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A POSTERIORI ERROR ESTIMATES FOR THE 3D STABILIZED MORTAR FINITE ELEMENT METHOD APPLIED TO THE LAPLACE EQUATION *

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Abstract. We consider a non-conforming stabilized domain decomposition technique for the discretization of the three-dimensional Laplace equation. The aim is to extend the numerical analysis of residual error indicators to this model problem. Two formulations of the problem are considered and the error estimators are studied for both. In the first one, the error estimator provides upper and lower bounds for the energy norm of the mortar finite element solution whereas in the second case, it also estimates the error for the Lagrange multiplier.

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1. Introduction

We consider the Dirichlet problem

$$\begin{cases} -\Delta u = f, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^3$, with a Lipschitz continuous boundary $\partial\Omega$ and f is a given function in $L^2(\Omega)$. We are interested in the mortar finite element discretization of such model problem.

The mortar element method is a non-overlapping domain decomposition technique which allows for using different discretizations in the sub-domains. So, as it was already observed in [6], it is well adapted for mesh adaptivity. Indeed, the possibility of working with non-matching grids leads to an efficient algorithm. In order to take full advantage of the mortar method, it is necessary to define the error indicators associated to this approach. The *a posteriori* analysis with various error indicators has been extensively studied (see [14] and the references therein), however it is still insufficient for the mortar method. We refer to [16] for first estimates concerning the residual type error indicators in the mortar framework. The extension of these error indicators to the mortar finite element discretization in the two-dimensional case together with the derivation of optimal estimates has been carried out in [4] without any saturation assumption (*i.e.*, without assuming that finite elements of higher order provide better approximation).

Keywords and phrases. Mortar finite element method, *a posteriori* estimates, mixed variational formulation, stabilization technique, non-matching grids.

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The aim of this paper is to extend the *a posteriori* analysis with residual type error indicators to a three-dimensional Laplace equation (as a typical elliptic second-order problem). Relying on a hybrid formulation which gives rise to a saddle point problem, residual error indicators are proposed for two formulations of the model problem: in the first formulation, which is the constrained one, they are associated to the error on the mortar finite element solution. In the second case they are associated to the unconstrained formulation and also enable us to estimate the Lagrange multiplier error. For both cases, we derive estimates which allow one to compare the error estimators with the energy-norm of the error, without any saturation assumption. These estimates are independent of the discretization parameters.

In order to perform the *a posteriori* analysis, we have to describe the appropriate mortar finite element method in this framework. In three-dimensions, the approximation of the Laplace equation by using the finite element method in the context of the domain decomposition with non-matching grids involves many difficulties. The mortar element method has allowed to circumvent some of these difficulties and provides an efficient tool for solving such problems. However, in the case of the tetrahedral affine finite element method the construction of the Lagrange multiplier spaces (which express the matching condition across the interfaces) in the earlier mortar approach is not an easy task (see [3]). A new formulation combining the saddle point formulation of the mortar method and stabilization techniques is proposed in [1]. It allows the construction of a variant for the mortar approach well adapted to the numerical simulation by the tetrahedral affine finite element method for the three-dimensional Laplace equation. Based on this formulation, we review the *a priori* analysis and perform the *a posteriori* analysis by residual error indicators.

The outline of the paper is as follows: in Section 2, we define the continuous setting and the discrete problems. We also prove their well-posedness and derive some *a priori* estimates. In Section 3 we introduce the error estimators and perform the *a posteriori* analysis by proving that they provide upper and lower bounds to the energy norm of the mortar finite element solution. Section 4 is devoted to the *a posteriori* analysis of the unconstrained formulation for which the error estimators also enable us to estimate the Lagrange multiplier.

2. Stabilized mortar finite element method

2.1. Variational formulation and regularity

We introduce a decomposition of Ω into a finite number of disjoint open sub-domains Ω^ℓ, $1 \le \ell \le L$. In view of the discretization, we consider a mixed variational formulation (known as the primal hybrid formulation of the Poisson–Dirichlet problem (1), [11]). We denote by X the space of functions

$$X = \Big\{ v \in L^2(\Omega),\ v_\ell = v_{|\Omega^\ell} \in H^1\left(\Omega^\ell\right),\ 1 \le \ell \le L \Big\},$$

equipped with the broken H^1-norm

$$\|v\|_X = \left(\sum_{\ell=1}^{L} \|v_\ell\|^2_{H^1(\Omega^\ell)} \right)^{\frac{1}{2}}.$$

We consider the space

$$H(\mathrm{div}, \Omega) = \Big\{ \mathbf{q} \in \left(L^2(\Omega)\right)^3, \mathrm{div}\, \mathbf{q} \in L^2(\Omega) \Big\},$$

and the associated norm

$$\|\mathbf{q}\|_{H(\mathrm{div},\Omega)} = \Big(\|\mathbf{q}\|^2_{(L^2(\Omega))^3} + \|\mathrm{div}\, \mathbf{q}\|^2_{L^2(\Omega)} \Big)^{\frac{1}{2}}.$$

It is well known that for any $\mathbf{q} \in H(\mathrm{div}, \Omega)$ the normal components $\left(\mathbf{q}_{|\Omega^\ell} \cdot \mathbf{n}_\ell\right)$ are in $H^{-\frac{1}{2}}(\partial\Omega^\ell)$. We now define the Lagrange multiplier space

$$M = \left\{ \psi = (\psi_\ell)_\ell \in \prod_{\ell=1}^{L} H^{-\frac{1}{2}}\left(\partial\Omega^\ell\right),\ \exists \mathbf{q} \in H(\mathrm{div}, \Omega), \quad \text{such that } \psi_\ell = \mathbf{q}_{|\Omega^\ell} \cdot \mathbf{n}_\ell,\ 1 \le \ell \le L \right\},$$

equipped with the norm

$$\|\psi\|_M = \left(\sum_{\ell=1}^{L} \|\psi_\ell\|^2_{H^{-\frac{1}{2}}(\partial\Omega^\ell)}\right)^{\frac{1}{2}}.$$

We also introduce the two bilinear forms $a(.,.)$ and $b(.,.)$, defined on the product spaces $X \times X$, and $X \times M$ respectively, by

$$a(u,v) = \sum_{\ell=1}^{L} \int_{\Omega^\ell} \mathbf{grad}\, u \cdot \mathbf{grad}\, v \,\mathrm{d}x,$$

$$b(v,\psi) = -\sum_{\ell=1}^{L} \langle \psi_\ell, v_\ell \rangle_{\frac{1}{2},\partial\Omega^\ell},$$

where $\langle\, ,\rangle_{\frac{1}{2},\partial\Omega^\ell}$ denotes the duality product between $H^{\frac{1}{2}}(\partial\Omega^\ell)$ and its dual $H^{-\frac{1}{2}}(\partial\Omega^\ell)$. The mixed variational formulation associated with problem (1) reads: find $(u,\varphi) \in X \times M$, such that

$$\begin{cases} a(u,v) \ + b(v,\varphi) = (f,v), & \forall v \in X, \\ \qquad\qquad b(u,\psi) = 0, & \forall \psi \in M, \end{cases} \tag{2}$$

where $(.,.)$ is the usual L^2 scalar product. Note that the space $H^1_0(\Omega)$ can be characterized as

$$H^1_0(\Omega) = \{v \in X, \quad b(v,\psi) = 0, \quad \forall \psi \in M\},$$

hence, we retrieve the standard variational formulation of problem (1) that reads: find $u \in H^1_0(\Omega)$, such that

$$a(u,v) = (f,v), \quad \forall v \in H^1_0(\Omega). \tag{3}$$

The well-posedness of problem (3) is guaranteed by the Lax–Milgram lemma. Moreover, according to the Babuska–Brezzi theory, the well-posedness of problem (2) relies on an inf-sup condition which is standard in this case.

Proposition 2.1. *For any data $f \in L^2(\Omega)$, problem* (2) *has a unique solution $(u,\varphi) \in X \times M$. Moreover, the following stability inequality holds*

$$\|u\|_X + \|\varphi\|_M \le C\|f\|_{L^2(\Omega)}. \tag{4}$$

In addition, u is the solution of problem (3) *and we have*

$$\varphi_\ell = \frac{\partial u}{\partial n_\ell}, \qquad 1 \le \ell \le L.$$

2.2. The mortar discrete spaces

Since we are interested in adaptivity, we will consider particular decompositions into sub-domains obtained dynamically from successive refined meshes.

More precisely, let $(\mathcal{T}_{h^0})_{h^0}$ be a family of "coarse" triangulations of Ω. Each triangulation $\mathcal{T}_{h^0}$ consists of elements which are tetrahedra with a maximum size h^0 satisfying the usual admissibility assumption, *i.e.*, the intersection of two different elements is either empty, a vertex, a whole edge or a whole face. In addition, $\mathcal{T}_{h^0}$ is assumed regular, *i.e.*, the ratio of the diameter of any element $K \in \mathcal{T}_{h^0}$ to the diameter of its largest inscribed ball is bounded by a constant σ independent of K and h^0.

Starting from this family $(\mathcal{T}_{h^0})_{h^0}$, we build iteratively new families of refined triangulations as follows. We assume that $(\mathcal{T}_{h^{n-1}})_{h^{n-1}}$ is known for each h^{n-1}.

- For arbitrary positive integers k, we cut some elements of $\mathcal{T}_{h^{n-1}}$ into k tetrahedra.

- We denote by $\mathcal{T}_{h^{n,\ell}}(K)$ the set of tetrahedra obtained by cutting ℓ times the element K of $\mathcal{T}_{h^0}$, in n refining iterations.
- We denote by $\Omega^{n,\ell}$ the union over $K \in \mathcal{T}_{h^0}$ of $\mathcal{T}_{h^{n,\ell}}(K)$.
- We set $\mathcal{T}_{h^n}$ the union over ℓ of $\mathcal{T}_{h^{n,\ell}}$.

