# IDENTIFICATION OF CRACKS WITH NON LINEAR IMPEDANCES

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**Abstract.** We consider the inverse problem of determining a crack submitted to a non linear impedance law. Identifiability and local Lipschitz stability results are proved for both the crack and the impedance.

## 1. Introduction

The inverse problem of finding out a crack from boundary measurements has been thoroughly studied since the pioneering paper by Friedmann and Vogelius [15], where the authors showed that two measurements are needed to identify a crack with no ambiguity, since this latter may be part of a current line (for insulating cracks) which makes a single measurement unable to discriminate it from another one lain along the same or any other current line. *Two measurements taken from a pair of proper fluxes, that generate families of intersecting current lines, in order that no smooth crack may lie in the same time along one of each*, have later on been proved to be sufficient to provide identifiability for any family of cracks [3, 10].

*Provided some further information is available on the location of the crack, for example in the case this latter is a breaking surface line segment one, it has also been proved that a single identifying flux can be explicitly set up for the Laplace equation as well as the elasticity system [6, 7]. Alternatively, replacing prior information on the crack by an appropriate condition on the prescribed flux, is another way to get identifiability. However, one cannot expect such a condition to be verifiable by only checking the flux and the measurements it produces, if no restriction on the crack itself is a priori set, since – for any given flux – moving the crack from one current line to another doesn't change in any way the measurements.*

Plane or line segment cracks have focused lots of interest, not only for the reason they are easier to solve, but also because the situation described is not as restrictive as it looks to be: actually, cracks mostly initiate in fragile parts of the structures, for instance those that have been welded or stuck by any means, and usually their geometries are simple if not plane. Moreover, fracture mechanics theories claim that cracks are unlikely to change directions when propagating, unless material inhomogeneities are met, which makes straight cracks a wide enough class to deserve attention. Identifiability from one single explicitly designed flux has been proved for breaking surface line segment cracks [6, 7], and local Lipschitz stability results have also been obtained. The setting up of algorithms based on the reciprocity gap also provides with some identifiability results [5]. Finally,

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stability results have been obtained for planar cracks in several works by Alessandrini *et al.* [1, 2, 4]. Most of the above mentioned papers have dealt with insulating cracks, or perfectly conducting ones.

In the present one, we are interested in the stable recovery of an arbitrary crack, meaning a crack that might be non straight, submitted to a non linear impedance boundary condition, from *a single* boundary measurement. *The reason is that our concern is, beyond identifiability, the stability which means – given an identifying flux – the continuity of the recovered crack and impedance with respect to the measured data.*

*Requiring the flux to generate singularities at both crack tips, in order to be identifying and to furthermore provide the recovery process with local Lipschitz stability, has been more than once proved to be sufficient [6, 7]. Though not verifiable, such a condition is critical for the stability task, without bringing serious restriction with respect to the identifiability one. Two kinds of difficulties arise at this stage.*

*The first one is that, whereas the singular parts of the solutions are explicitly known for Neumann and Dirichlet boundary conditions [16], they here need to be investigated. As expected, the first singularities are those of the Neumann problem, and the above condition again turns out to ensure identifiability, both for the crack and the impedance law.*

*The second difficulty is that the method used in [6, 7, 14] to prove stability in the case of line segment cracks, including those submitted to a linear impedance boundary condition, doesn't work anymore for arbitrarily shaped cracks. Actually, that method is based on the ability – due to the simple geometry of the crack – to explicitly exhibit a wide enough class of harmonic functions verifying appropriate boundary conditions on it. Instead, point source solutions have been used here, and the situation has been handled by using the single and double layer potentials features on the crack. It turns out that the method also addresses, at the price of a few slight additional difficulties, the widest possible range of impedance laws, i.e. all those ensuring the forward problem well posedness. There is therefore no use to limiting our study to the linear impedance case, which is definitely not the only relevant one.*

The outline of the paper is the following. In Section 2, the forward problem is set up, its variational formulation recalled, and expansions of the solution in a singular part and a regular one are investigated. Section 3 is then devoted to the identifiability *result from one single measurement*, and Section 4 to the study of the longitudinal and transverse stability.

## 2. The direct problem

Let $\Omega$ be a bounded connected open set of the plane with a Lipschitz boundary $\Gamma$. We suppose that $\Omega$ contains exactly one crack $\sigma$ strictly included into $\Omega$. In the whole paper, a crack is supposed to be a $\mathcal{C}^2$ non self-intersecting compact curve with a finite length and $\Omega_\sigma$ will mean the domain $\Omega \setminus \sigma$. The extremities of $\sigma$ will be denoted by $S_1$ and $S_2$.

To describe the direct problem we have in mind we suppose given a current flux $\phi \in H^{-1/2}(\Gamma)$ such that

$$\langle \phi, 1 \rangle = 0.$$

We further fix an increasing and continuous mapping $r$ from $\mathbb{R}$ into itself with the properties:

$$r(0) = 0, \tag{1}$$
$$|r(x)| \leq M(1 + |x|^\alpha), \ \forall x \in \mathbb{R}, \tag{2}$$

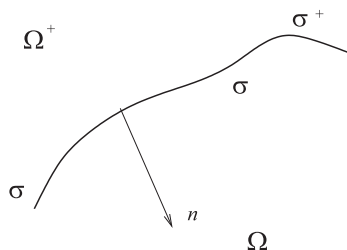for some $\alpha \geq 1$ and $M \geq 0$.

FIGURE 1. Normal to the crack.

The direct problem consists in finding $u \in H^1(\Omega_\sigma)$ solution of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_\sigma, \\ \partial_n u + r([\,u\,]) = 0 & \text{on } \sigma, \\ \partial_n u = \phi & \text{on } \Gamma, \end{cases} \tag{3}$$

that we normalize by requiring that

$$\int_\Gamma u(s)\,\mathrm{d}s = 0. \tag{4}$$

On $\Gamma$, $\partial_n u$ means the outward normal derivative of $u$, while on $\sigma$ it means the normal derivative in one fixed normal direction, let us say from the "upper part" $\Omega^+$ to the "lower part" $\Omega^-$ (see Fig. 1). With this choice the jump of $u$ through $\sigma$ is defined by $[\,u\,] := u^+ - u^-$ on $\sigma$, where $u^\pm = u_{|\Omega^\pm}$.

To set problem (3) in a variational form, let us define

$$V = \{v \in H^1(\Omega_\sigma) \text{ satisfying } (4)\},$$

equipped with the standard semi-norm

$$|v|_{1,\Omega_\sigma} = \left( \int_{\Omega_\sigma} |\nabla v(x)|^2\,\mathrm{d}x \right)^{1/2},$$

which is a norm in $V$, that we denote later on by $\|\cdot\|_V$ for shortness.

