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A THEOREM ON POLARISED PARTITION RELATIONS FOR SINGULAR CARDINALS

BY

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RÉSUMÉ. — Si $\varkappa$ est un nombre cardinal singulaire et une limite des nombres cardinales mesurable, pour chaque $\alpha<\varkappa^+$ est valide :

$$\binom{\varkappa^+}{\varkappa}\to\binom{\alpha}{\varkappa}.$$

ABSTRACT. — If $\varkappa$ is a measurable limit cardinal then

$$\binom{\varkappa^+}{\varkappa}\to\binom{\alpha}{\varkappa}\quad\text{for any}\quad \alpha<\varkappa^+.$$

In [EHR], ERDŐS, HAJNAL and RADO discussed polarizes partition relations for cardinal numbers. By an easy counterexample it is shown that for all cardinals $\varkappa$ we have $\binom{\varkappa}{\varkappa}\nrightarrow\binom{\varkappa}{\varkappa}$. So it is a natural question to ask for which cardinals $\varkappa$ the following relation is valid:

$$\binom{\varkappa}{\varkappa^+}\to\binom{\varkappa}{\varkappa}.$$

PRIKRY proved [PR] that the negation of the partition relation is consistent for all successor cardinals $\varkappa$. The first auther proved the partition relation

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$\binom{\varkappa}{\varkappa^+} \to \binom{\varkappa}{\varkappa}$ for all measurable cardinals in [CH], the second auther proved the theorem for weakly compact cardinals [WO 2].

For singular cardinals, there is a positive result in [EHR] for cardinals with cofinality ω. Here we want to show the relation for measurable limit cardinals.

1. 1. Notation.

The set theoretical notations are standard, see [DR]. Small Greek letters denote ordinals, $\varkappa$, λ are infinite cardinals.

An ultrafilter U is $\varkappa$-complete iff for all n. e. sets $X \subset U$, $|X| < \varkappa$, $\cap X \in U$.

The cardinal $\varkappa$ is measurable iff there exists a $\varkappa$-complete non-principal ultrafilter on $\varkappa$; $\varkappa$ is a measurable limit cardinal iff there exists a strongly monotone increasing sequence of cardinals $(\varkappa_\nu \mid \nu < \mathrm{cf}\,\varkappa)$, such that all $\varkappa_\nu$ are measurable and $\lim_{\nu < \mathrm{cf}} \varkappa_\nu = \varkappa$.

Let $\mathscr{P} = (P, \leqslant)$ be a partially ordered set. $\mathscr{P}$ is a forcing set iff there exists a $O_P \in P$ such that for all $p \in P$, $O_P \leqslant p$. If p, $q \in P$ and $p \leqslant q$ than q is called an extension of p. A subset D of P is P-dense in $\mathscr{P}$ iff every $p \in P$ has an extension in D.

Let $\mathscr{P} = (P, \leqslant)$ be a forcing set, and $\mathscr{D}$ be a family of dense subsets of $\mathscr{P}$. A subset G of P is $\mathscr{D}$-generic iff:

(i) for all $p \in G$ and $q \leqslant p$, $q \in G$;

(ii) for all p, $q \in G$, p and q have a common extension in G;

(iii) for all dense sets $D \in \mathscr{D}$, $G \cap D \neq \emptyset$.

A subset K of P is an α-chain iff $(K, \leqslant)$ is a total ordering and has order type α.

For ordinals α, β, γ and δ, the polarised partition relation:

$$\binom{\alpha}{\beta} \to \binom{\gamma}{\delta}, \tag{1}$$

has the following meaning:

(1′) Let $\alpha \times \beta = I_0 \cup I_1$. Then there exists a subset $A \subset \alpha$, type $(A) = \gamma$ and a subset $B \subset \beta$, type $(B) = \delta$ and $A \times B \subset I_0$ or $A \times B \subset I_1$.

2. Two simple remarks

Using the axiom of choice we can trivially prove the following:

PROPOSITION 1. — *Let $\mathscr{P}=(P;\leqslant)$ be a forcing set and $\varkappa$ be an infinite cardinal. Let for all $\xi<\varkappa$ and ξ-chain $K\subset P$ there exists a $p\in P\setminus K$ such that for all $q\in K, q\leqslant p$ ($\mathscr{P}$ is closed under unions of chains of length $<\varkappa$). If $\mathscr{D}$ is a system of P-dense sets and $|\mathscr{D}|\leqslant\varkappa$, then there exists a $\mathscr{D}$-generic set $G\subset P$.*

The following proposition is also clear:

PROPOSITION 2. — *We have:*

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\to\begin{pmatrix}\gamma\\ \delta\end{pmatrix},$$

iff for every family $(X_\nu:\nu<\beta)$ of subsets of α, there exists an $I\subset\beta$, type $(I)=\delta$ such that type $(\bigcap_{\nu\in I}X_\nu)\geqslant\gamma$ or type $(\bigcap_{\nu\in I}(\alpha-X_\nu))\geqslant\gamma$.

