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ON Σ-GROUPS

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Recently, IRWIN and WALKER [2] introduced the notion of Σ-groups. One question left open is whether or not the property of being a Σ-group is heriditary, (i. e.) whether every subgroup of a Σ-group is again a Σ-group. We will show that this is not always the case by constructing an easy example. Further, it is asked in [3] whether every high subgroup of a Σ-group G is an endomorphic image of G. We will give an affirmative answer to this question. Lastly, we investigate when a cotorsion group is a Σ-group.

All groups considered in this note are abelian. If G is any group, then G^1 denotes the subgroup of elements of infinite height in G, that is, $G^1 = \bigcap_{n=1}^{\infty} nG$. We will sometimes refer to it as the *Radical* of G. A subgroup maximally disjoint with G^1 is known as a high subgroup of G. A group is called a Σ-group, if all its high subgroups are direct sums of cyclic groups. It is known [3] that if G is a Σ-group, then all its high subgroups are isomorphic. A group G is called cotorsion if it is reduced and $\operatorname{Ext}(Q, G) = 0$, where Q is the additive group of rational numbers. For general properties of high subgroups we refer to [2]. If G is any group, then G_t denotes the torsion part of G.

LEMMA 1.

(i) *Every torsion group G contains a Σ-group R such that $G^1 = R^1$, R is pure in G and G/R divisible.*

(ii) *Every torsion-free group G contains a Σ-group R such that $G^1 = R^1$ and R is pure in G.*

PROOF. — Assertion (i) is the content of theorem 10 of [2], while (ii) follows on noting that for the torsion free group G, the radical G^1 is divisible and so is itself a Σ-group.

LEMMA 2. — *Every reduced torsion group G is isomorphic to the radical of some reduced torsion Σ-group R.*

PROOF. — Appealing to the assertion (i) of lemma 1, we see that it is enough if we show that G is a subgroup of a torsion group H, with $H^1 = G$. To each $a \in G$, associate a sequence of elements $(a_2, a_3, \ldots, a_n, \ldots)$ with the conditions that $na_n = a$. Let H be the group obtained by adjoining to G all such sequences of elements with the stated conditions. It is then immediately checked up that $H^1 = G$ and that H is torsion.

THEOREM 1 — *Not every subgroup of a Σ-group need again be a Σ-group* (1).

PROOF. — Since every divisible group is a Σ-group, the assertion follows trivially, when we consider a divisible group containing a group G which is not a direct sum of cyclic groups and for which $G^1 = 0$.

We could even give an example of a reduced group which is a Σ-group and not all subgroups of which are Σ-groups.

Let G be an unbounded closed p-group. Then it follows from [1] (p. 116) that G is not a Σ-group. But, by lemma 2, there is a reduced Σ-group R which contains G (even as its radical). This R is the required counter example.

The following theorem answers a question raised in [3].

THEOREM 2. — *Let G be a Σ-group. Then every high subgroup of G is an endomorphic image of G.*

PROOF. — We assume, without loss in generality, that G is reduced. If G is torsion, then every high subgroup of G is a basic subgroup of G and hence is an endomorphic image of G. If G is torsion free, since G is reduced, $G^1 = 0$ and so, G is its own high subgroup, so that it clearly has the required property. Let now G be a mixed group. Then we distinguish two cases :

Case 1. — Let G/G_t be reduced. Now every high subgroup H of G is a direct sum of cyclic groups and hence H splits. Then, by theorem 2 of [3], G also splits; $G = G_t + F$. Since G_t and F are Σ-groups, all their high subgroups are endomorphic images. Then we readily check up that every high subgroup of G is an endomorphic image of G.

(1) The author thanks Prof. E. A. WALKER for offering comments towards the simplification of the example. He also thanks him for having given the benefit of papers [2] and [3] long before their publication.

Case 2. — Let G/G_t be not reduced. Let M/G_t be the maximal divisible subgroup of G/G_t. Let $H = H_t + H_f$ be a high subgroup of G. Then by theorem 1 of [3], $G = M + H_f$.

Now H_t is high in G_t and $G_t \subseteq M$. We show that H_t is high in M. Clearly, $H_t \cap M^1 = 0$. If H_t is not high in M, let $K \supseteq H_t$ be high in M. Then, $K + H_f$ includes $H_t + H_f = H$ and is high in G, which contradicts the maximality of H. Hence $K = H_t$ so that H_t is high in M.