Then the variational formulation is to find $u \in V$ solution of

$$\int_{\Omega_\sigma} \nabla u(x) \cdot \nabla v(x)\,\mathrm{d}x + \int_\sigma r([u(x)])[v(x)]\,\mathrm{d}s(x) = \langle \phi, v \rangle, \quad \forall v \in V. \tag{5}$$

Let us notice that the assumption (2) and the embedding $H^{1/2}(\sigma) \hookrightarrow L^p(\sigma)$, for all $1 < p < \infty$ give a meaning to the left-hand side of (5). We start with an existence and uniqueness result:

**Theorem 2.1.** *The problem (5) has a unique solution $u \in V$ that satisfies*

$$\|\,u\,\|_V \leq C\|\,\phi\,\|_{H^{-1/2}(\Gamma)}, \tag{6}$$

*for some positive constant $C$.*

*Proof.* To prove the existence result we introduce the nonlinear functional

$$I : V \to \mathbb{R} : u \to \frac{1}{2} \int_{\Omega_\sigma} |\nabla u(x)|^2\,\mathrm{d}x + \int_\sigma R([u(x)])\,\mathrm{d}s(x) - \int_\Gamma \phi(x)u(x)\,\mathrm{d}s(x),$$

where $R(x) := \int_0^x r(t)\,dt$, for all $x \in \mathbb{R}$. Observe that the assumptions on $r$ imply that $R$ is a nonnegative function on $\mathbb{R}$. The growth condition (2) on $r$ and the embedding $H^{1/2}(\sigma) \hookrightarrow L^p(\sigma)$, for all $1 < p < \infty$, lead to the well posedness of $\int_\sigma R([u(x)])\,ds(x)$, since

$$\int_\sigma R([u(x)])\,ds(x) \leq M \int_\sigma |[u(x)]|(1 + |[u(x)]|^\alpha)\,ds(x) \leq M \left( \|[u(x)]\|_{L^1(\sigma)} + \|[u(x)]\|_{L^{1+\alpha}(\sigma)}^{1+\alpha} \right).$$

Moreover $I$ is lower bounded due to the nonnegativeness of $R$.

Now consider a minimizing sequence $(u_n)_{n\geq 1}$, which has a weakly convergent subsequence $(u_{n_k})_{k\geq 1}$. Let us denote by $u$ its weak limit. A similar argument than above shows that $(R([u_{n_k}]))_{k\geq 1}$ is bounded in $L^2(\sigma)$ and thus uniformly integrable. By Theorem 21 of [12], $R([u_{n_k}])$ tends to $R([u])$ in $L^1(\sigma)$. This allows to prove that the nonlinear functional $I$ attains its minimum at $u$. Consequently $I'(u) = 0$, which shows that $u$ is a solution of (5).

Let us pass to the uniqueness: Let $u_1 \in V$ and $u_2 \in V$ be two solutions of (5). Then, taking their difference as a test function, we get

$$\int_{\Omega_\sigma} |\nabla(u_1 - u_2)|^2\,dx + \int_\sigma \{r([u_1]) - r([u_2])\}[u_1 - u_2]\,ds = 0.$$

Since the monotonicity of $r$ implies that

$$\int_\sigma \{r([u_1]) - r([u_2])\}[u_1 - u_2]\,ds \geq 0,$$

the above identity yields

$$\int_{\Omega_\sigma} |\nabla(u_1 - u_2)|^2\,dx = 0,$$

and consequently $u_1 = u_2$.

Taking $v = u$ as test function in (5), using the estimate

$$\int_{\Omega_\sigma} |\nabla u|^2\,dx + \int_\sigma r([u])[u]\,ds \geq |u|_{1,\Omega_\sigma}^2,$$

following from the property $r([u])[u] \geq 0$, and a standard trace theorem, we obtain the estimate (6). $\qquad \square$

For our future purposes we need the following regularity results for the solution $u \in V$ of (5):

**Theorem 2.2.** *Let $u \in V$ be the unique solution of problem (5). Then it satisfies*

$$[u] \in \mathcal{C}(\bar{\sigma}). \tag{7}$$

*If moreover $r$ is locally Lipschitz, i.e., for all $\rho > 0$ there exists $M_\rho > 0$ such that*

$$|r(x) - r(y)| \leq M_\rho |x - y|, \quad \forall |x|, |y| < \rho,$$

*then $u$ admits the following decomposition into a regular part and a singular one*

$$u = u_{reg} + \sum_{i=1,2} c_i r_i^{\frac{1}{2}} \cos\left(\frac{\phi_i}{2}\right) \quad in\ \omega \setminus \sigma, \tag{8}$$

*where $\omega$ is a neighbourhood of $\sigma$, $u_{reg} \in H^{2+\varepsilon}(\omega \setminus \sigma)$ for a small enough $\varepsilon > 0$ is the regular part, $c_i$ is the coefficient of singularity related to the extremity $S_i$ (the so-called stress intensity factor) and $(r_i, \phi_i)$ are polar coordinates centred at $S_i$ such that the half-lines $\phi_i = 0$ and $\phi_i = 2\pi$ are tangent to $\sigma$ at $S_i$.*

*Proof.* By the growth condition (2) on $r$ and the embedding $H^{1/2}(\sigma) \hookrightarrow L^p(\sigma)$, for all $1 < p < \infty$, $r([u])$ belongs to $L^{p'}(\sigma)$, for all $p' > 1$. Since for all $\varepsilon \in (0, \frac{1}{2})$ $L^2(\sigma) \hookrightarrow H^{\varepsilon-1/2}(\sigma)$ we deduce that

$$\partial_n u = -r([u]) \in H^{\varepsilon-\frac{1}{2}}(\sigma), \quad \text{for all } \varepsilon \in \left(0, \frac{1}{2}\right). \tag{9}$$

Looking at $u$ as solution of the Neumann problem near $\sigma$, by Theorem 23.3 of [11] (a standard reflexion argument allows to reduce the problem to a mixed Dirichlet–Neumann boundary value problem and a pure Neumann problem in a flat domain, both problems being in the scope of the above mentioned theorem), $u$ satisfies

$$u \in H^{1+\varepsilon}(\omega \setminus \sigma), \quad \forall \varepsilon \in \left(0, \frac{1}{2}\right), \tag{10}$$

where $\omega$ is a neighbourhood of $\sigma$. By the Sobolev embedding theorem we conclude that $u$ is continuous on $\omega \setminus \sigma$ with a finite limit from above and from below on $\sigma$. Consequently the jump of $u$ satisfies (7).

For the second assertion we remark that the locally Lipschitz continuity of $r$ and the regularity (7) of $u$ imply

$$|r([u(x)]) - r([u(y)])| \leq M_{||[u]||_\infty} |[u(x)] - [u(y)]|, \quad \forall x, y \in \sigma,$$

which implies that

$$\int_{\sigma \times \sigma} \frac{|r([u(x)]) - r([u(y)])|^2}{|x - y|^{2+2\varepsilon}} \, ds(x) \, ds(y) \leq M_{||[u]||_\infty}^2 \int_{\sigma \times \sigma} \frac{|[u(x)] - [u(y)]|^2}{|x - y|^{2+2\varepsilon}} \, ds(x) \, ds(y).$$

This clearly implies the property

$$r([u]) \in H_0^{\frac{1}{2}+\varepsilon}(\sigma), \quad \forall \varepsilon \in \left(0, \frac{1}{2}\right),$$

since (10) implies $[u] \in H_0^{\frac{1}{2}+\varepsilon}(\sigma)$, for all $\varepsilon \in (0, \frac{1}{2})$.

By the boundary condition on $\sigma$, we may see $u$ as solution of the Neumann problem (in a neighbourhood of $\sigma$) with a Neumann datum in $H_0^{\frac{1}{2}+\varepsilon}(\sigma)$. The decomposition (8) then follows from Theorem 23.7 of [11]. $\qquad \square$

## 3. Identifiability for the inverse problem

The inverse problem we are now interested in is the following: setting a current flux $\phi$ on the external part of the boundary $\Gamma$, and measuring the induced potential $u$ on some open subset with positive measure $M$ of the same boundary, try to recover the unknown crack $\sigma$ and impedance $r$. The first issue arising is identifiability, which means: is the pair $(\phi, u|_M)$ we are holding enough an information to recover the desired unknowns? This can also be seen as injectivity of the operator $(\sigma, r) \mapsto (\phi, u|_M)$.