3. Our main result is the following theorem, which generalizes results in [CH] and [WO 1].

THEOREM. — *Let $\varkappa$ be a singular, measurable limit cardinal. Then for any $\alpha<(\mathrm{cf}\,\varkappa)^+.\varkappa$:*

$$\begin{pmatrix}\varkappa\\ \varkappa^+\end{pmatrix}\to\begin{pmatrix}\varkappa\\ \alpha\end{pmatrix}$$

Proof. — Let $\varkappa$ be a singular measurable limit cardinal, $\mathrm{cf}\,\varkappa<\varkappa$ and let $(\varkappa_\nu:\nu<\mathrm{cf}\,\varkappa)$ be a monotonic strictly increasing sequence of measurable cardinals such that:

$$\mathrm{cf}\,\varkappa<\varkappa_0<\ldots<\varkappa_\nu<\ldots<\varkappa:\qquad \nu<\varkappa;$$

$$\lim_{\mu<\nu}\varkappa_\mu<\varkappa_\nu\qquad\text{for any}\quad\nu<\mathrm{cf}\,\varkappa$$

and:

$$\lim_{\nu<\mathrm{cf}\,\varkappa}\varkappa_\nu=\varkappa.$$

Let $\varkappa=\bigcup_{\nu<\mathrm{cf}\varkappa}M_\nu$, where $M_\nu=\varkappa_\nu$ for all $\nu<\mathrm{cf}\,\varkappa$. Let U_ν be a $\varkappa_\nu$-complete non-principal ultrafilter on M_ν for any $\nu<\mathrm{cf}\,\varkappa$ and $\mathscr{D}_0$ be an uniform ultrafilter on $\mathrm{cf}\,\varkappa$.

We define a product ultrafilter $\mathscr{D}$ on $\varkappa$:

$$\mathscr{D}=\{X\subset\varkappa:\{\nu<\mathrm{cf}\,\varkappa:X\cap M_\nu\in U_\nu\}\in\mathscr{D}_0\}.$$

Let $(X_\rho : \rho<\varkappa^+)$ be an arbitrary family of subsets of $\varkappa$. According to Proposition 2 it is necessary to show that there exists such an $I\subset\varkappa^+$, type $(I)=\alpha$ that:

$$|\bigcap_{\zeta\in I} X_\zeta|=\varkappa \qquad \text{or} \qquad |\bigcap_{\zeta\in I}(\varkappa\setminus X_\zeta)=\varkappa.$$

As $\mathscr{D}$ is an ultrafilter on $\varkappa$; we can suppose without loss of generality that:

$$E_0=\{\zeta<\varkappa^+ : X_\zeta\in\mathscr{D}\} \text{ has power } \varkappa^+.$$

Thus for any $\zeta\in E_0$, $C_\zeta=\{\gamma<\mathrm{cf}\,\varkappa : X_\zeta\cap M_\gamma\in U_\gamma\}\in\mathscr{D}_0$ and so C_ζ has power $\mathrm{cf}\,\varkappa$ for any $\zeta\in E_0$. Because $2^{\mathrm{cf}\,\varkappa}<\varkappa$, there exists such $E_1\subset E_0$, $|E_1|=\varkappa^+$ that for all $\zeta_1, \zeta_2\in E_1$:

$$C_{\zeta_1}=C_{\zeta_2}=C, \qquad |C|=\mathrm{cf}\,\varkappa.$$

We denote for simplicity $A=\bigcup_{\mu\in C} M\,\mu$ and $Y_\zeta=X_\zeta\cap A$ for $\zeta\in E_1$. Without loss of generality we can suppose that $C=\mathrm{cf}\,\varkappa$. Thus:

$$A=\bigcup_{\mu<\mathrm{cf}\,\varkappa} M_\mu \tag{2}$$

and for all $\rho\in E_1$

(3) $Y_\xi\subseteq A$ and $Y_\zeta\cap M_\nu\in U_\nu$ for all $\nu<\mathrm{cf}\,\varkappa$, where $|E_1|=\varkappa^+$.