For each n, there exists an integer L^n, such that

$$\overline{\Omega} = \bigcup_{\ell=0}^{L^n} \overline{\Omega}^{n,\ell}, \qquad \overline{\Omega}^{n,\ell} \cap \overline{\Omega}^{n,\ell'} = \emptyset,\ 0 \leq \ell < \ell' \leq L^n. \tag{5}$$

In addition, the parameters $h^{n,\ell}$ and h^n are defined in an obvious way as the maximal diameters of the elements of $T_{h^{n,\ell}}$ and T_{h^n}, respectively, and they satisfy

$$h^{n,\ell} \leq k^{-\ell} h^0 \quad \text{and} \quad h^n = \max_{0\leq \ell \leq L^n} h^{n,\ell}. \tag{6}$$

In the sequel we will assume given such a decomposition of Ω. We will also perform the analysis for a fixed iteration n, so we omit the index n.

Finally, we define the skeleton of the partition as follows

$$S = \bigcup_{\ell=1}^{L} \partial\Omega^{\ell} \backslash \partial\Omega, \tag{7}$$

and we fix a decomposition of it into disjoint (open) mortars [7]

$$\overline{S} = \bigcup_{m=1}^{M^*} \overline{\gamma}^m, \quad \gamma^m \cap \gamma^{m'} = \emptyset, \quad 1 \leq m < m' \leq M^*.$$

We make the assumption that each $\overline{\gamma}^m$, $1 \leq m \leq M^*$, is a whole face of a tetrahedron of the triangulation $\mathcal{T}_{h^\ell}$, located on one side of γ^m, and thanks to the assumptions on the decomposition, it is the union of faces of tetrahedra in $\mathcal{T}_{h^{\ell_1}} \cup \mathcal{T}_{h^{\ell_2}} \cup \cdots \cup \mathcal{T}_{h^{\ell_p}}$, where $\ell_i > \ell$. We will denote by $\ell(m), \ell_1(m), \ldots, \ell_p(m)$ the corresponding indices $\ell, \ell_1, \ldots, \ell_p$ and by $p(m)$ the number p.

Let us denote by $\mathcal{P}_1(K)$ the space of affine functions over K, and set the local approximation spaces, for $1 \leq \ell \leq L$,

$$Y_h^\ell = \left\{ v_{h,\ell} \in \mathcal{C}^0\left(\overline{\Omega}^\ell\right), \quad v_{h,\ell|K} \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_{h^\ell}, \quad v_{h,\ell} = 0 \text{ on } \partial\Omega \cap \partial\Omega^\ell \right\}.$$

In order to enforce the matching condition we have to define the discrete mortar spaces associated with γ^m, $1 \leq m \leq M^*$. However, the residual based error estimators in the two-dimensional case show that the standard choices of these mortar spaces may lead to non optimal results for affine finite element approximations [4]. In addition, the difficulty (in practice) of constructing the Lagrange multipliers space (at least with tetrahedral meshes) and the lack of the inf-sup condition when dealing with affine finite elements yield to consider a modified mortar method. This approach introduced in [1], combines the saddle point formulation of the mortar element method with stabilization techniques [8].

Let now γ^m, $1 \leq m \leq M^*$, be one of the mortars. Choosing one side of γ^m to be the "master" (in the mortars terminology [7]), the other side being the slave (the Lagrange) one. With every element T in the 2D triangulation of the non mortar side of γ^m, $1 \leq m \leq M^*$, we associate the bubble function φ_T^m, equal to the product of the three barycentric coordinates on T. Denoting by K the (unique) tetrahedron having T as a face and by $\{x_{K,1}, x_{K,2}, x_{K,3}\}$ (resp. $\{\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}\}$) the common vertices of K and T (resp. the associated barycentric coordinates), we set

$$\varphi_T^m(x) = \frac{60}{|T|}\lambda_{K,1}(x)\lambda_{K,2}(x)\lambda_{K,3}(x), \quad \forall x \in K,$$

extended by zero elsewhere. We denote by $|T|$ the (two-dimensional) Lebesgue measure of T. The coefficient ensures that $\int_T \varphi_T^m \mathrm{d}\sigma = 1$. By scaling, it is easy to prove that

$$|T||\varphi_T^m|^2_{H^1(K_T)} + \|\varphi_T^m\|^2_{L^2(K_T)} \le c|T|^{-\frac{1}{2}}, \tag{8}$$

with the constant c not depending on the shape of T.

The stabilization technique consists of enriching the discrete spaces Y_h^ℓ by the above bubble functions leading to the new local approximation spaces

$$X_h^\ell = Y_h^\ell \oplus \left(\oplus_{T\in\mathcal{T}_h^{\gamma^m}} \mathbb{R}\varphi_T^m\right),$$

where $\mathcal{T}_h^{\gamma^m}$ is the triangulation on the Lagrange side of γ^m and $\mathbb{R}$ the set of real numbers.

Two possible choices exist for defining the local Lagrange multiplier space used to enforce the matching condition on γ^m.

- The coarse space $M_C(\gamma^m) = \mathbb{P}_0(\gamma^m)$ (in this case $\mathcal{T}_h^{\gamma^m}$ reduce to a triangle). We denote by $\mathcal{E}^m$ the set $\{\gamma^m\}$.
- Otherwise, we define $\mathcal{E}^m$ as the set of the open connected components $\Gamma^{\ell_i} = \overline{\gamma}^m \cap \partial\Omega^{\ell_i(m)}, 1 \le i \le p(m)$. Each of this component being provided with a regular 2D triangulation $\mathcal{T}_h^{\Gamma^{\ell_i},\ell}$, and we define $\mathcal{T}_h^{\gamma^m}$ as the union of $\mathcal{T}_h^{\Gamma^{\ell_i},\ell}$, $1 \le i \le p(m)$. We set

$$M_F(\gamma^m) = \left\{\psi_{h,m} \in L^2(\gamma^m), \quad \forall e \in \mathcal{E}^m,\ \psi_{h,m|T} \in \mathcal{P}_0(T), \forall T \in \mathcal{T}_h^e\right\}\cdot$$

We denote by $M_h(\gamma^m)$ one of these two spaces according to the corresponding choice of the master side on γ^m. The global space of approximation X_h is now defined in the standard way

$$X_h = \left\{v_h = (v_{h,\ell})_\ell \in L^2(\Omega), \quad v_{h,\ell} \in X_h^\ell, \quad 1 \le \ell \le L\right\},$$

and the Lagrange multiplier space is the subspace of M, defined by

$$M_h(S) = \left\{\psi_h = (\psi_{h,\ell})_\ell \in \prod_{\ell=1}^L L^2\left(\partial\Omega^\ell\right),\ \psi_{h,\ell(m)} \in M_h(\gamma^m),\ 1 \le m \le M^*,\ \psi_{h,\ell(m)} + \psi_{h,\ell_i(m)} = 0,\ 1 \le i \le p(m)\right\}\cdot$$

2.3. Discrete problem

The discrete problem associated with equation (1) is built from the mixed variational formulation (2). For a fixed datum $f \in L^2(\Omega)$, find $(u_h, \varphi_h) \in X_h \times M_h$ such that

$$\begin{cases} a(u_h, v_h) + b(v_h, \varphi_h) = (f, v_h), & \forall v_h \in X_h, \\ b(u_h, \psi_h) = 0, & \forall \psi_h \in M_h. \end{cases} \tag{9}$$

We must now check the well-posedness of problem (9). Let V_h be the space

$$V_h = \left\{v_h = (v_{h,\ell})_\ell \in X_h, \quad b(v_h, \psi_h) = 0,\ \forall \psi_h \in M_h(S)\right\},$$

that is

$$V_h = \left\{ v_h = (v_{h,\ell})_\ell \in X_h, \quad \int_{\gamma^m} [v_h]\, \psi_h \mathrm{d}\tau = 0,\ \forall m \in \{1, \ldots, M^*\},\ \forall \psi_h \in M_h(\gamma^m) \right\},$$

where $[v_h]$ stands for the jump of v_h at γ^m. We introduce the reduced problem: find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{10}$$

Note that since $M_h(S)$ is a proper subspace of M, the space V_h is not contained in $H_0^1(\Omega)$ and the method is always non-conforming. The bilinear form $a(.,.)$ is obviously continuous with respect to the broken norm $\|.\|_X$, and its ellipticity on V_h can be checked by exactly the same argument as in [6]. In view of the *a posteriori* analysis, we will prove a stronger version of the uniform ellipticity of $a(.,.)$. We only state the result in Lemma 2.3, which requires the following assumption.

Assumption A.1. *For $1 \le m \le M^*$, either $p(m) = 1$ or the space $M_h(\gamma^m)$ coincides with $M_F(\gamma^m)$.*

Remark 2.2. Assumption A.1 is also needed in the two-dimensional case.

Lemma 2.3. *Under Assumption A.1, there exists a constant α only depending on the geometry of Ω such that the following ellipticity property holds*

$$\forall v_h \in V_h, \quad a(v_h, v_h) \ge \alpha \|v_h\|_X^2. \tag{11}$$

From the Lax–Milgram lemma, we obtain the well-posedness of the reduced problem (10). Furthermore, the bilinear form $b(.,.)$ is continuous on $X_h \times M_h$, and by using the same argument [1, Lem. 5.1], we obtain the following inf-sup condition (12), namely:

Lemma 2.4. *There exists a constant β depending only on the geometry of Ω such that the following inf-sup condition holds*

$$\forall \psi_h \in M_h(S), \quad \sup_{v_h \in X_h} \frac{b(v_h, \psi_h)}{\|v_h\|_h} \ge \beta \|\psi_h\|_M. \tag{12}$$

This leads to the well-posedness and stability properties of problem (9) stated in the following corollary.