We are actually going to prove that two different pairs of cracks and impedances – belonging to proper classes – may not produce the same measurements on the boundary, provided the prescribed current flux indeed generates singularities at both tips of the actual crack. Impedances need only to belong to the class insuring well posedness of the forward problem. As for the cracks, we are able to discriminate only between those holding coherent directions, in the sense of the following:

**Definition 3.1.** Two cracks $\sigma_1$ and $\sigma_2$ will be said to hold coherent directions if both of them can be parameterized with respect to the same frame $(X, Y)$:

$$(X, Y) \in \sigma_i \iff Y = \varphi_i(X); \quad X \in [\alpha_i, \beta_i]; \ i = 1, 2$$

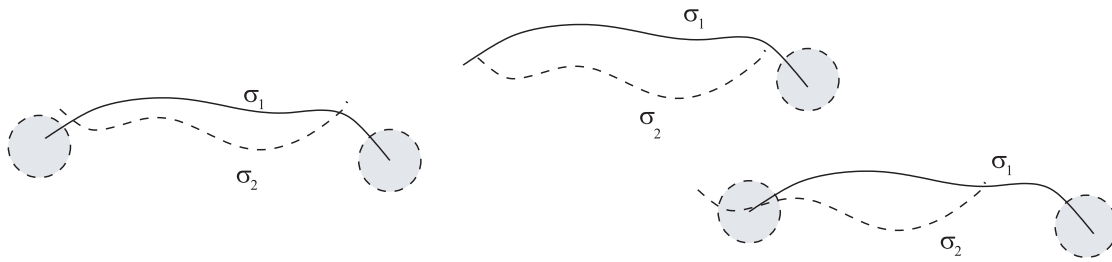$\varphi_1$ and $\varphi_2$ being $\mathcal{C}^2$ functions.

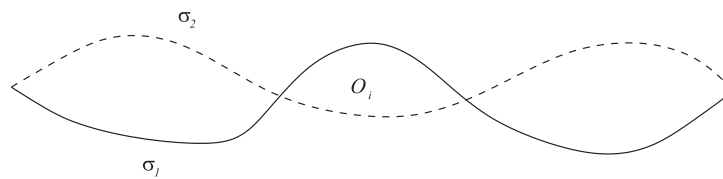FIGURE 2. Intersecting cracks with different cracktips.



FIGURE 3. Intersecting cracks with same cracktips.

**Theorem 3.2** (Identifiability). *Let $\sigma_1$ and $\sigma_2$ be two cracks with coherent directions. Assume a prescribed flux $\phi$ generating singularities at both extremities of the actual crack (say $\sigma_1$), also generates equal measurements on M. Therefore $\sigma_1 \equiv \sigma_2 := \sigma$ and $u_1 \equiv u_2 := u$ on $\Omega_\sigma$. Furthermore $r_1 \equiv r_2$ on the whole of the range of $x \mapsto [u(x)]$ ($x \in \sigma$), which means the impedance laws are the same.*

*Proof.*
• Let us first prove geometrical identifiability. Let $u_1$ solve the forward problems in $\Omega \setminus \sigma_1$ with $r_1$ as an impedance, and $u_2$ solve it in $\Omega \setminus \sigma_2$, with $r_2$ as an impedance. Let $w := u_1 - u_2$ be their difference, hence solving:

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \setminus (\sigma_1 \cup \sigma_2) \\ \partial_n w = 0 & \text{on } \Gamma \\ \quad w = 0 & \text{on } M. \end{cases} \tag{11}$$

By Holmgren's theorem, we derive that $w \equiv 0$ in the external connected component $\Omega_e$ of $\Omega \setminus (\sigma_1 \cup \sigma_2)$, *i.e.* the one having $M$ as part of its boundary. Assuming $\sigma_1 \neq \sigma_2$, two situations may occur:

($a$) The cracks are disconnected: In that case, $\Omega_e = \Omega \setminus (\sigma_1 \cup \sigma_2)$. But $u_2$ is continuous across $\sigma_1$, whereas $u_1$ is not (because of its singular parts). This situation is therefore not possible.

($b$) The cracks are intersecting: Then, because of the singular parts of $u_1$, the cracks cannot have different endpoints (see Fig. 2). Otherwise, $u_2$ would be singular at the vicinity of an endpoint of $\sigma_1$ which is an internal point either to $\Omega \setminus \sigma_2$ or to $\sigma_2$. In the first case, $u_2$ is smooth, whereas it does not behave like $u_1$ in the second case. $\sigma_1$ and $\sigma_2$ have hence the same cracktips. Since both cracks have the same main direction, they can be parameterized by

$$(X, Y) \in \sigma_i \iff Y = \varphi_i(X); \quad X \in [\alpha, \beta]; \quad i = 1, 2$$

this situation has been pictured in Figure 3.

The boundary of $\Omega_e$ is composed by $\Gamma$ and upper or lower parts of $\sigma_1$ and $\sigma_2$. Let $\mathcal{O}_i$ be any connected component of $\Omega \setminus (\sigma_1 \cup \sigma_2 \cup \Omega_e)$, the boundary of which is necessarily composed by parts of $\sigma_1$ and $\sigma_2$.

Across $\sigma_2$, $u_1$ is continuous, and so are $\partial_n u_1$ and $\partial_n u_2$, this latter because of the boundary condition on the crack. On the other hand, $u_1 \equiv u_2$ in $\Omega_e$, which yields $\partial_n u_1 = \partial_n u_2$ on $\partial\Omega_e$, including the external parts of $\sigma_1$

and $\sigma_2$. It comes out therefore:

$$\partial_n w = 0 \text{ on } \sigma_2$$

and for the same reason $\partial_n w = 0$ on $\sigma_1$, and $w$ is harmonic thus constant on $\mathcal{O}_i$. Let $\kappa_i$ be that constant. We have then:

$$w = \kappa_i \text{ on } \sigma_1 \cap \partial \mathcal{O}_i$$

and since $w$ is null in $\Omega_e$, this yields:

$$[u_1] \equiv [w] = \pm \kappa_i \text{ on } \sigma_1 \cap \overline{\mathcal{O}_i}$$

$[u_1]$ is therefore piecewise constant on $\sigma_1$, which is not possible unless it is constant on the whole of $\sigma_1$ since no discontinuity is allowed to functions in $H^{\frac{1}{2}}(\sigma_1)$. Hence:

$$[u_1] = \kappa \quad \text{on } \sigma_1.$$

Now, because $[u_1] \in H_{00}^{\frac{1}{2}}(\sigma_1)$, and thus vanishes at the endpoints of $\sigma_1$, the constant $\kappa$ cannot be other than zero, making $u_1$ continuous across $\sigma_1$ and hence not singular, which contradicts the assumption made on the flux. The cracks cannot thus intersect either.

This leads to $\sigma_1 \equiv \sigma_2 := \sigma$ and accordingly to $u_1 = u_2 := u$ on $\Omega \setminus \sigma$.

