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ON THE CANONICAL TOPOLOGY OF AN ANALYTIC ALGEBRA AND OF AN ANALYTIC MODULE ;

BY

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Introduction. — An analytic algebra A over a commutative field k with a complete non-discrete valuation is a k-algebra A isomorphic to a non-zero quotient of an algebra $\Lambda = k\{X_1, \ldots, X_n\}$ of convergent power series. If A is an analytic algebra, an A-module M is called analytic if it is of finite type over A.

In this paper, we define and study the canonical topology on an analytic algebra and on analytic module. The work has its origin in a tentative to explain and generalize the following theorem of H. CARTAN [1] : If Λ is the algebra of all germs of holomorphic functions at $0 \in \mathbf{C}^n$, then for any integer $r > 0$ and any submodule H of Λ^r, there exists a stricts continuous epimorphism $\Lambda^p \to H$ for the uniform convergence of germs. A first but incomplete variant was exposed in Stoilow's Seminar [3].

In paragraph 1, we define the canonical topology on the algebra $\Lambda = k\{X_1, \ldots, X_n\}$ and we give the principal properties of this topology. We use this topological structure of Λ to give a proof of the preparation lemma by successive approximations. In paragraph 3, we define the canonical topology of an analytic algebra and of an analytic module. The fundamental theorem asserts that any algebraically exact sequence $0 \to M' \to M \to M'' \to 0$ of analytic A-modules splits as sequence of topological vector k-spaces. A corollary is the following generalization of Cartan's theorem : any homomorphism $u : M \to N$ of analytic A-modules is continuous strict for the canonical topologies of M an N. For $A = k$, the analytic A-modules are exactly the vector k-spaces of finite dimension, and the canonical topology of an analytic A-module M is then the unique Hausdorff topology compatible with the structure of vector k-space of M. Paragraph 4 details with bounded sets in analytic algebras and modules. In fact, only the case of the algebra $\Lambda = k\{X_1, \ldots, X_n\}$ is treated because the extension of the theory to the general case is straightforward.

1. The algebra $\Lambda = k\{X_1, \ldots, X_n\}$.

Let $\mathbf{N}$ be the additive monoid of integers $n \geqq 0$. If $n \in \mathbf{N}$ and if M is an object in a category with direct products and with a final object O, we define M^n to be the direct product $M \times \ldots \times M$ (n times) when $n > 0$ and $M^n = O$ when $n = 0$. For any $m \in \mathbf{N}^n$, we set $|m| = m_1 + \ldots + m_n$.

Let k be a fixed *commutative field with a complete non-discrete valuation*, and let $n \in \mathbf{N}$. We shall denote by $k[[X_1, \ldots, X_n]]$ the k-algebra of formal power series in the variables $X_1, \ldots, X_n$ with coefficients in k, and by $k[X_1, \ldots, X_n]$ the subalgebra of polynomials in $X_1, \ldots, X_n$ with coefficients in k; for $n = 0$, these algebras are defined to be $= k$.

Let $\hat{\Lambda} = k[[X_1, \ldots, X_n]]$. For each $m \in \mathbf{N}^n$, we shall denote by $\hat{\pi}_m$ the canonical projection of index m of $\hat{\Lambda}$ $\left(\text{if } f = \sum_{m \in \mathbf{N}^n} a_m X_1^{m_1} \ldots X_n^{m_n}, \text{ then } \hat{\pi}_m(f) = a_m\right)$; $\hat{\pi}_m$ is k-linear. We define the *weak topology* of $\hat{\Lambda}$ to be the least fine topology of $\hat{\Lambda}$ for which all maps $\hat{\pi}_m$ are continuous. When k is locally compact, clearly the weak topology of $\hat{\Lambda}$ is Montel (i. e. any bounded set in $\hat{\Lambda}$ is relatively compact).

For any $f \in \hat{\Lambda}$, we shall denote by $\mathbf{o}(f)$ the order of f. The ring $\hat{\Lambda}$ is local, its maximal ideal is

$$\hat{\mathfrak{m}} = \operatorname{Ker} \hat{\pi}_0 = \{ f \in \hat{\Lambda} \mid \mathbf{o}(f) > 0 \},$$

and its residue field is $\approx k$. The $\hat{\mathfrak{m}}$-adic topology of $\hat{\Lambda}$ is strictly finer as the weak topology because the valuation of k is non-discrete.

Let $\mathbf{R}_+^*$ be the set of all positive real numbers, and let $n \in \mathbf{N}$, $n > 0$. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbf{R}_+^{*n}$, we shall write $\beta \leqq \alpha$ when $\beta_i \leqq \alpha_i$ for all i, and $\beta \ll \alpha$ when $\beta_i < \alpha_i$ for all i.

Definition. — For any $f \in k[[X_1, \ldots, X_n]]$ and any $\alpha \in \mathbf{R}_+^{*n}$, we define $\| f \|_\alpha$ by

$$\| f \|_\alpha = \sum_{m \in \mathbf{N}^n} | a_m | \alpha_1^{m_1} \ldots \alpha_n^{m_n},$$

where $a_m = \hat{\pi}_m(f)$.

Theorem 1.1. — $\| f \|_\alpha = 0 \Leftrightarrow f = 0,$

$$\| f + g \|_\alpha \leqq \| f \|_\alpha + \| g \|_\alpha, \qquad \| fg \|_\alpha \leqq \| f \|_\alpha \| g \|_\alpha,$$
$$\| 1 \|_\alpha = 1, \qquad \| tf \|_\alpha = | t | . \| f \|_\alpha \textit{ for } t \in k,$$

and

$$|\hat{\pi}_m(f)| \leq \frac{\|f\|_\alpha}{\alpha_1^{m_1}\dots\alpha_n^{m_n}} \quad \textit{for all } m \in \mathbf{N}^n.$$

The proof is trivial and will be omitted.

Let $\hat{\Lambda} = k[[X_1, \dots, X_n]]$. For any $\alpha \in \mathbf{R}_+^{*n}$, we consider the set

$$\Lambda_\alpha = \left\{ f \in \hat{\Lambda} \,\middle|\, \|f\|_\alpha < +\infty \right\}.$$

Clearly $\Lambda_\alpha \subset \Lambda_\beta$ for $\beta \leq \alpha$.

COROLLARY. — *Λ_α is a subalgebra of $\hat{\Lambda}$, $k[X_1, \dots, X_n] \subset \Lambda_\alpha$, $\|.\|_\alpha$ is a norm on Λ_α, and the maps $\hat{\pi}/\Lambda_\alpha$ are continuous with respect to this norm.*

For any $\alpha \in \mathbf{R}_+^{*n}$, Λ_α will be considered as a normed k-algebra with the canonical norm $\|.\|_\alpha$, and in particular as a topological k-algebra.

THEOREM 1.2.

1° *The closed balls of Λ_α are closed subsets of $\hat{\Lambda}$ for the weak topology.*

2° *Λ_α is complete* (1).

3° *If α, $\beta \in \mathbf{R}_+^{*n}$ and $\beta \ll \alpha$, then the canonical topology of Λ_β and the weak topology of $\hat{\Lambda}$ induce the same topology on any bounded set of Λ_α.*

Proof.

1° Let M be a closed ball in Λ_α with centre $g \in \Lambda_\alpha$ and radius $\rho (< +\infty)$ and let $(f_i)_{i \geq 0}$ be a sequence of elements $f_i \in M$, weakly convergent to an element $f \in \hat{\Lambda}$. Let $a_m = \hat{\pi}_m(f)$, $a_m^i = \hat{\pi}_m(f_i)$ and $b_m = \hat{\pi}_m(g)$. Then

$$\sum_{m \in \mathbf{N}^n} |a_m^i - b_m| \alpha_1^{m_1}\dots\alpha_n^{m_n} \leq \rho$$

for all $i \geq 0$. Since $\lim_i a_m^i = a_m$ for all $m \in \mathbf{N}^n$, it follows that

$$\sum_{|m| < q} |a_m - b_m| \alpha_1^{m_1}\dots\alpha_n^{m_n} \leq \rho$$

for all $q > 0$, and therefore

$$\|f - g\|_\alpha = \sum_{m \in \mathbf{N}^n} |a_m - b_m| \alpha_1^{m_1}\dots\alpha_n^{m_n} \leq \rho,$$

that is $f \in M$.

2° Let $(f_i)_{i \geq 0}$ be a Cauchy sequence in Λ_α. Since the canonical injection $\Lambda_\alpha \to \hat{\Lambda}$ is continuous for the weak topology of $\hat{\Lambda}$, the sequence $(f_i)_{i \geq 0}$

(1) The proof of completeness of Λ_α is due to C. FOIAS.

is weakly Cauchy and consequently weakly convergent to an element $f \in \hat{\Lambda}$. Let $\varepsilon > 0$. Then there exists an integer $i_0(\varepsilon) > 0$ such that

$$\| f_{i+p} - f_i \|_\alpha \leqq \varepsilon$$

for all $i \geqq i_0(\varepsilon)$ and all $p \geqq 0$. Since $(f_i)_{i \geqq 0}$ is weakly convergent to f, it follows from 1° that

$$\| f - f_i \|_\alpha \leqq \varepsilon$$

for all $i \geqq i_0(\varepsilon)$. Also $f = f_i + (f - f_i) \in \Lambda_\alpha$.

