VERTICAL COMPACTION IN A FAULTED SEDIMENTARY BASIN

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Abstract. In this paper, we consider a 2D mathematical modelling of the vertical compaction effect in a water saturated sedimentary basin. This model is described by the usual conservation laws, Darcy's law, the porosity as a function of the vertical component of the effective stress and the Kozeny-Carman tensor, taking into account fracturation effects. This model leads to study the time discretization of a nonlinear system of partial differential equations. The existence is obtained by a fixed-point argument. The uniqueness proof, by Holmgren's method, leads to work out a linear, strongly coupled, system of partial differential equations and boundary conditions.

1. INTRODUCTION

Extracted of [15]: "The constitution of a sedimentary basin during the geologic history implies processes of: sedimentation, erosion, compaction, eviction and transfer of fluids, thermic transfer and of diagenesis, of which outcome is a geologic structure capable of establishing a reservoir of hydrocarbons or a deposit of mineral resources." The modelling, at a geological scale, of these various mechanisms and their numerical simulations establish a promising tool for the evaluation of the oil potential of basins (see [5, 23] and [24]).

The simulation of the genesis and the migration of hydrocarbons in the sedimentary coverage has to take into account sedimentation and erosion phenomena, and so compaction of sediments. From then on, it is necessary to consider poromechanical models.

The reader interested in similar problems, outside the framework of hydrocarbons, will be able to consult Sciarra *et al.* [25] who consider a binary mixture where a dilatation of pores is observed under extremal pressure. One can also see the importance of compaction in the dynamics of large ice masses as mentioned by Gödert *et al.* in [13].

In this first approach, we shall suppose that the mechanics of cliffs (sediments) deformation ensues from the vertical compaction. The other phenomena of deformation, such as the gliding of the sedimentary layers for example, will be untidy, either, supposed known in advance and implicitly contained in the data of the problem. It will be enough then to consider a rheological model allowing to express the porosity variations according to
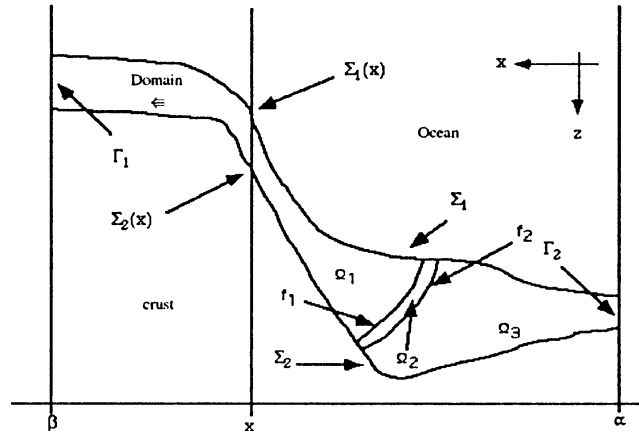
FIGURE 1. Faulted domain.

the only vertical constraint. This simplifying working hypothesis does not remove anything in the capability of the model as it is noticed by Luo *et al.* [17], Perez [22] or Wangen *et al.* [26] and [27].

The current study concerns a 2D monophasic model in a faulted porous medium. It takes into account the vertical compaction and the fracturing according to the state of the effective constraint, with an effect of threshold of release.

There are few mathematical publications on compaction models in sedimentary basins. Most of them concerns numerical aspects (see for example Badea [2], Fowler *et al.* [8], Ismail-zade *et al.* [14], Wangen [26], Wangen *et al.* [27] or Zakarian *et al.* [29]).

Our goal is to give some mathematical tools in order to analyse such models. Numerical aspects are actually studied at the Institut Français de Pétrole (Schneider *et al.* [23, 24]).

2. MATHEMATICAL MODELLING

As mentioned in the previous section, the anisotropy is mainly vertical, then a 2D model is considered in the plane Oxy with origin O, horizontal axis x and vertical axis z pointing to the direction of gravity. In this theoretical study, we consider that the physical domain Ω is decomposed into three sub-domains (*cf.* Fig. 1): sub-domains Ω_1 and Ω_3 represent two parts of a sedimentary basin Ω, separated by a fault Ω_2. The thickness of this fault is small compared with the size of Ω_1 and Ω_3, but it is not unimportant compared with the physical phenomena which it generates. So, we shall consider Ω_2 as a separate sub-domain with appropriate characteristics, *i.e.* allowing a free passage in compulsory directions or causing obstructions.

This working hypothesis becomes delicate in numerical analysis, where it can be preferable to model this fault as an interface (so without thickness). From then on, this has to lead to artificial boundary conditions in order to restore the real role of this fault in the water – sediments traffic. One finds then, in previous studies, different subtleties.

In his approach, [22] imposes a continuance of the stream through the fault and introduces a parameter δ to take into account the discontinuity of the pressure between the superior wall and the lower wall. This discontinuity is necessary, otherwise, it would mean that this fault, with a null Lebesgue measure, is neglected in the system of equations and so would not exist physically.

As indicated by the author, the problem is that the determination of δ is purely empirical. Furthermore, this internal discontinuity prevents the representative function of the water pressure from being in a Sobolev's space of first order on Ω, that does not receive any convincing explanations.