• Identifiability for impedances: From the above conclusions, we derive that $\partial_n u_1 - \partial_n u_2 = 0$ on $\sigma$ and hence

$$(r_1 - r_2)([u(x)]) = 0 \quad \forall x \in \sigma.$$

This means $r_1 - r_2$ vanishes on the range of $[u(x)])$, which is enough to derive the impedance laws are the same: actually, what the impedance law is outside that set does not impact in any way *the state, and can thus not be derived from the measurements that the flux has produced.* $\square$

**Remark 3.3.** Current fluxes that generate no singularities at both crack tips actually span a proper closed subset of $H^{-\frac{1}{2}}(\Gamma)$, since the singularities coefficients $c_i$ are given by the following continuous functions of the flux [17]

$$c_i(\phi) = \int_\sigma \partial_n u [K_i] \, \mathrm{d}s + \langle \phi, \, K_i \rangle,$$

where $K_i$ is the so-called dual singular function given by

$$K_i = 2\eta_i \, r_i^{-1/2} \cos\left(\frac{\phi_i}{2}\right) - v_i,$$

with $v_i \in V$ such that $\Delta K_i = 0$ in $\Omega_\sigma$ and $\partial_n K_i = 0$ on $\Gamma \cup \sigma$; $\eta_i$ is a smooth cut-off function such that $\eta_i \equiv 1$ in a neighbourhood of $S_i$ and $\eta_i \equiv 0$ outside a small but larger neighbourhood of $S_i$.

This feature makes such fluxes unlikely to meet, *since the subset they belong to is not dense in $H^{-\frac{1}{2}}(\Gamma)$.* Should *moreover one be met*, computational errors would *anyway* draw it away from that subset, hence making the requirement on the singular parts of the solution be fulfilled. Making this assumption is therefore not serious restriction.

**Remark 3.4.** In order to prove identifiability, it would have been sufficient to require the flux to generate a solution with a non vanishing discontinuity on the crack $\sigma$, instead of non vanishing first singular coefficients. The whole argument works the same way but the condition, if weaker, is no more verifiable. And the singular behaviour is critical for stability purposes, as will be pointed out in Section 4.

## 4. Local Lipschitz stability

In this section, we are going to investigate how "small" perturbations on the measured data may impact the recovered crack and impedance. Actually, we are trying to show up a Lipschitz dependence of the unknowns to be recovered with respect to the measured data. However, such a result cannot be obtained with no additional information on the unknowns, for inverse problems are well known to be ill posed. This is the reason why we restrict our investigation to local Lipschitz stability, meaning we shall prove such a dependence only in some neighbourhood of the actual crack and impedance. Following [6, 13, 18], the tool we shall be using to this end is the Lagrangian derivative.

### 4.1. **Lagrangian derivative**

Consider a family of mappings

$$F_h = \mathrm{Id} + h\,\theta,$$

where Id is the identity mapping on $\mathbb{R}^2$ and $\theta \in \left(\mathcal{C}^2(\bar{\Omega})\right)^2$ is such that $F_h(\Omega_\sigma) = \Omega \backslash \sigma_h$ for some crack $\sigma_h := F_h(\sigma)$ and $\theta = 0$ on $\Gamma$, $h$ being a "small" positive real number. Clearly there exists $h_0 > 0$ small enough such that for all $0 < h \le h_0$, $F_h$ is a diffeomorphism from $\Omega_\sigma$ into $\Omega_h := F_h(\Omega_\sigma)$. $F_h$ is a virtual kinematics describing the cracks move in the direction $\theta$. Actually, only the value of $\theta$ on the crack $\sigma$ is meaningful, though we need to define it on the whole $\Omega$ for the calculations.

Similarly, we shall be considering perturbations of the impedance obtained by

$$r_h = r + hr_1$$

where $r$ and $r_1$ are $\mathcal{C}^2$ mappings, the perturbation direction $r_1$ is actually an impedance itself fulfilling the same conditions (1, 2).

The Lagrangian derivative of the solution $u$ of problem (3, 4) with respect to the domain (*i.e.* the parameter $h$) in the directions $(\theta, r_1)$ is therefore given by the asymptotic expansion in the following theorem.

**Theorem 4.1.** *Let $u^h = u_h \circ F_h$, $u_h \in V_h := \left\{v_h \in H^1(\Omega \setminus \sigma_h); \int_\Gamma v_h = 0\right\}$ being the solution of:*

$$\int_{\Omega_h} \nabla u_h(y) \cdot \nabla v(y)\,\mathrm{d}y + \int_{\sigma_h} r_h([u_h(x_h)])[v(x_h)]\,\mathrm{d}s_h = \langle \phi, v \rangle, \ \forall v \in V_h. \tag{12}$$

*Then there exists $h_0 > 0$ small enough such that for all $h \in ]0, h_0[$, $u^h$ admits the expansion*

$$u^h = u^0 + hu^1 + \mathcal{O}_h^2, \tag{13}$$

*where $u^0 = u \in V$ is the unique solution of (5) and $u^1 \in V$ is the unique solution of*

$$\int_{\Omega_\sigma} \nabla u^1 \cdot \nabla v\,\mathrm{d}x + \int_\sigma r'([u^0])[u^1][v]\,\mathrm{d}s = \int_{\Omega_\sigma} \left[(\mathrm{D}\theta + \mathrm{D}\theta^\top)\nabla u^0\right] \cdot \nabla v\,\mathrm{d}x$$

$$- \int_{\Omega_\sigma} (\nabla u^0 \cdot \nabla v)\mathrm{div}\,\theta\,\mathrm{d}x - \int_\sigma \left(r_1([u^0]) + r([u^0])\left(t^\top \mathrm{D}\theta\, t\right)\right)[v]\,\mathrm{d}s, \ \forall v \in V \tag{14}$$

*where $\mathrm{D}\theta$ is the Jacobian matrix of $\theta$ and $\mathcal{O}_h^2 \in V$ with*

$$\lim_{h \to 0} \frac{\|\mathcal{O}_h^2\|_V}{h} = 0. \tag{15}$$

*Proof.* We first remark that the assumptions on $r$ and $r_1$ guarantee that (12) has a unique solution $u_h \in V_h$ satisfying (see Th. 2.1)

$$| u_h |_{1,\Omega_h} \leq C\|\phi\|_{H^{-1/2}(\Gamma)}, \tag{16}$$

for some positive constant $C$ (independent on $h$). Indeed fix a subdomain $D$ of $\Omega$ such that $\sigma_h$ is included into $\Omega \setminus \bar{D}$ for all $h$ small enough and such that the boundary of $D$ contains $\Gamma$. By the positiveness of $r_1$ and a standard trace theorem in $D$, we get

$$| u_h |_{1,\Omega_h} \leq \|\phi\|_{H^{-1/2}(\Gamma)}\| u_h \|_{H^{1/2}(\Gamma)} \leq C_1 \|\phi\|_{H^{-1/2}(\Gamma)}\| u_h \|_{1,D},$$

for some positive constant $C_1$ (independent on $h$). Since $u_h$ has a mean zero on $\Gamma$, we may write

$$\| u_h \|_{1,D} \leq C_2 | u_h |_{1,D},$$

for some positive constant $C_2$ (independent on $h$). The two above estimates yields (16) since we clearly have $| u_h |_{1,D} \leq | u_h |_{1,\Omega_h}$.