Let $w\subseteq A$ and $w\subseteq\bigcup_{\nu<\mu} M_\nu$ for some $\mu<\mathrm{cf}\,\varkappa$. We denote:

$$T_w=\{\zeta\in E_1 : w\subset Y_\zeta\}$$

and call w exceptional (symbolically, $w\in Ex$), if $|T_w|\leqq\varkappa$. Since $\varkappa$ is a strong limit cardinal, $\sum_{\alpha<\varkappa}2^\alpha=\varkappa$, the number of all sets $w\subseteq\bigcup_{\nu<\mu} M_\nu$ for some $\mu<\mathrm{cf}\,\varkappa$, is at most $\varkappa$.

Thus for:

$$E_2=E_1\setminus\bigcup_{w\in Ex} T_w,$$

we have:

$$|E_2|=\varkappa^+.$$

We can assume without loss of generality that:

$$E_2=\varkappa^+.$$

In particular, if $w\subseteq A$ and w is bounded in $\varkappa$ and $w\subseteq Y_\zeta$ for some $\zeta<\varkappa^+$, then w is not exceptional and $|\{\eta<\varkappa^+ : w\subseteq Y_\eta\}|=\varkappa^+$.

Now we define a forcing set $\mathscr{P}=(P; \leqslant)$. Let P be the set of all pairs $\tau=(C; D)$ such that:

(i) there exists such $\xi<\mathrm{cf}\,\varkappa$, that $\xi\geqslant 1$:

$$C\subset\bigcup_{\mu<\xi} M_\mu \tag{4}$$

and:

(5) $$|C \cap M_\mu| = \varkappa_\mu \quad \text{for all } \mu < \xi;$$

(ii) $$D \subseteq \varkappa^+;$$

(iii) $$C \subseteq \bigcap_{\zeta \in D} Y_\zeta;$$

(iv) $$|D| \leqslant |C|;$$

(v) $$M_0 \cap Y_0 \subseteq C \text{ and } O \in D.$$

Because $M_0 \cap Y_0 \in U_0$, $|M_0 \cap Y_0| = \varkappa_0$, C is infinite. By (iii) and (v), $C \subset Y_0$, so if $(C; D) \in P$, then C is not exceptional.

For $\tau_i = (C_i; D_i) \in P$ we put $\tau_0 \leqslant \tau_1$ iff $C_0 \subseteq C_1$, $D_0 \subseteq D_1$. Then $\mathscr{P} = (P; \leqslant)$ is a forcing set with a minimal element $(M_0 \cap Y_0; \{0\}) \in P$.

LEMMA 3. — *The set $\mathscr{P}$ is closed under union of chains of length $< \mathrm{cf}\, \varkappa$.*

Proof of lemma 3. — Let $\{\tau_\nu : \nu < \xi\}$ be a chain in $\mathscr{P}$ of lenght ξ, $\xi < \mathrm{cf}\, \varkappa$, i. e. $\tau_{\nu_1} \leqslant \tau_{\nu_2}$ for $\nu_1 \leqslant \nu_2 < \xi$.

Case 1: $\xi = \eta + 1$. So $\tau_\eta = (C: D)$ is the greatest element in the chain. There exists such an $\rho < \mathrm{cf}\, \varkappa$, $1 \leqslant \rho$, that $C \subseteq \bigcup_{\mu < \rho} M_\mu$ and $|C \cap M_\mu| = \varkappa_\mu$ for all $\mu < \rho$ and $C \subseteq \bigcap_{\zeta \in D} Y_\zeta$. As C is not exceptional by definition of $\mathscr{P}$ and $|D| < \varkappa$, there exists such $a\, \zeta_0 \in \varkappa^+ \setminus D$, that $C \subseteq Y_{\zeta_0}$. Then $(C; D \cup \{\xi_0\})$ is an element of P and $(C; D) \leqslant (C; D \cup \{\zeta_0\})$, where $(C; D) \neq (C; D \cup \{\zeta_0\})$.

Case 2: ξ is a limit ordinal. If $\tau_\nu = (C_\nu;\ D_\nu) : \nu < \xi$, then $(\bigcup_{\nu < \xi} C_\nu; \bigcup_{\nu < \xi} D_\nu)$ belongs to P as $|\bigcup_{\nu < \xi} D_\nu| < \varkappa$ for $|D_\nu| < \varkappa : \nu < \xi$ and $\xi < \mathrm{cf}\, \varkappa$. For all $\mu < \xi$, $\tau_\mu \leqslant (\bigcup_{\nu < \xi} C; \bigcup_{\nu < \xi} D_\nu)$ and Lemma 3 is proved. □

We shall now define a family of dense subsets of $\mathscr{P}$ in order to apply Proposition 1.

