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ON THE HILBERT-SAMUEL PARTITION OF STABLE MAP-GERMS

BY

James DAMON (*) and André GALLIGO (*)

RÉSUMÉ. — « Sur la partition de Hilbert-Samuel des germes d'applications stables ». Soit $f: k^n, 0 \to k^p, 0$ ($k = \mathbf{R}$ ou $\mathbf{C}$) un germe d'application stable de type Σ_2 i. e. dim (Ker df_0) $= 2$. On démontre premièrement la lissité des strates de la partition de Hilbert-Samuel de l'espace source, deuxièmement que cette partition est un invariant topologique si $p-n$ est assez grand.

ABSTRACT. — "On the Hilbert-Samuel partition of stable map-germs".

For a stable germ $f: k^n, 0 \to k^p, 0$ ($k = \mathbf{R}$ or $\mathbf{C}$) of type Σ_2 i. e. dim Ker $df = 2$. We prove first that the Hilbert-Samuel partition of the source space is a partition by smooth manifolds and secondly, if $p-n$ is not too small, this partition is a topological invariant and must be preserved under topological equivalence.

CONTENTS

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Introduction

For a germ $f: k^n \to k^p$, $k=\mathbb{R}$ or $\mathbb{C}$ and f smooth or holomorphic, one important problem is to understand the structure of the versal unfolding (*see* § 1) of f. This can be approached by decomposing the versal unfolding into strata defined by certain intrinsic invariants. One such method of decomposition (or stratification) has been carried out by Thom and Mather ([20], [21]) for their construction of topologically stable germs. However, the problem of explicitly determining these strata has remained largely intractable, and work has concentrated on the cases where the singularities can be listed (e. g. [3], [6], [10]).

In this paper we begin to address this problem in the first situation where the preceding methods don't generally apply, namely for finite germs f with $n \leqslant p$ and of type Σ_2, i. e. $\dim_k \operatorname{Ker} df_0 = 2$.

We consider the Hilbert-Samuel partition of the source space for the versal unfolding (*see* § 1), then we prove two main results about the partition. First it is a partition by smooth manifolds. Secondly if $p-n$ is not too small (in a sense to be made precise) this partition is a topological invariant and must be preserved under topological equivalence.

In preprint form this paper was originally entitled "The Hilbert-Samuel Partition of Σ_2" and has been referred to by this title in other papers.

0. Preliminary definitions and notations

We let $k[[x_r]]$ denote $k[[x_1, \ldots, x_r]]$ with maximal ideal $\mathfrak{m}_r$. Also we denote the algebra of C^∞ or holomorphic germs k^n, $0 \to k^p$, 0 by $\mathscr{C}_n$ with maximal ideal also denoted by $\mathfrak{m}_n$. If $f: k^n$, $0 \to k^p$, 0 then f induces $f^*: \mathscr{C}_p \to \mathscr{C}_n$ and the local algebras $Q(f)$ and $Q_l(f)$ are defined by:

$$Q(f) \xrightarrow{\sim} \mathscr{C}_n / f^* \mathfrak{m}_p \mathscr{C}_n,$$

$$Q_l(f) \xrightarrow{\sim} \mathscr{C}_n / f^* \mathfrak{m}_p \mathscr{C}_n + \mathfrak{m}_n^{l+1}.$$

In fact $Q_l(f)$ only depends on the l-jet $j^l(f)(0)$. As usual $J^l(n,p)$ denotes the l-jets of germs $k^n, 0 \to k^p, 0$. By [19], IV; 2.1, there is an algebraic group $\mathscr{K}^l$ (the contact group) acting on $J^l(n, p)$ such that two l-jets f and g are in the same $\mathscr{K}^l$-orbit if and only if $Q_l(f) \xrightarrow{\sim} Q_l(g)$.

We let $\Sigma_i = \{ z \in J^l(n, p) : \dim_k \operatorname{Ker}(dz)_0 = i \}$, and $\Sigma_i^l \subset J^l(n, p)$ denotes the l-jets whose associated 1-jets belong to Σ_i.

We consider only finite map germs $f : k^n, 0 \to k^p, 0$ *so that* $n \leqslant p$ *and* f *has finite singular type* i. e. $\dim Q(f) < +\infty$; *we let* $\delta(Q) = \dim_k Q(f)$.

If f is of type Σ_r then $Q(f) \xrightarrow{\sim} k[[x_r]]/I$ with $I \subset \mathfrak{m}_r^2$.

The *order of an ideal* I, denoted by $\nu(I)$, is defined by $\nu(I) = \min\{ j \in \mathbb{N} : I \not\subset \mathfrak{m}^{j+1} \}$. We also define *the order of* f *or* $Q(f)$, f as above, by $\nu(f) = \nu(Q(f)) = \nu(I)$. That this is independent of the representation follows from the alternative description:

$$\nu(Q(f)) = \min\{ j : \dim_k Q(f)/\mathfrak{m}_r^{j+1} \neq \dim_k t[[x_r]]/\mathfrak{m}_r^{j+1} \}$$

For a function $h : \mathbb{N} \to \mathbb{N}$ we define the order of h :

$$\nu(h) = \min\{ j : h(j) \neq \dim_k k[[x_r]]/\mathfrak{m}_r^{j+1} \text{ where } h(1) = r+1 \}.$$

The Hilbert-Samuel function h of a local k-algebra Q with maximal ideal $\mathfrak{m}$ is a function:

$$h : \mathbb{N} \to \mathbb{N} \quad \text{defined by } h(j) = \dim_k(Q/\mathfrak{m}^{j+1}).$$

If h is the Hilbert-Samuel function of Q then $\nu(h) = \nu(Q)$. Lastly, we will also denote the Hilbert function of Q by h_Q.

We will consider the decomposition of a set (or set germ) X into a union of distinct sets (or set germs) U_α. (We describe the situation for sets. For set germs it is analogous, *see* e. g. [6], I.) If the decomposition is locally finite we will refer to such a decomposition as a *partition.* If moreover the X and U_α are smooth manifolds and the $\{U_\alpha\}$ satisfy the axiom of the frontier: $U_\alpha \cap \bar{U}_\beta \neq \emptyset \Rightarrow U_\alpha \subset U_\beta$, then the decomposition is a *stratification.* If the U_α are only topological manifolds the decomposition will be called a *topological stratification.*

Finally, we recall that an unfolding of $f : k^n, 0 \to k^p, 0$ is a germ $F : k^{n+q}, 0 \to k^{p+q}, 0$ of the form $F(x, u) = (\bar{F}(x, u), u)$ such that $\bar{F}(x, 0) = f(x)$. Then, F is versal if any other unfolding $F_1(x, v)$ of f can be obtained from $F(x, u)$ by a mapping $u = \lambda(v)$ up to $\mathscr{K}$-equivalence (*see* Martinet [18]). For our purposes the important property of a versal unfolding is that as a germ F is infinitesimally stable in the sense of Mather [19], this implies that the germ $j^l f : k^{n+2}, 0 \to J^l(n+q, p+q)$ is transverse to all $\mathscr{K}^l$-orbits (Mather [19], V).

1. Statement of Results

We first describe how to partition Σ_r^l by the Hilbert-Samuel functions. Given $\mathfrak{h} : \mathbb{N} \to \mathbb{N}$, we define:

$$\Sigma_{\mathfrak{h}} = \{ z \in J^l(n, p) : \mathfrak{h}_{Q_l(z)} = \mathfrak{h} \}.$$

We note that the decomposition of $J^l(n,p)$ by $\Sigma_{\mathfrak{h}}$ respects the Σ_r^l since if f is of type Σ_r, $\mathfrak{h}_{Q(f)}(1) = r+1$. Thus, we obtain a partition of Σ_r^l by those $\Sigma_{\mathfrak{h}} \subset \Sigma_r^l$. Such a $\Sigma_{\mathfrak{h}}$ will be referred to as a *Hilbert-Samuel stratum.* For a germ $f : k^n, 0 \to k^p, 0$ this leads to a partition of k^n by the set germs $\{ \Sigma_{\mathfrak{h}}(f) = j^l(f)^{-1}(\Sigma_{\mathfrak{h}}) \}$. We will refer to this as the *Hilbert-Samuel partition of* k^n.

The first result concerns the partition by these strata.

THEOREM 1′. — *If* $\Sigma_{\mathfrak{h}} \subset \Sigma_2^l \subset J^l(n, p)$, *then* $\Sigma_{\mathfrak{h}}$ *is a smooth submanifold.*

Then, for infinitesimally stable germs, it follows that:

THEOREM 1. — *If* $f : k^n, 0 \to k^p, 0$ *is an infinitesimally stable germ of type* Σ_2 *then the Hilbert-Samuel partition of* k^n *is a partition by (germs of) smooth manifolds.*

Remark. — For Σ_1^l the result follows because the Hilbert-Samuel strata are the $\mathcal{K}^l$-orbits.

To describe how topological properties of an infinitesimally stable germ determines its Hilbert-Samuel partition we consider the multiplicity $m(f)$ of a germ $f : k^n, 0 \to k^p, 0$. The definition of real multiplicity given in [7] did not depend on $k = \mathbb{R}$, thus, the definition can be equally given for $k = \mathbb{C}$.

$m(f) = \min \{ j :$ for any representative
$f : U, 0 \to k^p$ and neighborhood of
$0 \in V \subset k^p$ there is a $y \in V$ so that $|f^{-1}(y) \cap U| = j \}$.

($|A|$ = cardinality of a set A). We call $m(f)$ the *local multiplicity.* Using the local multiplicity we define a partition of k^n. Let $f_{(x)}$ denote the germ of f at x. We let $T_j = \{ x \in k^n : m(f_{(x)}) = j \}$. Then, $\mathcal{T} = \{ T_j \}$ is a partition of the set germ k^n at 0. From $\mathcal{T}$, we obtain the minimal C^0-refinement $\mathcal{T}^0$ of $\mathcal{T}$. The strata of $\mathcal{T}^0$ consist of equivalence classes where x_1 is equivalent to x_2 if there is a germ of a homeomorphism $\varphi : (k^n, x_1) \to (k^n, x_2)$ preserving the strata of $\mathcal{T}$. By [6], II, $\mathcal{T}^0$ is a topological stratification refined by the Thom-Mather stratification. The topological structures of $\mathscr{T}^0(f)$ and $\mathscr{T}(f)$ are related to the algebraic properties of $Q(f)$ by:

THEOREM 2. — *If $f: k^n, 0 \to k^p, 0$ is an infinitesimally stable germ of type Σ_2 with $p-n \geqslant \nu(f)-1$ then:*

(i) *the topological type of $\mathscr{T}(f)$ determines the Hilbert-Samuel function of f.*

(ii) *the topological stratification $\mathscr{T}^0(f)$ refines the partition $\{\Sigma_b(f)\}$. (i. e. each strata of $\mathscr{T}^0(f)$ is contained in a strata $\Sigma_b(f)$).*

As a consequence of this result, we can state for C^0-stable germs (finitely $\mathscr{K}$-determined germs all of whose unfoldings are topologically equivalent to trivial unfoldings).

THEOREM 3. — *The Hilbert-Samuel function of $Q(f)$ is a topological invariant for all C^0-stable germs $f: k^n, 0 \to k^p, 0$ of type Σ_2 with $p-n \geqslant \nu(f)-1$.*

Lastly, there is a consequence for generic stairs of local ideals (*see* § 4).