In the variational problem (12), performing the change of variables $y = F_h(x)$, $u^h$ is then the unique solution of

$$a_h \left( u^h, v \right) = \langle \phi, v \rangle, \ \forall v \in V, \tag{17}$$

where

$$a_h(u,v) = \int_{\Omega_\sigma} ((I + hD\theta)^{-1} \nabla u) \cdot ((I + hD\theta)^{-1} \nabla v)(1 + h \operatorname{div} \theta + h^2 \det D\theta) \, \mathrm{d}x$$

$$+ \int_\sigma r_h([u])[v] \left\{ 1 + h \left( t^\top D\theta \, t \right) + O(h^2) \right\} \, \mathrm{d}s,$$

since $\mathrm{d}s_h = \left\{ 1 + h \left( t^\top D\theta \, t \right) + O(h^2) \right\} \mathrm{d}s$. Furthermore for $h$ small enough, the estimate (16) is equivalent to

$$\| u^h \|_V \leq C\|\phi\|_{H^{-1/2}(\Gamma)}, \tag{18}$$

for some positive constant $C$ (independent on $h$). As the matrix $(I + hD\theta)^{-1}$ admits the expansion

$$(I + hD\theta)^{-1} = \sum_{k=0}^{\infty} (-1)^k \, h^k (D\theta)^k, \tag{19}$$

for $h$ small enough, the (nonlinear) form $a_h$ admits the expansion

$$a_h(u,v) = a_0(u,v) + ha_1(u,v) + \mathcal{R}_h(u,v), \ \forall u,v \in V,$$

where $a_0, a_1, \mathcal{R}_h$ are (nonlinear) forms satisfying

$$a_0(u,v) = \int_{\Omega_\sigma} \nabla u(x) \cdot \nabla v(x) \, \mathrm{d}x + \int_\sigma r([u])[v] \, \mathrm{d}s, \ \forall u,v \in V, \tag{20}$$

$$|a_1(u,v)| \leq C(\|u\|_V)\|v\|_V, \ \forall u,v \in V, \tag{21}$$

$$|\mathcal{R}_h(u,v)| \leq h^2 C(\|u\|_V)\|v\|_V, \ \forall u,v \in V, \tag{22}$$

where $C(\|u\|_V) \geq 0$ depends continuously on $\|u\|_V$.

Taking into account the above expansion of $a_h$, the difference between (17) and (5) yields

$$a_0 \left( u^h, v \right) - a_0 \left( u^0, v \right) = -ha_1 \left( u^h, v \right) - \mathcal{R}_h \left( u^h, v \right), \ \forall v \in V.$$

For $v = u^h - u^0$, we get

$$\int_{\Omega_\sigma} \left| \nabla \left( u^h - u^0 \right) \right|^2 \, \mathrm{d}x + \int_\sigma \left\{ r \left( [u^h] \right) - r \left( [u^0] \right) \right\} \left[ u^h - u^0 \right] \, \mathrm{d}s = -ha_1 \left( u^h, u^h - u^0 \right) - \mathcal{R}_h \left( u^h, u^h - u^0 \right).$$

The monotonicity of $r$ and the estimates (18), (21) and (22) lead to

$$\left\| u^h - u^0 \right\|_V \leq Ch, \tag{23}$$

for some positive constant $C$ depending on $\|\phi\|_{H^{-1/2}(\Gamma)}$. This estimate means that $u^h$ tends to $u^0$ as $h$ goes to $0$ but further means that $u_h^1 := \frac{u^h - u^0}{h}$ is uniformly bounded in $V$. The definition of $u_h^1$ is equivalent to

$$u^h = u^0 + hu_h^1.$$

Inserting this expression in (17) and using the fact that $u^0$ satisfies (5), we get

$$\int_{\Omega_\sigma} \left( \nabla u_h^1 - \left( \mathrm{D}\theta + \mathrm{D}\theta^\top \right) \nabla u^0 + \mathrm{div}\,\theta\,\nabla u^0 \right) \cdot \nabla v \, \mathrm{d}x + \int_\sigma \left( r'([u^0])[u_h^1] + r_1([u^0]) + r([u^0]) \left( t^\top \mathrm{D}\theta\, t \right) \right) [v] \, \mathrm{d}s$$
$$= -h^{-1} \mathcal{R}_h \left( u^h, v \right) + \mathcal{R}_h' \left( u^0, u_h^1, v \right), \ \forall v \in V,$$

where $\mathcal{R}_h'$ is a remainder satisfying (thanks to Th. 2.2 and the properties on $r$ and $r_1$)

$$\left| \mathcal{R}_h' \left( u^0, u_h^1, v \right) \right| \leq Ch \| v \|_V, \forall v \in V, \tag{24}$$

for some positive constant $C$ depending on $\|\phi\|_{H^{-1/2}(\Gamma)}$. Comparing this problem with (14) we see that

$$\int_{\Omega_\sigma} \nabla \left( u_h^1 - u^1 \right) \cdot \nabla v \, \mathrm{d}x + \int_\sigma r'([u^0]) \left[ u_h^1 - u^1 \right] [v] \, \mathrm{d}s = -h^{-1} \mathcal{R}_h \left( u^h, v \right) + \mathcal{R}_h' \left( u^0, u_h^1, v \right), \ \forall v \in V.$$

As before taking $v = u_h^1 - u^1$, using the fact that $r'(x) \geq 0$ and the estimates (18), (22) and (24), we obtain

$$\left\| u_h^1 - u^1 \right\|_V \leq Ch,$$

for some positive constant $C$ depending on $\|\phi\|_{H^{-1/2}(\Gamma)}$. The conclusion follows by setting $\mathcal{O}_h^2 = h \left( u_h^1 - u^1 \right)$.
□

In the following, some additional regularity on $u^1$ in a neighbourhood of the crack tip will be needed. More precisely, we need:

**Lemma 4.2.** *If $r$ is a $\mathcal{C}^2$ mapping satisfying the conditions (1, 2). Then there exists a neighbourhood $\omega$ of the extremities of the crack $\sigma$ and $\varepsilon > 0$ such that $u^1$ belongs to $H^{1+\varepsilon}(\omega \setminus \sigma)$. Consequently, $u^1$ is bounded near the crack tips.*

*Proof.* Taking test function in $\mathcal{D}(\Omega_\sigma)$ in (14) we can see that $u^1$ satisfies (in the whole proof $u \in V$ is the unique solution of (5))

$$\Delta u^1 = -\mathrm{div} \left( \left( \mathrm{D}\theta + \mathrm{D}\theta^\top \right) \cdot \nabla u \right) + \mathrm{div} \left( \mathrm{div}\,\theta(\nabla u) \right) \text{ in } \Omega_\sigma. \tag{25}$$

Let us notice that the above right-hand side belongs to $H^{-1+\varepsilon}(\Omega_\sigma)$, for all $\varepsilon \in (0, 1/2)$ thanks to (10) and the smoothness of $\theta$.