Corollary 2.5. *Under Assumption A.1, for any data $f \in L^2(\Omega)$ and for any h, problem* (9) *has a unique solution (u_h, φ_h). Moreover, this solution satisfies, for a constant c independent of h,*

$$\|u_h\|_X + \|\varphi_h\|_M \le c\|f\|_{L^2(\Omega)}. \tag{13}$$

2.4. *A priori* analysis

The *a priori* analysis of problem (9) is performed in [1] in the simpler case of a conforming decomposition, however many results extend to the present case. If Assumption A.1 holds, with the discrete uniform inf-sup condition (12), we derive two abstract error estimates which follow easily from [10] (*II*. Th. 1.1 and condition 1.18), and read

$$\|u - u_h\|_X \le c \left(\inf_{v_h \in V_h} \|u - v_h\|_X + \sup_{v_h \in V_h} \frac{1}{\|v_h\|_X} \sum_{\ell=1}^{L} \left\langle \frac{\partial u}{\partial n}, v_{h,\ell} \right\rangle_{\frac{1}{2}, \partial\Omega^\ell} \right), \tag{14}$$

$$\|\varphi - \varphi_h\|_M \le c \left(\inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{\psi_h \in M_h} \|\varphi - \psi_h\|_M \right). \tag{15}$$

Moreover, from [10] (*II*. 1.16) and the inf-sup condition (12), the error estimate on φ_h is a consequence of that on u_h we thus focus on estimate (14). Note that the first term in (14) is the approximation error and the second term is the consistency error (which is zero in the conforming approximation case). The evaluation of the consistency error relies on the matching condition, which implies, for $1 \le m \le M^*$,

$$\forall \chi \in M_h(\gamma^m), \quad \int_{\gamma^m} \frac{\partial u}{\partial n}\,[v]\,\mathrm{d}\tau = \int_{\gamma^m} \left(\frac{\partial u}{\partial n} - \chi\right)[v]\,\mathrm{d}\tau,$$

where the integral over γ^m must be replaced by the duality product if u is not sufficiently smooth. From the approximation properties on the orthogonal projection operator π_h which maps $L^2(\gamma^m)$ into $M_h(\gamma^m)$ [2], and with smoothness assumptions on u, namely $u_{|\Omega^\ell} \in H^{s_\ell}(\Omega^\ell)$, $1 < s_\ell \le 2$, we obtain

$$\sup_{v_h \in V_h} \frac{1}{\|v_h\|_X} \sum_{\ell=1}^{L} \left\langle \frac{\partial u}{\partial n}, v_{h,\ell} \right\rangle_{\frac{1}{2},\partial\Omega^\ell} \le c \left(\sum_{\ell=1}^{L} h_\ell^{2(s_\ell-1)} \|u\|^2_{H^{s_\ell}(\Omega^\ell)} \right)^{\frac{1}{2}}, \tag{16}$$

with the constant c being independent of h.

The estimate of the approximation error is more technical. We first assume $s_\ell > \frac{3}{2}$, $1 \le \ell \le L$, and associate with u the Lagrange interpolation operator (in each sub-domain) $v_{h,\ell} = \mathcal{I}_{h,\ell} u$ in Y_h^ℓ. We derive

$$\|u - v_{h,\ell}\|_{H^1(\Omega^\ell)} \le c h^{s_\ell - 1} \|u\|_{H^{s_\ell}(\Omega^\ell)}. \tag{17}$$

The function $v_h = (v_{h,\ell})_\ell$ does not satisfy the matching condition and we therefore add a correction term to obtain an approximation which belongs to V_h. Consider a fixed mortar γ^m, then

- either the master side is a single face of a tetrahedron K of a sub-domain, say Ω^ℓ, then we set

$$\tilde{v}_{h,\ell_i} = v_{h,\ell_i} + \sum_{T \in \mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} \left(\int_T (v_{h,\ell} - v_{h,\ell_i})\,\mathrm{d}\tau \right) \varphi_T^m, \quad 1 \le i \le p(m);$$

- or the master side is a union of faces, then we set

$$\tilde{v}_{h,\ell} = v_{h,\ell} + \sum_{i=1}^{p(m)} \left(\int_T (v_{h,\ell_i} - v_{h,\ell})\,\mathrm{d}\tau \right) \varphi_T^m.$$

We agree to set $\tilde{v}_{h,\ell} = v_{h,\ell}$ on the master side. It is readily checked that $\tilde{v}_h$ fulfills the matching condition across γ^m. It remains to bound the correction term. Denoting by S_m the term added to v_{h,ℓ_i} or $v_{h,\ell}$ and using (8), we have in the first case $1 \le i \le p(m)$,

$$\begin{aligned}
|S_m|^2_{H^1(\Omega^{\ell_i})} &\le c \sum_{T \in \mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} \left(\int_T (v_{h,\ell} - v_{h,\ell_i})\,\mathrm{d}\tau \right)^2 |\varphi_T^m|^2_{H^1(\Omega^{\ell_i})} \\
&\le c \sum_{T \in \mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}} \|v_{h,\ell} - v_{h,\ell_i}\|^2_{L^2(T)} \\
&\le c \sum_{T \in \mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}} \left(\|u - v_{h,\ell_i}\|^2_{L^2(T)} + \|u - v_{h,\ell}\|^2_{L^2(T)} \right).
\end{aligned}$$

The first term in the sum is easily bounded thanks to the trace theorem and the Lagrange interpolation operator properties. This yields

$$\sum_{T \in \mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}} \|u - v_{h,\ell_i}\|^2_{L^2(T)} \le c h_{\ell_i}^{2(s_{\ell_i}-1)} \|u\|^2_{H^{s_{\ell_i}}(\Omega^{\ell_i})}. \tag{18}$$

For the second term, denoting by Δ'_T the triangle on the master side, we have

$$\begin{aligned}\sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}}\|u-v_{h,\ell}\|^2_{L^2(T)} &\le c \sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}}\|u-v_{h,\ell}\|^2_{L^2(\Delta'_T)}\\ &\le c \sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}}|\Delta'_T|^{s_\ell-\frac{1}{2}}\|u\|^2_{H^{s_\ell-\frac{1}{2}}(\Delta'_T)}\\ &\le c \sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |\Delta'_T|^{s_\ell-1}\|u\|^2_{H^{s_\ell-\frac{1}{2}}(\Delta'_T)}\\ &\le ch_\ell^{2(s_\ell-1)} \sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} \|u\|^2_{H^{s_\ell-\frac{1}{2}}(\Delta'_T)},\end{aligned}$$

where we have used the Lagrange interpolation operator estimates and the comparison between adjacent mesh sizes. We obtain

$$\sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} |T|^{-\frac{1}{2}}\|u-v_{h,\ell}\|^2_{L^2(T)} \le ch_\ell^{2(s_\ell-1)}\|u\|^2_{H^{s_\ell}(\Omega^\ell)}. \tag{19}$$

The second case is handled in a similar way.

In the case where $1 < s_\ell \le \frac{3}{2}$, the same approximation error bounds can be achieved by replacing the Lagrange interpolant by the quasi-interpolant operator studied in [12]. Inserting (16)–(19) in (14, 15) leads to the *a priori* estimate.

Theorem 2.6. *Let the solution of problem* (1) *be such that each* $u_{|\Omega^\ell}$, $1 \le \ell \le L$, *belongs to* $H^{s_\ell}(\Omega^\ell)$, $1 < s_\ell \le 2$. *Under Assumption A.1, there exists a constant* c *independent of* h *such that the following error estimate holds*

$$\|u-u_h\|_X + \|\varphi-\varphi_h\|_M \le c\left(\sum_{\ell=1}^{L}(h_\ell)^{2(s_\ell-1)}\|u\|^2_{H^{s_\ell}(\Omega^\ell)}\right)^{\frac{1}{2}}. \tag{20}$$

3. *A posteriori* analysis. Case I

For the *a posteriori* analysis, we assume that $f \in L^2(\Omega)$ and fix f_h, a finite element approximation of it associated with $\mathcal{T}_h$. The residual error indicators in the framework of the mortar element method are of two kinds [4].

- Error indicators linked to the elements of the mesh

  For each $K \in \mathcal{T}_h$, we denote by $\mathcal{E}_K$ the set of faces of K which are not contained in $\partial\Omega$. We use h_K to denote the diameter of K and h_e to denote the diameter of any $e \in \mathcal{E}_K$. Then the residual indicator η_K associated with a tetrahedron K is defined in a standard way, [14]:

$$\eta_K = h_K\|f_h+\Delta u_h\|_{L^2(K)} + \frac{1}{2}\sum_{e\in\mathcal{E}_K} h_e^{\frac{1}{2}}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(e)}, \tag{21}$$

  where $\frac{\partial}{\partial n}$ denotes the normal derivative at the face e and [.] the jump across e.
- Error indicators linked to the faces of the skeleton.

Following [4], for $1 \le m \le M^*$, we associate with each $e \in \mathcal{E}^m$ the indicator η_e defined as

$$\eta_e = h_e^{-\frac{1}{2}}\|\,[u_h]\,\|_{L^2(e)}. \tag{22}$$

Remark 3.1. The term Δu_h in the definition of η_K vanishes except if K intersects a mortar γ^m and is located on the slave side. This fact is due to the stabilization (by adding the bubble function) and its computation is easy.

3.1. An upper bound for the error

The ellipticity of the form $a(.,.)$ on $H_0^1(\Omega)$ is an obvious consequence of the Poincaré–Friedrichs inequality and the uniform ellipticity on V_h is stated in Lemma 2.3; however, this does not yield the ellipticity on the sum of $H_0^1(\Omega)$ and V_h. Let V be the space of functions v such that

- their restrictions to each Ω^ℓ, $1 \leq \ell \leq L$, belongs to $H^1(\Omega^\ell)$;
- they vanish on $\partial\Omega$;
- for any γ^m, $1 \leq m \leq M^*$;

$$\forall e \in \mathcal{E}^m, \quad \int_e [v] \, \mathrm{d}\tau = 0. \tag{23}$$

We now state a stronger property which yields the ellipticity on the space V containing $H_0^1(\Omega)$ and V_h that we prove in Appendix A.