3° We have only to prove that, if a bounded sequence $(f_i)_{i \geqq 0}$ in Λ_α is weakly convergent to an $f \in \hat{\Lambda}$, then $f \in \Lambda_\alpha$ and $(f_i)_{i \geqq 0}$ converges to f in Λ_β. The first assertion follows from 1°. Further since $(f_i)_{i \geqq 0}$ is bounded, there exists a number $\rho > 0$ such that

$$\| f_i \|_\alpha = \sum_{m \in \mathbf{N}^n} | a^i_m | \alpha_1^{m_1} \ldots \alpha_n^{m_n} \leqq \rho$$

for all $i \geqq 0$. Hence, for $\beta \ll \alpha$,

$$\| f - f_i \|_\beta = \sum_{m \in \mathbf{N}^n} | a_m - a^i_m | \beta_1^{m_1} \ldots \beta_n^{m_n} \leqq 2\rho \sum_{m \in \mathbf{N}^n} \left(\frac{\beta_1}{\alpha_1} \right)^{m_1} \cdots \left(\frac{\beta_n}{\alpha_n} \right)^{m_n}.$$

Let $\varepsilon > 0$. It follows that there exists an integer $q > 0$ such that

$$\sum_{|m| > q} | a_m - a^i_m | \beta_1^{m_1} \ldots \beta_n^{m_n} \leqq \frac{\varepsilon}{2}$$

for all $i \geqq 0$. Since (f_i) is weakly convergent to f, there existe an integer $i_0(\varepsilon) > 0$ such that

$$\sum_{|m| \leqq q} | a_m - a^i_m | \beta_1^{m_1} \ldots \beta_n^{m_n} \leqq \frac{\varepsilon}{2}$$

for all $i \geqq i_0(\varepsilon)$. Thus

$$\| f - f_i \|_\beta = \sum_{m \in \mathbf{N}^n} | a_m - a^i_m | \beta_1^{m_1} \ldots \beta_n^{m_n} \leqq \varepsilon$$

for all $i \geqq i_0(\varepsilon)$, which completes the proof of the theorem.

Since Λ_α is complete, any normally convergent series of elements $f_i \in \Lambda_\alpha$ is convergent in Λ_α and its sum f satisfies

$$\| f \|_\alpha \leqq \sum_{i \geqq 0} \| f_i \|_\alpha ;$$

moreover the sum of a convergent series in Λ_α coincides with its weak sum and with its $\hat{\mathfrak{m}}$-adic sum. For instance, if $f \in \Lambda_\alpha$ and $a_m = \hat{\pi}_m(f)$, the series $\sum_{m \in \mathbf{N}^n} a_m X_1^{m_1} \dots X_n^{m_n}$ is normally convergent in Λ_α and its sum is f.

COROLLARY 1. — *Assume k locally compact. Then, for $\beta \leqslant \alpha$, the canonical injection $\Lambda_\alpha \to \Lambda_\beta$ is compact (more precisely : any closed ball of Λ_α is a compact subset of Λ_β).*

Proof. — Let M be a closed ball in Λ_α. By Theorem 1.2, 1°, M is closed in $\hat{\Lambda}$ for the weak topology. As k is locally compact, $\hat{\Lambda}$ is Montel, consequently M is compact in $\hat{\Lambda}$ for the weak topology. Hence, by Theorem 1.2, 3°, M is compact in Λ_β.

Let E be a vector k-space. A non-empty set $M \subset E$ is called *absolutely convex* if, for any couple of elements $f, g \in M$ and any couple of elements $s, t \in k$ such that $|s| + |t| \leqq 1$, $sf + tg \in M$. (Note that for any absolutely convex set $M \subset E$, $0 \in M$.) A topological vector k-space E is called a *locally convex k-space* if $0 \in E$ has a fundamental system of absolutely convex neighbourhoods.

Locally convex k-spaces and continuous k-linear maps form an additive category; we shall denote it by **ELC**. Also, we shall denote by $\mathcal{F}$ the full subcategory of **ELC** formed by all spaces $\in$ **ELC** which are metrizable complete, and by $\mathcal{LF}$ the full subcategory of **ELC** formed by all Hausdorff spaces $\in$ **ELC** which are inductive limits in **ELC** of sequences of spaces $\in \mathcal{F}$.

A topological k-algebra A is called *locally convex* if so is the subjacent topological vector k-space of A. Similarly, if A is a locally convex k-algebra, a topological A-module M is called locally convex if the subjacent topological vector k-space of M is locally convex.

DEFINITION. — The set

$$\Lambda = \bigcup_{\alpha \in \mathbf{R}_+^{*n}} \Lambda_\alpha$$

is a subalgebra of $\hat{\Lambda} = k[[X_1, \dots, X_n]]$, is called *the algebra of convergent power series* in the variables $X_1, \dots, X_n$ with coefficients in k, and is denoted by $k\{X_1, \dots, X_n\}$; also we set $\mathfrak{m} = \Lambda \cap \hat{\mathfrak{m}}$ and $\pi_m = \hat{\pi}_m | \Lambda$.

If $f \in \Lambda = k\{X_1, \dots, X_n\}$, we define the " convergence domain " of f to be

$$D_f = \bigcup_{\substack{\alpha \in \mathbf{R}_+^{*n} \\ f \in \Lambda_\alpha}} \{ x \in k^n \,|\, |x_i| < \alpha_i \}.$$

Then D_f is an open set in k^n, and when k is such that the discs of k are connected, D_f is also connected.

DEFINITION. — The finest locally convex topology on $\Lambda = k\{X_1, ..., X_n\}$ such that all canonical injections $\Lambda_\alpha \rightarrow \Lambda$ are continuous is called the *canonical topology of* Λ.

We already know that the canonical injections $\Lambda_\alpha \rightarrow \hat{\Lambda}$ are continuous, hence the canonical topology of Λ is finer as that induced by the weak topology of $\hat{\Lambda}$; in particular the *canonical topology of* Λ *is Hausdorff* (2).

The canonical topology of Λ is compatible with the structure of vector k-space of Λ, and

$$\Lambda = \varinjlim_{\alpha} \Lambda_\alpha$$

in **ELC**. It follows that a k-linear map from Λ to a locally convex k-space E is continuous if an only if all maps $u|\Lambda_\alpha$ are continuous. By convexity reasons, for $\Lambda = k\{X_1, \ldots, X_n\}$ and $\Gamma = k\{Y_1, \ldots, Y_p\}$, the product topology of $\Lambda \times \Gamma$ coincides with the locally convex direct sum topology, consequently

$$\Lambda \times \Gamma = \varinjlim_{\alpha} (\Lambda_\alpha \times \Gamma_\alpha)$$

in **ELC**. It follows that the multiplication $\Lambda \times \Lambda \rightarrow \Lambda$ is also continuous. Thus Λ *is a locally convex k-algebra* (for its canonical topology); clearly the subalgebra $k[X_1, \ldots, X_n]$ is a dense subset of Λ.

Unless otherwise stated, Λ will be considered as a locally convex k-algebra with the canonical topology.

COROLLARY 2. — *For any integer* $r \geqq 0$, Λ^r *is a k-space* $\in \mathcal{LF}$.

COROLLARY 3. — *If* $E \in \mathcal{F}$, *a k-linear map* $u : E \rightarrow \Lambda$ *is continuous if and only if there exists an* $\alpha \in \mathbf{R}_+^{*n}$ *such that* $u(E) \subset \Lambda_\alpha$ *and such that the induced map* $E \rightarrow \Lambda_\alpha$ *is continuous.*

Proof. — By *Théorème* 1 (p. 268, in [2]).

From Corollary 3, it follows that a k-linear map

$$u : \quad k\{X_1, \ldots, X_n\} \rightarrow k\{Y_1, \ldots, Y_p\}$$

is continuous if and only if, for any $\alpha \in \mathbf{R}_+^{*n}$, there exists $\beta \in \mathbf{R}_+^{*n}$ and $c > 0$ such that

$$\| u(f) \|_\beta \leqq c \| f \|_\alpha$$

for all $f \in \Lambda_\alpha$.

(2) It is easily seen that the canonical topology of Λ is strictly finer than the *weak topology* of Λ (i. e. the topology induced on Λ by the weak topology of $\hat{\Lambda}$). As a matter of fact it will be proved (Corollary to Theorem 4.2) that the canonical topology of Λ is not metrizable.

Remark. — If A is a local k-algebra with residue field $\approx k$, and if $\mathfrak{m}$ is the maximal ideal of A, then $x \in \mathfrak{m}$ if and only if $1 - tx$ is invertible in A for any $t \in k$. Hence if A and B are local k-algebras with residue field $\approx k$, then any k algebra homomorphism $u : A \rightarrow B$ is local, i. e. $u(\mathfrak{m}) \subset \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of A and B.

