = Z_{c-1} = 0$ be projective equations of L.

We denote by $V_R(L)$ the closed neighbourhood of L in $\mathbb{P}^N$ defined by:

$$V_R(L)=\{\sum_{i=c}^{N}|Z_i|^2 \geqslant R(\sum_{j=0}^{c-1}|Z_j|^2)\}.$$

2.1.2. THEOREM (Deligne's conjecture). — Let X_0 be a non singular (connected) quasi-projective variety of complex dimension n and let $f: X_0 \to \mathbb{P}^N$ be an algebraic morphism with *finite fibers*. For any $R>0$ large enough, the pair $(X_0, f^{-1}(V_R(L)))$ is n^*-connected.

2.1.3. *Remarks.* — (i) Using the theory of good neighbourhoods in the sense of D. PRILL [P] one may prove that the theorem holds if $V_R(L)$ is replaced by $V(\varepsilon) := \{z, d(z, L) \leqslant \varepsilon\}$ where the distance d is given by any Riemannian metric on $\mathbb{P}^N$.

(ii) Actually the theorem holds for all $R>0$ except a finite number of them $R_1, \ldots, R_l$, as our proof will be similar to the one given in the first paragraph.

(iii) Our proof will give that for every $R>0$ except $R_1, \ldots R_l$, the space X_0 is diffeomorphic to the interior of a manifold with boundary obtained from $f^{-1}(V_R(L)))$ by adding handles of indexes $\geqslant n^*+1$.

(iv) Obviously Theorem 2.1.2 (in this version) implies Theorem 1.1.1 of paragraph I.

2.1.4. THEOREM. — *Let X_0 be a quasi-projective variety. Suppose the rectified homotopical depth of X_0 to be $\geqslant m$* (see *the definition below*). *Let $f: X_0 \to \mathbb{P}^N$ be an algebraic morphism with* finite fibers. *For all $R>0$ except a finite number of them, the pair $(X_0, f^{-1}(V_R(L)))$ is m^*-connected.*

2.1.5. DEFINITION (Compare to [G], Exp. XIII, § 6, def. 2). — Let A be a reduced complex analytic space of complex dimension n. Let $(A_i)_{0\leqslant i\leqslant n}$ be its canonical Whitney stratification (*see* for instance [L-T 1], § 6.1 or [Te], VI, § 3) with $\dim_{\mathbb{C}} A_i = i$.

Let $X \in A_i$. We say that A has rectified homotopical depth at x greater or equal to m if, for a local embedding of (A, x) into $\mathbb{C}^N$ and for any open ball $B_\varepsilon(x)$ centered at x with a sufficiently small radius $\varepsilon>0$, the pair $(B_\varepsilon(x) \cap A, B_\varepsilon(x) \cap A - A_i)$ is $(m-i-1)$-connected. We say that A has rectified homotopical depth $\geqq m$ if it is so at any $x \in A$.

2.1.6. *Remarks.* If $x \in A$ is non singular, then A has rectified homotopical depth $\geqslant \dim_{\mathbb{C}}(A, x)$ at x.

If the germ of A at x is a complete intersection, then A has rectified homotopical depth $\geqslant \dim_{\mathbb{C}}(A, x)$ at x (*cf.* [H 3]).

2.2. Proof of Deligne's conjecture

2.2.1. The quasi-projective variety $X_0 \subset \mathbb{P}^r$ is a Zariski open dense subset of some complex projective variety $\tilde{X}_0$. Let $\bar{X}_0$ be the closure of the graph of f in $\tilde{X}_0 \times \mathbb{P}^N$. It is a complex projective variety and we have a natural open immersion of X_0 into $\bar{X}_0$. Let $\bar{f}$ be the projection from $\tilde{X}_0 \times \mathbb{P}^N$ onto $\mathbb{P}^N$ restricted to $\bar{X}_0$. To avoid heavy notations we shall identify X_0 with an open dense Zariski subset of $\bar{X}_0$ such that $\bar{f}|_{X_0} = f$.

We shall denote $\bar{X}_0$ by X and $\bar{X}_0 - X_0$ by Z.