We put for $\xi < \mathrm{cf}\, \varkappa$:

$$\Delta_\xi = \{(C; D) \in P : |C \cap M_\xi| = \varkappa_\xi\}.$$

LEMMA 4. — *For any $\xi < \mathrm{cf}\, \varkappa$, Δ_ξ is dense in $\mathscr{P}$.*

Proof of lemma 4. — Let $\tau_0 = (C; D) \in P$ and $(C; D) \notin \Delta_\xi$. Then there exists an $\eta \leqslant \xi$ such that $C \subseteq \bigcup_{\mu < \eta} M_\mu$ and since $|D| \leqslant |C|$, $|D| \leqslant \sum_{\mu < \eta} \varkappa_\mu < \varkappa$. Because $Y_\zeta \cap M_\rho \in U_\rho$ for all $\zeta \in D$, $\eta \leqslant \rho \leqslant \xi$ and all U_ρ are $\varkappa_\eta$-complete and uniform on $M_\rho : \eta \leqslant \rho \leqslant \xi$, we have:

$$\bigcap_{\zeta \in D} (Y_\zeta \cap M_\rho) \in U_\rho, \qquad \eta \leqslant \rho \leqslant \xi,$$

$$\bigcap_{\zeta \in D} (Y_\zeta \cap M_\rho)| = \varkappa_\rho \qquad \text{for} \quad \eta \leqslant \rho \leqslant \xi.$$

Thus $\tau_1=(C\cup(\bigcap_{\zeta\in D} Y_\zeta\cap(\bigcup_{\eta\leqslant\rho\leqslant\xi} M_\rho)), D)$ belongs to P and $\tau_1\in\Delta_\xi$ by construction. Also $\tau_0=(C; D)\leqslant\tau_1$ and Δ_ξ is dense. □

Let's take a sequence $(\alpha_\lambda : \lambda<\varkappa^+)$ such that $\alpha_\lambda\in\{\varkappa_\nu : \nu<\mathrm{cf}\,\varkappa\}$ and each $\varkappa_\nu$ has $\varkappa^+$ many appearences in the sequence $(\alpha_\lambda : \lambda<\varkappa^+)$.

LEMMA 5. — *There exists a sequence* $(\xi_\lambda : \lambda<\varkappa^+)$ *of elements* $\varkappa^+$, *such that the sets:*

$$\nabla_\lambda=\{(C; D)\in P : |D\cap\{\zeta : \xi_\lambda\leqslant\zeta<\xi_{\lambda+1}\}|\geqslant\alpha_\lambda\}$$

are dense in $\mathscr{P}$ *for all* $\lambda<\varkappa^+$.

Proof of lemma 5. — Let $\xi_0<\varkappa^+$-be arbitrary. We shall construct the sequence $(\xi_\lambda : \lambda<\varkappa^+)$ by induction.

Suppose that we have constructed $(\xi_\lambda : \lambda\leqslant\delta)$ for $\delta<\varkappa^+$ such that all $\nabla_\zeta : \zeta<\delta$ are dense in $\mathscr{P}$. Let us suppose however that there are no $\xi_{\delta+1}$ such that ∇_δ is dense in $\mathscr{P}$. Then for any $\beta>\xi_\delta$, $\beta<\varkappa^+$, there is $\tau_\beta=(C_\beta; D_\beta)\in P$ such that τ_β is not extended by a menber of ∇_δ, where (in definition of ∇_δ) we put $\xi_{\delta+1}=\beta$.

For each C_β : $\xi_\delta<\beta<\varkappa^+$ there exists an $\eta<\mathrm{cf}\varkappa$ with $C_\beta\subseteq\bigcup_{\mu<\eta} M_\mu$. Because $\varkappa$ is a strong limit, $\sum_{\alpha<\varkappa} 2^\alpha\leqslant\varkappa$, we can find $E\subset\varkappa^+$, $|E|=\varkappa^+$ with:

$$C_{\beta_1}=C_{\beta_2}=C \quad \text{for all } \beta_1, \beta_2\in E. \tag{6}$$

Case 1: Let $|C|\geqslant\alpha_\delta$. Since C is not an exceptional set (by (6) and definition of P), $|\{\zeta>\xi_\delta : C\subseteq Y_\zeta\}|=\varkappa^+$. Let us take $M\subseteq\varkappa^+$ with:

$$M\subseteq\{\zeta>\zeta_\delta : C\subseteq Y_\zeta\} \qquad \text{and} \qquad |M|=\alpha_\delta.$$