In an other approach, one imposes the continuity of the traces on both sides of the fault. But then, as mentioned above, the system of equations alters the physical reality. Then, in the modelling of the fault,

one has to use a term of order 1, and one may have jumps across the interface. It gives, in the formulation of equations, a supplementary term: a measure with support in the interface-fault, as if it was a well. Furthermore, it does not seem realistic that the pressure is the same on both sides of the fault.

In our approach, we adopt the principles of continuity of traces and of fluxes to write the relations which govern the water pressure in each interface. So, we take into account the physical process in Ω_2 and jumps exist from an extreme edge to the other.

If for numerical motivations, one has to consider that the fault is an interface, a last approach could be based on an asymptotic analysis with regard to ε (the parameter of the fault thickness) in order to find one of the above mentioned configurations.

The above proposed method, which represents a simplified model (one fault which cuts in two the studied domain) can be generalized to a sedimentary basin with several faults and several types of sediments. The main consideration is to be able to decompose the domain in a certain number of sub-domains with known geologic, rheologic, ..., characteristics.

In a first part, we are interested in the mathematical analysis of the model, where compactions are of weak amplitude. We study then the time semi-discretisation of the system. The existence of a solution results from Schauder–Tychonoff's fixed point theorem, in the separable hilbertian framework. Then, uniqueness is proved with a technique of transposition, inspired by Antontsev and Domansky's works [1] on the analytical study of diphasic filtration system. This technique, generalizing the method of pivot space changing [16], classically returns the study of the uniqueness of the primal problem to the study of the existence for the dual problem. This last one admits at least a solution due to Lax–Milgram's theorem.

In a second part, we consider a model where compaction is of general amplitude. To study this problem of predictive-corrective type, we do not make reference to the narrow-mindedness of certain coefficients in the state laws. Uniqueness is obtained by applying Fredholm's alternative, associated to a weak maximum principle.

2.1. **Notations**

Let $\alpha \in \{s, w\}$ (w label for water and s label for sediments) and $i \in \{1, 2, 3\}$ (for domain Ω_i):

$\overrightarrow{V}_\alpha$: speed of phase α, ρ_α : volumic mass, p_w : water presure,
P_{atm} : atmospheric pressure, K_i : permeability tensor, σ_z : total constraint,
λ_a^i and λ_s^i : coefficients of anisotropy, k_i : absolute permeability, σ : effective constraint,
S_0^i : specific area of the porous media, h_i : state law of fracturation, ϕ_i : porosity,
$\phi_r^i, \phi_a^i, \phi_b^i, \sigma_a^i,\ \sigma_b^i$: being characteristic parameters of the deposited sediment nature,
g_s : sedimentation speed at the bottom of the ocean, g : gravity acceleration, H_w : height of water,
$F_i = [\rho_w \phi_i + \rho_s(1 - \phi_i)]g,$ $^t\overrightarrow{B} = (0, \rho_w g).$

2.2. **Conservation laws**

In each sub-domain Ω_i, the conservation laws are the following ones:

1) mass conservation of the sediment and water:

$$\frac{\partial}{\partial t}(\rho_s^i(1 - \phi_i)) + div(\rho_s^i(1 - \phi_i)\overrightarrow{V}_s) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho_w \phi_i) + div(\rho_w \phi_i \overrightarrow{V}_w) = 0. \tag{2}$$

2) momentum conservation (equilibrium equation)

$$\frac{\partial \sigma_z}{\partial z} = (\phi_i \rho_w + (1 - \phi_i)\rho_s^i)g = F_i(\phi_i). \tag{3}$$

One supposes in the sequel that ρ_s and ρ_w are constant on each sub-domain.

2.3. Behavior laws

These conservation laws have to be completed by phenomenologic behavior laws. We held:

$$\phi_i(\overrightarrow{V}_w - \overrightarrow{V}_s) = -K_i\,(\overrightarrow{\nabla p_w} - \overrightarrow{B}) \qquad\qquad \text{(Darcy's law)}, \qquad\qquad (4)$$

$$\sigma = \sigma_z - p_w \qquad\qquad \text{(Terzaghi's relation)}, \qquad\qquad (5)$$

$$\phi_i(\sigma) = \phi_r^i + \phi_a^i \exp[-\frac{\sigma}{\sigma_a^i}] + \phi_b^i \exp[-\frac{\sigma}{\sigma_b^i}] \qquad\qquad \text{(Elastoplastic rheological law)}, \qquad\qquad (6)$$

$${}^t\overrightarrow{V}_s = (0, v_s) \qquad\qquad \text{(Vertical compaction hypothesis)}, \qquad\qquad (7)$$

$$K_i(\sigma, p_w) = k_i(\phi_i)\begin{pmatrix} \lambda_s^i & 0 \\ 0 & \lambda_a^i \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & h_i(\sigma, p_w) \end{pmatrix} \qquad \text{(Permeability law)}, \qquad (8)$$

where the tensor, expressed in a strate – antistrate base, rests on Kozeny–Carman's law with $k_i(\phi) = \frac{0{,}2\,\phi_i^3}{S_0^{i2}(1-\phi_i)^2}$. It is completed with the consideration of the fracturation, by means of a supplementary term h_i, which intervenes in the physical phenomenon as soon as certain critical threshold is reached.