Using a result similar to the main theorem of paragraph 3 in [Hi], we have a Whitney stratification $(X_i)_{i \in I}$ of X and a Whitney stratification $(S_j)_{j \in J}$ of $\mathbb{P}^N$ such that:

(i) all strata X_i and S_j are non singular connected locally closed algebraic sets;

(ii) the spaces Z and $\bar{f}(X)$ are unions of strata respectively in X and $\mathbb{P}^N$;

(iii) for any $i \in I$, there is a $j \in J$ such that $\bar{f}$ induces a surjective submersive mapping $X_i \to S_j$.

2.2.2. Let Z_{l+1} denote the union of Z and of the strata X_i contained in X_0 with $\dim_{\mathbb{C}} X_i \leqslant l$:

$$Z_{l+1} = Z \cup (\cup_{\dim_{\mathbb{C}} X_i \leqslant l,\ X_i \subset X_0} X_i), \qquad l = 0, \ldots, n.$$

One notices that Z_l is closed in $\bar{X}_0 = X$. We may suppose that $\mathbb{P}^r \times \mathbb{P}^N$ and hence X are embedded in $\mathbb{P}^M$. Let

$$F_1^{(l)} = \ldots = F_{r_l}^{(l)} = 0,$$

be homogeneous equations of Z_l in $\mathbb{P}^M$. Let $d_{l,\,j}$ be the degree of $F_j^{(l)}$.

Let Z_i^0 denote a homogeneous function which extends $Z_i \circ \bar{f}$ in $\mathbb{P}^M$. In $\mathbb{P}^M$ the degree of Z_i^0 is supposed to be λ_i.

We consider the function $-\theta_l$ restriction to $X - \bar{f}^{-1}(L)$ of the function :

$$\sum_{j=1}^{r_l} \frac{|F_j^{(l)}|^{2\lambda}}{\sum_{i=0}^{c-1} |Z_i^0|^{2d_{l,\,j}\lambda/\lambda_i}} \times \left(\frac{\sum_{j=c}^{N} |Z_j|^2}{\sum_{i=0}^{c-1} |Z_i|^2} \circ \bar{f} \right),$$

where $\lambda = \prod_{i=0}^{c-1} \lambda_i$. Let ψ denote the restriction to $X_0 - f^{-1}(L)$ of

$$\left(\sum_{j=c}^{N} |Z_j|^2 / \sum_{i=0}^{c-1} |Z_i|^2\right) \circ \bar{f}.$$

2.2.3. Using again Corollary 2.8 of [M 2], for almost every $R>0$ except a finite number $R_1, \ldots, R_q$ of them, the manifold $\partial V_R(L)$ is transverse to all the strata of X_0. *Let us fix such a* $R>0$. Moreover the same result of J. Milnor gives that the restrictions of the functions θ_l ($l=0, \ldots, n$) have no critical values in $[-\varepsilon_l, 0[$ for some $\varepsilon_l>0$, on any stratum of $\mathbb{P}^N$ and on its intersection by $V_R(L)$. Let us fix ε, $\inf_{0\leq l\leq n} \varepsilon_l \geqslant \varepsilon > 0$. Let $\mathscr{T}_\varepsilon(Z_l)$ denote the set $\{\theta_l \geqslant -\varepsilon\}$.

2.2.4. Doing as in paragraph I, we use Morse theory with θ_n on X_0 "modulo $f^{-1}(V_r(L))$" and we prove that $(X_0 - \mathscr{T}_\varepsilon(Z_n)) \cup f^{-1}(V_R(L))$ is obtained from f^{-1} $(V_R(L))$ by adding handles of indexes $\geqq \dim_{\mathbb{C}} X_0 - c + 1 = n^* + 1$. If $c=1$, the arguments used are the same as in paragraph I. If $c>1$, the only point to be checked is that, at each point inside of $X_0 - \mathscr{T}_\varepsilon(Z_n) \cup f^{-1}(V_R(L))$, the Levi form of θ_n has index $\geqslant n^*+1$ and at each point on the boundary f $(\partial V_R(L))$ the index fo $\theta_n - \lambda\psi$, $\lambda>0$, is $\geqslant n^*+1$.