COROLLARY 4. — *Any homorphism of k-algebras*

$$u:\quad k[[X_1, \ldots, X_n]] \rightarrow k[[Y_1, \ldots, Y_p]]$$

such that $u(X_i) = g_i \in k\{Y_1, \ldots, Y_p\}$ induces a continuous homomorphism

$$k\{X_1, \ldots, X_n\} \rightarrow k\{Y_1, \ldots, Y_p\}.$$

Proof. — Be the preceding remark, u is local and therefore $\mathbf{o}(g_i) > 0$. Hence, if $f \in k[[X_1, \ldots, X_n]]$ and $\hat{\pi}_m(f) = a_m$, then

$$u(f) = \sum_{m \in \mathbf{N}^n} a_m g_1^{m_1} \ldots g_n^{m_n} = f(g_1, \ldots, g_n)$$

(here the second equality is a definition).

Assume now $f \in k\{X_1, \ldots, X_n\}$ and let $\alpha \in \mathbf{R}_+^{\star n}$. Since $\mathbf{o}(g_i) > 0$, there exists $\beta \in \mathbf{R}_+^{\star n}$ such that

$$\| g_i \|_\beta \leqq \alpha_i$$

for all $i = 1, \ldots, n$. It follows that the series $\sum_{m \in \mathbf{N}^n} a_m g_1^{m_1} \ldots g_n^{m}{}_n$ is normally convergent in Λ_β and therefore

$$\| u(f) \|_\beta \leqq \sum_{m \in \mathbf{N}^n} \| a_m g_1^{m_1} \ldots g_n^{m_n} \|_\beta \leqq \sum_{m \in \mathbf{N}^n} | a_m | \alpha_1^{m_1} \ldots \alpha_n^{m_n} = \| f \|_\alpha.$$

COROLLARY 5. — *The partial derivations in $\Lambda = k\{X_1, \ldots, X_n\}$ are continuous. More precisely : for $\beta \leqslant \alpha$ and all $m \in N^n$,*

$$\left\| \frac{\partial^{|m|} f}{\partial X_1^{m_1} \ldots \partial X_n^{m_n}} \right\|_\beta \leqq \frac{m\,!}{\alpha_1^{m_1} \ldots \alpha_n^{m_n}} \cdot \frac{\| f \|_\alpha}{\left(1 - \frac{\beta_1}{\alpha_1}\right)^{m_1+1} \cdots \left(1 - \frac{\beta_n}{\alpha_n}\right)^{m_n+1}},$$

where $m\,! = m_1\,! \ldots m_n\,!$.

Proof. — For $f \in \Lambda$, we write $f \prec F$ if $F \in \mathbf{R}\{X_1, \ldots, X_n\}$ and $|\pi_m(f)| \leqq \pi_m(F)$ for all $m \in \mathbf{N}^n$. Clearly $f \prec F$ implies $\| f \|_\alpha \leqq \| F \|_\alpha$ for any $\alpha \in \mathbf{R}_+^{\star n}$, and

$$\frac{\partial^{|m|} f}{\partial X_1^{m_1} \ldots \partial X_n^{m_n}} \prec \frac{\partial^{|m|} F}{\partial X_1^{m_1} \ldots \partial X_n^{m_n}}.$$

Also

$$f \prec \frac{\|f\|_\alpha}{\left(1-\dfrac{X_1}{\alpha_1}\right)\cdots\left(1-\dfrac{X_n}{\alpha_n}\right)} \quad \text{for} \quad f\in\Lambda_\alpha,$$

and

$$\left\| 1-\frac{X_i}{\alpha_i}\right\|_\beta = 1-\frac{\beta_i}{\alpha_i},$$

and the corollary follows.

By BOURBAKI, if G and G' are topological abelian groups, a continuous homomorphism $u: G \to G'$ is called *strict* if the canonical map

$$\bar{u}: \quad \frac{A}{u^{-1}(0)} \to u(A)$$

is a homeomorphism. For instance, if u has a left or a right inverse, then u is strict. A continuous homomorphism of topological rings (or algebras or modules) is called strict if it is strict for the subjacent topological abelian groups.

COROLLARY 6. — *Let* $\Lambda = k\{X_1, \ldots, X_n\}$ *and* r *an integer* > 0. *The* Λ*-linear map* $u: \Lambda^r \to k\{X_1, \ldots, X_{n+1}\}$ *defined by*

$$u(f_0, \ldots, f_{n-1}) = \sum_{i=0}^{r-1} f_i X_{n+1}^i$$

is continuous strict.

Proof. — Clearly u is continuous. Further let $v: k\{X_1, \ldots, X_{n+1}\} \to \Lambda^r$ be the map defined as follows. Any $f \in k\{X_1, \ldots, X_{n+1}\}$ has a unique expansion

$$f = \sum_{i\geq 0} f_i X_{n+1}^i$$

with $f_i \in \Lambda$; then we set $v(f) = (f_0, \ldots, f_{r-1})$. Clearly v is Λ-linear continuous and $vu = 1_{\Lambda^r}$.

From Corollary 6 it follows that, for $n < p$, the canonical injection

$$k\{X_1, \ldots, X_n\} \to k\{X_1, \ldots, X_p\}$$

is strict.

2. The preparation lemma.

In this paragraph we shall consider two examples of " successive approximations " for power series. We begin with the preparation lemma.

Assume $n > 0$, and let s be an integer > 0. We define the endomorphism u_s and v_s of $\hat{\Lambda} = [[X_1, \ldots, X_n]]$ in the following way. Any $f \in \hat{\Lambda}$ has a unique expansion

$$f = \sum_{i \geq 0} f_i X_n^i$$

with $f_i \in k[[X_1, \ldots, X_{n-1}]]$; we set

$$u_s(f) = \sum_{i=0}^{s-1} f_i X_n^i, \qquad v_s(f) = \sum_{i \geq s} f_i X_n^{i-s}.$$

Clearly u_s and v_s are $k[[X_1, \ldots, X_{n-1}]]$-linear and in particular k-linear. Also, for any $f \in k[[X_1, \ldots, X_n]]$,

$$f = u_s(f) + X_n^s v_s(f)$$

and, for any $\alpha \in \mathbf{R}_+^{*n}$,

$$\| f \|_\alpha = \| u_s(f) \|_\alpha + \alpha_n^s \| v_s(f) \|_\alpha,$$

so that

$$\| u_s(f) \|_\alpha \leq \| f \|_\alpha, \qquad \| v_s(f) \|_\alpha \leq \frac{\| f \|_\alpha}{\alpha_n^s}.$$

Thus u_s and v_s induce continuous endomorphisms of $k\{X_1, \ldots, X_n\}$.

For any $\alpha \in k[[X_1, \ldots, X_n]]$ we define

$$\omega(f) = \begin{cases} \min\limits_{\hat{\pi}_m(f) \neq 0} (m_1 + \ldots + m_{n-1}) & \text{if } f \neq 0, \\ +\infty & \text{if } f = 0. \end{cases}$$

Then

$$\omega(f) = +\infty \iff f = 0, \qquad \omega(fg) = \omega(f) + \omega(g)$$

and

$$\omega(f+g) \geq \min(\omega(f), \omega(g)).$$

Also, $\omega(u_s(f)) \geq \omega(f)$ and $\omega(v_s(f)) \geq \omega(f)$.

Let

$$\hat{\mathfrak{a}} = \{ f \in \hat{\Lambda} \mid \omega(f) > 0 \}.$$

Then $\hat{\mathfrak{a}}$ is an ideal of $\hat{\Lambda}$, $\hat{\mathfrak{a}} \subset \hat{\mathfrak{m}}$, and clearly the $\hat{\mathfrak{a}}$-adic topology of $k[[X_1, \ldots, X_n]]$ is Hausdorff complete.

THEOREM 2.1 (The preparation lemma). — *Let $g \in \hat{\mathfrak{m}}$ such that $g(0, \ldots, 0, X_n) \neq 0$, and let s be the order of the series $g(0, \ldots, 0, X_n) \in k[[X_n]]$. For any $f \in k[[X_1, \ldots, X_n]]$ there exists a unique $\lambda = \lambda_f \in k[[X_1, \ldots, X_n]]$ such that*

(1) $$v_s(f - \lambda g) = 0.$$

Moreover the map $f \to \lambda_f$ is $k[[X_1, \ldots, X_{n-1}]]$-linear and is continuous for the weak, $\hat{\mathfrak{m}}$-adic and $\hat{\mathfrak{a}}$-adic topologies of $k[[X_1, \ldots, X_n]]$.

If $g \in \Lambda = k\{X_1, \ldots, X_n\}$ then, for any $f \in \Lambda$, $\lambda_f \in \Lambda$ and the map $f \to \lambda_f$ is continuous for the canonical topology of Λ (more precisely : there exists a cofinal subset I_g of $\mathbf{R}_+^{\star n}$ and for each $\alpha \in I_g$ a $c_\alpha > 0$ such that, if $f \in \Lambda_\alpha$, $\lambda_f \in \Lambda_\alpha$ and $\| \lambda_f \|_\alpha \leq c_\alpha \| f \|_\alpha$).