Lemma 3.2. *Under Assumption A.1, there exists a constant c depending only on the geometry of Ω such that*

$$\forall v \in V, \quad \|v\|_{L^2(\Omega)} \leq c \left(\sum_{\ell=1}^{L} |v|^2_{H^1(\Omega^\ell)} \right)^{\frac{1}{2}}. \tag{24}$$

We now want to prove that (21, 22) yield an upper bound for the error. Note that from Lemma 3.2 we deduce the following inequality

$$\alpha \|u - u_h\|_X^2 \leq a(u - u_h, u - u_h).$$

For brevity we set $v = u - u_h$ and let v_h be an approximation of v in V_h. Multiplying the first line of (1) by v_h, integrating by parts and subtracting (9) we obtain

$$\alpha \|u - u_h\|_X^2 \leq a(u - u_h, v - v_h) + \int_S \frac{\partial u}{\partial n} [v_h] \, \mathrm{d}\tau.$$

Next, we integrate by parts the first term on the right-hand side on each element K of T_h. This leads to

$$\alpha \|u - u_h\|_X^2 \leq \sum_{K \in T_h} \left(\int_K (f + \Delta u_h)(v - v_h) \mathrm{d}x + \int_{\partial K} \frac{\partial (u - u_h)}{\partial n} (v - v_h) \mathrm{d}\tau \right) + \int_S \frac{\partial u}{\partial n} [v_h] \, \mathrm{d}\tau.$$

Adding and subtracting f_h and introducing the jump of $\frac{\partial(u-u_h)}{\partial n}(v - v_h)$ on each edge e of ∂K gives

$$\alpha \|u - u_h\|_X^2 \leq \sum_{K \in T_h} \left(\int_K (f_h + \Delta u_h)(v - v_h) \mathrm{d}x + \int_K (f - f_h)(v - v_h) \mathrm{d}x \right.$$
$$\left. + \frac{1}{2} \sum_{e \in \mathcal{E}_K} \int_e \left[\frac{\partial (u - u_h)}{\partial n} (v - v_h) \right] \mathrm{d}\tau \right) + \int_S \frac{\partial u}{\partial n} [v_h] \, \mathrm{d}\tau.$$

Note that if $e \in \mathcal{E}_K$ is not contained in the skeleton, we have

$$\int_e \left[\frac{\partial (u - u_h)}{\partial n} (v - v_h) \right] \mathrm{d}\tau = - \int_e \left[\frac{\partial u_h}{\partial n} \right] (v - v_h) \mathrm{d}\tau,$$

otherwise, we obtain

$$\int_e \left[\frac{\partial(u-u_h)}{\partial n}(v-v_h)\right] \mathrm{d}\tau = -\int_e \left[\frac{\partial u_h}{\partial n}\right](v-v_h)\mathrm{d}\tau + \int_e \frac{\partial(u-u_h)}{\partial n}\,[v-v_h]\,\mathrm{d}\tau.$$

Hence

$$\alpha\|u-u_h\|_X^2 \le \sum_{K\in T_h}\left(\int_K (f_h+\Delta u_h)(v-v_h)\mathrm{d}x + \int_K (f-f_h)(v-v_h)\mathrm{d}x - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e \left[\frac{\partial u_h}{\partial n}\right](v-v_h)\mathrm{d}\tau\right) + \sum_{m=1}^{M}\sum_{e\in\mathcal{E}^m}\left(-\int_e \frac{\partial(u-u_h)}{\partial n}\,[v-v_h]\,\mathrm{d}\tau + \int_e \frac{\partial u}{\partial n}\,[v_h]\,\mathrm{d}\tau\right).$$

Taking v_h to be a conforming approximation of v, *i.e.*, $v_h \in V_h \cap H^1(\Omega)$ (the construction of v_h will be discussed further on) and using the Cauchy–Schwarz inequality three times yields

$$\alpha\|u-u_h\|_X \le c\Bigg(\sum_{K\in T_h}\Bigg(\|f_h+\Delta u_h\|_{L^2(K)}\frac{\|v-v_h\|_{L^2(K)}}{\|v\|_X} + \|f-f_h\|_{L^2(K)}\frac{\|v-v_h\|_{L^2(K)}}{\|v\|_X} + \frac{1}{2}\sum_{e\in\mathcal{E}_K}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(e)}\frac{\|v-v_h\|_{L^2(e)}}{\|v\|_X}\Bigg) + \left|\sum_{m=1}^{M}\sum_{e\in\mathcal{E}^m}\int_e \frac{\partial(u-u_h)}{\partial n}\,[u_h]\,\mathrm{d}\tau\right|^{\frac{1}{2}}\Bigg). \quad (25)$$

To achieve our goal we have to evaluate the ratios and the last term in the right-hand side.

Bounding the first three ratios depends on the construction of a quasi-interpolant operator such as those studied in [5, 13]. We introduce the new approximation space

$$\tilde{X}_h = \left\{v_h \in H_0^1(\Omega),\ 1\le \ell \le L,\ v_{h,\ell}\in X_h^\ell,\ \forall \ell\right\}\cdot \quad (26)$$

We have the following proposition which we prove in Appendix B.

Proposition 3.3. *There exists an operator R_h from V into $\tilde{X}_h$ and a constant c independent of h such that the following estimate holds for all v in V,*

$$\sum_{K\in\mathcal{T}_h}\left(h_K^{-2}\|v-R_hv\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_K} h_e^{-1}\|v-R_hv\|_{L^2(e)}^2\right) \le c\|v\|_X^2. \quad (27)$$

In order to evaluate the last term in the right-hand side of (25), we need Assumption A.1. For any s, $0 < s < \frac{1}{2}$, we define $\lambda_{h,s}(u)$ as the smallest constant such that

$$\left(\sum_{K\in\mathcal{T}_h} h_K^{2s}|u-u_h|_{H^{1+s}(K)}^2\right)^{\frac{1}{2}} \le \lambda_{h,s}(u)\|u-u_h\|_X. \quad (28)$$

Proposition 3.4. *Under Assumptions A.1, for any s, $0 < s < \frac{1}{2}$, the following estimate holds for $1 \le m \le M^*$ and $e \in \mathcal{E}^m$,*

$$\left|\int_e \frac{\partial(u-u_h)}{\partial n}\,[u_h]\,\mathrm{d}\tau\right| \le c\eta_e\Bigg(h_e^s|u-u_h|_{H^{1+s}(\mathcal{K}_e)} + \Bigg(\sum_{K\in\mathcal{T}_h, K\in\mathcal{K}_e} h_K^2\left(\|f_h+\Delta u_h\|^2_{L^2(K)} + \|f-f_h\|^2_{L^2(K)}\right) + \frac{1}{2}\sum_{e\in\mathcal{E}_K} h_e\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|^2_{L^2(e)}\Bigg)^{\frac{1}{2}}\Bigg), \tag{29}$$

where $\mathcal{K}_e$ is a union of elements sharing the face e.

Note that $\mathcal{K}_e$ is a tetrahedron obtained by cutting $\ell(m)$ times an element of $\mathcal{T}_{h^0}$.

Proof. To prove (29), we set for each $e \in \mathcal{E}^m$, $v = (u-u_h)$ and we introduce the L^2-projection operator π_h^e from $H^{-\frac{1}{2}}(\gamma^m)$ into $M_h(\gamma^m)$. Thanks to the matching condition, we have

$$\left|\int_e \frac{\partial v}{\partial n}\,[u_h]\,\mathrm{d}\tau\right| = \left|\int_e \left(\frac{\partial v}{\partial n} - \pi_h^e\left(\frac{\partial v}{\partial n}\right)\right)[u_h]\,\mathrm{d}\tau\right| \le c\left\|\frac{\partial v}{\partial n} - \pi_h^e\left(\frac{\partial v}{\partial n}\right)\right\|_{H^{-\frac{1}{2}}(e)} \|\,[u_h]\,\|_{H^{\frac{1}{2}}(e)}.$$

The second term on the right-hand side is evaluated from an inverse inequality [9]

$$\|\,[u_h]\,\|_{H^{\frac{1}{2}}(e)} \le c h_e^{-\frac{1}{2}}\|\,[u_h]\,\|_{L^2(e)}. \tag{30}$$

The first term uses the following property of the operator π_h^e [2, Th. 2.4] which can be established by a duality argument

$$\left\|\frac{\partial v}{\partial n} - \pi_h^e\left(\frac{\partial v}{\partial n}\right)\right\|_{(H^{-\frac{1}{2}}(e))} \le c h_e^s\left\|\frac{\partial v}{\partial n}\right\|_{H^{s-\frac{1}{2}}(e)}.$$

It remains to estimate the term

$$\left\|\frac{\partial v}{\partial n}\right\|_{H^{s-\frac{1}{2}}(e)} = \sup_{g\in H^{\frac{1}{2}-s}(e)} \frac{\langle \frac{\partial v}{\partial n}, g\rangle}{\|g\|_{H^{\frac{1}{2}-s}(e)}}.$$

For any g in $H^{\frac{1}{2}-s}(e)$, there exists a lifting w_g of g in $H^{1-s}(\mathcal{K}_e)$ which vanishes on $\partial K_e \setminus e$ and satisfies

$$|w_g|_{H^{1-s}(\mathcal{K}_e)} + h_{\mathcal{K}_e}^{s-1}\|w_g\|_{L^2(\mathcal{K}_e)} \le c\|g\|_{H^{\frac{1}{2}-s}(e)}. \tag{31}$$

Let us assume that $w_g \in H^1(\mathcal{K}_e)$, then we have

$$\left\langle \frac{\partial v}{\partial n}, g\right\rangle = \int_{\mathcal{K}_e} \mathbf{grad}\, v\cdot \mathbf{grad}\, w_g\,\mathrm{d}x + \sum_{K\in\mathcal{K}_e}\int_K \Delta v\, w_g \mathrm{d}x - \frac{1}{2}\sum_{K\in\mathcal{K}_e}\sum_{e\in\mathcal{E}_K}\int_e \left[\frac{\partial v}{\partial n}\right] w_g\,\mathrm{d}\tau,$$

therefore, combining (31) and the trace theorem (58) we obtain

$$\left\|\frac{\partial v}{\partial n}\right\|_{H^{s-\frac{1}{2}}(e)} \le c|v|_{H^{1+s}(\mathcal{K}_e)} + c'\Bigg(\sum_{K\in\mathcal{K}_e} h_K^{2(1-s)}\big(\|f_h+\Delta u_h\|^2_{L^2(K)} + \|f-f_h\|^2_{L^2(K)}\big) + h_K^{-2s}\Bigg(\frac{1}{2}\sum_{e\in\mathcal{E}_K} h_e^{-1}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|^2_{L^2(e)}\Bigg)\Bigg)^{\frac{1}{2}}.$$

A density argument and (30) achieve the proof of (29). □

Next, taking the sum over all m in (29), using Hölder inequality together with the following inequality: for $\gamma > 0$ (chosen small enough)

$$\lambda_{h,s}\left(\sum_{m=1}^{M^*}\sum_{e\in\mathcal{E}^m}\eta_e^2\right)^{\frac{1}{2}}\|u-u_h\| \le \gamma\|u-u_h\|^2 + \frac{\lambda_{h,s}^2}{4\gamma}\left(\sum_{m=1}^{M^*}\sum_{e\in\mathcal{E}^m}\eta_e^2\right),$$

and Proposition 3.3, we get the following result.