2.4. Description of the domain and of the boundary conditions

• One supposes (geometrical regularity) that there are four Lipschitzian functions γ_1, γ_2, f_1 and f_2, such that (*cf.* Fig. 1):

$$\Omega = \{(x,z) \in \mathbb{R}^2,\ \alpha < x < \beta,\ \gamma_1(x) < z < \gamma_2(x)\},\quad \Omega_1 = \{(x,z) \in \Omega,\ z > f_1(x)\},$$
$$\Omega_2 = \{(x,z) \in \Omega,\ f_1(x) > z > f_2(x)\},\qquad \Omega_3 = \{(x,z) \in \Omega,\ z < f_2(x)\},$$
$$\Gamma_1 = \{(\alpha,z),\ \gamma_1(\alpha) < z < \gamma_2(\alpha)\},\qquad \Gamma_2 = \{(\beta,z),\ \gamma_1(\beta) < z < \gamma_2(\beta)\},$$
$$\Sigma_i = \{(x,z) \in \mathbb{R}^2,\ \alpha < x < \beta,\ z = \gamma_i(x)\},\qquad f_i = \{(x,z) \in \Omega,\ z = f_i(x)\},\ i \in \{1,2\}\cdot$$

• Conditions on Σ_1 : $p_w = \sigma_z = P_{\text{atm}} + \rho_w g H_w \underset{\text{notation}}{=} P_{\Sigma_1}$, in particular, according to (5), one has

$$p_w = P_{\Sigma_1}, \quad \sigma = 0. \qquad\qquad (9)$$

In fact, Domain Ω evolves during time because of the erosion and of the sedimentation by gravitation on the free boundary Σ_1 (bottom of the ocean), according to the law: $v_s = g_s$ where, for example, $g_s = \frac{\overrightarrow{Q_s.n}}{n_z}$, $\overrightarrow{Q_s}$ ensuing from the sedimentary load of the ocean, the direction, the intensity of maritime currents...

• Condition on Σ_2 : $\overrightarrow{V_w}.\overrightarrow{n} = 0$ and, according to (4), one gets

$$-{}^t\overrightarrow{n}\,K\,(\overrightarrow{\nabla p_w} - \overrightarrow{B}) = -\phi(\sigma)v_s n_z, \qquad\qquad (10)$$

where n_z represents the vertical part of the normal vector $\overrightarrow{n}$.

Furthermore, Domain Ω evolves, *a priori*, also during time by the motion of crusts. So Σ_2 is also a free boundary. We shall suppose it fixed in this study, in order to seriate the difficulties.

• Conditions on Γ_1 and Γ_2 (artificial free boundaries): $\overrightarrow{V_w}.\overrightarrow{n} = 0$. In particular, according to (4) and by noticing that ${}^t\overrightarrow{n} = (+/-1, 0)$, $\overrightarrow{V_s}.\overrightarrow{n} = 0$, so that one has

$$-{}^t\overrightarrow{n}\,K\,(\overrightarrow{\nabla p_w} - \overrightarrow{B}) = 0. \qquad\qquad (11)$$

• Conditions in the internal interfaces of the domain f_i, $i = 1, 2$:

$$\begin{cases} \sigma_{\Omega_i} = \sigma_{\Omega_{i+1}}, \quad v_{s,\Omega_i} = v_{s,\Omega_{i+1}}, \quad p_{w,\Omega_i} = p_{w,\Omega_{i+1}}, \\ -\,^t\overrightarrow{n}\,K_i\left(\overrightarrow{\nabla p}_{w,\Omega_i} - \overrightarrow{B}\right) = -\,^t\overrightarrow{n}\,K_{i+1}\left(\overrightarrow{\nabla p}_{w,\Omega_{i+1}} - \overrightarrow{B}\right). \end{cases} \tag{12}$$

• Initial conditions (for $t = 0$, there are no sediments):

$$\Sigma_1 = \Sigma_2, \quad \phi_i = \phi_r^i + \phi_a^i + \phi_b^i, \quad \sigma_z = p_w = P_{atm} + \rho_w g H_w, \text{ so } \sigma = 0.$$

2.5. Presentation of the system of equations

First, we introduce some notations: Considering $\mathbb{I}_{\Omega_i}$ the caracteristic function of Domain Ω_i, one sets,

$$K(x, z, \sigma, p) = \sum_i K_i(\sigma, p)\,\mathbb{I}_{\Omega_i}(x, z), \quad \phi(x, z, \sigma) = \sum_i \phi_i(x, z, \sigma)\,\mathbb{I}_{\Omega_i}(x, z), \quad F(x, z, \sigma) = \sum_i F_i(\sigma)\,\mathbb{I}_{\Omega_i}(x, z).$$

It is important to notice that each of these functions is regular on Ω_i and that it has a trace on $\partial\Omega_i$ (thus on Σ_i, Γ_i and f_i).