THEOREM 4. — *If $f: k^n, 0 \to k^p, 0$ is an infinitesimally stable germ of type Σ_2 with $p-n \geqslant \nu-1$, then the generic stairs of the local ideals will be constant on the strata of the Thom-Mather stratification.*

These last two results suggest a possibly very close relation between $\mathscr{T}^0$ and the Thom-Mather stratification $\mathscr{S}$. If $\mathscr{H}$ denotes the Hilbert-Samuel partition, then an elementary argument (*see* [5] or [6], II) implies $\mathscr{H}^0 = \mathscr{T}^0$. Thus, any differences in the deformation theory of the Hilbert-Samuel function is detected by $\mathscr{T}^0$. Already $\mathscr{T}^0$ agrees with the C^∞-stratification in the nice dimensions; and for general $n \leqslant p$, they agree on the complement of a set of codim $\geqslant 6(p-n)+8$ (by [6], II). Thus, it is not so unreasonable to ask.

Question: How closely does $\mathscr{T}^0$ approximate the Thom-Mather stratification? In particular, is it possible that with only minor corrections $\mathscr{T}^0 = \mathscr{S}$?

2. Remarks on the proofs

To prove the smoothness of the Hilbert-Samuel strata, we give in paragraph 5 a modified local version of a result in [4] which constructs blow-ups of the closures of contact class orbits in jet-bundles. In this local version, the blow-up is constructed using the local Hilbert scheme (Theorem 15 of paragraph 5). However BRIANÇON [1] and IARROBINO [16] have proven the smoothness of the strata of the Hilbert-Samuel partition in the Hilbert scheme of $k[[x,y]]$. We give a precise description of this result in paragraph 4. Then the blow-up construction enables us to transplant their results to the jet space to obtain smoothness there.

The blow-up construction is also used to determine the normal bundle to the Hilbert-Samuel strata in the jet-bundles. This permits us to examine the local structure of the partition in a tubular neighborhood of a stratum. In paragraph 7 we use this analysis to prove that the topological structure of $\mathscr{T}(f)$ detects the Hilbert-Samuel function of f. This is done by relating both the Hilbert-Samuel function of f and the topological structure of $\mathscr{T}(f)$ with the corresponding data for near-by germs types g with $\delta(g)<\delta(f)$.

Specifically, using the Hilbert-Samuel function of such near-by germs we define in paragraph 6 a deformed Hilbert-Samuel function $\tilde{\mathfrak{h}}_{Q(f)}$ of f. Using properties of the generic stairs and results of Briançon, we show that $\mathfrak{h}_{Q(f)}$ can be recovered from $\tilde{\mathfrak{h}}_{Q(f)}$ and an invariant $\alpha(Q(f))$, which measures the size of the last increase of $\mathfrak{h}_{Q(f)}$. Using induction on the topological invariant $\delta(g)=m(g)$, we know that the topological type of $\mathscr{T}(g)$ determine $\mathfrak{h}_{Q(g)}$; however for such near-by germs g, $\mathscr{T}(g)$ occurs as a germ of $\mathscr{T}(f)$ at points arbitrarily close to 0. Thus, the topological structure of $\mathscr{T}(f)$ allows us to recover $\tilde{\mathfrak{h}}_{Q(f)}$. By induction, the closure $\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}})$ of the deformed Hilbert-Samuel stratum $\Sigma_{\tilde{\mathfrak{h}}}$ is determined by $\mathscr{T}(f)$, then it is enough to prove that $\alpha(Q(f))$ can be determined by the topological structure of $\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}})$.

In paragraph 7, we shall explicitly determine the normal structure of $\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}})$ near $\Sigma_{\mathfrak{h}}$ to be topologically equivalent to a cone obtained by collapsing the zero-section of a line bundle on a product of projective spaces (Proposition 21). Then, from the computation in paragraph 3 of the local cohomology of $\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}})$ at a point of $\Sigma_{\mathfrak{h}}$ using coefficients C_k ($=\mathbb{Q}$ if $k=\mathbb{C}$ or $\mathbb{Z}/2\mathbb{Z}$ if $k=\mathbb{R}$), we obtain a topological expression for $\alpha(Q(f))$:

$$\dim_{C_k} H^*(\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}}(f)); C_k)_{\mathrm{loc}}=2\,\alpha(Q(f))-1.$$

Once Theorems 1 and 2 are proven, the remaining results follow in a straightforward manner using results about the Thom-Mather stratification, C^0-stable germs and the relation between generic stairs and the Hilbert-Samuel function, this is given in paragraph 8.

Remark 1. — One of the reasons that the proof only works for germs of type Σ_2 is that the equality $\delta(f)=m(f)$ is known *not* to hold in general for germs of type Σ_r, $r\geqslant 3$ by results of IARROBINO [15]. Thus there is no known way to give an induction argument using topological invariants.

Remark 2. — It could also be suggested that other invariants such as the Thom-Boardman symbols might be better invariants to investigate for topological invariance. However, LOOIJENGA [17] has shown that this seems not to be so.

Remark 3. — It is also possible to go in the other direction and obtain information about the Hilbert-scheme from properties of stable map germs. This has been done by GAFFNEY [9] and GRANGER [11].

3. Extending previous results from $\mathbb{R}$ to $\mathbb{C}$

Several results proven in [6], I, and [7] for $k=\mathbb{R}$ have extensions to $k=\mathbb{C}$. We will need these extensions in later sections. The first such extension is an algebraic computation of the local multiplicity $m(f)$ for an infinitesimally stable germ.

THEOREM 5. — *If $f: \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $n \leqslant p$, is an infinitesimally stable germ of type Σ_2 (or discrete algebra type) then*

$$m(f)=\delta(f)\ (=\dim_{\mathbb{C}} Q(f)).$$

Since Mather's classification of infinitesimally stable germs [19], IV, is valid for holomorphic germs (and the classification of discrete algebra types is inclusive for complex germs), the proof follows exactly the proof in [7] for $k=\mathbb{R}$.

We also wish to be able to extend to $k=\mathbb{C}$ the results of [6], I, describing topological properties of infinitesimally stable germs. We summarize these with:

PROPOSITION 6. — *If $f: k^n, 0 \to k^p, 0$ is an infinitesimally stable germ with $k=\mathbb{R}$ or C, then the Σ_i-type of f is a topological invariant. If moreover $p-n \geqslant \binom{i}{2}$, then the $\Sigma_{i,(j)}$ type of f is a topological invariant.*

More importantly, for Σ_2-type germs f we have, using the notation of [6], I.

PROPOSITION 7. — *For infinitesimally stable germs $f: k^n, 0 \to k^p, 0$ ($k=\mathbb{R}$ or $\mathbb{C}$, $n \leqslant p$) of type Σ_2, the $\Sigma_{\underbrace{2,2,\ldots,2,(j)}_{l}}$ type is a topological invariant as long as $l \leqslant p-n+1$.*

However, we note that f is of type $\Sigma_{\underbrace{2,2,\ldots,2,(j)}_{l}}$ with $j \neq 3$ if and only if $\nu(f) = l+1$. Thus, we also have:

COROLLARY 8. — *For infinitesimally stable germs $f: k^n, 0 \to k^p, 0$ of type Σ_2 ($k = \mathbb{R}$ or $\mathbb{C}$, $n \leqslant p$), if $\nu(f) \leqslant p-n+1$ then $\nu(f)$ is a topological invariant, and those f with $\nu(f) \leqslant p-n+1$ are topologically distinct from those f with $\nu(f) > p-n+1$.*

Thus, the condition $\nu(f) \leqslant p-n+1$ which we impose in our results can be detected topologically.

The analysis carried out in [6], I, for $k = \mathbb{R}$ can be repeated verbatum for $k = \mathbb{C}$ with the exception of one step.

NOTATION. — For V and W finite dimensional vector spaces let: Z denote the set of linear maps of rank $\leqslant 1$ in $\mathrm{Hom}_k(V, W)$; V^* denote the dual of V; $P(V^*)$, $P(W)$ denote the associated projective spaces; γ' and γ denote the canonical line bundles over $P(V^*)$ and $P(W)$; E denote the line bundle $\gamma' \otimes \gamma$ over $P(V^*) \times P(W)$ (we also denote by γ' and γ the pull-back of γ' and γ). Also, $\mathbb{S}(E)$ denotes the sphere bundle of E and Cone ( ) denotes the cone construction.

PROPOSITION 9. — $(Z, 0) \xrightarrow{\sim} (\mathrm{Cone}(\mathbb{S}(E)), *)$.

The proof is almost identical to that given in [6], I, for $k = \mathbb{R}$, except that here we avoid introducing a Riemannian metric. The map $E \to \mathrm{Hom}_k(V, W)$ is given by $h \otimes w \mapsto \varphi$ where $\varphi(v) = h(v)\, w$. As a corollary of this proposition, we can describe the topological structure of Σ_{i-1} near Σ_i in $\mathrm{Hom}_k(V, W)$ where:

$$\Sigma_r = \{ h \in \mathrm{Hom}_k(V, W);\ \dim_k \ker h = r \}.$$

COROLLARY. — *Near Σ_i, $\mathrm{Cl}(\Sigma_{i-1})$ is locally homeomorphic to Cone $(S^r(\mathbb{S}\, E))$, where S is the suspension operator and $E = \gamma \otimes \gamma'$ for γ and γ' the canonical line bundles over $\mathbb{P}(K^*) \times \mathbb{P}(C)$ where K and C are the canonical kernel and cokernel bundles over Σ_i.*

In the case $k = \mathbb{R}$, $\mathbb{S}\, E$ is a double covering of $P(V^*) \times P(W)$; in the case $k = \mathbb{C}$, $\mathbb{S}\, E$ is an orientable S^1-bundle. In either case, we can compute the local cohomology. We will use coefficients $C_k = \mathbb{Q}$ if $k = \mathbb{C}$ and $C_k = \mathbb{Z}/2\mathbb{Z}$ if $k = \mathbb{R}$, we also use local cohomology of set germs $(X, *)$:

$$H^*(X)_{\mathrm{loc}} = H^*(X, X - \{ * \}).$$

Then we have the following lemma which will be also used in paragraph 7.

LEMMA 10. — *For k-vector spaces V and W and line bundle E as above, we have:*

$$\dim_{C_k} H^*(\mathrm{Cone}(\mathbb{S}\,E);\, C_k)_{\mathrm{loc}} = 2\min(\dim_k V, \dim_k W) - 1.$$

Proof:

$$\begin{aligned} H^*(\mathrm{Cone}(\mathbb{S}\,E);\, C_k)_{\mathrm{loc}} &= H^*(\mathrm{Cone}(\mathbb{S}\,E), \mathrm{Cone}(\mathbb{S}\,E) - \{*\};\, C_k) \\ &\simeq \tilde{H}^{*-1}(\mathbb{S}\,E;\, C_k). \end{aligned}$$

Then we can compute:

$$\dim_{C_k} H^*(\mathrm{Cone}(\mathbb{S}\,E);\, C_k)_{\mathrm{loc}} = \dim_{C_k} H^*(\mathbb{S}\,E;\, C_k) - 1.$$

Then, E is an oriented vector bundle using C_k-coefficients. Thus, we may use the Gysin sequence in the form (*cf.* e. g. [22]):

$$\to H^{i-l}(X) \to H^i(X) \to H^i(\mathbb{S}\,E) \to H^{i-l+1}(X) \to \dots$$

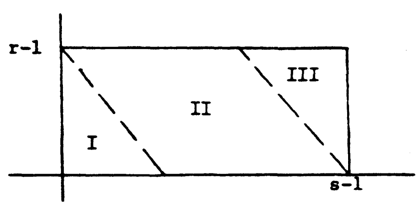
where $l=1$ if $k=\mathbb{R}$ or $l=2$ if $k=\mathbb{C}$. Also, $X=\mathbb{P}(V^*)\times\mathbb{P}(W)$.

If $x=b_1(\gamma)$ and $y=b_1(\gamma')$ denote the first Chern classes or Stiefel-Whitney classes (depending on whether $k=\mathbb{C}$ or $\mathbb{R}$) then:

$$H^*(X;\, C_k) \simeq C_k[x]/(x^r)\otimes C_k[y]/(y^s),$$

where $r=\dim_k V$ and $s=\dim_k W$. Also, the map $H^{*-l}(X) \to H^*(X)$ is given by multiplication by the Euler class (or mod 2 Euler class) $e(E)=b_1(\gamma)+b_1(\gamma')=x+y$. Suppose for example that $r\leqslant s$.