Taking now test-functions $v \in V$ such that $v^- \equiv 0$ on $\sigma$ we get

$$\partial_n u^{1+} + r'([u])[u^1] = \left\{ \left( \mathrm{D}\theta + \mathrm{D}\theta^\top \right) \nabla u^+ \right\} \cdot n - (\partial_n u)\,\mathrm{div}\,\theta - r_1([u]) - r([u]) \left( t^\top \mathrm{D}\theta\, t \right) \text{ on } \sigma.$$

Similarly taking test-functions $v \in V$ such that $v^+ \equiv 0$ on $\sigma$ we get

$$\partial_n u^{1-} + r'([u])[u^1] = \left\{ \left( D\theta + D\theta^\top \right) \nabla u^- \right\} \cdot n - (\partial_n u) \operatorname{div} \theta - r_1([u]) - r([u]) t^\top D\theta \, t \text{ on } \sigma.$$

By Theorem 2.2, $r'([u])$ is bounded and

$$r'([u])[u^1] \in H^{\varepsilon - \frac{1}{2}}(\sigma), \quad \forall \varepsilon \in (0, 1/2).$$

Moreover the regularity (10) of $u$ and the smoothness of $\theta$ imply that

$$\left\{ \left( D\theta + D\theta^\top \right) \nabla u^\pm \right\} \cdot n - (\partial_n u) \operatorname{div} \theta \in H^{\varepsilon - \frac{1}{2}}(\sigma), \quad \forall \varepsilon \in (0, 1/2).$$

Similarly (9) and the smoothness of $\theta$ imply that

$$r([u]) \left( t^\top D\theta \, t \right) \in H^{\varepsilon - \frac{1}{2}}(\sigma), \quad \forall \varepsilon \in (0, 1/2).$$

In the same way the growth condition (2) on $r_1$ yields (see the proof of (9))

$$r_1([u]) \in H^{\varepsilon - \frac{1}{2}}(\sigma), \quad \forall \varepsilon \in (0, 1/2).$$

All together this means that $u^1$ may be seen as a solution of a Neumann problem in a neighbourhood $\omega$ of $\sigma$ with interior datum in $H^{-1+\varepsilon}(\omega \setminus \sigma)$, for all $\varepsilon \in (0, 1/2)$ and Neumann data in $H^{\varepsilon - 1/2}(\sigma)$. Consequently by Theorem 23.3 of [11] we deduce the announced regularity for $u^1$. $\qquad \square$

**Remark 4.3.** Following [8], local Lipschitz stability is achieved if one can prove that

$$u^1 \not\equiv 0 \ \text{ on } M.$$

Indeed, this means the measured data are sensitive "at the first order" to the local crack and impedance moves. On the other hand, $h$ is the parameter "measuring" the magnitude of the unknowns moves in prescribed directions $\theta$ and $r_1$, and we have

$$u^1 = \lim_{h \to 0} \frac{u^h - u^0}{h}.$$

Therefore, for $h$ small enough, and provided $u^1$ does not vanish on the whole of $M$, we get

$$h \leq c |u^h - u^0|_{0,M} \tag{26}$$

with $c = 2/|u^1|_{0,M}$ for example. Equation (26) is the expected local ($h$ needs to be small) and directional ($u^1$ is a directional derivative) Lipschitz stability result.

There is no ambiguity for what regards the impedance virtual moves. As for the crack, we shall need to distinguish the stability with respect to the length (extensions or contractions), from the transverse one, for they are not exactly proved the same way.

## 4.2. Stability with respect to the length

Longitudinal virtual moves of the crack will be described by taking a direction $\theta$ verifying $\theta \cdot n = 0$ on $\sigma$. Actually, only the values of $\theta$ at the crack tips are meaningful (see Fig. 4).

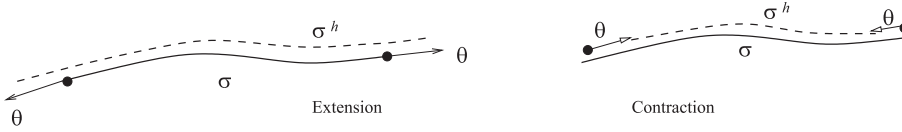Let us first start with a useful identity that we shall need later on.

FIGURE 4. Longitudinal virtual moves of the crack.

**Lemma 4.4.** *Let $u$ and $u^1$ be the respective solution of (5) and (14). Assume that $u^1$ is identically equal to zero on $\Gamma$. Then for all $v \in H^1(\Omega)$ such that*

$$\Delta v = 0 \ in \ \Omega,$$

*it holds*

$$\int_\sigma \{([u^1] - \theta \cdot t[\partial_t u]) \, \partial_n v + \theta \cdot n[\partial_t u] \, \partial_t v\} \mathrm{d}s = 0. \tag{27}$$

*Proof.* Taking test functions $v$ as in the statement of the Lemma in the identity (14) we get

$$\lim_{\delta \to 0} \left\{ \int_{\Omega_{\sigma,\delta}} \nabla u^1 \cdot \nabla v \, \mathrm{d}x + \int_{\sigma_\delta} r'([u(x)])[u^1(x)] \, [v(x)] \, \mathrm{d}s \right.$$
$$- \int_{\Omega_{\sigma,\delta}} \left\{ \left( \mathrm{D}\theta + \mathrm{D}\theta^\top \right) \nabla u \right\} \cdot \nabla v \, \mathrm{d}x + \int_{\Omega_{\sigma,\delta}} (\nabla u \cdot \nabla v) \, \mathrm{div} \, \theta \, \mathrm{d}x$$
$$\left. + \int_{\sigma_\delta} \left( r_1([u]) + r([u]) \left( t^\top \mathrm{D}\theta \, t \right) \right) [v] \, \mathrm{d}s \right\} = 0,$$

where we have set $\Omega_{\sigma,\delta} = \Omega_\sigma \setminus \bigcup_{i=1,2} B(S_i, \delta)$ and $\sigma_\delta = \sigma \setminus \bigcup_{i=1,2} \sigma \cap B(S_i, \delta)$ (see Fig. 5). Some integrations by parts in $\Omega_{\sigma,\delta}$ and the harmonicity of $u$ and $v$ in $\Omega_{\sigma,\delta}$ as well as the nullity of $u^1$ on $\Gamma$ lead to

$$\lim_{\delta \to 0} \left\{ \int_{\sigma_\delta^+ \cup \sigma_\delta^-} \{u^1 \partial_n v - (\theta \cdot \nabla u) \, \partial_n v - (\theta \cdot \nabla v) \, \partial_n u + (\nabla u \cdot \nabla v) \, \theta \cdot n\} \, \mathrm{d}s \right.$$
$$\left. -I_\delta + \int_{\sigma_\delta} \left( r'([u]) \, [u^1] + r_1([u]) + r([u])(t^\top \mathrm{D}\theta t) \right) [v] \, \mathrm{d}s \right\} = 0, \quad (28)$$

where $\sigma^+$ (resp. $\sigma^-$) is the "upper part" (resp. "lower part") of $\sigma$; on $\sigma^+$ (resp. $\sigma^-$), $n$ means the normal vector directed from $\sigma^+$ to $\sigma^-$ (resp. $\sigma^-$ to $\sigma^+$) and

$$I_\delta := \sum_{i=1,2} \int_{\partial B(S_i,\delta)} \{u^1 \partial_n v - (\theta \cdot \nabla u) \, \partial_n v - (\theta \cdot \nabla v) \, \partial_n u + (\nabla u \cdot \nabla v) \, \theta \cdot n\} \, \delta \, \mathrm{d}\phi,$$

where $n$ means here the outward normal vector on $\partial B(S_i, \delta)$.