A triple preliminary analysis of the partial differential equations introduced by the model, of the behavior laws which structure them and of the boundary and interfaces conditions imposed by the experimentation, leads to the following choice for the main unknown

$$\sigma, v_s \text{ and } p_w,$$

and to the following choice for the system of equations on Ω

$$\frac{\partial\sigma}{\partial z} = F(., ., \sigma) - \frac{\partial p_w}{\partial z}, \tag{13}$$

$$-\frac{\partial}{\partial t}\phi(., ., \sigma) + \frac{\partial}{\partial z}(1 - \phi(., ., \sigma))v_s = 0, \tag{14}$$

$$-div\{K(., ., \sigma, p_w)\,(\overrightarrow{\nabla p_w} - \overrightarrow{B})\} + \frac{\partial}{\partial z}v_s = 0. \tag{15}$$

Equation (13) comes from the equilibrium equation (3) and Terzaghi's relation (5), equation (14) comes from the mass conservation of the sediment (1) and equation (15) comes from the mass conservation laws (1) and (2) and Darcy's law (4).

In the sequel, the analysis concerns the study of a time discretisation, reasonable method for the approach of slow evolution processes.

We do not plan to study the time continuous system. Indeed, contribution of sedimentation and erosion effects prevent a good control of the evolution of Ω during time, in order to pass easily to the limit. Here, the hypothesis of monotonicity for example of the application $t \mapsto \Omega(t)$, according to the ideas of [16] (p. 415), is not justifiable. However, we can propose a correction of Domain Ω at each iteration. Indeed, boundary Σ_1 is a free boundary, subjected to the oceanic phenomena of deposits or erosion.

Referring to a model which authorizes only vertical deformations, one transcribes the principle of the conservation of sedimentary material quantity on the vertical line, over the point x of Σ_2, by

$$\int_{\Sigma_2(x)}^{z^*(x)} (1 - \widetilde{\phi}(., ., \sigma(x, z)))\,dz = \int_{\Sigma_2(x)}^{\Sigma_1(x)} (1 - \phi(., ., \sigma_0(x, z)))\,dz + \left(\overrightarrow{Q}_s . \overrightarrow{n}\right)_x . h,$$

where: $(\overrightarrow{Q_s}.\overrightarrow{n})_x$ represents the contribution or the volumic loss *via* the ocean, $z^*(x)$ is the new quotation for the boundary Σ_1 at the top of the point x of Σ_2, supposed indeformable

$$\widetilde{\phi}(.,.,\sigma(x,z)) = \phi(.,.,\sigma(x,z)) \quad \text{if} \quad z \leq \Sigma_1(x), \quad \phi(.,.,0) \quad \text{if} \quad z \geq \Sigma_1(x).$$

Function h_x (for any x) which gives, for any z^*, $\int_{\Sigma_2(x)}^{z^*}(1 - \widetilde{\phi}(.,.,\sigma(x,z))\,\mathrm{d}z$, is increasing over $\mathbb{R}^+$. For any x, this equation defines $z^*(x)$ in a unique way. So, it supplies the Cartesian equation of the new profile of Σ_1.

By keeping in memory the value of the maximal constraint punctually reached [22], let us notice that this process of rectification can be enriched with the irreversibility consideration of the compaction.

In the next section, we present the mathematical analysis of the first iteration of (13, 14) and (15) in the case of an implicit discretisation. Moreover, weak amplitude compaction (*i.e.* with a small variation of ϕ and K with respect to the unknown σ and p_w) is considered.

In Section 4, we present the mathematical analysis of the first iteration of (13, 14) and (15) in the case of a predictive-corrective type discretisation. General amplitude compaction is considered.

3. MATHEMATICAL ANALYSIS OF A FIRST CASE: WEAK AMPLITUDE COMPACTION

Let $h = \Delta t$ be the iteration step and σ_0 be the datum of σ at the previous iteration. The proposed discretised scheme is based on equations (13) and (15) for the unknown σ and p_w and on

$$\frac{\phi(.,.,\sigma_0) - \phi(.,.,\sigma)}{h} + \frac{\partial}{\partial z}\left((1 - \phi(.,.,\sigma))v_s\right) = 0, \tag{16}$$

for the unknown v_s.

3.1. **Notations of functional analysis**

Then, one has to look for the solutions in an adapted cartesian product of first order Sobolev's spaces. In order to do so, one denotes

$$V^f = \{u \in H^1(\Omega), \ u_{|\Sigma_1} = f\}, \tag{17}$$

where f is given on Σ_1, regular enough so that V^f is not empty.

$$W = \{u \in L^2(\Omega), \frac{\partial u}{\partial z} \in L^2(\Omega)\} = L^2(\alpha < x < \beta, H^1[\gamma_1(x), \gamma_2(x)]). \tag{18}$$

W is a separable Hilbert space for its natural norm

$$\forall u \in W, \ ||u||_W^2 = \int_\Omega u^2 \,\mathrm{d}x + \int_\Omega \left(\frac{\partial u}{\partial z}\right)^2 \,\mathrm{d}x.$$

It is provided with a trace operator, linear and continuous for the natural topologies

$$\gamma : W \to L^2(\Sigma_1) \times L^2(\Sigma_2), \ u \mapsto (u_{|\Sigma_1}, u_{|\Sigma_2}),$$

so that the following notation is coherent: $W^g = \{u \in W, \ u_{|\Sigma_1} = g\}$.