Then, representing the cohomology of X as the integer lattice with (i, j) corresponding to $x^i\otimes y^j$, we see that in region I:



the image under multiplication by $e(E)$ has codimension 1 in each degree and in region III, its kernel has dim = 1 in each degree. In region II, multiplication by $e(E)$ is an isomorphism. Then, by the exact sequence:

$$\dim_{C_k} H^*(\mathbb{S}\, E;\, C_k) = \text{codim (image)} + \dim (\ker)$$
$$= r+(r+s-1)-(s-1) = 2r.$$

Thus, from $\dim H^*(\text{Cone}\, \mathbb{S}\, E, C_k)_{\text{loc}}$ we can recover $\min(\dim V, \dim W)$. Moreover, if $\dim V, \dim W > 1$, then $(\text{Cone}(\mathbb{S}\, E), *)$ is not the germ of a topological manifold.

4. Generic stairs and the Hilbert-Samuel strata in $\text{Hilb}^l(k[[x_2]])$

We let $\text{Hilb}^l(k[[x_r]])$ denote the set of ideals $I \subset k[[x_r]]$ such that $\dim_k k[[x_r]]/I = l$. To endow this set with geometric structure, we embed it in a Grassmannian: by Nakayama's lemma, $\dim_k k[[x_r]]/I = l$ implies $\mathfrak{m}_r^l \subset I$. Thus, we can identify such an I in a canonical way with $\bar{I} = I/\mathfrak{m}_r^{l+1} \subset k[[x_r]]/\mathfrak{m}_r^{l+1}$. We denote $k[[x_r]]/\mathfrak{m}_r^{l+1}$ by $A_l(r)$, or just A_l if there is no confusion. Thus, if $\mathfrak{m} = \dim_k A_l(r) - l$ then we can embed:

$$\text{Hilb}^l(k[[x_r]]) \hookrightarrow G_{\mathfrak{m}}(A_l(r)),$$
$$I \mapsto I' = I/\mathfrak{m}_r^{l+1},$$

where $G_{\mathfrak{m}}(A_l(r))$ denotes the Grassmannian of $\mathfrak{m}$-planes in $A_l(r)$. We identify $\text{Hilb}^l(k[[x_r]])$ with its image in $G_{\mathfrak{m}}(A_l(r))$. It is a standard fact that $\text{Hilb}^l(k[[x_r]])$ is a Zariski closed subset of $G_{\mathfrak{m}}(A_l(r))$ in the case $k = \mathbb{C}$. Also, $\text{Hilb}^l(\mathbb{R}[[x_r]])$ is obtained by intersecting $\text{Hilb}^l(\mathbb{C}[[x_r]])$ with real Grassmannian; it is also Zariski closed for $k = \mathbb{R}$.

To define the Hilbert-Samuel partition of $\text{Hilb}^l(k[[x_r]])$ we consider a function $\mathfrak{h} : \mathbb{N} \to \mathbb{N}$ and define:

$$S_{\mathfrak{h}} = \{ I \subset \text{Hilb}^l(k[[x_r]]) : \mathfrak{h} \text{ is the Hilbert-Samuel function of } k[[x_r]]/I \}.$$

Such an $S_{\mathfrak{h}}$ will be called a Hilbert-Samuel stratum. Then $\text{Hilb}^l(k[[x_r]])$ has a partition by the $\{S_{\mathfrak{h}}\}$. Furthermore, each $S_{\mathfrak{h}}$ is a constructible subset of $\text{Hilb}^l(k[[x_r]])$. In fact, if we define:

$$V_{\mathfrak{h}} = \{ I \subset G_{\mathfrak{m}}(A_l(r)) : \dim_k (I \cap \mathfrak{m}_r^{j+1})$$
$$= m - (\dim_k A_j(r) - \mathfrak{h}(j)),\ 0 \leqslant j \leqslant l \}$$

then V_r is a Schubert cell and hence constructible. Also:

$$S_{\mathfrak{h}} = V_{\mathfrak{h}} \cap \text{Hilb}^l(k[[x_r]]);$$

and thus $S_{\mathfrak{h}}$ is also constructible. However, by a result of Briançon [1] and Iarrobino [12], we can say more, namely.

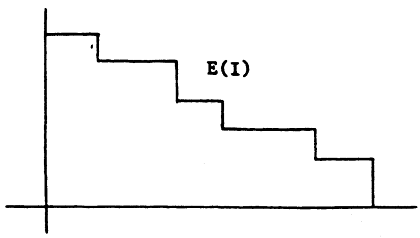
THEOREM 11. — *In* $\mathrm{Hilb}^l(k[[x_2]])$ *with* $k=\mathbb{R}$ *or* $\mathbb{C}$, *the Hilbert-Samuel strata are smooth manifolds.*

To say more we describe the notion of generic stairs of an ideal. We state the key properties for ideals $I\subset k[[x,y]]$. *See* [1], or [8] for more details. We consider the lexicographical ordering on the monomials $x^\alpha y^\beta$ so that $x^{\alpha_1} y^{\beta_1} \prec x^{\alpha_2} y^{\beta_2}$ if $\alpha_1<\alpha_2$ or $\alpha_1=\alpha_2$ and $\beta_1<\beta_2$. Then, given $f\in k[[x,y]]$ with

$f=\sum_{(\alpha,\beta)} a_{(\alpha,\beta)} x^\alpha y^\beta$

$$\exp(f)=\min\{(\alpha,\beta) \text{ in lexicographical ordering so that } a_{(\alpha,\beta)}\neq 0).$$

Then, $E(I)=\{\exp(f) : f\in I\}$ is called the stairs of I.



Also, $E(I)$ has a boundary $F(I)$, the minimal subset of $E(I)$ such that:

$$E(I)=\bigcup_{(\alpha,\beta)\in F(I)} (\alpha,\beta)+\mathbb{N}^2.$$

We denote the elements of $F(I)$ by (α_i,β_i) with $\alpha_0<\alpha_1<\ldots<\alpha_k$ and $\beta_0>\beta_1>\ldots>\beta_k$. Then, a theorem of Grauert implies the following about the structure of E (*see* [1] or [8]).

THEOREM 12. — *For a generic set of formal coordinates for* $k[[x,y]]$, $E(I)$ *is constant for a fixed* I. *In this case,* $\beta_i-\beta_{i+1}=1$ *for all* i *and* $\beta_0=\nu(I)$; *thus* $\beta_i=\nu(I)-i$. $E(I)$ *is called the generic stairs. Furthermore we can choose* $f_i\in I$ *with* $\exp(f_i)=(\alpha_i,\nu-i)$ *so that* $f\in \mathfrak{m}_2^{\alpha_i+\nu-i}$.

One consequence in [1] or [8] is that:

PROPOSITION 13. — *Ideals* I_1 *and* $I_2\subset k[[x,y]]$ *have the same Hilbert-Samuel functions if and only if they have the same generic stairs.*

Furthermore, by the results of [1], if $E(I)$ is the generic stairs of I, and $I\subset S_b$, then there is a neighborhood U of I in S_b such that if $I'\in U$, then $E(I)$ is the generic stairs of I'. Then, we can describe the tangent space to S_b at I as follows. We may pick $f_i\in I$ so that $\exp(f_i)=(\alpha_i,\nu-i)$,

$f_i \in m_2^{\alpha_i+v-i}$ so that $f_i = x^{\alpha_i} g_i$ with $g_i = y^{v-i} + a_1^{(i)}(x)\, y^{v-i-1} + \ldots + a_{v-i}^{(i)}(x)$ and ord $a_j^{(i)}(x) \geqslant j$ while deg $a_j^{(i)}(x) < \alpha_{i+j} - \alpha_i$. Let $\Delta(I)$ be subspace spanned by monomials $x^\alpha y^\beta$ with $(\alpha, \beta) \notin E(I)$.

To describe the tangent space we use a local representation of S_b near I, due to Briançon [III, 3.1, 1]. In a neighborhood V of I we can represent $I' \in V$ by the standard generators for the generic stairs, $f_i' = x^{\alpha_i} g_i'$, where:

$$g_i' = (y + u'_{i,i+1})\, g'_{i+1} + \sum_{j=i+1}^{v} u'_{i,j}\, g'_j$$

and $u'_{i,j} \in k[x]$ satisfies:

$$\operatorname{ord}(u'_{i,j}) \geqslant \max\{\alpha_j - \alpha_{i+1}, j-i\}$$

and deg $u'_{i,j} < \alpha_j - \alpha_i$.

From this we can compute the tangent space. Let $c_{l,m}^{(k)}$ be the coefficient of x^k in $u_{l,m}$. Then we compute inductively:

$$\frac{\partial f_i}{\partial c_{l,m}^{(k)}} = \begin{cases} 0, & i > l, \\ x^{\alpha_l + k}\, g_m, & i = l, \\ x^{\alpha_i}\left\{(y + u_{i,i+1})\, \dfrac{\partial g_{i+1}}{\partial c_{l,m}^{(k)}} + \sum_{j=i+2}^{l} u_{i,j}\, \dfrac{\partial g_j}{\partial c_{l,m}^{(k)}}\right\}, & i < l. \end{cases}$$

Then, if we let p denote the projection onto $\Delta(I)$ along I, $T_I S_b$ is spanned by elements of $\operatorname{Hom}_k(k^{v+1}, \Delta(I))$ corresponding to triples (l, m, k) with $\max\{\alpha_m - \alpha_{l+1}, m-l\} \leqslant k < \alpha_m - \alpha_l$ $(0 \leqslant l < m)$. The map corresponding to (l, m, k) is defined for the standard basis $\{e_i, 1 \leqslant i \leqslant v+i\}$ for k^{v+1} by:

$$e_i \mapsto p\left(\frac{\partial f_{i-1}}{\partial c_{l,m}^{(k)}}\right).$$

5. Blowing up singular submanifolds of $J(n, p)$ using Hilb^m

In this section, to describe a local version of a result in [4], we will have to consider images of mappings. For the complex case it is enough to consider constructible sets and use Chevalley's Theorem [12], but for the real case the situation is more difficult and we will use a theorem of Hironaka [13].

We define singular manifolds to be either:

(i) constructible sets in the case $k=\mathbb{C}$ or (ii) subanalytic sets of analytic manifolds in the case $k=\mathbb{R}$. We recall (*see* e. g. [13] or [14]) that subanalytic sets X are sets whose germs at points x (not necessarily in X) belong to the Boolean algebra of germs of sets at x generated by images of bounded semi-analytic sets by local analytic maps. (We could obtain similar results using instead semi-algebraic sets.) Both types X have the property that they contain dense open sets X_0, such that X is a smooth manifold in a neighborhood of each point of X_0. Then

$$\dim X=\max\{\dim \text{ of components of } X_0\}.$$

A mapping $f: X \to Y$ between singular manifolds will mean:

(i) for $k=\mathbb{C}$, f is defined on all of X by rational functions; and

(ii) for $k=\mathbb{R}$ a mapping which extends to an analytic mapping on some neighborhoods of X and Y in the ambient space. Then a blow-up is a proper surjective mapping $f: \tilde{X} \to X$ of singular manifolds such that for open dense smooth submanifolds $\tilde{U}\subset\tilde{X}$ and $U\subset X$, $f: \tilde{U} \to U$ is a diffeomorphism.