Proof. — We write u and v instead of u_s and v_s. Then we have

$$g = u(g) + X_n^s v(g)$$

with $v(g)$ invertible and $\omega(u(g)) > 0$. Let $p = u(g)(v(g))^{-1}$. Then it is immediate that a $\lambda \in k[[X_1, \ldots, X_n]]$ satisfies the equation (1) if and only if $h = \lambda\, v(g)$ satisfies the equation

$$h = v(f) - v(ph). \tag{2}$$

This reduction of (1) to (2) is due to ZARISKI and SAMUEL ([5], p. 140).

We define the sequence $(h_i)_{i \geq 0}$ by the conditions

$$h_0 = 0,$$
$$h_{i+1} = v(f) - v(ph_i).$$

Since $\omega(p) > 0$, we have

$$\omega(h_{i+1} - h_i) > \omega(h_i - h_{i-1})$$

and so, by induction on i,

$$\omega(h_{i+1} - h_i) \geq i + \omega(f).$$

Thus the sequence $(h_i)_{i \geq 0}$ is Cauchy and therefore convergent for the $\hat{\mathfrak{a}}$-adic topology of $k[[X_1, \ldots, X_n]]$ and clearly $h = \lim_i h_i$ satisfies (2). If h' is another solution of (2), then

$$\omega(h' - h) \geq i$$

for any $i \geq 0$, whence $h' = h$. Thus the solution $h = h_f$ of the equation (2) exists and is unique. Unicity of h_f implies that the map $f \to h_f$ is $k[[X_1, \ldots, X_{n-1}]]$-linear (and in particular k-linear). Moreover, we have

$$\omega(h) \geq \omega(v(f)) \geq \omega(f),$$

so that the map $f \to h_f$ is continuous for the $\hat{\mathfrak{a}}$-adic topology of $k[[X_1, \ldots, X_n]]$. Also, for any $m \in \mathbf{N}^n$, $\hat{\pi}_m(h) = \hat{\pi}_m(h_i)$ for a sufficiently large i, whence the map $f \to h_f$ is weakly continuous. But the requirement k non-discrete does not play any role in the proof, and since for k discrete the weak topology coincides with the $\hat{\mathfrak{m}}$-adic topology, the map $f \to h_f$ is continuous also for the $\hat{\mathfrak{m}}$-adic topology of $k[[X_1, \ldots, X_n]]$. Thus $\lambda_f = h_f(v(g))^{-1}$ has the properties required in the first part of the theorem.

Assume now $g \in \Lambda = k\{X_1, \ldots, X_n\}$, and let I_g be the set of all $\alpha \in \mathbf{R}_+^{\star n}$ such that g and $(v(g))^{-1} \in \Lambda_\alpha$ and such that

$$\| p \|_\alpha < \alpha_n^s.$$

By the definition of p and since $u(g)$ is a polynomial in X_n with coefficients in the maximal ideal of $k\{X_1, \ldots, X_n\}$, it is clear that I_g is cofinal with $\mathbf{R}_+^{\star n}$. Let $\alpha \in I_g$, and let

$$\theta = \frac{\| p \|_\alpha}{\alpha_n^s}.$$

Then $\theta < 1$, and, given $f \in \Lambda_\alpha$, it follows from the inductive definition of h_i that all $h_i \in \Lambda_\alpha$ and that

$$\| h_{i+1} \|_\alpha \leq \frac{\| f \|_\alpha}{\alpha_n^s} + \theta \| h_i \|_\alpha.$$

Hence, by induction on i,

$$\| h_i \|_\alpha \leq \frac{\| f \|_\alpha}{(1-\theta)\alpha_n^s}$$

for all i. Thus, the sequence $(h_i)_{i \geq 0}$ is bounded in Λ_α, and is $\hat{\mathfrak{a}}$-convergent (therefore also weakly convergent). By Theorem 1.2, $(h_i)_{i \geq 0}$ is convergent to $h = h_f$ in Λ_α, and

$$\| h_f \|_\alpha \leq \frac{\| f \|_\alpha}{(1-\theta)\alpha_n^s}.$$

As $\lambda_f = h_f(v(g))^{-1}$, the theorem is completely proved.

Remark. — For any $f \in k\{X_1, \ldots, X_n\}$ and $x \in D_f$, we define

$$f(x) = \sum_{m \in \mathbf{N}^n} a_m x_1^{m_1} \ldots x_n^{m_n}.$$

Assume $k = \mathbf{C}$, and let D be a closed polydisc centred at 0 in $\mathbf{C}^n$. If $D \subset D_f$, we define

$$\| f \|_D = \max_{x \in D} | f(x) |.$$

Cauchy's inequalities and the maximum principle yield

$$\| u(f) \|_D \leq s \| f \|_D \qquad \text{and} \qquad \| v(f) \|_D \leq \frac{s+1}{\alpha_n^s} \| f \|_D.$$

Thus in the above proof we may use $\|.\|_D$ instead of $\|.\|_\alpha$. Also, by means of an exhaustion with closed polydiscs, we may go on to open polydiscs. We then obtain the following theorem of H. Cartan [1] :

There exist a fundamental system of open polydiscs D (with centre o *in $\mathbf{C}^n$) and for each such a D a $c_D > 0$ with the property that if f is holomorphic on D then λ_f is holomorphic on D and*

$$\sup_{x\in D} |\lambda_f(x)| \leq c_D \sup_{x\in D} |f(x)|.$$

Next we shall consider another example of " successive approximations " for power series.

Let $\hat{\Lambda} = k[[X_1, \ldots, X_n]]$. For $f = (f_1, \ldots, f_n) \in \hat{\Lambda}^n$, we define

$$\mathbf{o}(f) = \min_i \mathbf{o}(f_i).$$

If $h, f \in \hat{\Lambda}^n$ and if $\mathbf{o}(f) > 0$, we set

$$h(f) = (h_1(f_1, \ldots, f_n), \ldots, h_n(f_1, \ldots, f_n)).$$

Theorem 2.2. — *Let $g, h \in \hat{\Lambda}^n$ such that $\mathbf{o}(g) > 0$ and $\mathbf{o}(h) > 1$. Then there exists a unique $f \in \hat{\Lambda}^n$ such that*

$$\left\{\begin{array}{l}\mathbf{o}(f) > 0,\\ f = g + h(f).\end{array}\right. \tag{3}$$

Proof. — Since $\mathbf{o}(h) > 1$, there exists a $n \times n$-matrix $M(X, Y)$ with elements $M_{ij}(X, Y)$ in the maximal ideal of $k[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$ such that

$$h(X) - h(Y) = (X - Y)\,M(X, Y)$$

with a matrix product in the right side; in particular, we have

$$h(X) = X M(X, 0).$$

We define the sequence $(f_{(\nu)})_{\nu\geq 0}$ inductively by the conditions

$$f_{(0)} = 0$$
$$f_{(\nu+1)} = g + h(f_{(\nu)}).$$

Then

$$f_{(\nu+1)} - f_{(\nu)} = (f_{(\nu)} - f_{(\nu-1)})\,M(f_{(\nu)}, f_{(\nu-1)})$$

and so

$$\mathbf{o}(f_{(\nu+1)} - f_{(\nu)}) > \mathbf{o}(f_{(\nu)} - f_{(\nu-1)})$$

because we have, by induction on ν,

$$\mathbf{o}(M(f_{(\nu)}, f_{(\nu-1)})) > 0.$$

Thus $(f_{(\nu)})_{\nu\geq 0}$ is convergent for the $\hat{\mathfrak{m}}$-adic topology of $\hat{\Lambda}^n$, necessarily to a solution $f \in \hat{\Lambda}^n$ of (3).

If f' is another solution of (3), then

$$\mathbf{o}(f'-f) > \mathbf{o}(f'-f)$$

so that $\mathbf{o}(f'-f) = +\infty$, i. e. $f' = f$.

Next suppose that the component series of g and h are convergent. Then we may choose the series $M_{ij}(X, Y)$ to be also convergent. We define

$$\| g \|_\alpha = \max_i \| g_i \|_\alpha \quad \text{and} \quad \| M(X, 0) \|_\alpha = \max_{i,j} n \| M_{ij}(X, 0) \|_\alpha.$$

Let $\alpha \in \mathbf{R}_+^{*n}$ such that

$$\| M(X, 0) \|_\alpha = \theta < 1,$$

and let $\beta \leqq \alpha$ such that

$$\frac{\| g \|_\beta}{1-\theta} = \varepsilon \leqq \min_i \alpha_i.$$

Then it is easily seen, by induction on ν, that

$$\| f_{(\nu)} \|_\beta \leqq \varepsilon$$

for all $\nu \geqq 0$. (Indeed, if $\| f_{(\nu)} \|_\beta \leqq \varepsilon$, then, by the proof of Corollary 4 to Theorem 1.2, we have

$$\| M(f_{(\nu)}, 0) \|_\beta \leqq \| M(X, 0) \|_\beta \leqq \| M(X, 0) \|_\alpha = \theta,$$

whence, by the inductive definition of $f_{(\nu+1)}$, $\| f_{(\nu+1)} \|_\beta \leqq \varepsilon(1-\theta) + \varepsilon\theta = \varepsilon$). Thus the sequence $(f_{(\nu)})_{\nu \geqq 0}$ is bounded in Λ_β^n. Hence, again by Theorem 1.2, $(f_{(\nu)})_{\nu \geqq 0}$ is convergent to f in Λ_β^n and

$$\| f \|_\beta \leqq \varepsilon = \frac{\| g \|_\beta}{1-\theta},$$

Q. E. D.

We remark that any system of „ implicit functions “ can be easily reduced to the equation (3).