Let x be an interior point of $X_0 - \mathscr{T}_\varepsilon(Z_n) \cup f^{-1}(V_R(L)))$. Suppose that its image $\tilde{x}$ by f is in the chart where $Z_0 \neq 0$. Dividing the homogeneous coordinates of $\mathbb{P}^M$ at x by a λ_0-th root of Z_0^0, $-\theta_n$ near x is equal to the restriction of a function of the form:

$$\left(\sum_{j=1}^{r_n} \frac{|f_j^{(n)}|^{2\lambda}}{1+\sum_{i=1}^{c-1} |x_i^0|^{2d_{n,j}\lambda/\lambda_i}}\right) \times \left(\frac{\sum_{j=c}^{N} |x_j|^2}{1+\sum_{i=1}^{c-1} |x_i|^2} \circ f\right).$$

We notice that the Levi form of θ_n is negative definite on the linear subspace of vectors v of $T_x X_0$ where :

$$(df_x)(v) \in \{x_1 = \ldots = x_{c-1} = 0\} \subset T_x \mathbb{P}^N$$

because (df_x) is injective as f *has finite fibres*. The dimension of this subspace is at least $\dim_{\mathbb{C}} X^0 - c + 1$.

On the boundary $f^{-1}(\partial V_R(L))$, we do a similar reasoning because the Levi form of the function $\sum_{i=c}^{N} |x_i|^2 / 1 + \sum_{j=1}^{c-1} |x_j|^2$ defined near $\tilde{x}$ is positive semi-definite on the linear subspace

$$T_{f(x)}(f(X_0)) \cap \{x_1 = \ldots = x_{c-1} = 0\} \quad \text{and } (df_x),$$

is injective. Thus the index of the Levi form of $\theta_n - \lambda\psi$, $\lambda>0$, is at least n^*+1 on $T_{f(x)}(f(X_0))$.

2.2.5. We shall prove by descending induction on k that, for any ε' small enough and any ε'', $\varepsilon' \gg \varepsilon'' > 0$, the space $(X_0 - \mathring{\mathscr{T}}_{\varepsilon''}(Z_k)) \cup f^{-1}(V_R(L))$ is diffeomorphic to a space obtained from $(X_0 - \mathring{\mathscr{T}}_{\varepsilon'}(Z_{k+1})) \cup f^{-1}(V_R(L))$ by attaching handles of indexes $\geqslant n^* + 1$; here $Z_0 := Z$.

Notice that $Z_{n+1} = X$ so that

$$(X_0 - \mathring{\mathscr{T}}_{\varepsilon}(Z_{n+1})) \cup f^{-1}(V_R(L)) = f^{-1}(V_R(L))$$

and our assertion has been proved for $k = n$. Moreover for $k = 0$, $Z_0 = Z$ and for $\varepsilon > 0$ small enough we can find as in 1.2.3 that $(X_0 - \mathscr{T}_{\varepsilon}(Z)) \cup f^{-1}(\mathring{V}_R(L))$ is diffeomorphic to X_0. Thus X_0 will be diffeomorphic to the interior of a space obtained from $f^{-1}(V_R(L))$ by adding handles of indexes $\geqslant n^* + 1$.

Obviously this will end the proof of Deligne's conjecture in its stronger form as stated in (2.1.3) (ii) and (iii).

2.2.6. To prove our induction we shall do Morse theory on X_0 "modulo $(X_0 - \mathring{\mathscr{T}}_{\varepsilon'}(Z_{k+1})) \cup f^{-1}(V_R(L)$", when $\varepsilon' > 0$ is small enough and start with the function θ_k.

We choose $\tilde{\varepsilon}$, $0 < \tilde{\varepsilon} \leqslant \varepsilon$, small enough such that there is no $x \in X_0 - (Z_{k+1} \cup f^{-1}(\mathring{V}_R(L)))$, $\lambda \leqslant 0$ such that

$$\theta_k(x) \geqslant -\tilde{\varepsilon} \qquad \text{and} \qquad d(\theta_k)_x = \lambda\, d(\theta_{k+1})_x,$$

or

$$x \in f^{-1}(\partial V_R(L)), \qquad d(\theta_k \,|\, f^{-1}(\partial V_R(L)))_x = \lambda d(\theta_{k+1} \,|\, f^{-1}(\partial V_R(L)))_x.$$

The proof of the existence of such a $\tilde{\varepsilon}$ proceeds as in the case of [H3] Lemma 2.13 (note that $\theta_k = 0$ implies $\theta_{k+1} = 0$).