We will examine a procedure for canonically constructing blow-ups of singular submanifolds of $J^l(n,p)$ using Hilb^m. To begin, we let $\mathrm{Aut}_l(r)$ denote the algebra of automorphisms of $A_l(r)$. Then $\mathrm{Aut}_l(r)$ is an algebraic group and acts algebraically on $A_l(r)$. There is an induced action of $\mathrm{Aut}_l(r)$ on $G_m(A_l(r))$ ($m=\dim A_l(r)-l$). We further note that $\mathrm{Aut}_l(r)$ preserves $\mathrm{Hilb}^m(k[[\mathrm{x}_r]])$. Here we consider singular submanifolds $S\subset\mathrm{Hilb}^m(k[[\mathrm{x}_r]])$ which are invariant under $\mathrm{Aut}_l(r)$.

Given such an $S\subset\mathrm{Hilb}^m\, k[[\mathrm{x}_r]]$ we define:

$$\Sigma_S=\{\, z\in J^l(n,p) \,:\, Q_l(z) \xrightarrow{\sim} k[[\mathrm{x}_r]]/I, \text{ some } I\in S\,\}.$$

We first describe the blow-up in the special case when $n=r$. We assume that for $I\in S$, $\mathrm{m}_r^l\subset I$ so that the problem is really one involving l-jets. In $A_l(n)$ we have the maximal ideal $\mathrm{m}_n/\mathrm{m}_n^{l+1}$. Then, any proper ideal I in $k[[\mathrm{x}_n]]$ and containing m_n^{l+1} corresponds to $\bar{I}\subset\mathrm{m}_n/\mathrm{m}_n^{l+1}$. Thus, we can just as well embed

$$\mathrm{Hilb}^m(k[[\mathrm{x}_n]])\subsetneq G_m(\mathrm{m}_n/\mathrm{m}_n^{l+1}).$$

If γ_m is the canonical m-plane bundle over $G_m(\mathfrak{m}_n/\mathfrak{m}_n^{l+1})$ and $k_\cdot^{p^*}$ denotes the trivial vector bundle with fiber k^{p^*} over the indicated space, then we have a diagram:

$$\begin{tikzcd}
\mathrm{Hom}(k_\cdot^{p^*}, \gamma_m \mid S) \arrow[r] \arrow[d] & \mathrm{Hom}(k_\cdot^{p^*}, \gamma_m) \arrow[r] \arrow[d] & J^l(n,p) \\
S \arrow[r, hook] & G_m(\mathfrak{m}_n/\mathfrak{m}_n^{l+1}) &
\end{tikzcd}$$

where we identify $J^l(n,p)$ with $\mathrm{Hom}(k^{p^*}, \mathfrak{m}_n/\mathfrak{m}_n^{l+1})$. Then, the composition $\mathrm{Hom}(k_\cdot^{p^*}, \gamma_m \mid S) \to J^l(n,p)$ is a diffeomorphism from an open dense set of $\mathrm{Hom}(k^{p^*}, \gamma_m | S)$ (viewed as a manifold) onto Σ_s. If S is closed, then this is a blow-up of $\mathrm{Cl}(\Sigma_S)$. These statements follow from the more general version given in Theorem 15. While the case of $n=r$ can be given this especially simple form, the general case $r<n$ requires some gymnastics with constructions of bundles to first reduce the problem to one over $\Sigma_r \subset J^1(n,p)$. Hopefully, this will clarify the general construction which we turn to next.

We make several additional restrictions on S : if $I \in S$ then $\mathfrak{m}_r^l \subset I$ and $I \subset \mathfrak{m}_r^2$. This guarantees that $\Sigma_S \subset \Sigma_r^l$ and that we are not really ignoring part of S in constructing Σ_S.

We first describe a canonical construction of the blow-up. Over Σ_r^l are canonical bundles K and C such that if $z \in \Sigma_r^l$, $K=\ker(dz)_0$, $C=\mathrm{coker}(dz)_0$. If W is a k-vector space, we let $W_\cdot$ denote the trivial vector bundle $W \times X \to X$ over a topological space X. The space X will be clear from the context. Then $K \subset k_\cdot^n$ and C is a quotient bundle of $k_\cdot^p$. Furthermore, if $S^i V$ denotes the ith-symmetric product of either vector spaces or vector bundles, we use the following notation suggesting for bundles the situation analogous to that for ideals in $A_l(n)$:

$$M_n^j/M_n^{l+1} = \textstyle\sum_{i=j}^l S^i(k_\cdot^{n^*}),$$

$$M_r^j/M_r^{l+1} = \textstyle\sum_{i=j}^l S^i(K^*).$$

First, we give a standard blow-up of $\mathrm{Cl}(\Sigma_r)$. Over $G_r(k^n) \times G_{n-r}(k^p)$ (which we will denote by $G_r \times G_{r'}$, $r'=n-r$) we have canonical bundles γ_r and $\gamma_{r'}$ (pull-backs of the canonical bundles on each factor). There is a natural map

$$\begin{tikzcd}
\mathrm{Hom}(k_\cdot^n/\gamma_r, \gamma_{r'}) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}(k^n, k^p) \arrow[r, "\sim"] & J^1(n,p) \\
G_r \times G_{r'} & &
\end{tikzcd}$$

sending $h \overset{\theta}{\mapsto} i \circ h \circ \pi$ where $\pi : k^n_\cdot \to k^n_\cdot/\gamma_r$ and $i : \gamma_{r'} \hookrightarrow k^p_\cdot$. Let

$$U = \{ h \in \mathrm{Hom}(k^n_\cdot/\gamma_r, \gamma_{r'}) : \mathrm{rank}\, h = n-r \}.$$

This is an open dense subbundle and it is a standard result that $\theta : U \to \Sigma_r$ is a diffeomorphism. Then, θ is proper (it is the composition of an inclusion of vector bundles over $G_r \times G_{r'}$ together with the projection $\mathrm{Hom}(k^n_\cdot, k^p_\cdot) \to \mathrm{Hom}(k^n, k^p)$ which has compact fibers $G_r \times G_{r'}$). Thus, θ is a closed mapping so $\theta(\mathrm{Hom}(k^n_\cdot/\gamma_r, \gamma_{r'}))$ is closed and contains Σ_r so it contains $\mathrm{Cl}(\Sigma_r)$. As $\theta^{-1}(\mathrm{Cl}(\Sigma_r))$ is closed and contains u, it is all of $\mathrm{Hom}(k^n_\cdot/\gamma_r, \gamma_{r'})$; thus θ is a blow-up. We build upon this blow-up to obtain one for $\mathrm{Cl}(\Sigma_S)$.

Over $G_r \times G_{r'}$ we have a mapping of vector bundles:

$$\begin{array}{c} \mathrm{Hom}(k^{p^*}_\cdot, M_n^2/M_n^{l+1}) \oplus \mathrm{Hom}(k^n_\cdot/\gamma_r, \gamma_{r'}) \to J^l(n,p) \\ \downarrow \\ G_r \times G_{r'} \end{array}$$

whose restriction yields a diffeomorphism:

$$\mathrm{Hom}(k^{p^*}_\cdot, M_n^2/M_n^{l+1}) \times U \to \Sigma_r^l.$$

The mapping is induced by:

$$\mathrm{Hom}(k^{p^*}_\cdot, M_n^2/M_n^{l+1}) \oplus \mathrm{Hom}(k^n_\cdot/\gamma_r, \gamma_{r'}) \to \mathrm{Hom}\left(\textstyle\sum_{i=2}^l S^i(k^n_\cdot), k^p_\cdot\right) \oplus \mathrm{Hom}(k^n_\cdot, k^p_\cdot).$$

Since we can identify an l-jet z with its derivatives $(dz)_0, (d^2 z)_0, \ldots, (d^l z)_0$:

$$J^l(n,p) \xrightarrow{\sim} \mathrm{Hom}(\Sigma_{i=1}^l S^i(k^n), k^p).$$

Next, consider $G_m(M_r^2/M_r^{l+1})$.

We denote it by G_m. It has a canonical bundle γ_m. By our assumptions on S, we can also view $S \subset G_m(\mathfrak{m}_r^2/\mathfrak{m}_r^{l+1})$. It is invariant under the action of $\mathrm{Aut}_l(r)$; thus we can form a subbundle $\mathscr{S} \subset G_m(M_r^2/M_r^{l+1})$ with fibers S. Then, the fibers in $\gamma_m | \mathscr{S}$ are just the ideals in S.

Then we have the diagram:

$$
\begin{tikzcd}
L \arrow[rr, "\lambda'"] \arrow[d, "\pi'"] \arrow[drr, phantom, "\boxed{1}"] & & \mathrm{Hom}\,(k^{p*}, M_n^2/M_n^{l+1}) \arrow[r, "\theta'"] \arrow[d, "\pi"] & \mathrm{Cl}\,(\Sigma_r^l) \arrow[dd] \\
\mathrm{Hom}\,(C^*, \gamma_m|) \arrow[rr] \arrow[d, "\rho'"] \arrow[drr, phantom, "\boxed{2}"] & & \mathrm{Hom}\,(C^*, M_r^2/M_r^{l+1}) \arrow[d, "\rho"] & \\
H \arrow[rr, "\lambda''"] \arrow[d, "\pi_1'"] & & \mathrm{Hom}\,(k^n/\gamma_r, \gamma_{r'}) \arrow[r, "\theta"] \arrow[d, "\pi_1"] & \mathrm{Cl}\,(\Sigma_r) \\
S \arrow[r] & G_m \arrow[r] & G_r \times G_{r'} &
\end{tikzcd}
$$

where L is the pull back of Hom(,) by λ, which is induced by the inclusion of $\gamma_m|$ (the pull-back of the restriction of γ_m) in M_r^2/M_r^{l+1}. Also, H is the pull-back of $\mathrm{Hom}\,(k^n/\gamma_r, \gamma_{r'})$ over $\mathscr{S}$, and π is a projection map of vector bundles both pulled over $\mathrm{Hom}\,(k^n/\gamma_r, \gamma_{r'})$ induced by the projection $M_n^2/M_n^{l+1} \to M_r^2/M_r^{l+1}$ along the ideal generated by $\mathrm{Im}\,(C^{*\perp})$ in M_n/M_n^{l+1}. Here C^* can be identified with the subbundle of k^{p*} which annihilates $\ker\,(k^p \to C)$, and $C^{*\perp}$ denotes the complement with respect to the natural inner product on k^{p*}.

Now we can define:

$$L_S = \{ f \in L : \pi'(f) + \cdot : C^* \oplus (M_r/M_r^{l+1} \otimes \gamma_m|) \to \gamma_m| \text{ is onto and } \rho \circ \pi \circ \lambda'(f) \in U \}.$$

Here " . " $M_r/M_r^{l+1} \otimes \gamma_m| \to \gamma_m|$ is induced by multiplication in the symmetric algebra. Then, L_S consists of those f such that the ideal generated by $\pi'(f)(C^*)$ is $\pi_1' \circ \rho' \circ \pi'(f)$.

LEMMA 14. — *If S is a singular submanifold of $G_m(\mathfrak{m}_r/\mathfrak{m}_r^{l+1})$ then L and L_S are singular manifolds.*

Proof. — $\mathcal{S}$ be a sub-bundle of $G_m \to G_r \times G_{r'}$; hence by the local triviality, $\mathcal{S}$ is also a singular manifold in either case. Then, both L and L_S are subfiber bundles of algebraic vector-bundles over G_m (and hence $\mathcal{S}$) whose fibers are Zariski open-subsets. Thus, they are either constructible or subanalytic.

We have a map $L_S \to \Sigma_S$; however it is not *a priori* clear that Σ_S is a singular manifold. This, in fact, is included in the following theorem.