3. Analytic algebras and analytic modules.

Definition. — An *analytic algebra* over k is a k-algebra A such that

1° $A \neq 0$, and

2° there exists a k-algebra epimorphism $\varphi : k\{X_1, \ldots, X_n\} \to A$.

Any analytic algebra (over k) is local and has residue field $\approx k$. It follows that if A and B are analytic algebras (over k), then any k-algebra homomorphism $u : A \to B$ is local.

LEMMA 3.1. — *Let A and B be analytic algebras, $u : A \to B$ a homomorphism of k-algebras, and $\varphi : \Lambda \to A$, $\psi : \Gamma \to B$ epimorphisms of k-algebras, where $\Lambda = k\{X_1, \ldots, X_n\}$, $\Gamma = k\{Y_1, \ldots, Y_p\}$. Then there exists a homomorphism of k-algebras $v : \Lambda \to \Gamma$ such that the following diagram is commutative*

$$\begin{array}{ccc} \Lambda & \xrightarrow{v} & \Gamma \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ A & \xrightarrow{u} & B \end{array}$$

Proof. — Since u is local, the elements $u(\varphi(X_i))$ belong to the maximal ideal of B, hence there exist elements g_i in the maximal ideal of Γ such that

$$\psi(g_i) = u(\varphi(X_i)).$$

Let $v : \Lambda \to \Gamma$ be the homomorphism of k-algebras such that $v(X_i) = g_i$. Then clearly the restrictions of $u\varphi$ and ψv to $k[X_1, \ldots, X_n]$ are equal. Since $k[X_1, \ldots, X_n]$ is dense in $\Lambda = \{X_1, \ldots, X_n\}$ for the $\mathfrak{m}$-adic topology, while B is Hausdorff for its (maximal ideal)-adic topology, we have $u\varphi = \psi v$,

Q. E. D.

Let A be an analytic algebra over k and $\varphi : \Lambda \to A$ a k-algebra epimorphism, where $\Lambda = k\{X_1, \ldots, X_n\}$. We consider the quotient topology of A by φ, that is the finest topology on A for which φ is continuous. This topology is compatible with the k-algebra structure of A, and moreover φ is strict. Also this topology is independent of the choice of φ. Indeed, let $\psi : \Gamma \to A$ be another k-algebra epimorphism, where $\Gamma = k\{Y_1, \ldots, Y_p\}$. By Lemma 3.1, there exists a k-algebra homomorphism $v : \Lambda \to \Gamma$ such that $\varphi = \psi v$. By Corollary 4 to Theorem 1.2, v is continuous, hence φ strict implies ψ strict, etc.

DEFINITION. — Let A be an analytic algebra and $\varphi : \Lambda \to A$ a k-algebra epimorphism, where $\Lambda = k\{X_1, \ldots, X_n\}$. The quotient topology of A by φ (which is independent of the choice of φ) is called *the canonical topology* of A.

Any analytic algebra A will be considered as a topological k-algebra for its canonical topology; and clearly any analytic algebra is locally convex.

COROLLARY. — *Let A and B be analytic algebras over k. Any k-algebra homomorphism $u : A \to B$ is continuous (for the canonical topologies of A and B).*

Proof. — With the notations of Lemma 3.1, $u\varphi = \psi v$ is continuous (since ψ and v are continuous), hence u is continuous because φ is strict.

DEFINITION. — Let A be an analytic algebra. An A-module M is said to be an *analytic module* over A if it is finite over A, i. e. if there exists an A-module epimorphism.

$$\alpha: \quad A^p \to M$$

with $p \in \mathbf{N}$.

Any submodule of an analytic module over A is analytic over A, and any quotient of an analytic module over A is analytic over A; also any finite direct sum (or direct product) of analytic modules over A is analytic over A. For $A = k$, the analytic modules over A are exactly the vector k-spaces of finite dimension.

Let A be an analytic algebra, M an analytic module over A and $\alpha : A^p \to M$ an A-module epimorphism. It is clear that for any topology on M compatible with the A-module structure of M, α is continuous. Thus the quotient topology of M by α is the finest topology on M compatible with its A-module structure, hence it is independent of the choice of α. Clearly this topology is locally convex.

DEFINITION. — Let A be an analytic algebra, M an analytic module over A, and $\alpha : A^p \to M$ an A-module epimorphism. The quotient topology of M by α (which is independent of the choice of α) is called the *canonical topology of* M.

For instance, the canonical topology of A^p is the product topology. For $A = k$, any analytic module M over A is a vector k-space of finite dimension and the canonical topology of M is the unique Hausdorff topology compatible with the structure of vector k-space of M.

Any analytic module M over an analytic algebra A will be considered as a locally convex A-module with its canonical topology.

LEMMA 3.2.

a. Let M and N be analytic modules over A; then any A-module homomorphism $u : M \to N$ is continuous (for the canonical topologies of M and N).

b. If L is any submodule of Λ^r then L is analytic over Λ and is a k-space $\in \mathcal{LF}$ (for its canonical topology).

Proof. — To prove a, let $\alpha : A^p \to M$ be an A-module epimorphism. Then α is continuous strict and $u\alpha$ is continuous, hence u is continuous.

To prove b, let $i : L \to \Lambda$ be the canonical injection of L. By a, i is continuous. Since Λ^r is Hausdorff, L is Hausdorff, hence L, being a quotient of a Λ^p, is a k-space $\in \mathcal{LF}$ by Corollary 2 to Theorem 1.2.

LEMMA 3.3. — *Let A be an analytic algebra over k and $\varphi : \Lambda \to A$ a k-algebra epimorphism, where $\Lambda = k\{X_1, \dots, X_n\}$. Then any analytic A-module M is also an analytic Λ-module with $\lambda x = \varphi(\lambda)x$ for $\lambda \in \Lambda$*

and $x \in M$; *moreover the vector k-space structures and the canonical topologies of M as an analytic A-module and as an analytic* Λ*-module are identical.*

Proof. — Let $\alpha : A^p \to M$ be an A-module epimorphism. The composite map

$$\Lambda^p \xrightarrow{\varphi \times \ldots \times \varphi} A^p \xrightarrow{\alpha} M$$

is continuous strict both for the canonical topology of M as an analytic A-module and for the canonical topology of M as an analytic Λ-module, etc.

Definition. — Let $\mathcal{C}$ be an additive category. A sequence

$$S: \quad 0 \to E' \xrightarrow{i} E \xrightarrow{q} E'' \to 0$$

in $\mathcal{C}$ is said *to be split* or *to split* if there exist morphisms $p : E \to E'$ and $j : E'' \to E$ such that $pi = 1_{E'}$, $pj = 0$, $qi = 0$, $qj = 1_{E''}$ and $ip + jq = 1_E$, i. e. E is a direct sum of E' and E'' with canonical injections i, j and canonical projections p, q.

If S splits, obviously $i = \operatorname{Ker} q$ and $q = \operatorname{Coker} i$. Conversely if $i = \operatorname{Ker} q$ and $q = \operatorname{Coker} i$, the following statements on S are equivalents :

1° S splits;

2° i has a left inverse;

3° q has a right inverse.

In fact, assume 2° and let $p : E \to E'$ such that $pi = 1_{E'}$. Then $(1_E - ip)i = i - i(pi) = 0$. Since $q = \operatorname{Coker} i$, $qi = 0$ and there exists $j : E'' \to E$ such that $1_E - ip = jq$. It follows that

$$qjq = q(1_E - ip) = q - (qi)p = q,$$

hence $qj = 1_{E''}$. Also

$$pjq = p(1 - ip) = p - (pi)p = 0,$$

hence $pj = 0$. Thus $2° \Rightarrow 1°$ Also, if we have a $j : E'' \to E$ such that $qj = 1_{E''}$, then we see that there exists $p : E \to E'$ such that $1_E - jq = ip$, etc.

Lemma 3.4. — *Let* $\mathcal{C} = \mathcal{LF}$ *and suppose* S *is algebraically exact. Then the following statements on* S *are equivalent :*

(i) *S splits;*

(ii) *i has a left inverse in* $\mathcal{C}$;

(iii) *q has a right inverse in* $\mathcal{C}$.