Now let ε', $0 < \varepsilon' \leqslant \varepsilon$, be chosen so small that

$$\theta_{k+1} \,|\, \partial\mathscr{T}_{\tilde{\varepsilon}}(Z_k) \cap X_0 - f^{-1}(L)$$

$$\text{and} \quad \theta_{k+1} \,|\, \partial\mathscr{T}_{\tilde{\varepsilon}}(Z_k) \cap X_0 \cap f^{-1}(\partial V_R(L)),$$

have no critical values in $[-\varepsilon', 0[$; this is possible since θ_{k+1} is real algebraic (*see* [M2], Corollary 2.8).

Finally, let ε'', $0 < \varepsilon'' \leqslant \varepsilon$, be chosen so small that $\mathscr{T}_{\varepsilon''}(Z_k) - f^{-1}(\mathring{V}_R(L))$ is contained in $\mathscr{T}_{\varepsilon'}(Z_{k+1})$.

Now we have to compare the sets:

$$M^{-\beta} = (X_0 - \mathring{\mathscr{T}}_{\varepsilon'}(Z_{k+1}) \cap \mathring{\mathscr{T}}_{\beta}(Z_k)) \cup f^{-1}(V_R(L)),$$

for different values of β. We consider them as manifolds with boundary (for general β) by straightening the corner $\partial\mathcal{T}_{\varepsilon'} \cap f^{-1}(\partial V_R(L))$ first and the remaining corner afterwards (*see* Appendix).

For $\beta \gg 0$ we have $M^{-\beta} = (X_0 - \mathring{\mathcal{T}}_{\varepsilon'}(Z_{k+1})) \cup f^{-1}(V_R(L)))$; on the other hand: $M^{-\tilde{\varepsilon}}$ is diffeomorphic to (and a strong deformation retract of)

$$M^{-\varepsilon''} = (X_0 - \mathring{\mathcal{T}}_{\varepsilon'\backslash}(Z_k)) \cup f^{-1}(V_R(L)).$$

This is due to the choice of ε and $\tilde{\varepsilon}$ (*see* Appendix).

Now let x be a point of $Z_{k+1} - \mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k) \cup f^{-1}(\mathring{V}_R(L))$ (which is compact). There is a neighbourhood U_x of x in X_0 and a system of local complex analytic coordinates $\{\xi_1, \ldots, \xi_n\}$ of X_0 in U_x such that:

$$Z_{k+1} \cap U_x = \{\xi_1 = \ldots = \xi_{n-k} = 0\}$$

$$\text{and in} \quad U_k: \quad \theta_{k+1} = \theta_{k+1}(x) + \textstyle\sum_{i=1}^{n-k} |\xi_i|^{2\lambda}.$$

This implies that, for ε′ small enough, the space

$$\mathcal{T}_{\varepsilon'}(Z_{k+1}) - \mathring{\mathcal{T}}_{\varepsilon}(Z_k) \cup f^{-1}(\mathring{V}_R(L))$$

is a disc bundle over

$$Z_{k+1} - (\mathcal{T}_{\varepsilon}(Z_k) \cup f^{-1}(\mathring{V}_R(L))$$

with fibres of real dimension 2 $(\dim_{\mathbb{C}} X_0 - k)$.

By the arguments of 2.2.4 we know that

$$(Z_{k+1} - \mathring{\mathcal{T}}_{\varepsilon}(Z_k)) \cup f^{-1}(V_R(L))$$

is diffeomorphic to a space obtained from f^{-1} $(V_R(L))$ by attaching handles of index $\geqslant k-c+1$: actually let $R_* > R$, sufficiently near to R then the space $Z_{k+1} - (\mathring{\mathcal{T}}_{\varepsilon}(Z_k) \cup f^{-1}(\mathring{V}_{R_*}(L)))$ is diffeomorphic to a space obtained from $f^{-1}(\mathring{V}_R(L) - \mathring{V}_{R_*}(L))) \cap Z_{k+1}$ by attaching such handles, the diffeomorphism being the identity on a neighbourhood of $f^{-1}(\partial V_{R_*}(L)) \cap Z_{k+1}$. Thus for $\varepsilon'_* > \varepsilon'$ sufficiently near to ε′, $\mathcal{T}_{\varepsilon'_*}(Z_{k+1}) - f^{-1}(\mathring{V}_{R_*}(L)) \cup \mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k)$ is diffeomorphic to a space obtained from

$$\mathcal{T}_{\varepsilon'_*}(Z_{k+1}) - f^{-1}(\mathring{V}_{R_*}(L)) \cup \mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k) - [\mathring{\mathcal{T}}_{\varepsilon'}(Z_{k+1}) - f^{-1}(V_R(L))],$$

by attaching handles of index $\geqq (k-c+1) + 2(\dim_{\mathbb{C}} X_0 - k) \geqslant n^* + 1$, the diffeomorphism being the identity on a neighbourhood of the boundary of

$$\mathcal{T}_{\varepsilon'_*}(Z_{k+1})-f^{-1}(\mathring{V}_{R_*}(L))\cup\mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k).$$