Furthermore, for any Lipschitzian function f from $\mathbb{R}$ into $\mathbb{R}$, the following chain rule holds for the weak derivatives: $\forall u \in W, \ f(u) \in W$ and

$$\frac{\partial f(u)}{\partial z} = f'(u)\frac{\partial u}{\partial z} \quad \text{a.e. in } \Omega,$$

where f' indicates a bounded Borelian representative of f derivative (it exists in the classic sense almost everywhere according to Rademacher's theorem).

3.2. **Hypothesis**

One assumes that: $\exists c, C, M > 0, \quad 0 < c < C < 1,$

$$c \leq \phi_i \leq C, \quad 0 \leq h \leq M \tag{19}$$

$$\forall \sigma_1, \sigma_2 \in \mathbb{R}, \quad \sigma_1 < \sigma_2 \Rightarrow 0 \leq \phi(.,.,\sigma_2) - \phi(.,.,\sigma_1) \leq M(\sigma_2 - \sigma_1) \tag{20}$$

$$\forall \sigma_1, \sigma_2, p_1, p_2 \in \mathbb{R}, \quad |h_i(\sigma_1, p_1) - h_i(\sigma_2, p_2)| \leq M(|\sigma_2 - \sigma_1| + |p_2 - p_1|), \quad i = 1, 2, 3. \tag{21}$$

$$\sigma_0 \in W^0, \quad \exists q > 2, V^{P_{\Sigma_1}} \cap W^{1,q}(\Omega) \neq \emptyset, \quad g_s \in L^\infty(\Sigma_1), W^{g_s} \neq \emptyset. \tag{22}$$

3.3. **Definition of a solution**

Definition 3.1. One calls solution to System (13, 16, 15) for Conditions (9)–(11) and (12), any (σ, v_s, p_w) in $W^0 \times W^{g_s} \times V^{P_{\Sigma_1}}$ such that

(13) and (16) are satisfied a.e. in Ω,

and, for any φ in V^0,

$$\int_\Omega {}^t\overrightarrow{\nabla\varphi} K(.,.,\sigma, p_w) \left(\overrightarrow{\nabla p_w} - \overrightarrow{B}\right) \, \mathrm{d}x + \int_\Omega \frac{\partial v_s}{\partial z} \varphi \, \mathrm{d}x = \int_{\Sigma_2} \phi(\sigma) v_s n_z \varphi \, \mathrm{d}\sigma. \tag{23}$$

One has to remark that (12) is implicitly contained in the fact that the solutions belong to W^0, W^{g_s} and $V^{P_{\Sigma_1}}$. Indeed, these functions possess the property of traces countinuity through interfaces f_i. It is also contained in the formulation (23) for the property of flux countinuity.

3.4. **Existence of a solution**

Proposition 3.2. *There exists at least one solution to the problem in the sense of Definition 3.1.*

In order to establish the existence of a solution, we suggest to use Schauder–Tikhonov's fixed point theorem, in the context of hilbertian separable spaces [10]. Therefore, one fixes p_w in $V^{P_{\Sigma_1}}$. Thus, integration of equations (13) and (16) gives a solution $\sigma(p_w)$ in W^0 and a solution $v_s(p_w)$ in W^{g_s}. It is enough then to inject these two solutions in the following paralinearised version of equation (23)

$$\int_\Omega {}^t\overrightarrow{\nabla\varphi} K(.,.,\sigma(p_w), p_w) \left(\overrightarrow{\nabla p_w} - \overrightarrow{B}\right) \, \mathrm{d}x + \int_\Omega \frac{\partial v_s(p_w)}{\partial z} \varphi \, \mathrm{d}x = \int_{\Sigma_2} \phi(.,.,\sigma) v_s(p_w) n_z \varphi \, \mathrm{d}\sigma, \tag{24}$$

to obtain a unique solution $\overline{p_w}$ in $V^{P_{\Sigma_1}}$ and so, to build an application $\mathcal{S}$ from $V^{P_{\Sigma_1}}$ into himself, defined for any p_w by $\mathcal{S}(p_w) = \overline{p_w}$.

It is immediate to notice that the solutions of System (13, 16, 15), for the boundary conditions (9)–(11) and the interface condition (12), are the fixed points of $\mathcal{S}$.

One shows then that $\mathcal{S}$ keeps a non empty bounded closed convex set of $H^1(\Omega)$. It relies mainly on the fact that v_s is bounded in $L^\infty(\Omega)$. From then on, with the help of *a priori* estimations, of subsequences extraction and of weak compactness arguments, one shows that $\mathcal{S}$ is weak-sequentially continuous. So, there exists a fixed point. The reader interested in technical details of these results is referred to [11].

3.5. L^p **regularity of the obtained solution: an extension of N.G. Meyers's principle**

For many coupled non linear systems, the study of solution uniqueness passes by the treatment of trilinear term integration. From then on, it is necessary to improve the regularity knowledge of the obtained solution. This wellknown result, relaying on a disturbance of the Poisson equation, was initially introduces by [18]. Used again by [4, 19] and recently by [20] and [9] within the framework of diphasic models. We propose in [11] an adapted version to our problem, namely, an elliptic equation in a Lipschitzian domain, and of which a corollary is

Proposition 3.3. *There exists $p_0 > 2$, $k_0 > 0$, independent of solutions σ and v_s so that: $p_w \in W^{1,p_0}(\Omega)$ and $||p_w||_{W^{1,p_0}} \leq k_0$.*

3.6. Uniqueness of the solution

3.6.1. *Presentation of the duality method*

Suggested by S.N. Antontsev's works [1] concerning the analytical study of diphasic filtration systems. The proposed method of uniqueness is based on a technique of transposition. It is inspired also by Holmgren's duality method, implemented by Oleĭnik [21].