Theorem 3.5. *Under Assumption A.1, for any s, $0 < s < \frac{1}{2}$, there exists a constant c independent of h such that*

$$\|u-u_h\|_X \le c\left(\sum_{K\in\mathcal{T}_h}\left(\eta_K^2 + h_K^2\|f-f_h\|^2_{L^2(K)}\right) + (1+\lambda_{h,s})\sum_{m=1}^{M^*}\sum_{e\in\mathcal{E}^m}\eta_e^2\right)^{\frac{1}{2}}. \tag{32}$$

Remark 3.6.

(1) It is possible to obtain optimal lower bounds, *i.e.*, with the constant $\lambda_{h,s}(u) = 1$ in (32), if we assume that the functions of V_h are continuous on the boundary of e as in the two-dimensional case. However, this assumption is not realistic in the three-dimensional case since it leads to numerical locking [3].

(2) Note that the constant $\lambda_{h,s}(u)$ tends to 1 when s tends to zero.

3.2. An upper bound for the indicator

In order to bound the indicators η_K and η_e both locally and globally as functions of the error, we take a test function w in $H_0^1(\Omega)$ and we rewrite (3) as follows

$$\int_\Omega \mathbf{grad}\,(u-u_h)\cdot\mathbf{grad}\,w\,\mathrm{d}x = \int_\Omega fw\,\mathrm{d}x - \int_\Omega \mathbf{grad}\,u_h\cdot\mathbf{grad}\,w\,\mathrm{d}x.$$

Integrating by parts yields

$$\sum_{K\in\mathcal{T}_h}\int_K \mathbf{grad}\,(u-u_h)\cdot\mathbf{grad}\,w\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\left(\int_K (f+\Delta u_h)w\,\mathrm{d}x + \int_{\partial K}\frac{\partial(u-u_h)}{\partial n}w\,\mathrm{d}\tau\right),$$

hence

$$\sum_{K\in\mathcal{T}_h}\int_K \mathbf{grad}\,(u-u_h)\cdot\mathbf{grad}\,w\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\left(\int_K (f_h+\Delta u_h)w\,\mathrm{d}x + \int_K (f-f_h)w\,\mathrm{d}x - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e\left[\frac{\partial u_h}{\partial n}\right]w\,\mathrm{d}\tau\right). \tag{33}$$

Now, as it is standard for residual type indicators, we will obtain the estimates by appropriate choices of the function w in (33).

With each K in $\mathcal{T}_h$, we associate the bubble function ψ_K equal to the product of the four barycentric coordinates on K. For each face T in $\mathcal{E}_K$, we associate the bubble function ψ_T equal to the product of the three barycentric coordinates on T. We now introduce an operator defined as follows: on a reference element $\hat{K}$, we fix a lifting operator $\hat{P}$ of polynomial traces on a face $\hat{T}$ of $\hat{K}$ that vanish on the boundary $\partial\hat{T}$, into polynomials on $\hat{K}$ that vanish on $\partial\hat{K}\setminus\hat{T}$. A similar operator is then obtained on each K by affine transformation.

Proposition 3.7. *There exists a constant c independent of h such that the following estimate holds for all K in T_h*

$$\eta_K \le c\left(\|u-u_h\|_{H^1(\mathcal{G}_K)} + \left(\sum_{K'\in\mathcal{G}_K} h_{K'}^2\|f-f_h\|^2_{L^2(K')}\right)^{\frac{1}{2}}\right), \tag{34}$$

where $\mathcal{G}_K$ is the union of K and all the tetrahedra that contain a face of K.

Note that thanks to the assumptions on the decomposition and the regularity of individual meshes in the sub-domains, the number of elements in $\mathcal{G}_K$ is bounded independently of h.

Proof. The proof is performed in two steps.

(1) We take w in (33) equal to

$$w = \begin{cases} (f_h + \Delta u_h)\psi_K & \text{on } K, \\ 0 & \text{elsewhere.} \end{cases}$$

Since ψ_K vanishes on ∂K, this yields

$$\left\|(f_h + \Delta u_h)\psi_K^{\frac{1}{2}}\right\|_{L^2(K)}^2 = \int_K \mathbf{grad}\,(u - u_h)\cdot\mathbf{grad}\,((f_h + \Delta u_h)\psi_K)\,\mathrm{d}x - \int_K (f - f_h)(f_h + \Delta u_h)\psi_K\,\mathrm{d}x.$$

It then follows that

$$\left\|(f_h + \Delta u_h)\psi_K^{\frac{1}{2}}\right\|_{L^2(K)}^2 \le |u - u_h|_{H^1(K)}|(f_h + \Delta u_h)\psi_K)|_{H^1(K)} + \|f - f_h\|_{L^2(K)}\|(f_h + \Delta u_h)\psi_K\|_{L^2(K)}.$$

By going back to the reference element, it can be checked that for any polynomial φ of degree at most k the following inequalities hold

$$\|\varphi\|_{L^2(K)} \le c\left\|\varphi\psi_K^{\frac{1}{2}}\right\|_{L^2(K)}, \quad |\varphi\psi_K|_{H^1(K)} \le c h_K^{-1}\|\varphi\|_{L^2(K)},$$

with the constants depending only on the degree k of the polynomial and on the shape parameters of K. Noting that $\psi_K \le 1$, we obtain

$$h_K\|(f_h + \Delta u_h)\|_{L^2(K)} \le c(|u - u_h|_{H^1(K)} + h_K\|f - f_h\|_{L^2(K)}). \tag{35}$$

(2) Let us denote by T a face of $\mathcal{E}_K$. We distinguish two cases. First, if T is not contained in S, it is a common face of two tetrahedra K and K' of the same triangulation $\mathcal{T}_{h^\ell}$. We take w in (33) equal to

$$w = \begin{cases} P_{K,T}(\left[\dfrac{\partial u_h}{\partial n}\right]\psi_T) & \text{on } K, \\ P_{K',T}(\left[\dfrac{\partial u_h}{\partial n}\right]\psi_T) & \text{on } K', \\ 0 & \text{elsewhere.} \end{cases}$$

This yields

$$\left\|\left[\frac{\partial u_h}{\partial n}\right]\psi_T^{\frac{1}{2}}\right\|_{L^2(T)}^2 \le \sum_{\kappa\in(K,K')}\left(|u - u_h|_{H^1(\kappa)}\left|P_{\kappa,T}\left(\left[\frac{\partial u_h}{\partial n}\right]\psi_T\right)\right|_{H^1(\kappa)}\right.$$
$$\left. + \left(\|f_h + \Delta u_h\|_{L^2(\kappa)} + \|f - f_h\|_{L^2(\kappa)}\right)\left\|P_{\kappa,T}\left(\left[\frac{\partial u_h}{\partial n}\right]\psi_T\right)\right\|_{L^2(\kappa)}\right).$$

The following inequalities, obtained by going back to the reference element, also hold

$$\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(T)} \le c\left\|\left[\frac{\partial u_h}{\partial n}\right]\psi_T^{\frac{1}{2}}\right\|_{L^2(T)},$$
$$\left|P_{\kappa,T}\left(\left[\frac{\partial u_h}{\partial n}\right]\psi_T\right)\right|_{H^1(\kappa)} + h_T^{-1}\left\|P_{\kappa,T}\left(\left[\frac{\partial u_h}{\partial n}\right]\psi_T\right)\right\|_{L^2(\kappa)} \le c h_T^{-\frac{1}{2}}\left\|\left[\frac{\partial u_h}{\partial n}\right]\right\|_{L^2(T)}.$$

Remarking that $ch_\kappa \leq h_T \leq h_\kappa$, we obtain

$$h_T^{\frac{1}{2}} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{L^2(T)} \leq c \sum_{\kappa \in (K,K')} \left(|u-u_h|_{H^1(\kappa)} + h_\kappa \|f-f_h\|_{L^2(\kappa)} + h_\kappa \|f_h + \Delta u_h\|_{L^2(\kappa)} \right). \tag{36}$$

The second case occurs when T is contained in γ^m.

- If K is contained in $\mathcal{T}_{h^{\ell(m)}}$, we denote by K' the tetrahedron on the other side of γ^m, *i.e.*, obtained by cutting an element of $\mathcal{T}_{h^0}$, and having T as a face. We proceed as above and use (6) to obtain (36).
- If K is contained in $\mathcal{T}_{h^{\ell_i(m)}}$, we denote by K' the tetrahedron in $\mathcal{T}_{h^{\ell(m)}}$ such that T is contained in a face T' of K'. We extend $\left[\frac{\partial u_h}{\partial n}\right] \psi_T$ to T' and we make the same choice as before for w which leads to (36).