If we extend this diffeomorphism by the identity on

$$(X_0-\mathring{\mathcal{T}}_{\varepsilon'_*}(Z_{k+1}))\cup f^{-1}(V_R(L)),$$

we obtain that $M^{-\tilde{\varepsilon}}$ is diffeomorphic to a space obtained from $(X_0-\mathring{\mathcal{T}}_{\varepsilon'}(Z_{k+1}))\cup f^{-1}(V_R(L))$ by attaching handles of index $\geqslant n^*+1$. This finishes the proof of our induction and thus the proof of Deligne's conjecture.

2.3. Proof of Theorem 2.1.4

The proof proceeds as in 2.2. The only essential change is that $\mathcal{T}_{\varepsilon'}(Z_{k+1})-\mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k)\cup f^{-1}(\mathring{V}_R(L))$ can no longer be regarded as a disc bundle but as a topological fibre bundle. By the assumption on the rectified homotopical depth we know that pair, consisting of the fibre and its "boundary", is $(m-k-1)$-connected. On the other hand, the pair which consists of $Z_{k+1}-(\mathring{\mathcal{T}}_{\varepsilon}(Z_k)\cup f^{-1}(\mathring{V}_R(L)))$ and its "boundary" is $(k-c)$-connected by 2.2. Since the restriction of the fibre bundle over cells (to be attached) is trivial, we can conlude by [H 1], Lemma 2 that the pair, which consists of $\mathcal{T}_{\varepsilon'}(Z_{k+1})-(\mathring{\mathcal{T}}_{\tilde{\varepsilon}}(Z_k)\cup f^{-1}(\mathring{V}_R(L)))$ and its "boundary", is m^*-connected. By the Blakers-Massey homotopy excision theorem the pair $(M^{-\varepsilon}, (X_0-\mathring{\mathcal{T}}_{\varepsilon'}(Z_{k+1}))\cup f^{-1}(V_R(L)))$ is also m^*-connected.

APPENDIX

A generalization of the classical morse theory

(i) Manifolds with corners considered as manifolds with boundary:

Let M be a paracompact C^∞ manifold without boundary. Let ψ_1, ψ_2 be C^∞ functions on M such that 0 is a regular value for ψ_1 aswell as for ψ_2 and for $\psi_2|\{x\in M\mid\psi_1(x)=0\}$. Let N_i denote the set $\{x\in M\mid\psi_i(x)\leqslant 0\}$, $i=1, 2$. Then $N_1\cup N_2$ is a "manifold with corners". We assume that $\partial N_1\cap\partial N_2$ is compact; we can consider $N_1\cup N_2$ as a C^∞ manifold with boundary as follows: there is an open neighbourhood U of $\partial N_1 \cap \partial N_2$ in M which is diffeomorphic to $\partial N_1 \cap \partial N_2 \times \{t \in \mathbb{R}^2 \mid . \|t\| < \varepsilon\}$ for some ε via a diffeomorphism h of the form $h(x) = (\pi(x), \psi_1(x), \psi_2(x))$. Let ρ be the homeomorphism of $\partial N_1 \cap \partial N_2 \times \{t \in \mathbb{R}^2 \mid \|t\| < \varepsilon\}$ onto itself defined by

$$\rho(y, r\cos\varphi, r\sin\varphi) = \begin{cases} (y, r\cos 2\varphi, r\sin 2\varphi), & 0 \leqslant \varphi \leqslant \dfrac{\pi}{2}, \\ \left(y, r\cos\dfrac{2(\varphi+\pi)}{3}, + r\sin\dfrac{2(\varphi+\pi)}{3}\right) & \dfrac{\pi}{2} \leqslant \varphi < 2\pi, \end{cases}$$

for $y \in \partial N_1 \cap \partial N_2$, $r \geqslant 0$. Then $\rho \circ h \mid U \cap (N_1 \cup N_2)$ is a homeomorphism onto

$$\partial N_1 \cap \partial N_2 \times \{t \in \mathbb{R}^2 \mid \|t\| < \varepsilon,\ t_2 \leqslant 0\}$$

which is a diffeomorphism on the complement of $\partial N_1 \cap \partial N_2$. Using $\rho \circ h$ we get a structure of a differentiable manifold with boundary for $N_1 \cup N_2$, which coincides with the natural one on the complement of $\partial N_1 \cap \partial N_2$.