Now M is a mixed group such that it has a high subgroup which is torsion. On the other hand, the group M, being a direct summand of G, is a Σ-group. Thus all high subgroups of M are isomorphic and hence torsion. Now if $y \in M$ and $o(y) = \infty$, then $\{y\} \cap M^1 \neq 0$ since otherwise, $\{y\}$ can be expanded to a high subgroup of M which will no longer be torsion. Hence there exists an integer k such that $ky \in M^1$. This implies, in particular, that $M^\star = M/M^1$ is a torsion group. Let $H^\star$ be the image of H_t in $M^\star$. Then $H^\star \cong H_t$ and hence is a direct sum of cyclic groups.

We show that $H^\star$ is pure in $M^\star$. Let $nx^\star = a^\star \in H^\star$, where $x^\star \in M^\star$. Then $nx = a + b$, where $x \in M$, $a \in H$, $b \in M^1$. Since b is in M^1, $b = ny$, $y \in M$. Therefore,

$$
\begin{aligned}
n(x - y) &= a \in H_t \\
&= nz, \quad z \in H_t, \quad \text{since } H_t \text{ is pure.}
\end{aligned}
$$

$$
\begin{aligned}
&\text{Hence } nx = nz + b \\
&\text{(i. e.) } nx^\star = nz^\star, \quad z^\star \in H^\star.
\end{aligned}
$$

This proves that $H^\star$ is pure in $M^\star$.

Thus $H^\star$ is a direct sum of cyclic groups and is pure in the torsion group $M^\star$. Then $H^\star$ is a direct summand of a basic subgroup $B^\star$ of $M^\star$ and so $H^\star$ is an endomorphic image of $M^\star$. Since $M^\star$ is an epimorphic image of M, it follows that $H^\star$ is an epimorphic image of M. But $H^\star$ is isomorphic to H_t so that we are assured of an epimorphism of M on H_t. That is, H_t is an endomorphic image of M. Let this mapping be θ.

Now $G = M + H_f$. Let π and π' be the corresponding projections, $\pi(G) = M$, $\pi'(G) = H_f$. Then define a mapping δ of G in to itself as $\delta = \theta\pi + \pi'$. Now we can readily check up that δ is an endomorphism and $\delta(G) = H$. Thus H is an endomorphic image of G and this completes the proof.

Remark. — It is worth noting that the subgroup M is fully invariant in G. This follows from the fact that M is the unique complementary summand of H_f in G which together with theorem 22.3 of [1] implies that M is fully invariant.

We now investigate when a cotorsion group is a Σ-group. Before that we consider two lemmas which are of independent interest.

LEMMA 3. — *A high subgroup H of a cotorsion group G is an endomorphic image of G if and only if $G = H$.*

PROOF. — We need only to prove the necessary part. If H is an endomorphic image of G, then clearly it should be cotorsion. Since H is high in G, G/H is divisible and so, the exact sequence

$$0 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 0$$

gives the exact sequence,

$$\text{Hom}(Q, G) = 0 \rightarrow \text{Hom}(Q, G/H) \rightarrow \text{Ext}(Q, H) \rightarrow \text{Ext}(Q, G) = 0,$$

where the first and the last terms are zero since G is cotorsion. Since G/H is divisible, $\text{Hom}(Q, G/H) \neq 0$. This means that $\text{Ext}(Q, H) \neq 0$ so that H is not cotorsion, which is a contradiction unless $G/H = 0$, (i. e.) $G = H$.

LEMMA 4. — *A cotorsion group is a direct sum of cyclic groups if and only if it is bounded.*

The proof follows from the observations :

(i) A direct summand of a cotorsion group is again cotorsion.
(ii) An infinite cyclic group S is cotorsion if and only if $S = 0$.
(iii) A torsion cotorsion group is bounded.

THEOREM 3. — *A cotorsion group is a Σ-group if and only if it is bounded.*

PROOF. — If G is a Σ-group, by theorem 2, H (high in G) is an endomorphic image of G. But then, by lemma 3, G coincides with H. Now H is a direct sum of cyclic groups and so lemma 4 settles that G is bounded.

On the other hand, if G is bounded, then clearly it is a Σ-group.

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