Combining (35) and (36) yields the desired bound for η_K. ☐

Taking the sum for $K \in \mathcal{T}_h$ of the square of estimate (34) leads to the following corollary.

Corollary 3.8. *There exists a constant c independent of h such that the following estimate holds*

$$\sum_{K\in\mathcal{T}_h} \eta_K^2 \leq c \left(\|u-u_h\|_X^2 + \sum_{K\in\mathcal{T}_h} h_K^2 \|f-f_h\|_{L^2(K)}^2 \right). \tag{37}$$

It now remains to bound η_e.

Proposition 3.9. *Under Assumption A.1, there exists a constant c independent of h such that the following estimate holds*

$$\eta_e \leq c\|u-u_h\|_{H^1(\Xi_e)}, \tag{38}$$

where Ξ_e is the union of all elements (from the two sub-domains) having a non null intersection with e.

Proof. We first consider the case where $M_h(\gamma^m)$ is taken equal to $M_F(\gamma^m)$, so that e coincides with a connected component of the intersection $\overline{\gamma}^m \cap \partial\Omega^{\ell_i(m)}$. Denoting by $\bar{u}_e$ the mean value of $[u_h]$ on e, and using the matching condition, we obtain

$$\|\,[u_h]\,\|_{L^2(e)}^2 = \int_e ([u_h]-\bar{u}_e)^2 \mathrm{d}\tau \leq ch_e|\,[u_h]\,|_{H^{\frac{1}{2}}(e)}^2.$$

This yields

$$h_e^{-\frac{1}{2}}\|\,[u_h]\,\|_{L^2(e)} \leq c|\,[u-u_h]\,|_{H^{\frac{1}{2}}(e)}.$$

By going back to the reference element we prove that the trace operator is continuous from $H^1(K)$ into $H^{\frac{1}{2}}(T)$, with its norm bounded independently of K, where T is a part of the boundary of K, $T \in e$. We therefore obtain

$$h_e^{-\frac{1}{2}}\|\,[u_h]\,\|_{L^2(e)} \leq c\left(\sum_{\kappa\in\Xi_e} |u-u_h|_{H^1(\kappa)}\right).$$

When $M_h(\gamma^m)$ is taken equal to $M_C(\gamma^m)$, by Assumption A.1, $p(m) = 1$ and the same argument yields the result. ☐

The global estimate follows directly from the local ones.

Corollary 3.10. *Under Assumption A.1, there exists a constant c independent of h such that the following estimate holds*

$$\sum_{m=1}^{M} \sum_{e\in\mathcal{E}^m} \eta_e^2 \leq c\|u-u_h\|_X^2. \tag{39}$$

4. The full estimator

In Section 3 we have studied a residual error estimator for the broken energy norm of the error $(u - u_h)$. In this section we introduce a residual error estimator for the saddle point formulation of the mortar approach.

The residual error indicators in the framework of this formulation are of two kinds. The error indicators linked to the elements and those linked to the faces of the skeleton. We will show that we can take the same error indicators as before, namely η_K defined in (21) and η_e defined in (22), and also obtain upper and lower bounds for the global error. To derive *a posteriori* estimates for the mixed formulation, we note first, that multiplying the first line of (1) by $v_h \in X_h$ and integrating by parts, yield for all $\varphi \in M$

$$a(u, v_h) + b(v_h, \varphi) = \int_\Omega f\, v_h \,\mathrm{d}x + \int_S \left(\frac{\partial u}{\partial n} - \varphi\right) [v_h] \,\mathrm{d}\tau.$$

Subtracting (9) and inserting into the previous line, we obtain

$$a(u - u_h, v_h) + b(v_h, \varphi - \varphi_h) = \int_S \left(\frac{\partial u}{\partial n} - \varphi\right) [v_h] \,\mathrm{d}\tau. \tag{40}$$

Next, we set $v = u - u_h$, then we deduce from Lemma 3.2 and (40)

$$\begin{aligned}\alpha\|u - u_h\|_X^2 &\le a(u - u_h, v - v_h) + a(u - u_h, v_h)\\ &\le a(u - u_h, v - v_h) - b(v_h, \varphi - \varphi_h) + \int_S \left(\frac{\partial u}{\partial n} - \varphi\right) [v_h] \,\mathrm{d}\tau.\end{aligned}$$

We expand the first term in the right-hand side as in the previous section and we obtain

$$\begin{aligned}\alpha\|u - u_h\|_X^2 \le \sum_{K\in T_h} &\left(\int_K (f_h + \Delta u_h)(v - v_h)\mathrm{d}x + \int_K (f - f_h)(v - v_h)\mathrm{d}x + \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e \left[\frac{\partial(u - u_h)}{\partial n}(v - v_h)\right] \mathrm{d}\tau\right)\\ &- b(v_h, \varphi - \varphi_h) + \int_S \left(\frac{\partial u}{\partial n} - \varphi\right) [v_h] \,\mathrm{d}\tau.\end{aligned}$$

Here also we have to consider two cases: when $e \in \mathcal{E}_K$ is not contained in the skeleton and when e belongs to S. Therefore, we have

$$\begin{aligned}\alpha\|u - u_h\|_X^2 \le \sum_{K\in T_h} &\left(\int_K (f_h + \Delta u_h)(v - v_h)\mathrm{d}x + \int_K (f - f_h)(v - v_h)\mathrm{d}x - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e \left[\frac{\partial u_h}{\partial n}\right](v - v_h)\mathrm{d}\tau\right)\\ &+ \sum_{m=1}^{M}\sum_{e\in\mathcal{E}^m}\int_e \frac{\partial(u - u_h)}{\partial n}[v - v_h]\,\mathrm{d}\tau - b(v_h, \varphi - \varphi_h) + \int_S \left(\frac{\partial u}{\partial n} - \varphi\right) [v_h] \,\mathrm{d}\tau. \end{aligned} \tag{41}$$

Note that from the definition of the bilinear form $b(.,.)$, we derive

$$\begin{aligned}b(v_h, \varphi - \varphi_h) &= -\sum_{\ell=1}^{L}\sum_{e\in\mathcal{T}_h^{\partial\Omega^\ell}}\int_e v_h(\varphi - \varphi_h)\,\mathrm{d}\tau\\ &= -\sum_{m=1}^{M^*}\sum_{e\in\mathcal{E}^m}\int_e (\varphi - \varphi_h)\,[v_h]\,\mathrm{d}\tau.\end{aligned}$$

We take v_h to be a conforming approximation of v, *i.e.*, $v_h \in X_h \cap H_0^1(\Omega)$, then, noting that

$$b(v_h, \varphi - \varphi_h) = 0,$$

and replacing in (41), lead to

$$\alpha\|u-u_h\|_X^2 \le \sum_{K\in\mathcal{T}_h}\left(\int_K (f_h+\Delta u_h)(v-v_h)\mathrm{d}x + \int_K (f-f_h)(v-v_h)\mathrm{d}x - \frac{1}{2}\sum_{e\in\mathcal{E}_K}\int_e \left[\frac{\partial u_h}{\partial n}\right](v-v_h)\mathrm{d}\tau\right) + \sum_{m=1}^{M}\sum_{e\in\mathcal{E}^m}\int_e \frac{\partial(u-u_h)}{\partial n}\left[v-v_h\right]\mathrm{d}\tau. \quad (42)$$

Next, we derive from the uniform inf-sup condition (12) and (40) the following inequality: for any $\psi_h \in M_h$

$$\|\varphi_h-\psi_h\|_M \le C\sup_{v_h\in X_h}\left(\frac{|a(u-u_h,v_h)+b(v_h,\varphi-\psi_h)-\int_S(\frac{\partial u}{\partial n}-\varphi)\left[v_h\right]\,\mathrm{d}\tau|}{\|v_h\|_X}\right). \quad (43)$$

The last term in (43) is the consistency error already bounded in the *a priori* analysis. Therefore, assuming for simplicity (see Rem. 4.1) u is such that $u_\ell \in H^2(\Omega^\ell)$, $1\le \ell\le L$, we deduce from (16)

$$\sup_{v_h\in X_h}\frac{|\int_S(\frac{\partial u}{\partial n}-\varphi)\left[v_h\right]\,\mathrm{d}\tau|}{\|v_h\|_X} \le ch\|u\|_X \le ch\|f\|_{L^2(\Omega)}. \quad (44)$$

Using triangular inequality, (43) and (44) yield

$$\|\varphi-\varphi_h\|_M \le c(\|u-u_h\|_X + \inf_{\psi_h\in M_h}\|\varphi-\psi_h\|_M + h\|f\|_{L^2(\Omega)}). \quad (45)$$

The second term in the right-hand side in inequality (45) depends only on the approximation properties of M_h. In fact, using the orthogonal projection operator π_h which maps $L^2(\gamma^m)$ into $M_h(\gamma^m)$, $1\le m\le M^*$ (see [2]) and assuming u is such that $u_\ell\in H^2(\Omega^\ell)$, we have

$$\begin{aligned}\inf_{\psi_h\in M_h}\|\varphi-\psi_h\|_M &\le ch\sum_{\ell=1}^{L}\|\varphi\|_{H^{\frac{1}{2}}(\partial\Omega^\ell)}\\ &\le ch\sum_{\ell=1}^{L}\|u\|_{H^2(\Omega^\ell)}\\ &\le ch\|f\|_{L^2(\Omega)}\\ &\le c\sup_{K\in\mathcal{T}_h}\left(\frac{h}{h_K}\right)\left(\sum_{K\in\mathcal{T}_h}h_K^2(\|f-f_h\|_{L^2(K)}^2+\|f_h\|_{L^2(K)}^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus, from the regularity of the triangulation we derive

$$\inf_{\psi_h\in M_h}\|\varphi-\psi_h\|_M \le c\left(\sum_{K\in\mathcal{T}_h}h_K^2\left(\|f-f_h\|_{L^2(K)}^2+\|f_h\|_{L^2(K)}^2\right)\right)^{\frac{1}{2}}. \quad (46)$$