Proof. — In the situation of (ii), q is strict by Banach's Theorem, hence $jq = 1_E - ip$ continuous implies j continuous. Also, in the situation of (iii), $ip = i_E - jq$ continuous implies p continuous by the Closed Graph Theorem; in fact, if $\lim_n x_n = 0$ in E and $\lim_n p(x_n) = x'$ in E', then

$$i(x') = \lim_n i(p(x_n)) = \lim_n (ip)(x_n) = (ip)(0) = 0,$$

hence $x' = 0$.

THEOREM 3.1. — *Any algebraically exact sequence $0 \to M' \to M \to M'' \to 0$ of analytic modules over an analytic algebra A splits in the category $\mathcal{LF}$.*

Proof. — By Lemma 3.3, it is enough to prove the theorem in the case $A = \Lambda = k\{X_1, \ldots, X_n\}$. In this case we shall use induction on n. For $n = 0$, M, M' and M'' are vector k-spaces of finite dimension, and the theorem is trivially true.

Next let $n > 0$, let $\Lambda' = k\{X_1, \ldots, X_{n-1}\}$, and assume, by the induction hypothesis, that the theorem is true for the analytic modules over Λ'. We shall then prove that the theorem is true for the analytic modules over Λ.

STEP 1. — For any ideal $\mathfrak{a}$ of Λ, there exists a k-linear continuous map $v : \Lambda \to \mathfrak{a}$ such that $v \mid \mathfrak{a} = 1_{\mathfrak{a}}$ ($\mathfrak{a}$ is an analytic module over Λ and has the canonical topology).

The case $\mathfrak{a} = \{0\}$ is trivial, hence we may assume $\mathfrak{a} \neq \{0\}$ Also, it suffices to prove the proposition for $\tau(\mathfrak{a})$ instead of $\mathfrak{a}$, where τ is any automorphism of Λ (by Corollary 4 to Theorem 1.2, τ is then topological). Hence we may suppose that $\mathfrak{a}$ contains a series g such that $g(0, \ldots, 0, X_n) \neq 0$. Let s be the order of the series

$$g(0, \ldots, 0, X_n) \in k[[X_n]],$$

and let H be the set of the elements $h \in \Lambda$ which are polynomials in X_n of degree $< s$. By the preparation lemma, for any $f \in \Lambda$,

$$f = \alpha(f) g + \beta(f),$$

where $\alpha : \Lambda \to \Lambda$ and $\beta : \Lambda \to \Lambda$ are Λ'-linear continuous and $\beta(\Lambda) \subset H$. Clearly H is an analytic module over Λ', canonically isomorphic to Λ'^s, and $\beta(\mathfrak{a})$ is a submodule of H. By the induction hypothesis, there exists a continuous k-linear map

$$v' : \quad H \to \beta(\mathfrak{a})$$

such that $v' \mid \beta(\mathfrak{a}) = 1_{\beta(\mathfrak{a})}$. We then define the map $v : \Lambda \to \mathfrak{a}$ by

$$v(f) = \alpha(f) g + v'(\beta(f)).$$

[Since $\beta(\mathfrak{a}) \subset \mathfrak{a}$, $v(f) \in \mathfrak{a}$ for all $f \in \Lambda$.] Clearly v is k-linear and clearly $iv : \Lambda \to \Lambda$ is continuous, where $i : \mathfrak{a} \to \Lambda$ is the canonical injection.

We know that Λ and $\mathfrak{a}$ are k-spaces $\in \mathcal{LF}$. Thus, by the Closed Graph Theorem, v is continuous. Finally, if $f \in \mathfrak{a}$, $\beta(f) \in \beta(\mathfrak{a})$, hence

$$v'(\beta(f) = \beta(f)$$

and therefore

$$v(f) = \alpha(f)\, g + \beta(f) = f,$$

Q. E. D.

STEP 2. — For any submodule L of Λ^r, there exists a continuous k-linear map $u' : \Lambda^r \to L$ such that $u'(f) = f$ when $f \in L$, i. e. $u'u = 1_L$ where $u : L \to \Lambda^r$ is the canonical injection.

In order to prove this proposition we shall make induction on r. We already know, by Step 1, that the proposition is true for $r = 1$. Next let $r > 1$. We assume, by the induction hypothesis, that the proposition is true for $r - 1$, and we shall prove it for r.

Let

$$\Lambda \underset{i}{\overset{p}{\leftrightarrows}} \Lambda^r \underset{q}{\overset{j}{\leftrightarrows}} \Lambda^{r-1}$$

be the canonical representation of Λ^r as direct sum of Λ and Λ^{r-1}. Let $\mathfrak{a} = i^{-1}(L)$ and $N = q(L)$, let $v : \mathfrak{a} \to \Lambda$ and $w : N \to \Lambda^{r-1}$ be the canonical injections, and let $\alpha : \mathfrak{a} \to L$ and $\psi : L \to N$ be the maps induced by i and q. Then $\mathfrak{a}$, L, N are analytic modules over Λ, and we know by Lemma 3.2 they are k-spaces $\in \mathcal{LF}$. We have a commutative diagram with (algebraically) exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{a} & \xrightarrow{\alpha} & L & \xrightarrow{\psi} & N & \to & 0 \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle w} & & \\ 0 & \to & \Lambda & \xrightarrow{i} & \Lambda^r & \xrightarrow{q} & \Lambda^{r-1} & \to & 0. \end{array}$$

Since the proposition is true for Λ and Λ^{r-1} there exist continuous k-linear maps

$$v' : \quad \Lambda \to \mathfrak{a} \qquad \text{and} \qquad w' : \quad \Lambda^{r-1} \to N$$

such that $v'v = 1_{\mathfrak{a}}$ and $w'w = 1_N$. From $qu = w\psi$ it follows that $w'qu = w'w\psi = \psi$. Let $\varphi = v'pu$. Then

$$\varphi\alpha = v'pu\alpha = v'piv = v'v = 1_{\mathfrak{a}},$$

hence φ is a continuous k-linear left inverse of α. By Lemma 3.4, the top row splits in the category $\mathcal{LF}$, hence there exists a continuous k-linear map $\beta : N \to L$ such that

$$\alpha\varphi + \beta\psi = 1_L.$$

We define

$$u' = \alpha v'p + \beta w'q;$$

then u' is k-linear continuous and

$$u'u = \alpha v' pu + \beta w' qu = \alpha\varphi + \beta\psi = 1_L,$$

Q. E. D.

From Step 2 it easily follows that any submodule of Λ^r is closed, hence *any analytic module is Hausdorff and therefore is a k-space* $\in \mathcal{LF}$; in particular this is true for any analytic algebra.

STEP 3. — Let $O \to M' \xrightarrow{u} M \xrightarrow{v} M'' \to O$ be an algebraically exact sequence of analytic modules over Λ, and let $\alpha : \Lambda^p \to M$ be a Λ-module epimorphism. By Step 2 and Lemma 3.4, $v\alpha$ has a continuous k-linear right inverse, say ε, hence v has a continuous k-linear right inverse namely $\alpha\varepsilon$. By Lemma 3.4, the sequence splits and the proof is complete.

COROLLARY 1. — *Let M and N be analytic modules over an analytic algebra A. Then any A-module homomorphism* $u : M \to N$ *is continuous strict.*

Proof. — We already know that u is continuous. Further we have $u = \beta\alpha$ with α an epimorphism and β a monomorphism (of analytic modules over A). Then α and β are continuous and, by Theorem 3.1, α has a continuous k-linear right inverse, and β has a continuous k-linear left inverse; hence α and β are strict, etc.

COROLLARY 2. — *Any analytic algebra and any analytic module are k-spaces* $\in \mathcal{LF}$.

COROLLARY 3. — *Any submodule of an analytic module M is a closed subset of M* (3).

4. Bounded sets.

DEFINITION. — We define $f \leq g$ in $\mathbf{R}\{X_1, \ldots, X_n\}$ if and only if $\pi_m(f) \leq \pi_m(g)$ for all $m \in \mathbf{N}^n$.

Clearly, this is an *order relation* on the $\mathbf{R}$-algebra $\mathbf{R}\{X_1, \ldots, X_n\}$ and makes it an ordered $\mathbf{R}$-algebra, i. e. $(f' \leq g'$ and $f'' \leq g'') \Rightarrow f' + f'' \leq g' + g''$ and $(f \leq g$ and $h \geq 0) \Rightarrow hf \leq hg$, in particular $(f \leq g$ and $t \geq 0$ in $k) \Rightarrow tf \leq tg$.