THEOREM 15. — *If S is a singular submanifold of $G_m(\mathrm{m}_r/\mathrm{m}_r^{l+1})$ satisfying the preceding restrictions, then:*

(i) *Σ_S is a singular submanifold of $J^l(n, p)$;*

(ii) *if S is closed, then $L \to \mathrm{Cl}(\Sigma_S)$ is a blow-up;*

(iii) *$L_S \to \Sigma_S$ restricts to a diffeomorphism between open dense smooth submanifolds;*

(iv) *if S is smooth then Σ_S is smooth.*

Proof. — First, we show that if S is closed then $L \to \mathrm{Cl}(\Sigma_r^l)$ is proper. For this we note that $\theta : \mathrm{Hom}(k_r^n/\gamma_r, \gamma_{r'}) \to \mathrm{Cl}(\Sigma_r)$ is proper, and if:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is a pull-back diagram with f proper then f' is proper. Then, θ' is equivalent to a pull-back diagram over θ, so θ' is proper. Also, $\boxed{1}$ is a pull-back diagram. Next λ can be factored as an inclusion of vector bundles $\mathrm{Hom}(C^*, \gamma_m|) \to \mathrm{Hom}(C^*, M_r^2/M_r^{l+1})$ over H and a pull-back of $\mathrm{Hom}(C^*, M_r^2/M_r^{l+1})$ over λ''. In turn λ'' is a pull back over $\mathcal{S} \to G_m \to G_r \times G_{r'}$; this composition is a projection of a bundle with compact fiber, and hence proper.

Thus (ii) will follow from (i) and (iii). For (i) we may apply Chevalley's theorem (*see* e.g. [12]) in the complex case to conclude image (L_S), image (L) are constructible. For the real case we may apply a result of Hironaka [13] or [14] to conclude that image(L_S) and image(L) are subanalytic sets. However, it is easily seen that image$(L_S) = \Sigma_S$. Then, by properness $\mathrm{Cl}(\Sigma_S) = \mathrm{image}(L)$.

For (iii) we define:

$$W_S = \{ h \in L_S : \pi_1' \circ \rho' \circ \pi'(h) \text{ is a smooth point of } \mathscr{S} \}.$$

This is the subfiber bundle $L_S \,|$ (smooth points of $\mathscr{S}$). The smooth points of $\mathscr{S}$ form an open dense set; thus, W_S is open and dense in L_S.

We claim $W_S \to \operatorname{Im}(W_S)$ is a diffeomorphism and $\operatorname{Im}(W_S)$ is dense in Σ_S. It is $1-1$ since $\operatorname{Im}(\pi'(h))$ uniquely determines I. It is trivially onto and continuous. To show it is a local diffeomorphism, we note that the maps are maps of fiber bundles. Thus, it is sufficient to prove it for a fiber.

We define an inverse for a fiber F of Σ_S over a $z_1 \in \Sigma_r$. Thus, we have a fixed K and C with $\ker(dz)_0 = K$ and $\operatorname{coker}(dz)_0 = C$ for $z \in F$. We further choose coordinates so that $K = k^r$, $\operatorname{Im}(dz_1)_0 = \{0\} \times k^{n-r}$ and $C = k^{p-n+r} \times \{0\}$. Then we define:

$$\psi : \quad F \to \operatorname{Hom}(k^{p^*}, M_n^2/M_n^{l+1})$$

by:

$$\psi(z) = (K, C, I(z), (dz)_0^*, \ldots, (d^l z)_0^*),$$

where $(d^j z)_0^* : k^{p^*} \to S^j k^{n^*}$ is the dual of $(d^j z)_0$. Also, $I(z)$ is the image of the ideal generated by $y_i \circ z$, $1 \leqslant i \leqslant p$, under the projection $\mathfrak{m}_n^2/\mathfrak{m}_n^{l+1} \to \mathfrak{m}_r^2/\mathfrak{m}_r^{l+1}$ along the ideal $\mathfrak{m}_n . (y_i \circ z, p-n+r < i \leqslant p)$. Then, ψ is an algebraic mapping and is clearly an inverse to $\theta' \circ \lambda' \,|$(fiber of W_S over $\theta^{-1}(z_1)$). Thus, $\theta' \circ \lambda' \mid W_S$ is a homeomorphism. Since in some neighborhood of any point, ψ can be extended to an algebraic mapping, it follows that $\theta' \circ \lambda' \,|$(fiber of W_S) has maximal rank (as composition with the extension of ψ is the identity). Thus, $\theta' \circ \psi' \mid W_S$ is a local diffeomorphism onto Σ_S and hence a global diffeomorphism.

To complete the proof of (iii) we must show that $\operatorname{Im}(W_S)$ is dense in Σ_S. We note that any $z \in \Sigma_S$ may be slightly perturbed to give z' in some $\mathscr{K}^l$-orbit with ideal I' a smooth point in S. Then z' is in the image of W_S.

Lastly, if S is smooth, and $z \in \Sigma_S$, then $L_S = W_S$ so that z is in the image of W_S. Now W_S is smooth and W_S maps onto a neighborhood of z in Σ_S. Thus, Σ_S is smooth near z.

As a corollary we have:

COROLLARY 17. — *The Hilbert-Samuel strata Σ_h are smooth submanifolds of Σ_2^l.*

Next, we use this result in the case S is smooth to determine the normal bundle to Σ_S.

PROPOSITION 18. — *The normal bundle of Σ_S in Σ_r^l pulled back to W_S is isomorphic to* $\mathrm{Hom}(C^*, \gamma_m^\perp)/TS$ *where $\gamma_m^\perp = (M_r^2/M_r^{l+1})/\gamma_m$ and TS denotes the pull-back of the bundle along the fiber of $\mathscr{S} \to G_r \times G_{r'}$.*

Proof. — As θ' is a diffeomorphism on $(\pi \circ \rho)^{-1}(U)$ *and* $\lambda \mid W_S$ is a diffeomorphism onto its image, it is enough to compute the normal bundle $\lambda'^* \, T\,\mathrm{Hom}(k^{p^*}, M_n^2/M_n^{l+1})/TL_S$ restricted to W_S. As $\boxed{1}$ is a pull-back diagram, it is enough to compute the normal bundle of λ. Now λ can be factored:

$$\begin{tikzcd}
\mathrm{Hom}(C^*, \gamma_m \mid) \arrow[r, "\lambda_1"] \arrow[d] \arrow[dr, phantom, "\boxed{4}"] & \mathrm{Hom}(C^*, \gamma_m) \arrow[r, "\lambda_2"] \arrow[d] \arrow[dr, phantom, "\boxed{3}"] & \mathrm{Hom}(C^*, M_r^2/M_r^{l+1}) \arrow[d] \\
\tilde{\mathscr{S}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & \tilde{G}_m \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{2}"] & \mathrm{Hom}(k^{n}/\gamma_r, \gamma_{r'}) \arrow[d] \\
\mathscr{S} \arrow[r] & G_m \arrow[r] & G_r \times G_{r'}
\end{tikzcd}$$

where $\boxed{1}$, $\boxed{2}$, and $\boxed{4}$ are pull-back diagrams. Then, the normal bundle of λ is the sum of the normal bundles of λ_1 and λ_2 pulled back to $\mathrm{Hom}(C^*, \gamma_m \mid)$. Then normal bundle of λ_1 is then the pull-back of the normal bundle of $\mathscr{S} \to G_m$; while the normal bundle of λ_2 can be computed since $\boxed{2}$ is a pull-back diagram. By looking in a fiber it is easy to see that the normal bundle of λ_2 is the pull-back of $\mathrm{Hom}(C^*, \gamma_m^\perp)/TG_m$ where TG_m denotes the bundle along the fiber of $G_m \to G_r \times G_{r'}$. The normal bundle of λ_1 is the pull back of TG_m/TS (again the quotient of bundles along the fibers). Thus, the normal bundle is given by:

$$\mathrm{Hom}(C^*, \gamma_m^\perp)/TG_m \oplus TG_m/TS$$

which is isomorphic to $\mathrm{Hom}(C^*, \gamma_m^\perp)/TS$. □

We describe how to compute this for S_b. By Briançon's proof of the smoothness of S_b, there is in a neighborhood of any point $I \subset S_b$, a vector bundle F spanned by the generators $\{x^{\alpha_i} g_i'\}$ of I' in the neighborhood. As we are only interested in elements of Σ_b up to $\mathscr{K}^l$-

equivalence, if $p-n \geqslant \nu(h)-1$, we may assume $z \in \Sigma_{\mathfrak{h}}$ is an l-jet so that the coordinate functions of z span F (which is generated by $\nu(h)+1$ elements). We may choose a complementary bundle $F^{\perp}$ to F in $M_r \cdot \gamma_m$ (restricted to the appropriate neighborhood). Thus, we may project the image of C^* onto F. For a small neighborhood near z', this mapping will be surjective. Then $\operatorname{Ker}(C^* \to F)$ is a sub-vector bundle. Let E be a complementary bundle. We have:

COROLLARY 19. — *The pull-back of the normal bundle of $\Sigma_{\mathfrak{h}}$ in Σ_2' is given in a neighborhood of z' by*

$$\operatorname{Hom}(C^*/E, \gamma_m^{\perp}) \oplus \operatorname{Hom}(E, \gamma_m^{\perp})/TS_{\mathfrak{h}}.$$

Proof. — It is immediate that:

$$\operatorname{Hom}(C^*, \gamma_m^{\perp}) \xrightarrow{\sim} \operatorname{Hom}(C^*/E, \gamma_m^{\perp}) \oplus \operatorname{Hom}(E, \gamma_m^{\perp}).$$

By the result of Briançon, $TS_{\mathfrak{h}}$ is a subbundle of $\operatorname{Hom}(E, \gamma_m^{\perp})$. $\square$

Now, exactly as in [6], I, we will identify this normal bundle with a tubular neighborhood of $\Sigma_{\mathfrak{h}}$; and to identify the Hilbert-Samuel partition near $\Sigma_{\mathfrak{h}}$, it is sufficient to identify it in the normal bundle.

6. The deformed Hilbert-Samuel function

To transform information about the Hilbert-Samuel partition in the normal bundle to topological form, we use a topological induction argument with induction on δ. For this, we shall relate information about the Hilbert-Samuel functions of near-by algebras Q' with $\delta(Q')<\delta(Q)$ with properties of the Hilbert-Samuel function of the algebra. To do so, we define the *deformed Hilbert-Samuel function of* Q, $\tilde{\mathfrak{h}}_Q$. We say the family Q_t' is near by Q if $Q_t' \xrightarrow{\sim} k[[\mathbf{x}_m]]/I_t$ with $I_t=(h_1(x,t), \ldots, h_s(x,t))$ for some $m, s>0$ and $h_i(x,t) \in k\{\mathbf{x}_m, t\}$ so that $Q_0' \xrightarrow{\sim} Q$. We define:

$$\tilde{\mathfrak{h}}_Q(j) = \max\{ r : \text{there is } Q_t' \text{ near-by } Q \text{ with } \delta(Q_t')<\delta(Q) \text{ for } t \neq 0 \text{ so that } \mathfrak{h}_{Q_t}(j)=r \text{ for } t\neq 0 \}.$$

Remark. — There is a potential problem in relating the algebra and geometry here. If $Q \xrightarrow{\sim} Q(f)$, $f: k^n, 0 \to k^p, 0$, then even though Q_t' occurs near-by Q, we may not be able to realize $Q_t' \xrightarrow{\sim} Q(f_{x(t)})$ for $x(t) \to 0$ as $t \to 0$. In fact, by [prop. 4.1; 6, I], the condition we need is that $p-n \geqslant -i(Q_t')$. However, using the structure of the generic stairs for $I \subset k[[x_2]]$, we have $-i(Q_t') \leqslant \nu(Q_t')-1$. If Q_t' occurs near-by Q then $\nu(Q_t') \leqslant \nu(Q) = \nu(Q(f))$. Thus, our assumption on $p-n$ yields $p-n \geqslant \nu f)-1 \geqslant \nu(Q_t')-1 \geqslant -i(Q_t')$. Hence, there is no problem.