We set $E_h(f) = \left(\sum_{K\in\mathcal{T}_h} h_K^2(\|f - f_h\|^2_{L^2(K)} + \|f_h\|^2_{L^2(K)})\right)^{\frac{1}{2}}$. Next combining (45) and (41) yield

$$\begin{aligned}\|u - u_h\|_X + \|\varphi - \varphi_h\|_M \leq C\Bigg(\sum_{K\in\mathcal{T}_h} \Bigg(\|f_h + \Delta u_h\|_{L^2(K)} \frac{\|v - v_h\|_{L^2(K)}}{\|v\|_X} \\ + \|f - f_h\|_{L^2(K)} \frac{\|v - v_h\|_{L^2(K)}}{\|v\|_X} + \frac{1}{2} \sum_{e\in\mathcal{E}_K} \left\| \left[\frac{\partial u_h}{\partial n} \right] \right\|_{L^2(e)} \frac{\|v - v_h\|_{L^2(e)}}{\|v\|_X} \Bigg) \\ + \left| \sum_{m=1}^{M^*} \sum_{e\in\mathcal{E}^m} \int_e \frac{\partial(u - u_h)}{\partial n} [u_h] \, \mathrm{d}\tau \right|^{\frac{1}{2}} + E_h(f) \Bigg). \quad (47)\end{aligned}$$

Remark 4.1. If u is such that $u_\ell \in H^{s_\ell}(\Omega^\ell)$, $1 < s_\ell \leq 2$, $1 \leq \ell \leq L$, the only modification we need is to multiply $E_h(f)$ by the factor $\sup_{1\leq\ell\leq L} h^{-s_\ell}$. Note that the perturbation term $E_h(f)$ is negligible for smooth data.

To evaluate the ratios in the first three terms we resort to the construction of a similar operator R_h as in Proposition 3.3, except that now R_h is defined on X.

Proposition 4.2. *There exists an operator R_h from X to $\tilde{X}_h$ and a constant c independent of h such that the following estimate holds for all v in X:*

$$\sum_{K\in\mathcal{T}_h} \left(h_K^{-2} \|v - R_h v\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_K} h_e^{-1} \|v - R_h v\|^2_{L^2(e)} \right) \leq c\|v\|^2_X. \quad (48)$$

Combining Proposition 4.2 and Proposition 3.4, and estimate (46) we can states the following theorem

Theorem 4.3. *Under Assumption A.1, for any s, $0 < s < \frac{1}{2}$, denoting $\lambda_{h,s}(u)$ the constant defined by* (28)*, there exists a constant c independent of h such that*

$$\|u - u_h\|_X + \|\varphi - \varphi_h\|_M \leq c \left(\sum_{K\in\mathcal{T}_h} \left(\eta_K^2 + h_K^2 \|f - f_h\|^2_{L^2(K)} \right) + (1 + \lambda_{h,s}) \sum_{m=1}^{M^*} \sum_{e\in\mathcal{E}^m} \eta_e^2 \right)^{\frac{1}{2}}. \quad (49)$$

To bound the error indicators we note that u_h, the solution of problem (2) is also the solution of problem (3). We have thus already obtained an upper bound for η_e, and η_K.

4.1. Conclusion

The bounds obtained for the residual error estimators considered here are quasi optimal in the sense that they are almost independent of the discretization parameters. In fact, up to the terms $h_K\|f - f_h\|_{L^2(K)}$ (which are negligible for smooth data), the Hilbertian sum of the indicators is equivalent to the error on the mortar finite element for the first formulation and also to the global error (with the Lagrange multiplier) in the unconstrained formulation.

Moreover, the *a priori* analysis shows that the stabilized mortar finite element method is well adapted, from the approximation and also from the practical point of view, for using affine tetrahedral meshes in the numerical simulation of three-dimensional problems in the framework of domain decomposition.

Combining these two essential ingredients of adaptivity enables us to have an efficient tool for solving elliptic second-order partial differential equations in three dimensions.

Moreover, once the strategy of cutting up (and also gluing back) elements is fixed, the implementation of an adaptive algorithm similar to the one used in the two-dimensional case [4] will be easy thanks to the stabilized mortar method.

Appendix A

Relying on an argument due to M. Crouzeix (cited in [4]), we now prove the V-ellipticity of the bilinear form $a(.,.)$ stated in Lemma 3.2.

Proof. We use a duality argument. Indeed, we have

$$\|v\|_{L^2(\Omega)} = \sup_{g\in L^2(\Omega)} \frac{\int_\Omega v(x)g(x)\mathrm{d}x}{\|g\|_{L^2(\Omega)}}. \tag{50}$$

For any $g \in L^2(\Omega)$, we consider the following homogeneous Dirichlet problem

$$\begin{cases} -\Delta\varphi = g, & \text{on } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

The regularity properties of the Laplace operator on a polyhedron yield the existence of a real number $s > \frac{1}{2}$ such that φ belongs to $H^{s+1}(\Omega)$ and satisfies

$$\|\varphi\|_{H^{s+1}(\Omega)} \le c\|g\|_{L^2(\Omega)}. \tag{51}$$

We have

$$\int_\Omega v(x)g(x)\,\mathrm{d}x = -\sum_{\ell=1}^{L}\int_{\Omega^\ell} v(x)\Delta\varphi(x)\,\mathrm{d}x.$$

By integration by parts we obtain

$$\int_\Omega v(x)g(x)\,\mathrm{d}x = \sum_{\ell=1}^{L}\int_{\Omega^\ell} \mathbf{grad}v(x)\cdot\mathbf{grad}\varphi(x)\,\mathrm{d}x - \sum_{m=1}^{M^*}\sum_{e\in\mathcal{E}^m}\int_e [v]\,\frac{\partial\varphi}{\partial n}\,\mathrm{d}\tau. \tag{52}$$

The first term in the right-hand side is easily bounded

$$\left|\int_{\Omega^\ell} \mathbf{grad}v(x)\cdot\mathbf{grad}\varphi(x)\,\mathrm{d}x\right| \le |v|_{H^1(\Omega^\ell)}|\varphi|_{H^1(\Omega^\ell)}. \tag{53}$$

To handle the second term, we fix a mortar γ^m, $1 \le m \le M^*$, and consider a face $e \in \mathcal{E}^m$ which is shared by two sub-domains Ω^ℓ and $\Omega^{\ell'}$. It follows from the definition of V and Assumption A.1 that $v_{|\Omega^\ell}$ and $v_{|\Omega^{\ell'}}$ have an equal mean value on e, that we denote by $\bar{v}_e$. Note that, in general, e is not the whole face of the two sub-domains. We therefore assume that e is a whole face of Ω^ℓ and only a part of the corresponding face of $\Omega^{\ell'}$. With obvious notation we have

$$\int_e [v]\,\frac{\partial\varphi}{\partial n}\mathrm{d}\tau = \int_{e^-} (v_\ell - \bar{v}_e)\,\frac{\partial\varphi}{\partial n^-}\,\mathrm{d}\tau + \int_{e^+} (v_{\ell'} - \bar{v}_e)\,\frac{\partial\varphi}{\partial n^+}\,\mathrm{d}\tau.$$

By going separately to the reference sub-domains $\hat{\Omega}$ and $\hat{\Omega}'$, and denoting (indifferently) by $\hat{e}$ the image of e^- (resp. e^+) by the piecewise affine transformation which maps Ω^ℓ into $\hat{\Omega}$ (resp. $\Omega^{\ell'}$ into $\hat{\Omega}'$), we derive

$$\left|\int_{e^-} (v_\ell - \bar{v}_e)\,\frac{\partial\varphi}{\partial n^-}\,\mathrm{d}\tau\right| \le c|e|h_{\hat{\Omega}}^{-1}\|\hat{v}_{|\hat{\Omega}} - \bar{v}_e\|_{L^2(\hat{e})}\|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}}\|_{L^2(\hat{e})},$$

and

$$\left|\int_{e^+} (v_{\ell'} - \bar{v}_e)\,\frac{\partial\varphi}{\partial n^+}\mathrm{d}\tau\right| \le c|\gamma^m|h_{\hat{\Omega}'}^{-1}\|\hat{v}_{|\hat{\Omega}'} - \bar{v}_e\|_{L^2(\hat{e})}\|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}'}\|_{L^2(\hat{e})}.$$

The equivalence of norms in finite-dimensional spaces and an easy consequence of the Bramble–Hilbert lemma yield

$$\|\hat{v}_{|\hat{\Omega}} - \bar{v}_e\|_{L^2(\hat{e})} \leq \hat{c}|\hat{v}|_{H^1(\hat{\Omega})}, \quad \|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}}\|_{L^2(\hat{e})} \leq \hat{c}\|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}}\|_{H^s(\hat{\Omega})},$$

and

$$\|\hat{v}_{|\hat{\Omega}'} - \bar{v}_e\|_{L^2(\hat{e})} \leq \hat{c}|\hat{v}|_{H^1(\hat{\Omega}')}, \quad \|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}'}\|_{L^2(\hat{e})} \leq \hat{c}\|\mathbf{grad}\hat{\varphi}_{|\hat{\Omega}'}\|_{H^s(\hat{\Omega}')}.$$