This method of duality returns classically the study of the uniqueness of the primal problem to the study of the existence of a solution to the dual problem, in compatible functional frameworks.

We denote (σ_1, p_1, v_1) and (σ_2, p_2, v_2) two solutions to system (13, 16, 15) with the boundary conditions (9)–(11) and (12).

After subtracting of equations (13, 16) and (23) verified by (σ_1, p_1, v_1) and (σ_2, p_2, v_2), we shall transport the derivations on the test functions, in order to introduce quantities $\overline{\sigma} = \sigma_1 - \sigma_2$, $\overline{v} = v_1 - v_2$ and $\overline{p} = p_1 - p_2$. In order to relieve the demonstration writing, we introduce some notations

$$
\begin{aligned}
&\phi' = \frac{\phi(.,.,\sigma_1) - \phi(.,.,\sigma_2)}{\sigma_1 - \sigma_2} && \text{if } \sigma_1 \neq \sigma_2, \quad \phi' = \frac{\partial \phi}{\partial \sigma}(.,.,\sigma_1) && \begin{array}{l}\text{on the } \mathcal{L}^2\text{-measurable}\\ \text{set } \{\sigma_1 = \sigma_2\},\end{array}\\
&F' = \frac{F(.,.,\sigma_1) - F(.,.,\sigma_2)}{\sigma_1 - \sigma_2} && \text{if } \sigma_1 \neq \sigma_2, \quad F' = \frac{\partial F}{\partial \sigma}(.,.,\sigma_1) && \text{if } \sigma_1 = \sigma_2,\\
&D_1 K = \frac{K(x,z,\sigma_2,p_1) - K(x,z,\sigma_2,p_2)}{\overline{p}} && \text{if } \overline{p} \neq 0, \quad D_1 K = \frac{\partial K(x,z,\sigma_2,p_2)}{\partial p} && \text{if } \overline{p} = 0,\\
&D_2 K = \frac{K(x,z,\sigma_1,p_1) - K(x,z,\sigma_2,p_1)}{\overline{\sigma}} && \text{if } \overline{\sigma} \neq 0, \quad D_2 K = \frac{\partial K(x,z,\sigma_1,p_1)}{\partial \sigma} && \text{if } \overline{\sigma} = 0,\\
&\overrightarrow{D_1}(\sigma_2) = {}^t[\overrightarrow{\nabla p_1} - \overrightarrow{B}] D_1 K, && \overrightarrow{D_2}(p_1) = {}^t[\overrightarrow{\nabla p_1} - \overrightarrow{B}] D_2 K.
\end{aligned}
$$

Since functions ϕ_i, F_i, k_i and h_i are Lipschitzian, ϕ' and F' are measurable functions. According to Rademacher's theorem, one is even able to choose them as bounded Borelian functions. Also, $\overrightarrow{D_2}(p_1)$ and $\overrightarrow{D_1}(\sigma_2)$ are $[L^2(\Omega)]^2$ functions. More exactly, thanks to Proposition 3.3, they are elements of $[L^{p_0}(\Omega)]^2$.

Right now, it is necessary for us to clarify the functional spaces required to look for the solutions to the dual problem and to legitimize Gauss–Green formulae. For it, one considers

$$
X = \left\{ u \in L^2(\Omega), \ \forall i, \ \frac{\partial u}{\partial z} \in L^2(\Omega_i) \right\}, \quad ||u||_X^2 = ||u||_{L^2(\Omega)}^2 + \sum_i \left|\left|\frac{\mathrm{d}u}{\mathrm{d}z}\right|\right|_{L^2(\Omega_i)}^2.
$$

It is a subspace of

$$
\overline{BV_z} = \left\{ u \in L^2(\Omega), \ \frac{\partial u}{\partial z} \in \mathcal{M}_b(\Omega) \right\}
$$

(where $\mathcal{M}_b(\Omega)$ indicates the set of bounded Radon measures on Ω). It has been introduced by [28] for the study of degenerated hyperbolic-parabolic problems. We suggest to denote by $\frac{\mathrm{d}u}{\mathrm{d}z}$ the absolutely continuous part, with respect to the Lebesgue measure $\mathcal{L}$, of the measure $\frac{\partial u}{\partial z}$. So, for any u of X,

$$
\frac{\mathrm{d}u}{\mathrm{d}z} = \sum_i \frac{\partial u_{|\Omega_i}}{\partial z} \mathbb{I}_{\Omega_i},
$$

while

$$
\frac{\partial u}{\partial z} = \frac{\mathrm{d}u}{\mathrm{d}z} \, \mathrm{d}\mathcal{L} + \sum_i (u^+ - u^-) n_z \, \mathcal{H}^1 {\llcorner}_{f_i},
$$

where u^+ and u^- represent the traces of u on the right and on the left of f_i (if one chooses the direction given by $\overrightarrow{n}$, the normal vector to f_i, from Ω_i to Ω_{i+1}) and $\mathcal{H}^1 \llcorner f_i$ represents the restriction of the 1-dimensionnal Hausdorff measure on f_i.