(ii) Let M be as above and N be a subset of the following from: for each $x \in \partial N (= \bar{N} - \mathring{N})$ there is a neighbourhood U of x in M and a differentiable function $\psi : U \to \mathbb{R}$ such that $(d\psi)_x \neq 0$ and $N \cap U = \{\xi \in M \mid \psi) \leqslant 0\}$. Obviously we can assume without loss of generality that $U = M$ and $(d\psi)_x \neq 0$ for all x with $\psi(x) = 0$.

$\overline{M-N}$ is (as well as N) a C^∞-manifold with boundary. Let $\varphi : M \to \mathbb{R}$ be a C^∞ function such that $\varphi \mid \overline{M-N}$ is a m-function in the sense of [H 2], p. 124. Let α and β be regular values for $\varphi \mid \overline{M-N}$ as well as for $\varphi \mid \partial(\overline{M-N})$, $\alpha < \beta$. Let M^α be the set $N \cup \{x \in M \mid \varphi(x) \leqslant \alpha\}$. Then M^α and M^β are "submanifolds with corners" which we consider as manifolds with boundary as explained in (i). On the other hand, we get a "manifold with corners" if we attach a handle to a manifold with boundary (*cf.* [S], p. 396); the whole can therefore be considered as a manifold with boundary by (i).

In [H 2], § 1 it is explained that there are three types of critical points to be considered if we consider the m-function $\varphi \mid \overline{M-N}$:

(a) critical points of $\varphi \mid M - N$;

(b) critical points of $\varphi \mid \partial(\overline{M-N})$ of type I;

(c) critical points of $\varphi \mid \partial(\overline{M-N})$ of type E.

Just recall that the critical point x of $\varphi|_{\partial(M-N)}$ is of type I if one has:

$$(d\varphi)_x = \lambda (d\psi)_x \qquad \text{with} \quad \lambda > 0,$$

-similarly the critical point x of $\varphi|_{\partial(M-N)}$ is of type E if one has:

$$(d\varphi)_x = \lambda (d\psi)_x \qquad \text{ith} \quad \lambda < 0.$$

(Compare to the Morse theory with boundary used in [H-L] where, on the contrary of what happens here, critical points of type I provoke the "morphogenesis", i. e. the adding of a cell; a similar classification of critical points appears in [Si] and might be linked with this Morse theory.)

THEOREM. — *M^β is diffeomorphic to a differentiable manifold with boundary obtained from M^α by attaching:*

(i) *a handle of index λ for each critical point $x \in \varphi^{-1}(]\alpha, \beta[)$ of $\varphi | M-N$ of index λ,*

(ii) *a handle of index $\lambda+1$ for each xritical point $x \in \varphi^{-1}(]\alpha, \beta[)$ of $\varphi | \partial(\overline{M-N})$ of index λ and type E.*

(*The critical points of $\varphi | \partial(\overline{M-N})$ of type I do not contribute.*)

Sketch of proof. — (*a*) We suppose first that $\varphi^{-1}(]\alpha, \beta[)$ contains only critical points of the type described under (*b*); in this case we must show that M^β and M^α are diffeomorphic. Outside the corner of M^β this diffeomorphism is obtained essentially by integration of a vector field (the corner of M^α makes some care necessary). We start by taking a diffeomorphism of $U \cap M^\beta$ onto $U \cap N$ (*cf.* the function ρ used above in (i)), U being a neighbourhood of $\{x \in \partial N | \varphi(x) = \beta\}$, which can be obtained outside this subset by integration of a smooth vector field v such that $d\varphi(v(x)) < 0$ and $d\psi(v(x)) < 0$. Now we note that we can replace -up to diffeomorphism- the differentiable structure ofM^β in such a way that the corners of M^α disappear, by new local charts as in (i). We extend the vector field v to a smooth vector field on $M^\beta - \{x \in \partial N | \varphi(x) = \beta\}$ (with respect to the new differentiable structure) such that $v(x) = 0$ on a neighbourhood of:

$$\{x \in \partial N | \varphi(x) > \beta\}, \qquad d\varphi(v(x)) < 0$$

on

$$M^\beta - \dot{M}^\alpha - \{x' \in \partial N | \varphi(x') = \alpha \quad \text{or} \quad \varphi(x') \geqslant \beta\},$$
$$d\psi(v(x)) < 0 \quad \text{on} \quad \partial N \cap \varphi^{-1}(]\alpha, \beta[),$$

v transversal to the boundary of M^{α} at $\{x \in \partial N \mid \varphi(x) = \alpha\}$. The conditions on v can be satisfied by the assumption about the critical points. Let us denote the corresponding flow by σ. If v has been chosen in a natural way, for each $x \in M^{\beta}$ with $v(x) \neq 0$ there are unique real numbers τ_x and τ'_x with:

$$\varphi(\sigma(\tau_x, x)) = \beta \qquad \text{and} \qquad \sigma(\tau'_x, x) \in \partial M^{\alpha},$$

depending smoothly on x. The desired diffeomorphism of M^{β} onto M^{α} is given for such a point x by $\sigma(\tau'_x - \tau_x, x)$, for the other points by the identity.

(*b*) As in usual Morse theory it is now sufficient to treat the case $\alpha = \gamma - \varepsilon$, $\beta = \gamma + \varepsilon$, where γ is a critical value corresponding to exactly one critical point of $\varphi \mid M - N$ or of $\varphi \mid \partial(\overline{M - N})$, in this case of type E. The first case is classical, *cf.* [S]. Therefore let x_0 be a critical point of $\varphi \mid \partial(\overline{M - N})$ of type E, $\varphi(x) = \gamma$, $\varepsilon > 0$ be so small that there are no other critical points in $\varphi^{-1}([\gamma - \varepsilon, \gamma + \varepsilon]) \cap \overline{M - N}$. Let $\delta > 0$ be small enough such that ψ and:

$$\psi \mid \{x \in \overline{M - N} \mid \varphi(x) = \gamma - \varepsilon \text{ or } \varphi(x) = \gamma + \varepsilon\},$$

have no critical value in $[0, \delta]$ and that the restriction of φ to the boundary of $\{x \in M \mid \psi(x) \leqslant \delta\}$ has no critical points of type E. Then the space $\{x \in M^{\beta} \mid \psi(x) \geqslant \delta\} \cup M^{\alpha}$: the proof is very similar to the proof in (*a*). (Note that the first space may have a corner of a different ype (namely at $\varphi = \beta$, $\psi = \delta$) but that considered before.)

Let λ be the index of the critical point x_0. Then M^{β} is diffeomorphic to a space obtained from $\{x \in M^{\beta} \mid \psi(x) \geqslant \delta\} \cup M^{\alpha}$ by attaching:

$$\{x \in M \mid 0 \leqslant \psi(x) \leqslant \delta,\ \alpha \leqslant \varphi(x) \leqslant \beta\}$$

along:

$$\{x \in M \mid 0 \leqslant \psi(x) \leqslant \delta,\ \varphi(x) = \alpha\} \cup \{x \in M \mid \alpha \leqslant \varphi(x) \leqslant \beta,\ \psi(x) \in \{0, \delta\}\}.$$

Now:

$$\{x \in M \mid 0 \leqslant \psi(x) \leqslant \delta,\ \alpha \leqslant \varphi(x) \leqslant \beta\}$$

is diffeomorphic to $\{x \in \partial N \mid \alpha \leqslant \varphi(x) \leqslant \beta\} \times [0, \delta]$, and by usual Morse theory, $\{x \in \partial N \mid \alpha' \leqslant \varphi(x) \leqslant \beta\}$ is obtained, for $\alpha' < \alpha$, α' near α, from $\{x \in \partial N \mid \alpha' \leqslant \varphi(x) \leqslant \alpha\}$ by attaching a handle of index λ (up to diffeomorphism).

Since the product of this handle by an interval is a handle of index $\lambda+1$, we obtain that M^β is diffeomorphic to a space obtained from $\{x \in M^\beta \mid \psi(x) \geq \delta\} \cup M^\alpha$ by attaching a handle of index $\lambda+1$.

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