Going back to the sub-domains Ω^ℓ and $\Omega^{\ell'}$, we get

$$\left|\int_{e^-} (v_\ell - \bar{v}_e)\, \frac{\partial\varphi}{\partial n^-}\mathrm{d}\tau\right| \leq \quad c|e|h_{\Omega^\ell}^{-1}|\Omega^\ell|^{-\frac{1}{2}}h_{\Omega^\ell}|v|_{H^1(\Omega^\ell)}|\Omega^\ell|^{-\frac{1}{2}}h_{\Omega^\ell}\|\mathbf{grad}\varphi_{|\Omega^\ell}\|_{H^s(\Omega^\ell)},$$

and

$$\left|\int_{e^+} (v_{\ell'} - \bar{v}_e)\, \frac{\partial\varphi}{\partial n^+}\mathrm{d}\tau\right| \leq c|\gamma^m|h_{\Omega^{\ell'}}^{-1}|\Omega^{\ell'}|^{-\frac{1}{2}}h_{\Omega^{\ell'}}|v|_{H^1(\Omega^{\ell'})}|\Omega^{\ell'}|^{-\frac{1}{2}}h_{\Omega^{\ell'}}\|\mathbf{grad}\varphi_{|\Omega^{\ell'}}\|_{H^s(\Omega^{\ell'})},$$

whence

$$\int_e [v]\, \frac{\partial\varphi}{\partial n}\mathrm{d}\tau \leq c\sum_{i=\ell,\ell'} |v|_{H^1(\Omega^i)}\|\varphi\|_{H^{s+1}(\Omega^i)}. \tag{54}$$

Since each sub-domain appears a finite number of times, the desired inequality follows by inserting (51–54) into (50) □

Appendix B

Here, we give the proof of Proposition 3.3. More precisely, to construct an operator R_h from V into $\tilde{X}_h$ which satisfies

$$\sum_{K\in\mathcal{T}_h}\left(h_K^{-2}\|v - R_hv\|_{L^2(K)}^2 + \sum_{e\in\mathcal{E}_K} h_e^{-1}\|v - R_hv\|_{L^2(e)}^2\right) \leq c\|v\|_X^2. \tag{55}$$

We first recall [12] the existence of an operator R^ℓ from the space of functions of $H^1(\Omega^\ell)$ vanishing on $\partial\Omega\cap\partial\Omega^\ell$ into Y_h^ℓ such that, for such a function v,

- the value of $R^\ell v$ at a corner $\mathbf{a}$ of an element K in $\mathcal{T}_h^\ell$ that does not belong to $\partial\Omega$ is equal to the mean value of v on the union $\overline{\Delta_{\mathbf{a}}^\ell}$ of all elements in $\mathcal{T}_h^\ell$ that contain $\mathbf{a}$,
- the following estimates hold for all K in $\mathcal{T}_h^\ell$ and all e in $\mathcal{E}_K$,

$$\|v - R^\ell v\|_{L^2(K)} \leq ch_K\|v\|_{H^1(\Delta_K^\ell)}, \quad \|v - R^\ell v\|_{L^2(e)} \leq ch_e^{\frac{1}{2}}\|v\|_{H^1(\Delta_e^\ell)}, \tag{56}$$

where $\overline{\Delta_K^\ell}$, resp. $\overline{\Delta_e^\ell}$, stands for the union of the $\overline{\Delta_{\mathbf{a}}^\ell}$ such that $\mathbf{a}$ is a corner of K, resp. an endpoint of e. Moreover, R^ℓ is continuous from $H^1(\Delta_K^\ell)$ into $H^1(K)$ with its norm bounded independently of h. The next estimate follows by a simple interpolation argument: for $0 < s < 1$,

$$\|v - R^\ell v\|_{H^s(K)} \leq ch_K^{1-s}\|v\|_{H^1(\Delta_K^\ell)}. \tag{57}$$

Following the argument of [4, Prop. 3.2] we built R_h in several steps. More precisely we will define $R_h = R_h^1 + R_h^2 + R_h^3 + R_h^4$. The operators R_h^j, $j \neq 3$, are defined as in the two-dimensional case while R_h^3 is build as in the proof of Theorem 2.6.

1) The operator R_h^1 is defined by

$$\left(R_h^1 v\right)_{|\Omega^\ell} = R^\ell v, \quad 1 \leq \ell \leq L.$$

Multiplying the first estimate in (56) by h_K^{-1} and summing its square on all K in $\mathcal{T}_h$ yields the first part of (55) for R_h^1. Note that the tetrahedra contained in Δ_K^ℓ appear at most a finite number of times in the sum, and this number is bounded as a function of the regularity of $\mathcal{T}_h^\ell$. The second part is handled in a similar way.

2) – We denote by $\tilde{\nu}^\ell$ the set of the corners of elements in $\mathcal{T}_h^\ell$ that belong to S and that are also corners of tetrahedra in $\mathcal{T}_h^{\ell'}$, $\ell' \neq \ell$. Each $\mathbf{a}$ in $\tilde{\nu}^\ell$ belongs to several $\overline{\Omega}^{\ell'}$ and we denote by $\ell(a)$ the largest such ℓ. Let φ_a be the Lagrange function associated with $\mathbf{a}$, we set

$$\left(\tilde{R}_h^2 v\right)_{|\Omega^\ell} = \sum_{\mathbf{a}\in\tilde{\nu}^\ell} \left(R^{\ell(a)} v(\mathbf{a}) - R^\ell v(\mathbf{a})\right) \varphi_{\mathbf{a}|\Omega^\ell}, \quad 1 \le \ell \le L.$$

The estimates for $\tilde{R}_h^2$ result from the same lines as in the two-dimensional case with obvious modifications.

– To each Ω^ℓ, we associate the set ν^ℓ of its corners which are inside a mortar γ^m, and we take

$$\left(R_h^2 v\right)_{|\Omega^\ell} = \sum_{\mathbf{a}\in\nu^\ell} \left[R_h^1 v + \tilde{R}_h^2 v\right](\mathbf{a}) \varphi_{\mathbf{a}|\Omega^\ell}, \quad 1 \le \ell \le L.$$

Note that if Ω^ℓ coincides with $\Omega^{\ell_i(m)}$, the jump $\left[R_h^1 v + \tilde{R}_h^2 v\right](\mathbf{a})$ means

$$\left(R_h^1 v + \tilde{R}_h^2 v\right)_{|\Omega^{\ell(m)}} - \left(R_h^1 v + \tilde{R}_h^2 v\right)_{|\Omega^{\ell_i(m)}}.$$

Following [4, Lem 2.3], we derive the estimate (55) for $R_h^1 + R_h^2$.

3) For each fixed mortar γ^m, $1 \le m \le M^*$, we denote by Ω^ℓ, Ω^{ℓ_i}, $1 \le i \le p(m)$, the sub-domains sharing this mortar. We define an operator R_h^3 such that,

– if the master side of γ^m is a single face (of Ω^ℓ), then

$$\left(R_h^3 v\right)_{|\Omega^{\ell_i}} = \sum_{T\in\mathcal{T}_h^{\Gamma^{\ell_i,\ell}}} \left(\int_T (v_\ell - v_{\ell_i})\,\mathrm{d}\tau\right) \varphi_T^m, \quad 1 \le i \le p(m);$$

– otherwise

$$\left(R_h^3 v\right)_{|\Omega^\ell} = \sum_{i=1}^{p(m)} \left(\int_T (v_{\ell_i} - v_\ell)\,\mathrm{d}\tau\right) \varphi_T^m,$$

where we have set $v_k = v_{|\Omega^k}$. We denote by $\Delta^{\ell_i(m)}$, $1 \le i \le p(m)$, and $\Delta^{\ell(m)}$ the union of elements K sharing the same mortar γ^m. We set $v = (R_h^1 + R_h^2)v$ in the definition of R_h^3. Easy computations (see the treatment of the approximation error in Sect. 2.4) yield

$$\sum_{K\in\Delta^{\ell_i(m)}} h_K^{-2} \left\|R_h^3 v\right\|_{L^2(K)}^2 \le c\left(\left\|R_h^1 v + R_h^2 v\right\|_{H^1(\Delta^{\ell_i(m)})}^2 + \|R_h^1 v + R_h^2 v\|_{H^1(\Delta^{\ell(m)})}^2\right).$$

The analogous estimate for the sum on the faces also holds thanks to the trace theorem [13, Lem. 3.2]

$$\|\varphi\|_{L^2(T)} \le c\left(h_T^{-\frac{1}{2}}\|\varphi\|_{L^2(K)} + h_T^{\frac{1}{2}}\|\varphi\|_{H^1(K)}\right). \tag{58}$$

4) For each m, the intersection of γ^m and $\partial\Omega^{\ell_i(m)}$, $1 \le i \le p(m)$, has a finite number, say $n(i)$, of open connected components, which we denote γ_{ij}^m, $1 \le j \le n(i)$. Let $\Delta_j^{\ell_i(m)}$ denote the set of tetrahedra

in $\mathcal{T}_h^{\ell_i(m)}$ that intersect γ_{ij}^m. Then, using the analogue operator of [5, Th. 5.1] (see also [15]) enables us to build the lifting operators $L^{m,ij}$ from the space of traces on γ_{ij}^m of functions in $X_h^{\ell_i(m)}$ vanishing at the endpoints of γ_{ij}^m, to the space $X_h^{\ell_i(m)}$, such that $L^{m,ij}\varphi$ vanishes on $\partial\Delta_j^{\ell_i(m)} \setminus \gamma^m$ and satisfies for $\frac{1}{2} < s < 1$,

$$\|L^{m,ij}\varphi\|_{H^s(\Delta_j^{\ell_i(m)})} \leq c\|\varphi\|_{H^{s-\frac{1}{2}}(\gamma_{ij}^m)}.$$

We define R_h^4 by

$$R_h^4 = \sum_{m=1}^{M^*}\sum_{i=1}^{p(m)}\sum_{j=1}^{n(i)} R_{ij}^m,$$

with

$$R_{ij}^m = L^{m,ij}\Big(\big((R_h^1 + R_h^2 + R_h^3)v\big)_{|\Omega^{\ell(m)}} - (R_h^1 + R_h^2 + R_h^3)v\big)_{|\Omega^{\ell_i(m)}}\Big).$$

The same computations as in the two-dimensional case are still valid and yield

$$\sum_{K\in\mathcal{T}_h} h_K^{-2}\|R_h^4 v\|_{L^2(K)}^2 \leq c\|v\|_X^2.$$

The analogous estimate for the sum on the faces also holds.

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