To relate $\tilde{\mathfrak{h}}_Q$ and $\mathfrak{h}_Q$, we also define the invariants:

$$\beta(Q) = \inf\{j : \mathfrak{h}_Q(j) = \delta(Q)\} - 1,$$
$$\alpha(Q) = \mathfrak{h}_Q(k+1) - \mathfrak{h}_Q(k) \qquad \text{where} \quad k = \beta(Q)$$

and

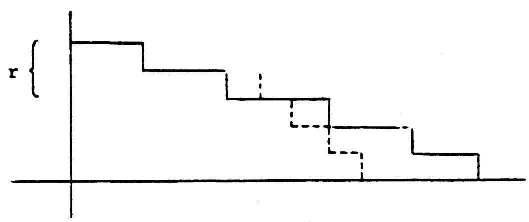
$$\tilde{\beta}(Q) = \inf\{j : \tilde{\mathfrak{h}}_Q(j) = \delta(Q) - 1\} - 1.$$

Then, $\beta(Q)$ is the last integer at which $\mathfrak{h}_Q$ increases; $\tilde{\beta}(Q)$ is (as we shall see in what follows) the last integer at which $\tilde{\mathfrak{h}}_Q$ increases; and $\alpha(Q)$ is the size of the last increase of $\mathfrak{h}_Q$. The Hilbert-Samuel functions can then be related by:

PROPOSITION 20. — *The functions $\mathfrak{h}_Q$ and $\tilde{\mathfrak{h}}_Q$ are related by:*

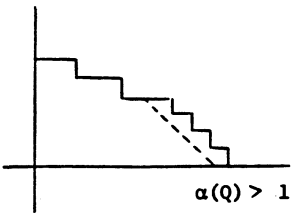
$$\mathfrak{h}_Q(j) = \begin{cases} \tilde{\mathfrak{h}}_Q(j), & j < \tilde{\beta}(Q), \quad \text{or } j = \tilde{\beta}(Q) \quad \text{and} \quad \alpha(Q) = 1, \\ \tilde{\mathfrak{h}}_Q(j) + 1, & j > \tilde{\beta}(Q), \quad \text{or } j = \tilde{\beta}(Q) \quad \text{and} \quad \alpha(Q) > 1. \end{cases}$$

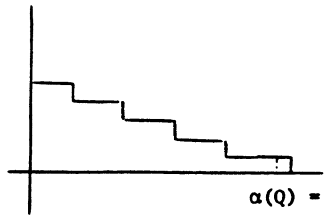
Proof. — By the upper semi-continuity of $\mathfrak{h}(j)$ (*see* e.g. [Prop. 2.3, 6, I]), $\tilde{\mathfrak{h}}_Q(j) \leqslant \mathfrak{h}_Q(j)$, $j \geqslant 0$. We consider the generic stairs of I where $Q = k[[x_2]]/I$ with $I \subset \mathfrak{m}_2^2$.



Suppose $j < \beta(Q)$. If there are r standard generators not in $\mathfrak{m}_2^{j+1}$ then $\dim\ I \cap \mathfrak{m}_2^{j+1}/\mathfrak{m}_2^{j+2} = j+2-(\nu-r)$. Thus, the remainning $\nu-r-1$ standard generators can be deformed to the remaining generators of $\mathfrak{m}_2^{j+1}$. Thus, there is a near-by family Q_t' with $\mathfrak{h}_{Q_t}(j) = \mathfrak{h}_Q(j)$, $j < \beta(Q)$. Hence $\tilde{\mathfrak{h}}_Q(j) \geqslant \mathfrak{h}_Q(j)$, $j < \beta(Q)$, so we must have equality.

For $j \geqslant \beta(Q)$ and any $\alpha(Q)$ we can deform





the standard generator x^{α_v} to $x^{\alpha_v - 1}$. We obtain a Q'_t with

$$\mathfrak{h}_{Q_t}(j) = \mathfrak{h}_Q(j) - 1\,(= \delta - 1) \quad \text{for } j \geqslant \beta(Q).$$

Thus, $\tilde{\mathfrak{h}}_Q(j) = \delta - 1$ for $j \geqslant \beta(Q)$ or $\mathfrak{h}_Q(j) = \tilde{\mathfrak{h}}_Q(j) + 1$ for $j \geqslant \beta(Q)$. Now it is a simple matter of observing that:

$$\text{if} \quad \alpha(Q) = 1, \qquad \tilde{\beta}(Q) = \beta(Q) - 1,$$

while

$$\text{if} \quad \alpha(Q) > 1, \qquad \tilde{\beta}(Q) = \beta(Q). \quad \square$$

Thus by the preceding Proposition, $\mathfrak{h}_{Q(f)}$ is determined by $\tilde{\mathfrak{h}}_{Q(f)}$, $\tilde{\beta}(Q(f))$, and $\alpha(Q(f))$. The first two invariants are by induction, topological invariants determined by the topological structure of $\mathcal{T}(f)$ or $\mathcal{T}^0(f)$. It remains to show that $\alpha(Q(f))$ is also determined by the topological structure of these partitions. In fact, we shall see that it is determined by the topological structure of the deformed Hilbert-Samuel stratum $\Sigma_{\mathfrak{h}}^{\sim}$ near $\Sigma_{\mathfrak{h}}$.

Remark. — $\nu(\tilde{\mathfrak{h}}) = \nu(\mathfrak{h})$ except when $\mathfrak{h}$ is the Hilbert-Samuel function of $\mathfrak{m}_2^j$, some j. In that case, $\mathfrak{m}_2^j$ can be characterized topologically as the ideal whose deformed Hilbert-Samuel function has lower order. The topological description given in the next section will be valid in these cases; however the proof must be slightly modified for $\mathfrak{m}_2^2$ for $\tilde{\mathfrak{h}}$ is then a Hilbert-Samuel function of a Σ_1-type (in fact, then $\mathrm{Cl}(\Sigma_{\tilde{\mathfrak{h}}})$ near $\Sigma_{\mathfrak{h}}$ is the same as $\mathrm{Cl}(\Sigma_1)$ near Σ_2).

7. Topological structure of the deformed Hilbert-Samuel strata

We have shown in paragraph 6 that we can topologically recover $\mathfrak{h}$ from $\tilde{\mathfrak{h}}$ if we know whether $\alpha(Q) = 1$ or > 1. We now show how to determine which condition α satisfies using the topological structure of $\Sigma_{\mathfrak{h}}^{\sim}$ near $\Sigma_{\mathfrak{h}}$.

PROPOSITION 21. — *If $f: k^n, 0 \to k^p, 0$, is an infinitesimally stable germ of type Σ_2 with Hilbert-Samuel fonction $\mathfrak{h}$ and $p-n \geqslant \nu(f)-1$, then as germs:*

$$\mathrm{Cl}\,(\Sigma_{\mathfrak{h}}^{-}(f), 0) \xrightarrow{\sim} (\mathrm{Cone}\,(\mathbb{S}\, E) \times k^c, (*, 0)).$$

Here the notation is the same as that of paragraph 3 where $E=\gamma' \otimes \gamma$ and γ' and γ are the pull-back to $P(k^{q*}) \times P(k^{\alpha})$ of the canonical bundles on the factors. Here q is an integer depending on $\mathfrak{h}$ and $q \geqslant \alpha$. Using local cohomology and the fact that, with S denoting suspension,

$$\mathrm{Cone}(X) \times \mathbb{R}^c \xrightarrow{\sim} \mathrm{Cone}(S^c X),$$

we have for total cohomology:

$$\begin{aligned}\dim H^*\,(\mathrm{Cl}\,\Sigma_{\mathfrak{h}}^{-}(f);\, C_k)_{\mathrm{loc}} \\ &= \dim H^*\,(\mathrm{Cone}\,(S^c\, \mathbb{S}\, E));\, C_k)_{\mathrm{loc}} = \dim H^*\,(\mathrm{Cone}\,(\mathbb{S}\, E);\, C_k)_{\mathrm{loc}}.\end{aligned}$$

As $q \geqslant \alpha$, by Lemma 10, the dimension equals $2\alpha-1$. Hence we have as a corollary, the topological determination of α.

COROLLARY 22. — *With f as in the preceding proposition:*

$$\dim_{C_k} H^*\,(\mathrm{Cl}\,(\Sigma_{\mathfrak{h}}^{-}(f));\, C_k)_{\mathrm{loc}} = 2\,\alpha\,(Q\,(f)) - 1.$$

In particular, $\mathrm{Cl}\,(\Sigma_{\mathfrak{h}}^{-}(f))$ has the local cohomology of a topological manifold at 0 exactly when $\alpha=1$.

Proof. — We begin by performing several reductions. We recall the following diagram from paragraph 5, for the special case of the Hilbert-Samuel strata $\Sigma_{\mathfrak{h}} \subset \Sigma_2^l$.

$$\begin{array}{ccccc} W_S = L_S \subsetneq L & \longrightarrow & \mathrm{Hom}\,(k^{p*}, M_n^2/M_n^{l+1}) & \longrightarrow & \mathrm{Cl}\,(\Sigma_2^l) \\ \downarrow & & \downarrow & & \\ H & \longrightarrow & U \subset \mathrm{Hom}\,(k^n/\gamma_2, \gamma_{n-2}) & \longrightarrow & \mathrm{Cl}\,(\Sigma_2) \\ \downarrow & & \downarrow & & \\ S_h & \longrightarrow & G_m \to G_2 \times G_{n-2} & & \end{array}$$

where U consists of linear maps of rank $n-2$.

We remark that the image of L *via* the top line is exactly $\Sigma_{\mathfrak{h}}$.

1) Let z be the l-jet of f and $I=I(z)$ be the ideal generated by z. We have $z\in\Sigma_b$ and $\mathfrak{m}^l\subset I$. To establish Proposition 21, it is enough to determine $\mathrm{Cl}(\Sigma_b^{\sim})$ in a tubular neighborhood of Σ_b which we can identify with the normal-bundle to Σ_b in Σ_2^l (*see* [6], I, for details).

2) Both Σ_b and $\Sigma_b^{\sim}\subset\Sigma_2^l$ are fiber-bundles over Σ_2 (except for the special case described in the Remark at the end of paragraph 6; however, this case already satisfies Proposition 21 by [6], I). Thus, to describe the closure of $\Sigma_b^{\sim}$ near Σ_b it is sufficient to consider fibers over a fixed 1-jet $z_1\in\Sigma_2$.

We remark that for fixed z_1 the fibers of M_2^2/M_2^{l+1} and M_n^2/M_n^{l+1} become the ideals $\mathfrak{m}_2^2/\mathfrak{m}_2^{l+1}$ and $\mathfrak{m}_n^2/\mathfrak{m}_n^{l+1}$.

3) We fix $z_1\in U$ and we consider the fiber S in S_b lying above $(\mathrm{Ker}\,dz_1, \mathrm{Im}\,dz_1)$. In this fiber, there is a neighborhood $V\subset S$ of I so that all ideals I' in V have the same generic stairs as I, with respect to some fixed set of coordinates.

Let $h\in L_S$ map to $z_1\in\Sigma_b$. Then, there is an open neighborhood V_1 of h, in the fiber of L_S over z_1, which maps onto a neighborhood of z_1 in Σ_b by Theorem 15. Also, we may suppose that V_1 lies over V.

4) We can choose local coordinates such that:

$$z_1(x_1, \ldots, x_n)=(x_3, \ldots, x_n, \underbrace{0, \ldots, 0}_{p-n+2\ \text{times}}).$$

This choice of coordinates gives us a splitting:

$$\mathfrak{m}_2^2/\mathfrak{m}_2^{l+1} \subsetneq \mathfrak{m}_n^2/\mathfrak{m}_n^{l+1}.$$

Note that x_1, x_2 are coordinates for $\mathrm{Ker}(dz_1)$, but we shall denote them by x, y.