$$Y_p = \left\{ v \in L^2(\Omega), \ \forall i, \ \frac{\partial v}{\partial z} \in L^q(\Omega_i) \right\}, \quad q = \frac{2p}{p+2}.$$

3.6.2. First step: superimposing of the balance states and transposition

Concerning equations verified by v_1 and v_2: the subtraction of the equations, the integration by parts on each Ω_i, by adding of the i together and by regrouping each terms, it comes, for any α of X,

$$\int_\Omega \overline{\sigma} \phi' \left(v_2 \frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{\alpha}{h} \right) \mathrm{d}x + \int_\Omega \overline{v} \left(\phi(.,.,\sigma_1) - 1 \right) \frac{\mathrm{d}\alpha}{\mathrm{d}z} \mathrm{d}x + \sum_i \int_{f_i} \overline{\sigma} v_2 \left[\phi'^+ \alpha^+ - \phi'^- \alpha^- \right] n_z \, \mathrm{d}\sigma$$

$$- \sum_i \int_{f_i} \overline{v}(1 - \phi^+(.,.,\sigma_1))\alpha^+ n_z \, \mathrm{d}\sigma + \sum_i \int_{f_i} \overline{v}(1 - \phi^-(.,.,\sigma_1))\alpha^- n_z \, \mathrm{d}\sigma$$

$$= \int_{\Sigma_2} \overline{\sigma} \phi' \, v_2 \, \alpha \, n_z \, \mathrm{d}\sigma - \int_{\Sigma_2} \overline{v}(1 - \phi(.,.,\sigma_1)) \, \alpha \, n_z \, \mathrm{d}\sigma. \quad (25)$$

Concerning equations verified by σ_1 and σ_2: in a same way, one has, for any β of Y_{p_0} (p_0 given by Property 3.3),

$$\int_\Omega \overline{\sigma} \left(\frac{\mathrm{d}\beta}{\mathrm{d}z} + F'\beta \right) \mathrm{d}x + \int_\Omega \overline{p} \frac{\mathrm{d}\beta}{\mathrm{d}z} \mathrm{d}x = \sum_i \int_{f_i} (\overline{p} + \overline{\sigma}) \, (\beta^+ - \beta^-) \, \eta_z \, \mathrm{d}\sigma + \int_{\Sigma_2} (\overline{p} + \overline{\sigma}) \, \beta \, \eta_z \, \mathrm{d}\sigma. \quad (26)$$

Concerning the variational equations verified by p_1 and p_2, a subtraction gives for any γ of V^0,

$$\int_\Omega \overline{\sigma} \overrightarrow{D}_2(p_1) \overrightarrow{\nabla \gamma} + \overline{p} \overrightarrow{D}_1(\sigma_2) \overrightarrow{\nabla \gamma} \, \mathrm{d}x + \int_\Omega {}^t \overrightarrow{\nabla \gamma} K(.,.,\sigma_2, p_2) \overrightarrow{\nabla p} \, \mathrm{d}x + \int_{\Sigma_2} \overline{v} \gamma \, n_z \, \mathrm{d}\sigma =$$

$$\int_\Omega \overline{v} \frac{\partial \gamma}{\partial z} \, \mathrm{d}x + \int_{\Sigma_2} \overline{\sigma} \phi' v_2 n_z \gamma \, \mathrm{d}\sigma + \int_{\Sigma_2} \overline{v} \phi(.,.,\sigma_1) \, n_z \gamma \, \mathrm{d}\sigma. \quad (27)$$

From then on, regrouping equations (25, 26) and (27) allows us to obtain the vectorial variational formulation on $X \times Y_{p_0} \times V^0$,

$$\int_\Omega \overline{\sigma} \left\{ \phi'(v_2 \frac{\mathrm{d}\alpha}{\mathrm{d}z} - \frac{\alpha}{h}) + \frac{\mathrm{d}\beta}{\mathrm{d}z} \right\} \mathrm{d}x + \int_\Omega \overline{\sigma} \left\{ F'\beta + \overrightarrow{D}_2(p_1) \overrightarrow{\nabla \gamma} \right\} \mathrm{d}x - \int_\Omega \overline{v} \left\{ (1 - \phi(.,.,\sigma_1)) \frac{\mathrm{d}\alpha}{\mathrm{d}z} + \frac{\partial \gamma}{\partial z} \right\} \mathrm{d}x$$

$$+ \int_\Omega \overline{p} \left\{ \frac{\mathrm{d}\beta}{\mathrm{d}z} + \overrightarrow{D}_1(\sigma_2) \overrightarrow{\nabla \gamma} \right\} \mathrm{d}x + \int_\Omega {}^t \overrightarrow{\nabla \gamma} K(.,.,\sigma_2, p_2) \overrightarrow{\nabla p} \, \mathrm{d}x$$