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(3) Prof. H. CARTAN kindly communicated me the following :

« Tout le paragraphe 3 pourrait aussi être traité pour la topologie faible (à partir de la topologie faible de Λ, on définit la topologie faible pour tout module de type fini sur une k-algèbre analytique A). On a les mêmes résultats, et l'on en déduit que la topologie faible d'un module analytique est *séparée*. En particulier, tout idéal de Λ est *fermé* pour la topologie faible; c'est là un résultat très utile, et plus intéressant que le résultat analogue pour la topologie canonique. »

DEFINITION. — Let $\Lambda = k\{X_1, \ldots, X_n\}$. We define the map

$$\Phi: \quad \Lambda \to \mathbf{R}\{X_1, \ldots X_n\}$$

by setting

$$\Phi(f) = \sum_{m \in \mathbf{N}^n} |a_m| X_1^{m_1} \ldots X_n^{m_n}$$

for any $f = \sum_{m \in \mathbf{N}^n} a_m X_1^{m_1} \ldots X_n^{m_n}$.

It will be convenient sometimes to use the notation $|f|$ instead of $\Phi(f)$.

LEMMA 4.1. — *The map Φ has the following properties :*

1° $\Phi(f) \geqq 0$, and $\Phi(f) = 0 \Leftrightarrow f = 0$;

2° $\Phi(f+g) \leqq \Phi(f) + \Phi(g)$;

3° $\Phi(fg) \leqq \Phi(f)\,\Phi(g)$;

4° $\Phi(tf) = |t|\,\Phi(f)$ for $t \in k$, and $\Phi(1) = 1$;

5° $\|\Phi(f)\|_\alpha = \|f\|_\alpha$.

The proof is trivial and will be omitted.

DEFINITION. — A subset D of Λ is called a *disc* (centred at $0 \in \Lambda$) if

1° D is absolutely convex, and

2° ($f \in D$ and $\Phi(g) \leqq \Phi(f)) \Rightarrow g \in D$.

Clearly any ball in a Λ_α is a disc in Λ.

COROLLARY. — *If D is a disc in $\mathbf{R}\{X_1, \ldots, X_n\}$, then $\Phi^{-1}(D)$ is a disc in Λ and $f + \Phi^{-1}(D) \subset \Phi^{-1}(|f| + D)$ for any $f \in \Lambda$.*

NOTATION. — We shall denote by **Top** the category of all topological spaces and continuous maps.

THEOREM 4.1. — *For k locally compact, the following statements hold :*

1° $\Lambda = \varinjlim_\alpha \Lambda_\alpha$ *in* **Top**, *i. e. a set $M \subset \Lambda$ is open (closed) in Λ if and only if $M \cap \Lambda_\alpha$ is open (closed) in Λ_α for all $\alpha \in \mathbf{R}_+^{\star n}$.*

2° *The discs D (centred at 0) such that $0 \in \mathring{D}$ form a fundamental system of neighbourhoods of 0 in Λ.*

3° *The sets $\Phi^{-1}(V)$ with V a neighbourhood of 0 in $\mathbf{R}\{X_1, \ldots, X_n\}$ form a fundamental system of neighbourhoods of 0 in Λ.*

The proof will be preceded by some lemmas.

LEMMA 4.2. — *Let $k = \mathbf{R}$ or $k = \mathbf{C}$. Then if D' and D'' are disc in Λ, so is $D = D' + D''$.*

Proof. — We begin with a remark : if a, b, $c \in k$ (where $k = \mathbf{R}$ or $k = \mathbf{C}$) and if $|c| \leqq |a| + |b|$, then there exist $x, y \in k$ such that $|x| \leqq |a|$, $|y| \leqq |b|$ and $x + y = c$.

Indeed, let $c = |c| e^{i\theta}$. Since $|c| \leqq |a| + |b|$, we may write $|c| = \varepsilon + \delta$ with $0 \leqq \varepsilon \leqq |a|$ and $0 \leqq \delta \leqq |b|$. We define $x = \varepsilon e^{i\theta}$ and $y = \delta e^{i\theta}$. Then $|x| \leqq |a|$, $|y| \leqq |b|$ and $x + y = c$. Moreover, if c is real, x and y are real.

Now let $f' \in D'$ and $f'' \in D''$, and let $f \in \Lambda$ such that $\Phi(f) \leqq \Phi(f' + f'')$. Then $\Phi(f) \leqq \Phi(f') + \Phi(f'')$, hence $|\pi_m(f)| \leqq |\pi_m(f')| + |\pi_m(f'')|$ for all $m \in \mathbf{N}^n$. By the preceding remark, there exist b'_m, $b''_m \in k$ such that $|b'_m| \leqq |\pi_m(f')|$, $|b''_m| \leqq |\pi_m(f'')|$ and $\pi_m(f) = b'_m + b''_m$. Let

$$g' = \sum_{m \in \mathbf{N}^n} b'_m X_1^{m_1} \ldots X_n^{m_n}, \qquad g'' = \sum_{m \in \mathbf{N}^n} b''_m X_1^{m_1} \ldots X_n^{m_n}.$$

Clearly g', $g'' \in \Lambda$, $\Phi(g') \leqq \Phi(f')$, $\Phi(g'') \leqq \Phi(f'')$ and $f = g' + g''$. As D' and D'' are discs, $g' \in D'$ and $g'' \in D''$, whence $f \in D$.

Q. E. D.

LEMMA 4.3. — *For k locally compact and $\Lambda = k\{X_1, \ldots, X_n\}$, let T be a topological space, U an open set in T and $\Psi : T \to \Lambda$ a map such that*

1° *Ψ is continuous;*

2° *for any $\alpha \in \mathbf{R}_+^{\star n}$ and any compact set $M \subset \Lambda_\alpha$, the set $\Psi^{-1}(M)$ is compact;*

3° *$\Psi^{-1}(0) \subset U$.*

Then there exists a neighbourhood D of 0 in Λ such that $\Psi^{-1}(D) \subset U$. Moreover, for $k = \mathbf{R}$ or $k = \mathbf{C}$, we may choose D to be a disc.

Proof. — For any integer $i \geqq 1$, let $\alpha(i) = \left(\frac{1}{i}, \ldots, \frac{1}{i}\right)$, and let $\Lambda_i = \Lambda_{\alpha(i)}$. Since k is locally compact, each canonical injection $\Lambda_i \to \Lambda_{i+1}$ is compact (by Corollary 1 to Theorem 1.2); also

$$\Lambda = \varinjlim_i \Lambda_i$$

in **ELC**. We shall construct, inductively, a sequence $(D_i)_{i \geqq 1}$ such that :

1° $D_1 = \{0\}$;

2° for $i > 1$, D_i is an absolutely convex bounded open neighbourhood of 0 in Λ_i,

3° $D_i \subset D_{i+1}$, and

4° $\Psi^{-1}(\overline{D}_i) \subset U$, where $\overline{D}_i$ is the closure of D_i in Λ_{i+1}.

In fact, let $p > 1$ and assume the construction made for $i < p$. Then $\overline{D}_{p-1}$ is compact in Λ_p and

$$\Psi^{-1}(\overline{D}_{p-1}) \subset U.$$

Clearly

$$\overline{D}_{p-1} = \bigcap_N (D_{p-1} + N)$$

for N in the set of all neighbourhoods of o in Λ_{p+1}. Hence

$$\Psi^{-1}(\overline{D}_{p-1}) = \bigcap_N \Psi^{-1}(D_{p-1} + N).$$

But, for N bounded, $D_{p-1} + N$ is relatively compact in Λ_{p+2}, hence $\Psi^{-1}(D_{p-1} + N)$ is relatively compact in T. It follows that there exists an $N = W_{p+1}$ such that

$$\Psi^{-1}(D_{p-1} + W_{p+1}) \subset U.$$

Let V_{p+1} be a neighbourhood of o in Λ_{p+1} such that

$$V_{p+1} + V_{p+1} \subset W_{p+1},$$

let B_p be a bounded open ball centred at o in Λ_p such that $B_p \subset U_{p+1}$ and let

$$D_p = D_{p-1} + B_p.$$

Then D_p is an absolutely convex bounded open neighbourhood of o in Λ_p,

$$D_{p-1} \subset D_p, \qquad \text{and} \qquad \overline{D}_p \subset D_{p-1} + B_p + V_{p+1} \subset D_{p-1} + W_{p+1},$$

hence

$$\Psi^{-1}(\overline{D}_p) \subset U.$$

This completes the induction.

Let $D = \bigcup_{i \geq 1} D_i$. Then D is an absolutely convex neighbourhood of o in Λ and

$$\Psi^{-1}(D) = \bigcup_{i \geq 1} \Psi^{-1}(D_i) \subset U.$$

Moreover, for $k = \mathbf{R}$ or $k = \mathbf{C}$, each $D_p = D_{p-1} + B_p$ is a disc by Lemma 4.2, hence D is a disc, and the proof of Lemma 4.3 is complete.

Remark. — Obviously, in Lemma 4.3, we may replace Λ with any $A = \varinjlim_i A_i$ (in **ELC**), where $(A_i)_{i \geq 1}$ is an increasing sequence of normed k-spaces with compact canonical injections $A_i \to A_{i+1}$. Then Lemma 4.3 can be interpreted as a generalisation of *Teorema* 1 of Sebastião e Silva in [4].