As $v$ is regular near $S_i$, $i = 1, 2$, the regularity of $u$ from Theorem 2.2 and the boundedness of $u^1$ near $S_i$ (Lem. 4.2) allow to conclude that

$$I_\delta \to 0 \text{ as } \delta \to 0.$$

FIGURE 5

Consequently taking the limit as $\delta \to 0$ in the above identity (28) we get

$$\int_{\sigma^+ \cup \sigma^-} \left\{ u^1 \partial_n v - (\theta \cdot \nabla u) \, \partial_n v - (\theta \cdot \nabla v) \, \partial_n u + (\nabla u \cdot \nabla v) \, \theta \cdot n \right\} \mathrm{d}s$$
$$+ \int_\sigma \left( r'([u]) \, [u^1] + r_1([u]) + r([u]) \left( t^\top \mathrm{D}\theta t \right) \right) [v] \, \mathrm{d}s = 0.$$

Since $v$ satisfies

$$[v] = [\partial_n v] = 0 \text{ on } \sigma,$$

the above identity becomes

$$\int_\sigma \left\{ [u^1] \, \partial_n v - (\theta \cdot [\nabla u]) \, \partial_n v + ([\nabla u] \cdot \nabla v) \, \theta \cdot n \right\} \mathrm{d}s = 0,$$

which is equivalent to (27) by expressing the gradient in the basis $(t, n)$ and using the fact that $\partial_n u$ is continuous across $\sigma$. $\qquad \square$

We are now ready to prove the stability with respect to the length:

**Theorem 4.5.** *Let $\theta$ satisfy the above assumptions as well as $\theta \cdot n = 0$ on $\sigma$ and $(\theta \cdot t)(S_i) \neq 0$ for $i = 1$ or 2. Then, under the assumption that one coefficient $c_i$ of the singularity of $u$ related to the extremity $S_i$ is different from 0, $u^1$ is not identically equal to zero on $M$.*

*Proof.* Assume that $u^1$ is identically equal to zero on $M$. Then as $\partial_n u^1 \equiv 0$ on $M$ and the fact that $u^1$ is harmonic in a neighbourhood of $\Gamma$, by Holmgren's unique continuation theorem $u^1$ vanishes in a neighbourhood of $\Gamma$.

By Lemma 4.4 the identity (27) here becomes

$$\int_\sigma \left( [u^1] - \theta \cdot t \, [\partial_t u] \right) \partial_n v \, \mathrm{d}s = 0,$$

for all $v \in H^1(\Omega)$ harmonic in $\Omega$.

Choosing as test functions the fundamental solution at some point $y \notin \bar{\Omega}$

$$v(x) = \ln |x - y|, \quad y \notin \bar{\Omega}$$

we get

$$\int_\sigma \left( [u^1] - \theta \cdot t \, [\partial_t u] \right) \frac{(x - y) \cdot n_x}{|x - y|^2} \, \mathrm{d}s_x = 0, \quad \forall y \notin \bar{\Omega}.$$

The left-hand side in the above equation is a double layer potential with $\left( [u^1] - \theta \cdot t \, [\partial_t u] \right)$ as dipole density, and it is thus an harmonic function in the whole of $\mathbb{R}^2 \setminus \sigma$. Since it vanishes in $\bar{\Omega}^c$, it also does in the whole of $\mathbb{R}^2 \setminus \sigma$, and is thus continuous across $\sigma$. Being a double layer potential, its jump across $\sigma$ at any point $x_0$ internal to
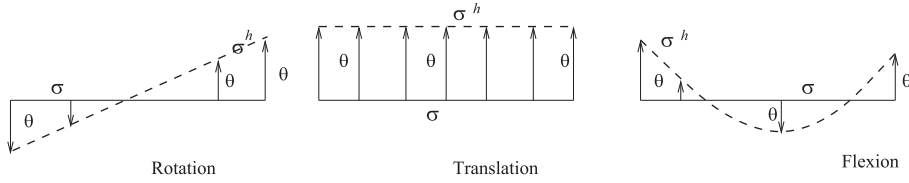
FIGURE 6. Transversal virtual moves of the crack.

$\sigma$ – where the density is smooth – is nothing else than a multiple of that density (see Lem. 4.8 below). This yields:

$$[u^1(x)] - \theta(x) \cdot t_x \, [\partial_t u(x)] = 0 \quad \text{for all } x \in \overset{\circ}{\sigma} .$$

This identity and Lemma 4.2 mean that $\theta \cdot t \, [\partial_t u]$ is bounded near $S_i$, which is impossible since $\theta \cdot t \, [\partial_t u]$ behaves like $2\theta \cdot t \, c_i \, r_i^{-\frac{1}{2}}$ near $S_i$, where $r_i$ is the distance to $S_i$.                                    $\square$

### 4.3. Transverse stability

We are now going to investigate the stability with respect to virtual transverse moves, which are described by directions $\theta$ verifying $\theta \cdot t = 0$ on $\sigma$. These directions may picture rotations ($\theta \cdot n(S_1)\, \theta \cdot n(S_2) < 0$) or translations ($\theta \cdot n$ is constant on $\sigma$), or any flexion deformation, provided at least one crack tip is concerned (see Fig. 6).

**Theorem 4.6.** *Let $\theta$ satisfy the above assumptions as well as $\theta \cdot t = 0$ on $\sigma$ and there exists a neighbourhood of $S_i$ where $(\theta \cdot n)$ does not vanish except eventually at $S_i$ for $i = 1$ or 2. Then, under the assumption that the coefficient $c_i$ of the singularity related to the extremity $S_i$ of $u$ is different from 0, $u^1$ is not identically equal to zero on $M$.*

*Proof.* We argue as in Theorem 4.5 by assuming that $u^1$ is identically equal to zero on $M$. Then by Holmgren's unique continuation theorem $u^1$ vanishes in a neighbourhood of $\Gamma$. Therefore Lemma 4.4 yields here

$$\int_\sigma \{[u^1] \, \partial_n v + \theta \cdot n[\partial_t u] \, \partial_t v\} \, \mathrm{d}s = 0, \tag{29}$$

for all $v \in H^1(\Omega)$ harmonic in $\Omega$. Lemma 4.7 below shows that this implies

$$\theta \cdot n \, [\partial_t u] \equiv 0 \ \text{ on } \sigma.$$

By the assumption on $\theta$ this means that

$$[\partial_t u] \equiv 0 \ \text{ on } \omega_i \cap \sigma,$$

where $\omega_i$ is a sufficiently small neighbourhood of $S_i$. Consequently

$$[u] \equiv \kappa \ \text{ on } \omega_i \cap \sigma,$$

for some constant $\kappa$ but since $[u](S_i) = 0$, we conclude that

$$[u] \equiv 0 \ \text{ on } \omega_i \cap \sigma,$$

which is impossible since the singular part of $[u]$ is not identically equal to zero.                                    $\square$

Let us now prove the lemma used to derive the above result.