5) Since everything is invariant under the action of $\mathscr{K}^l$, the normal structure is preserved by the $\mathscr{K}^l$-action, so it is enough to examine it for any point in the same $\mathscr{K}^l$-orbit of the given point. We shall use this later to simplify our choice of z.

6) We may replace V_1 by a section of V_1 over V because every fiber in V_1 over V lies in a single $\mathscr{K}^l$- orbit. Thus, the normal structure of Σ_b at a point in V_1 is independant of the point choosen in a given fiber.

Now we determine the normal bundle of Σ_b restricted to V_1 using Corollary 19.

Notice that $\gamma_m^{\perp}$ is a trivial bundle and can be identified with $V_1\times\Delta(I)$ since the generic stairs are constant. (Recall $\Delta(I)$ is the subspace spanned by monomials under the generic stairs). Also, C is a trivial bundle whose fiber is the subspace corresponding to the last $p-n+2$ coordinates of k^p. Also, the subbundle E is trivial over V; therefore, $E=V\times E_0$ for a vector space E_0. Then, if $\{e_i : 0\leqslant i\leqslant \nu\}$ is a basis for E_0, we define a section of E over V by $s(I')=h'$ where

$h'(e_i)=i$th element of the standard basis for the generic stairs of I'.

By corollary 19, the normal bundle to $\Sigma_{\mathfrak{h}}$ over $s(V)$ is a quotient of:

$$V\times \mathrm{Hom}(E_0, \Delta(I))\times \mathrm{Hom}(E_0^{\perp}, \Delta(I)).$$

7) To define the normal bundle map we first define a map:

$$V\times \mathrm{Hom}(E_0, \Delta(I))\times \mathrm{Hom}(E_0^{\perp}, \Delta(I)) \to J^l(n,p),$$
$$(I, h_1, h_2) \to h+h_1+h_2,$$

where $h=s(I)$ and where we extend h, h_1 and h_2 to be zero on the complements of E_0 or $E_0^{\perp}$.

Later we shall have to look at the induced map on the quotient; but first we examine which conditions must be imposed on (h_1, h_2) to ensure that the image lies in $\Sigma_{\mathfrak{h}}^{\sim}$. These will be given by series of simple Lemmas.

We let:

$$h(e_i)=f_i, \quad 0\leqslant i\leqslant \nu, \quad \text{and zero elsewhere},$$
$$h_1(e_i)=v_i, \quad 0\leqslant i\leqslant \nu, \quad \text{and zero elsewhere},$$
$$h_2(e_i)=v_i, \quad \nu\leqslant i\leqslant p-n+1, \quad \text{and zero elsewhere}.$$

Then, $h+h_1+h_2$ has an ideal generated by $\{f_i+v_i,\ 0\leqslant i\leqslant p-n+1\}$. We let $f_i'=f_i+v_i$ and also let Q' denote the associated quotient algebra of $k[[x_2]]$ by this ideal.

LEMMA 23. — *If* $\mathfrak{h}_{Q'}(j)=\mathfrak{h}_Q(j), j<\beta(Q)$, *then for* $i<\nu-\alpha$

$$\exp(f_i')=\exp(f_i) \quad \textit{and} \quad \mathrm{ord}(f_i')=\mathrm{ord}(f_i).$$

Proof. — The proof is by induction on $i<\nu-\alpha$. If $i=0$, then $f_0'=f_0+v_0$ and f_0 is a monic polynomial of degree ν. If $\mathrm{ord}(v_0)<\nu$, then $\mathfrak{h}_{Q'}(\nu-1)\neq\mathfrak{h}_Q(\nu-1)$. Also, if $\exp(f_0')<\exp(f_0)$, then f_0 must contain a term y^l, $l<\nu$, and again $\mathrm{ord}(f_0')<\mathrm{ord}(f_0)$.

Suppose by induction the result is true for $i<l<\nu-\alpha$. As $f_l'=f_l+v_l$, if $\exp(f_l')<\exp(f_l)$ then since $\exp(v_l)\notin E((f_i', i<l))$ we have

$v_l \notin (f_i', i<l) + \mathfrak{m}_2^{\alpha_l+\nu-l}$ (here $(f_i', i<l)$ denotes the ideal generated by the f_i', $i<l$). Thus, $\mathfrak{h}_{Q'}(\alpha_l+\nu-l-1) < \mathfrak{h}_Q(\alpha_l+\nu-l-1)$. Hence,

$$\operatorname{ord}(f_l') = \operatorname{ord}(f_l);$$

thus, ord $v_l \geqslant$ ord f_l. By the construction of the generic stairs, since $\exp(v_l) \notin E(f_i, i \leqslant l)$, $\exp(v_l) > \exp(f_l)$. □

Next, for the generators f_i, $\nu-\alpha<i\leqslant\nu$ we may choose $f_i = x^{\beta-i+1}\,y^{i-1}$ $(\beta=\beta(Q))$. Then,

LEMMA 24. — *Suppose* (h', h_1, h_2) *is as in lemma* 23 *and its image belongs to* $\Sigma_{\mathfrak{h}}^{\sim}$ *then either* $\operatorname{ord}(v_i)=\beta$ *or* $v_i=0$, *for* $\nu-\alpha\leqslant i\leqslant\nu$. *Also,* $\dim_k\langle v_i : \nu-\alpha\leqslant i\leqslant\nu\rangle = 1$.

Proof. — If for some i, $\operatorname{ord}(v_i)<\beta$, then by the choice of v_i, $\exp(v_i)\notin E(f_i, i<\nu-\alpha)$; thus we would have

$$\mathfrak{h}_{Q'}(\beta-1) < \mathfrak{h}_Q(\beta-1).$$

Thus, $\operatorname{ord}(v_i)=\beta$ (if it were greater $\exp(v_i)\in E(I)$). Also, if $\dim_k\langle v_i;\ \nu-\alpha\leqslant i\leqslant\nu\rangle>1$ then $\mathfrak{h}_{Q'}(\beta)<\mathfrak{h}_Q(\beta)-1$. Thus, the image of (h', h_1, h_2) would not belong to $\Sigma_{\mathfrak{h}}^{\sim}$. □

By Lemma 23, we have for $i<\nu-\alpha$, $f_i' = x^{\alpha_i}(g_i')$ where $g_i' = g_i + v_i'$ with $x^{\alpha_i} v_i' = v_i$. Thus, by our requirements that the $v_i\in\Delta(I)$ and our original conditions on g_i, it follows that:

$$\{\, x^i g_j',\ x^i g_l,\ 0\leqslant j<\nu-\alpha,\ \nu-\alpha\leqslant l\leqslant\nu,\ (i,j),\ (i,l)\notin E(I)\,\}$$

projects to a basis for Δ(I). Thus, we can write for $0\leqslant i<\nu-\alpha$.

$$\text{(A)}\qquad g_i' = (y+u_{i,i+1})\, g_{i+1}' + \textstyle\sum_{i+1<j<\nu-\alpha} u_{i,j}\, g_j' + \sum_{\nu-\alpha\leqslant j\leqslant\nu} u_{i,j}\, g_j$$

(where we recall $g_j = y^{\nu-j}$, $\nu-\alpha\leqslant j\leqslant\nu$); and for $i=\nu-\alpha-1$.

$$\text{(B)}\qquad g_i' = (y+u_{i,i+1})\, g_{i+1} + \textstyle\sum_{\nu-\alpha<j\leqslant\nu} u_{i,j}\, g_j.$$

Here $u_{i,j}\in k[x]$.

With this representation, we can say:

LEMMA 25. — 1) *For* $0\leqslant i<j<\nu-\alpha$, ord $u_{i,j}\geqslant\alpha_j-\alpha_{i+1}$, $j-i$ *and* 2) *for* $0\leqslant i<\nu-\alpha$ *and* $\nu-\alpha\leqslant j\leqslant\nu$, $\operatorname{ord}(u_{i,j})\geqslant\alpha_j-(\alpha_{i+1}+1)$ (*where for* 2) $\alpha_j-\alpha_{i+1}\geqslant j-i$ *since* $\alpha_{\nu-\alpha+1}-\alpha_{\nu-\alpha}\geqslant 2$).

Proof. — The stairs $E(I)$ and $E(I')$ agree beneath the stairs of $\mathfrak{m}_2^\beta$. However, $x^{\alpha_i} g_i' \in I'$, $0 \leqslant i < \nu - \alpha$. We consider the largest i and, for that i, the smallest j such that 1) or 2) fails. Then, multiply either (A) or (B) by $x^{\alpha_{i+1}}$. Then, by assumption $u_{i,j}\, x^{\alpha_{i+1}} g_{j'}' \in I'$, $j' < j$. Thus, for example, if $j < \nu - \alpha$:

$$h = \sum_{j \leqslant l < \nu - \alpha} u_{i,l}\, x^{\alpha_{i+1}} g_l' + \sum_{\nu - \alpha \leqslant l \leqslant \nu} u_{i,l}\, x^{\alpha_{i+1}} g_l \in I'.$$

However,

$$\exp(h) \leqq \exp(u_{i,j}\, x^{\alpha_{i+1}} g_j') = (\operatorname{ord}(u_{i,j}) + \alpha_{i+1}, \nu - j) < \exp(f_j)$$

contradicting $E(I) = E(I')$ beneath the stairs of $\mathfrak{m}_2^\beta$. A similar contradiction occurs if 2) is false. □

For h_1 and h_2 small, the $u_{i,j}$ will be in a sufficently small neighborhood of the $u_{i,j}$ associated to h. If we replace g_i' by $g_i' - w_i'$, where w_i' = terms of degree $\beta - \alpha_{i+1}$ in $\sum_{j=\nu-\alpha+1}^{\nu} u_{i,j} g_j$, then the $\{x^{\alpha_i} g_i'\}$ will be in a neighborhood of h given by Briançon's parametrization.

We can now define a map from a subbundle of the normal bundle so that the image contains $\Sigma_{\mathfrak{b}}^{-}$. We let V_1 be a neighborhood of h which projects to a neighborhood in $S_{\mathfrak{b}}$ which has a Briançon parametrization (here we assume that $\{f_i\}$ is a set of standard generators for the ideal associated to h). Also, we let N denote the vector space with basis $\{x^{\alpha_i - \alpha_{\nu-\alpha+1}} y^{\nu - i}, \nu - \alpha < i \leqslant \nu\}$. We define a map:

$$\psi : \quad V_1 \times \operatorname{Hom}(E, N) \times \operatorname{Hom}(E_0^{\perp}, N) \to J^l(n, p).$$

To define $\psi(h', \varphi_1, \varphi_2)$, we let the ideal associated to h' have standard generators $\{x^{\alpha_i} g_i\}$, then we define (*via* the natural identification $\operatorname{Hom}(k^{p^*}, M_n/M_n^{l+1}) \xrightarrow{\sim} J^l(n, p)$).

$$\psi(h', \varphi_1, \varphi_2) = \begin{cases} x^{\alpha_i} g_i', & 0 \leqslant i \leqslant \nu, \\ x^{\alpha_{\nu-\alpha+1}-1}\, \varphi_2(e_{i+1}), & \nu < i < p, \end{cases}$$

where g_i' is defined by:

$$g_i' = (y + u_{i,i+1})\, g_{i+1}' + \sum_{j=i+2}^{\nu-\alpha} u_{i,j}\, g_j' + x^{\alpha_{\nu-\alpha+1} - \alpha_{i+1} - 1}\, \varphi_1(e_i).$$

We then have:

LEMMA 26. — *The mapping ψ is an embedding in a neighborhood of h. Furthermore, in a neighborhood of $\psi(h)=\psi(h, 0, 0)$ in $J^1(n,p)$,* Im $(\psi) \supset \Sigma_{\mathfrak{h}}^{-}$.