Then we take $\beta\in E$ such that $\beta>M$, i. e. for any $\zeta\in M$, $\beta>\zeta$. We obtain by $|C|\geqslant\alpha_\delta$:

$$|D_\beta\cup M|\leqslant|C_\beta|+\alpha_\delta=|C|+\alpha_\delta=|C|;$$

and by our choice of M:

$$C\subseteq\bigcap_{\zeta\in(D_\beta\cup M)} Y_\zeta.$$

In other words, $(C; D_\beta\cup M)\in P$. Further, for $C_\beta=C : \beta\in E$, we have $\tau_\beta\leqslant(C; D_\beta\cup M)$. But $(C; D_\beta\cup M)\in\nabla_\delta$, where we put $\xi_{\delta+1}=\beta$, and so we come to a contradiction.

Case 2: $|C|<\alpha_\delta$, where $\alpha_\delta=\varkappa_\nu$. Let ρ be the least ordinal with $|C|<\varkappa_\rho$; $\rho\leqq\nu$.

Let $\beta_0\in E$ be arbitrary; since $|D_{\beta_0}|\leqslant|C|$ and $C\subseteq\bigcup_{\mu<\rho}M_\mu$, $|D_{\beta_0}|\leqslant\sum_{\mu<\rho}\varkappa_\mu<\varkappa_\rho$. Consequently, if:

$$C_1:=\bigcup_{\rho\leqslant\mu\leqslant\nu}\bigcap_{\xi\in D_{\beta_0}}(M_\mu\cap Y_\xi),$$

then $C\cup C_1$ is not exceptional as $C\cup C_1=C_{\beta_0}\cup C_1\in Y_\xi$ for any $\xi\in D_{\beta_0}$. From this it follows that:

$$|\{\zeta>\xi_\delta : C\cup C_1\subseteq Y_\zeta\}|=\varkappa^+;$$

we choose $M\subseteq\{\zeta>\xi_\delta : C\cup C_1\subseteq Y_\zeta\}$; $|M|=\varkappa_\nu=\alpha_\delta$ and $\beta\in E$ with $\beta>M$.

From $|D_\beta|\leqslant|C|$ it follows $|D_\beta|<\varkappa_\rho$ and $\bigcap_{\zeta\in D_\beta}(M_\mu\cap Y_\zeta)\in U_\mu$ for all μ, $\rho\leqslant\mu\leqslant\nu$. Then by definition of C_1, $(M_\mu\cap C_1)\in U_\mu\,\rho\leqslant\mu\leqslant\nu$. We put:

$$D_2=D_\beta\cup M;\qquad C_2=C\cup\bigcup_{\rho\leqslant\mu\leqslant\nu}\bigcap_{\xi\in D_\beta}(M_\mu\cap Y_\xi\cap C_1).$$

In these notations, $|D_2|\leqslant\varkappa_\nu\leqslant|C_2|$ and $C_2\subseteq\bigcap_{\zeta\in D_2}Y_\zeta$ and $(C_\beta;D_\beta)\leqslant(C_2;D_2)$. But $|D_2\cap\{\zeta : \zeta_\delta\leqslant\zeta<\beta\}|=\varkappa_\nu$ and this contradicts the construction of $\tau_\beta=(C_\beta; D_\beta)$. Lemma 5 ist proved. □

Now we can prove our theorem. Let $\alpha<(\mathrm{cf}\,\varkappa)^+.\varkappa$; then there is a sequence sum $\sum_{i\in I}\xi_{\lambda_i}\geqslant\alpha$.

We consider the following family:

$$\mathscr{D}=\{\Delta_\xi : \xi<\mathrm{cf}\,\varkappa\}\cup\{\nabla_{\lambda_i}\,|\,i\in I\},$$

of cf $\varkappa$ dense subsets of $\mathscr{P}$.

By Proposition 1 there exists a $\mathscr{D}$-generic set $G\subset P$. We put:

$$I=\bigcup rg\,G=\bigcup\{D : (C; D)\in G\};$$
$$J=\bigcup\mathrm{dom}\ G=\bigcup\{C : (C; D)\in G\}.$$

Then $I\subset\varkappa^+$, typ $(I)\geqslant\alpha$; $J\subset\varkappa$, $|J|=\varkappa$ and
by Proposition 2, the theorem is proved.

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