**Lemma 4.7.** *Let $\alpha_t \in L^p(\sigma) \cap \mathcal{C}^{0,\varepsilon}_{loc}(\sigma)$ for some $p > 1$ and $\alpha_n \in \mathcal{C}(\bar{\sigma}) \cap \mathcal{C}^{0,\varepsilon}_{loc}(\sigma)$ for some $\varepsilon > 0$ satisfy*

$$\int_\sigma \{\alpha_n \, \partial_n v + \alpha_t \, \partial_t v\} \, ds = 0, \tag{30}$$

*for all $v \in H^1(\Omega)$ harmonic in $\Omega$. Then*

$$\alpha_n = \alpha_t \equiv 0 \text{ on } \sigma. \tag{31}$$

*Proof.* Restricting ourselves as above to the fundamental solution at some $y \in \bar{\Omega}^c$ as test functions

$$v(x) = \ln|x - y| \quad \text{for } y \notin \bar{\Omega}$$

we get

$$\partial_n v(x) = \frac{(x - y) \cdot n_x}{|x - y|^2}$$

$$\partial_t v(x) = \frac{(x - y) \cdot t_x}{|x - y|^2} \qquad \forall y \notin \bar{\Omega}$$

and the identity (30) becomes

$$\int_\sigma \left\{ \alpha_n(x) \frac{(x - y) \cdot n_x}{|x - y|^2} + \alpha_t(x) \frac{(x - y) \cdot t_x}{|x - y|^2} \right\} \, ds(x) = 0, \quad \text{for all } y \notin \bar{\Omega}.$$

As the function of this left-hand side is harmonic in $\mathbb{R}^2 \setminus \sigma$, we finally obtain

$$\int_\sigma \left\{ \alpha_n(x) \frac{(x - y) \cdot n_x}{|x - y|^2} + \alpha_t(x) \frac{(x - y) \cdot t_x}{|x - y|^2} \right\} \, ds(x) = 0, \qquad \forall y \notin \sigma. \tag{32}$$

For the sake of shortness let us now introduce the following (integral) operators:

$$K_1\alpha(y) := \int_\sigma \alpha(x) \frac{(x - y) \cdot n_x}{|x - y|^2} \, ds(x),$$

$$K_2\alpha(y) := \int_\sigma \alpha(x) \frac{(x - y) \cdot t_x}{|x - y|^2} \, ds(x), \qquad \forall y \notin \sigma.$$

The first one is a double layer potential. With these notations the identity (32) is equivalent to

$$K_1\alpha_n(y) + K_2\alpha_t(y) = 0, \quad \forall y \notin \sigma. \tag{33}$$

By Lemma 4.8 below $K_2\alpha_t$ is continuous across $\sigma$ while $K_1\alpha_n$ has a jump across $\sigma$ equal to $-2\pi\alpha_n$, therefore the above identity (33) directly implies

$$-2\pi\alpha_n = 0 \text{ on } \sigma.$$

This proves the first assertion of the lemma.

For the second assertion, we take as test functions in (30)

$$w(x) = \arg(x - y),$$

for $y \in \bar{\Omega}^c$, the branch cut being chosen outside $\Omega$. Since $w$ is the harmonic conjugate of $v(x) = \ln|x - y|$, we get

$$\partial_n w = -\partial_t v, \ \partial_t w = \partial_n v.$$

Therefore (30) implies

$$K_2\alpha_t(y) = 0, \qquad \forall y \notin \bar{\Omega},$$

and the above arguments lead to $\alpha_t = 0$ on $\sigma$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

**Lemma 4.8.** *Let $\alpha \in L^p(\sigma) \cap \mathcal{C}_{loc}^{0,\varepsilon}(\sigma)$ for some $p > 1$ and some $\varepsilon > 0$. Then for all $x_0 \in \overset{\circ}{\sigma}$, we have*

$$K_1\alpha(y) \to -\pi\alpha(x_0) + K_1\alpha(x_0) \text{ as } y \to x_0, y \in \Omega^+, \tag{34}$$

$$K_1\alpha(y) \to \pi\alpha(x_0) + K_1\alpha(x_0) \text{ as } y \to x_0, y \in \Omega^-, \tag{35}$$

$$K_2\alpha(y) \to \ln\frac{|x_0 - S_2|}{|x_0 - S_1|}\alpha(x_0) + \int_\sigma (\alpha(x) - \alpha(x_0)) \frac{(x - x_0) \cdot t_x}{|x - x_0|^2} \, \mathrm{d}s(x) \text{ as } y \to x_0. \tag{36}$$

*Proof.* The proof of (34) and (35) is quite standard and consists in extending $\sigma$ into an appropriate closed $\mathcal{C}^2$ curve (see for instance [9,19,20]). More precisely for (34) (resp. (35)) we extend $\sigma$ into $\sigma_1^+$ (resp. $\sigma_1^-$) closed $\mathcal{C}^2$ curve so that $\sigma_1^+$ (resp. $\sigma_1^-$) is the boundary of a bounded domain $\Lambda^+$ (resp. $\Lambda^-$). In both cases, when $y$ tends to $x_0$ in $\Omega^+$ (resp. in $\Omega^-$), $y$ will be inside $\Lambda^+$ (resp. $\Lambda^-$). Therefore a standard property of the double layer potential on $\sigma_1^+$ (resp. $\sigma_1^-$) yields (see Sect. 26.3 of [21])

$$\int_{\sigma_1^\pm} \alpha(x)\frac{(x - y) \cdot n_x^\pm}{|x - y|^2} \, \mathrm{d}s(x) \to -\pi\alpha(x_0) + \int_{\sigma_1^\pm} \alpha(x)\frac{(x - x_0) \cdot n_x^\pm}{|x - x_0|^2} \, \mathrm{d}s(x) \text{ as } y \to x_0, y \in \Omega^\pm, \tag{37}$$

where $n_x^\pm$ means the normal vector at $x \in \Lambda^\pm$ directed towards the exterior of $\Lambda^\pm$. Since for $x \in \sigma_1^\pm \setminus \bar{\sigma}$ and $y$ in a sufficiently small neighbourhood of $x_0$, the distance from $x$ to $y$ is uniformly bounded from below, we clearly have

$$\int_{\sigma_1^\pm \setminus \bar{\sigma}} \alpha(x)\frac{(x - y) \cdot n_x^\pm}{|x - y|^2} \, \mathrm{d}s(x) \to \int_{\sigma_1^\pm \setminus \bar{\sigma}} \alpha(x)\frac{(x - x_0) \cdot n_x^\pm}{|x - x_0|^2} \, \mathrm{d}s(x), \text{ as } y \to x_0, y \in \Omega^\pm. \tag{38}$$

The difference between (37) and (38) yields (34) and (35) since $n_x^\pm = \pm n_x$ on $\sigma$.

For the property (36), we use the splitting

$$K_2\alpha(y) = I(y)\alpha(x_0) + \int_\sigma (\alpha(x) - \alpha(x_0)) \frac{(x - y) \cdot t_x}{|x - y|^2} \, \mathrm{d}s(x), \tag{39}$$

where we have set

$$I(y) = \int_\sigma \frac{(x - y) \cdot t_x}{|x - y|^2} \, \mathrm{d}s(x).$$

The second term of the identity (39) tends to the second term of (39) due to Lebesgue's bounded convergence theorem. For the first term we remark that

$$I(y) = \int_\sigma \partial_t \ln|x - y| \, \mathrm{d}s(x) = \ln|S_2 - y| - \ln|S_1 - y|.$$

We conclude since $x_0$ is different from $S_1$ and $S_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

## 5. CONCLUSION

In this paper, we have proved identifiability on cracks submitted to a non linear impedance condition. Except for the restriction on the direction of the cracks, the obtained result seems close to optimal: the assumption on the identifying fluxes is not really restrictive, and those on the impedances are also needed to ensure well posedness of the forward problem.

As for the local Lipschitz stability, we have somewhat adapted the usual techniques previously used [6, 7], by dropping the explicit construction of peculiar fields, which is not easy to achieve for non flat cracks. The so worked out proofs rely on properties of fundamental solutions, and others of single and double layer potentials, which make them likely to extend to several linear operators of mathematical physics. Another interesting feature is that longitudinal and transversal stabilities are processed similarly, though arguments in the proofs may of course vary.

Both identifiability and stability results have been obtained on the crack as well as on the impedance. Further interesting developments now regard the numerical aspect of the recovery problem.

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