Proof of Theorem 4.1. — Let $\tau_{\mathbf{Top}}$ the finest topology on Λ for which all canonical injections $\Lambda_\alpha \to \Lambda$ are continuous, and let T be the topological space Λ with the topology $\tau_{\mathbf{Top}}$; then $T = \varinjlim_\alpha \Lambda_\alpha$ in **Top**. Clearly the map $\Psi = 1_\Lambda$ satisfies conditions 1° and 2° of Lemma 4.3 and $\Psi^{-1}(0) = \{0\}$. From Lemma 4.3 it follows that any neighbourhood of 0 in T contains a neighbourhood of 0 in Λ (for the canonical topology), which proves 1°.

Also from Lemma 4.3 it follows that for $k = \mathbf{R}$ or $k = \mathbf{C}$, the discs D centred at 0 in Λ such that $0 \in \mathring{D}$ form a fundamental system of neighbourhoods of 0.

Before ending the proof of Theorem 4.1 we shall establish the following.

LEMMA 4.4. — *For k arbitrary, the map $\Phi : \Lambda \to \mathbf{R}\{X_1, \ldots, X_n\}$ is continuous. When k is locally compact, for any $\alpha \in \mathbf{R}_+^{\star n}$ and any compact set $M \subset \mathbf{R}\{X_1, \ldots, X_n\}_\alpha$, the set $\Phi^{-1}(M)$ is compact.*

Proof. — Let D be a disc in $\mathbf{R}\{X_i, \ldots, X_n\}$. Then $\Phi^{-1}(D)$ is a disc in Λ, in particular $\Phi^{-1}(D)$ is absolutely convex. For any α we have a commutative diagram

$$\begin{tikzcd} \Lambda_\alpha \arrow[r] \arrow[d, "\Phi_\alpha"'] & \Lambda \arrow[d, "\Phi"] \\ R\{X_1, \ldots, X_n\}_\alpha \arrow[r] & R\{X_1, \ldots, X_n\} \end{tikzcd}$$

with Φ_α continuous at 0 (since Φ_α is induced by Φ and since $\|\Phi(f)\|_\alpha = \|f\|_\alpha$). It follows that, for any α, $\Phi^{-1}(D) \cap \Lambda_\alpha$ is a neighbourhood of 0 in Λ_α. Hence $\Phi^{-1}(D)$ is a neighbourhood of 0 in Λ (for the canonical topology). Since the discs D centred at 0 in $\mathbf{R}\{X_1, \ldots, X_n\}$ such that $0 \in \mathring{D}$ form a fundamental system of neighbourhoods of 0, and by Corollary to Lemma 4.1, we conclude that Φ is continuous.

Now assume k locally compact. If M is a compact set in $\mathbf{R}\{X_1, \ldots, X_n\}_\alpha$, then M is bounded in $\mathbf{R}\{X_1, \ldots, X_n\}_\alpha$, hence $\Phi^{-1}(M)$ is bounded in Λ_α (since $\|\Phi(f)\|_\alpha = \|f\|_\alpha$) and consequently relatively compact in Λ (by Corollary 1 to Theorem 1.2). As M is also closed in $\mathbf{R}\{X_1, \ldots, X_n\}$ and Φ continuous, $\Phi^{-1}(M)$ is closed in Λ. Thus $\Phi^{-1}(M)$ is compact in Λ, which completes the proof of Lemma 4.4.

We now return to the proof of Theorem 4.1. Let $T = \Lambda$ and $\Psi = \Phi$. By Lemma 4.4, Φ satisfies conditions 1° and 2° of Lemma 4.3 and $\Phi^{-1}(0) = \{0\}$. From Lemma 4.3 it follows that any neighbourhood of 0 in Λ contains a set $\Phi^{-1}(V)$ with V a neighbourhood of 0 in $\mathbf{R}\{X_1, \ldots, X_n\}$. This proves 3°. Since for V disc, $\Phi^{-1}(V)$ is a disc, we see that 2° is also true. Thus Theorem 4.1 is completely proved.

COROLLARY. — *For k locally compact, the following statements hold :*

a. For any bounded set $M \subset \Lambda$, there exists an α such that $M \subset \Lambda_\alpha$ and M is bounded in Λ_α.

b. For any convergent sequence σ in Λ, there exists an α such that σ is " contained " in Λ_α and is convergent in Λ_α.

c. Λ is Montel (in particular : quasi-complete).

Proof. — Only the first statement needs to be proved. Suppose the contrary would hold for some bounded set $M \subset \Lambda$. Let $t \in k$ such that $0 < |t| < 1$. Then there exists a sequence of elements $f_i \in M$ such that

$$\| t^i f_i \|_{\alpha(i)} > 1 \qquad \text{and} \qquad t^i f_i \neq t^j f_j \quad \text{for } i \neq j.$$

Let N be the set $\{ tf_1, t^2 f_2, \ldots \}$. Since M is bounded in Λ,

$$\lim_i t^i f_i = 0,$$

hence N is not closed in Λ. However it is easily seen that each set $N \cap \Lambda_i$ is closed in Λ, which is a contradiction by Theorem 4.1.

Remark. — From the preceding Corollary and from Cauchy's inequalities it follows that, for $k = \mathbf{C}$, a sequence of elements $f_i \in \mathbf{C}\{X_1, \ldots, X_n\}$ is convergent if and only if there exists a closed polydisc D centred at 0 in $\mathbf{C}^n$ such that $D \subset \bigcap_i D_{f_i}$ and such that the sequence of functions $f_i(x)$ is uniformly convergent on D.

We shall now extend the preceeding corollary to the general case :

THEOREM 4.2 (4). — *For k arbitrary, the following statements hold :*

a. For any bounded set $M \subset \Lambda$, there exists an α such that $M \subset \Lambda_\alpha$ and M is bounded in Λ_α.

b. For any convergent sequence σ in Λ, there exists an α such that σ is " contained " in Λ_α and is convergent in Λ_α.

c. Λ is quasi-complete.

Proof. — Let $M \subset \Lambda$ be bounded. From Lemma 4.4 it follows that $\Phi(M)$ is bounded in $\mathbf{R}\{X_1, \ldots, X_n\}$. By Corollary to Theorem 4.1, there exist α and $c > 0$ such that

$$\| f \|_\alpha = \| \Phi(f) \|_\alpha \leq c \quad \text{for all } f \in M.$$

Also any Cauchy sequence in Λ is bounded and weakly Cauchy (therefore weakly convergent), etc.

(4) The idea of using the map Φ to extend Corollary of Theorem 4.1 to the general case and the proof of Theorem 4.2 are due to C. FOIAS.

COROLLARY. — Λ *is not metrizable.*

Proof (5). — Assume Λ would be metrizable. Then $\Lambda \in \mathcal{L}$ being quasi-complete. By Corollary 1.3, for $E = \Lambda$ and $u = 1_\Lambda$, $\Lambda = \Lambda_\alpha$ for some α, which is a contradiction (as is easily seen).

Remark. — Since the discs D centred at o in $\mathbf{R}\{X_1, \ldots, X_n\}$ such that $0 \in \mathring{D}$ form a fundamental system of neighbourhood of o, it is easily seen that the sets $\Phi^{-1}(V)$ with V a neighbourhood of o in $\mathbf{R}\{X_1, \ldots, X_n\}$ form a fundamental system of neighbourhoods of o in Λ for a locally convex topology τ_Φ on Λ, compatible with the structure of k-algebra of Λ. For k locally compact, we know, by Theorem 4.1, that $\tau_\Phi = \tau_{\mathbf{ELC}} = \tau_{\mathbf{Top}}$, where $\tau_{\mathbf{ELC}}$ is the canonical topology of Λ. In the general case we have

$$\tau_\Phi \subset \tau_{\mathbf{ELC}} \subset \tau_{\mathbf{Top}}.$$

It is also easily seen that the bounded sets in Λ for the topology τ_Φ are the same with those for the canonical topology $\tau_{\mathbf{ELC}}$. The question, if $\tau_\Phi = \tau_{\mathbf{ELC}}$ or if $\tau_{\mathbf{ELC}} = \tau_{\mathbf{Top}}$ in the general case, remains open.

REFERENCES.

[1] CARTAN (Henri). — Idéaux de fonctions analytiques de n variables complexes, *Ann. scient. Éc. Norm. Sup.*, 3e série, t. 61, 1944, p. 149-197.

[2] GROTHENDIECK (Alexander). — *Espaces vectoriels topologiques.* — São Paulo, Instituto de Matemática pura e aplicada do Conselho nacional de Pesquisas, 1954.

[3] JURCHESCU (Martin). — Teoria locală a functiilor de mai multe variabile complexe, Expuneri in *Seminarul Stoïlow*, 1960-1961, Bucuresti, Ed. Acad. R. P. R., 1965.

[4] SEBASTIÃO E SILVA (J.). — Su certe classi di spazi localmente convessi importanti per le applicazioni, *Rend. Math. e delle sue Appl.*, Roma, t. 14, 1955, p. 388-410.

[5] ZARISKI (O.) and SAMUEL (P.). — *Commutative algebra*, vol. 2. — Princeton, D. Van Nostrand Comp., 1960.

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