Proof. — In fact, the second statement follows from the three preceding Lemmas, since they have shown that in neighborhood of $\Sigma_{\mathfrak{h}}$, any element of $\Sigma_{\mathfrak{h}}^{-}$ must be of the form $\psi(h', \varphi_1, \varphi_2)$.

To show that ψ is a local embedding in a neighborhood of h, it is enough to examine the Jacobian of ψ. However, we already know that $\psi \mid V_1$ is locally a diffeomorphism onto a neighborhood of $\psi(h)$ in $\Sigma_{\mathfrak{h}}$. Thus, it is sufficient to show that the $d\psi_{(h,0,0)}$ restricted to $\mathrm{Hom}(k^{p^*}, N)$ is non-singular with image in the complement of $T\,\Sigma_{\mathfrak{h}(\psi(h))}$. Now $d\psi\,(\mathrm{Hom}\,(E_0^{\perp}, N)) \subset \mathrm{Hom}\,(E_0, \Delta\,(I))$, which is in the complement of $T\Sigma_{\mathfrak{h}}$. It is then sufficient to show that the projection of $d\psi(\mathrm{Hom}(E_0, N))$ in $\mathrm{Hom}(E_0, \Delta(I))/TS_{\mathfrak{h}}$ is non-singular. However, if $\mathrm{pr} : \Delta(I) \to x^{\alpha_\nu - \alpha + 1 - 1}.N$ is the projection along $\{x^i y^j \in \Delta(I) : i+j<\beta\}$ (here β is as in paragraph 6) then from the Briançon parametrization we obtain $\mathrm{pr}(TS_{\mathfrak{h}})=0$. However, by the definition of ψ,

$$\mathrm{pr}(d\psi(\varphi_1)(e_i)) = x^{\alpha_\nu - \alpha + 1 - 1}\,\varphi_1(e_i)$$

so

$$\mathrm{pr}(d\psi(\mathrm{Hom}(E_0, N))) = \mathrm{Hom}\,(E_0, x^{\alpha_\nu - \alpha + 1 - 1}\,.N). \quad \square$$

Lastly, we can identify $\Sigma_{\mathfrak{h}}^{-}$ in $\mathrm{Im}(\psi)$.

LEMMA 27. — *With the preceding notation, the Hilbert-Samuel function of $\psi(h', \varphi_1, \varphi_2)$ agrees with $\mathfrak{h}$ for $j \leqslant \beta$ if and only if there is a line $L \subset N$ such that $\varphi_1 \in \mathrm{Hom}(E_0, L)$, $\varphi_2 \in \mathrm{Hom}\,(E_0^{\perp}, L)$ and φ_1 and φ_2 are not both identically zero.*

Proof. — We consider for h' with standard generators $\{f_i = x^{\alpha_i} g_i\}$, the standard relations:

$$x^{\alpha_{i+1}-\alpha_i} f_i - (y+u_{i,i+1})\, f_{i+1} - \sum_{j=i+2}^{\nu} \bar{u}_{i,j}\, f_j = 0$$

(where $\bar{u}_{i,j} = x^{\alpha_{i+1}}\, u_{i,j}$). If we replace f_i by $f_i' = x^{\alpha_i} g_i'$ (as in the definition of ψ), then instead of 0 we obtain

$$x^{\alpha_{i+1}}\,(x^{\alpha_\nu - \alpha + 1 - \alpha_{i+1} - 1}\, \varphi_1(e_i)) = x^{\alpha_\nu - \alpha + 1 - 1}\, \varphi_1(e_i).$$

Similarly, from the $E_0^{\perp}$ terms we obtain $x^{\alpha_{\nu-\alpha+1}-1}\,\varphi_2(e_i)$. Then if $\mathfrak{h}'$ is the Hilbert-Samuel function of $\psi(h', \varphi_1, \varphi_2)$, then $\mathfrak{h}'(\beta)<\mathfrak{h}(\beta)-1$ unless

$$\dim_k \langle x^{\alpha_{\nu-\alpha+1}-1}\,\varphi_i(e_j)\rangle=1. \quad \text{Thus, } \varphi_i(e_j)\in L\subset N$$

for some line L.

Conversely, let $\{f_i\}$ be a standard basis for the ideal associated to h' and $\{f'_i=x^{\alpha_i}g'_i\}$ be the corresponding generators for the ideal of $\psi(h', \varphi_1, \varphi_2)$. We first consider the Hilbert-Samuel functions below degree β. By its construction, $\{f'_i, 0\leqslant i\leqslant \nu-\alpha;\ x^{\beta-1-j}y^j, 0\leqslant j<\alpha\}$ is a distinguished deformation of $\{f_i, 0\leqslant i\leqslant \nu-\alpha;\ x^{\beta-1-j}y^j, 0\leqslant j<\alpha\}$. Thus, by an argument of Briançon in [1], if J' and J denote the ideals generated by the respective sets of elements, then the Hilbert-Samuel functions of these ideals agree, and the sets consists of the standard generators. Now, these Hilbert-Samuel functions agree with those for $\psi(h', \varphi_1, \varphi_2)$ and h' below degree β. Thus, they agree below degree β.

Lastly, for degree β, the only relations for $\{f'_i, 0\leqslant i\leqslant \nu-\alpha;\ x^{\beta-1-j}y^j, 0\leqslant j<\alpha\}$ are generated by the standard relations. Suppose $w\in x^{\alpha_{\nu-\alpha+1}-1}\,.\,N$ and $w=\sum r_i f'_i+\sum a_i s_i$ with $\{s_i\}$ denoting:

$$\{\varphi_2(e_i),\ \nu+1<i<p;\ f'_i,\ \nu-\alpha<i\leqslant\nu\}.$$

Then:

$$\textstyle\sum_{i=0}^{\nu-\alpha} r_i f'_i=-\sum a_i s_i+w=w_1\in x^{\alpha_{\nu-\alpha+1}-1}\,.\,N \pmod{\mathfrak{m}_2^{\beta+1}}.$$

Thus, $\sum r_i f'_i-w_1$ is generated by the standard relations:

$$\textstyle\sum_{i=0}^{\nu-\alpha} r_i f'_i-w_1=\sum_{i=0}^{\nu-1} v_i\,\sigma_i$$

with $\{\sigma_i\}$ the standard relations for $\{f'_i,\ 0\leqslant i\leqslant\nu-\alpha;\ x^{\beta-1-j}y^j, 0\leqslant j<\alpha\}$. Then, if corresponding to σ_i, $0\leqslant i\leqslant\nu-\alpha$, we let:

$$\overline{\sigma}_i=\begin{cases} x^{\alpha_{i+1}} f'_i-(y+u_{i,i+1})\,f'_{i+1}-\sum_{j=i+2}^{\nu-\alpha} u_{i,j}\,f'_j, & i<\nu-\alpha,\\ x^{\alpha_{\nu-\alpha+1}}\,f'_{\nu-\alpha}, & i=\nu-\alpha.\end{cases}$$

Then

$$\textstyle\sum_{i=0}^{\nu-\alpha} r_i f'_i=\sum_{i=0}^{\nu-\alpha} v_i\,\overline{\sigma}_i$$
$$\textstyle\equiv\sum_{i=0}^{\nu-\alpha} v_{i0}\,x^{\alpha_{\nu-\alpha+1}-1}\,\varphi_1(e_{i+1})\in x^{\alpha_{\nu-\alpha+1}-1}\,.\,L \pmod{\mathfrak{m}_2^{\beta+1}}.$$

Also, by assumption $\Sigma a_i s_i \in x^{\alpha_\nu-\alpha+1-1} \cdot L \pmod{\mathfrak{m}_2^{\beta+1}}$; thus,

$$w \in x^{\alpha_\nu-\alpha+1-1} \cdot L \pmod{\mathfrak{m}_2^{\beta+1}}. \quad \square$$

Finally, to complete the proof of Proposition 21, we claim that the set of elements $\psi(h', \varphi_1, \varphi_2)$ with Hilbert-Samuel function $\mathfrak{h}' = \tilde{\mathfrak{h}}$ are, in fact, dense among those just having $\mathfrak{h}'(j) = \tilde{\mathfrak{h}}(j)$ for $j \leqslant \beta$. This extra condition just requires that the ideal of $\psi(h', \varphi_1, \varphi_2)$ contains $\mathfrak{m}_2^{\beta+1}$. However, there is an $(\alpha-1)$-dimensional subspace $K \subset N$, so that $x^{\alpha_\nu-\alpha+1} \cdot K$ is contained in the ideal. Then, a sufficient condition that the ideal contain $\mathfrak{m}_2^{\beta+1}$ is that:

$$\mathfrak{m}_2^\beta \subset \langle x^{\beta-j} y^j, 0 \leqslant j \leqslant \nu-\alpha \rangle + x^{\alpha_\nu-\alpha+1-1} \cdot (x L + y L + K) \pmod{\mathfrak{m}_2^{\beta+1}}.$$

This is true for generic choice of K and L. Thus, in a neighborhood of h, we obtain:

$$\mathrm{Cl}(\Sigma_{\mathfrak{h}}^{\sim}) \xrightarrow{\sim} V_1 \times \mathrm{Cl}\{\text{mappings in } \mathrm{Hom}(k^{p^*}, N) \text{ of rank } 1\}.$$

Hence by the results in paragraph 3 describing the local structure of the closure of the set of mappings of rank 1, the result follows in the jet space.

The argument now follows exactly as in [§ 4, 6, I]. An infinitesimally stable germ will have a jet extension transverse to $\Sigma_{\mathfrak{h}}^{\sim}$. Thus it will project submersively into a fiber of the tubular neighborhood. Then, locally $\mathrm{Cl}(\Sigma_{\mathfrak{h}}^{\sim}(f))$ will be the inverse image of $\mathrm{Cl}(\Sigma_{\mathfrak{h}}^{\sim})$ in the fiber of the normal bundle. We can then apply Lemma 10.

8. C^0-stable germs and the Thom-Mather stratification

Now, finishing the proofs of Theorems 3 and 4 is easy.

Proof of Theorem 3. — If f is a finitely $\mathscr{K}$-determined C^0-stable germ, then f has an infinitesimally stable unfolding F. However, F is a topologically trivial unfolding of f. Now we can use general arguments (again *see* e. g. [5]) to conclude that topological invariants for infinitesimally stable germs which are constant under trivial unfoldings are topological invariants for C^0-stable germs. Thus, $\delta(\ )$ is a topological invariant, and so is $\mathscr{T}$. Then we can inductively use $\mathscr{T}$ to prove the Hilbert-Samuel function is a topological invariant since the only invariant which was needed was $\dim \mathrm{H}^*(\ )_{\mathrm{loc}}$ which is constant under trivial unfoldings. $\square$

Proof of Theorem 4. — The key property of the Thom-Mather stratification which interests us is that the topological type of local germs is constant on strata. Thus, topological invariants for infinitesimally stable germs are constant on strata. By Theorem 2 the Hilbert-Samuel function is constant on strata. However, by Proposition 13, the Hilbert-Samuel function determines the generic stairs for ideals in $k[[\mathbf{x}_2]]$. □

We conclude with a remark in the case $p-n<\nu(f)-1$. We can still topologically separate germs $f_{(x)}$ with $\nu(f_{(x)})\leqslant p-n+1$ from those with $\nu(f_{(x)})>p-n+1$ by Corollary 8. If $\Gamma(f)=\{x : \nu(f_{(x)})>p-n+1\}$, then we can still conclude with:

COROLLARY 26. — *Even if $p-n<\nu(f)-1$ for an infinitesimally stable germ $f: k^n, 0 \to k^p, 0$ there is still a topologically defined partition of k^n given by:*

$$\{\Sigma_{\mathfrak{h}}(f) : \nu(h)\leqslant p-n+1\}\cup\Gamma(f).$$

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