$$+ \sum_i \int_{f_i} \overline{\sigma} v_2 [\phi'^+ \alpha^+ - \phi'^- \alpha^-] \, n_z \, \mathrm{d}\sigma - \sum_i \int_{f_i} \overline{\sigma} [\beta^+ - \beta^-] \, n_z \, \mathrm{d}\sigma$$

$$- \sum_i \int_{f_i} \overline{v} [1 - \phi^+(.,.,\sigma_1)] \alpha^+ n_z \, \mathrm{d}\sigma + \sum_i \int_{f_i} \overline{v} [1 - \phi^-(.,.,\sigma_1)] \alpha^- n_z \, \mathrm{d}\sigma - \sum_i \int_{f_i} \overline{p} \left\{ \beta^+ - \beta^- \right\} n_z \, \mathrm{d}\sigma$$

$$= \int_{\Sigma_2} \overline{\sigma} \{ \beta + \phi' v_2 (\alpha + \gamma) \} n_z \, \mathrm{d}\sigma + \int_{\Sigma_2} \overline{v} \{ (1 - \phi(.,.,\sigma_1))(\alpha + \gamma) \} \eta_z \, \mathrm{d}\sigma + \int_{\Sigma_2} \overline{p} \beta \, \eta_z \, \mathrm{d}\sigma. \quad (28)$$

Thus, it is enough to find (α, β, γ) in $X \times Y_{P_0} \times V^0$, solution to the formal dual (or transposed) problem in Ω

$$(1 - \phi(., ., \sigma_1)) \frac{d\alpha}{dz} + \frac{\partial \gamma}{\partial z} = 0, \tag{29}$$

$$\phi'\left(v_2 \frac{d\alpha}{dz} - \frac{\alpha}{h}\right) + \left(\frac{d\beta}{dz} + F'\beta\right) + \overrightarrow{D}_2(p_1)\overrightarrow{\nabla \gamma} = 0, \tag{30}$$

$$\frac{\partial \beta}{\partial z} + \overrightarrow{D}_1(\sigma_2)\overrightarrow{\nabla \gamma} - \mathrm{Div}\left\{K(., ., \sigma_2, p_2)\overrightarrow{\nabla \gamma}\right\} = \overline{p}, \tag{31}$$

with the interface condition on f_i

$$[1 - \phi^+(., ., \sigma_1)]\alpha^+ = [1 - \phi^-(., ., \sigma_1)]\alpha^-, \quad \beta^+ - \beta^- = v_2[\phi'^+\alpha^+ - \phi'^-\alpha^-], \tag{32}$$

and the boundary conditions

$$\gamma = 0 \text{ on } \Sigma_1, \quad \beta = 0, \ \alpha = -\gamma \text{ on } \Sigma_2, \quad -{}^t\overrightarrow{n}\mathrm{K}(\sigma_2, \mathrm{p}_2)\overrightarrow{\nabla \gamma} = 0 \text{ on } \partial\Omega\backslash\Sigma_1. \tag{33}$$

The dual system (29)–(31) is linear, but it is strongly coupled (by the equations, as well as by the boundary conditions).

3.6.3. *Second step: existence of a solution to the dual problem*

We are going to consider here a method by substitution. This method allows us to reduce the dual system to a linear equation concerning γ, the dual unknown of the pressure, in the framework of the classical Lax–Milgram's theorem. This theorem, which gives a sufficient condition of existence and uniqueness of a solution, can be too constraining. One can prefer Brezzi–Babuška's theorem ([7], p. 564 for example). But in our case, because of the coupling, the inf sup conditions do not seem immediate to obtain.

Let γ be fixed in V^0. By a simple quadrature of a linear differential equation on every Ω_i (and by using conditions on Σ_2 and on f_i), one obtains the existence and the uniqueness of the solution $\alpha[\gamma]$ in X to equation (29) with conditions (32) and (33).

Furthermore, the application $\alpha : V^0 \to X$, $\gamma \mapsto \alpha[\gamma]$ is linear and continuous [11].

The same work allows us to obtain a result of existence and uniqueness of $\beta[\gamma]$ in Y_{p_0} solution to

$$\frac{d\beta}{dz}[\gamma] + F'\beta[\gamma] + H[\gamma] = 0 \text{ in } \Omega \text{ where } H[\gamma] = \phi'(v_2\frac{d\alpha}{dz}[\gamma] - \frac{\alpha[\gamma]}{h}) + D_2(p_1)\nabla\gamma,$$
$$\beta^+ - \beta^- = v_2[\phi'^+\alpha^+ - \phi'^-\alpha^-] \text{ on } f_i, \qquad \beta[\gamma] = 0 \text{ on } \Sigma_2.$$

One obtains this solution in Y_{p_0} since, $\nabla p_1 \in L^p(\Omega)$, $\nabla \gamma \in [L^{p_0}(\Omega)]^2$ and $\overrightarrow{D}_2(p_1)\overrightarrow{\nabla \gamma} \in L^{q_0}(\Omega)$ where $q_0 = \dfrac{2p_0}{p_0 + 2}$.

Also, the application $\beta : V^0 \to Y_{p_0}$, $\gamma \mapsto \beta[\gamma]$ is linear and continuous [11].

We are henceforth able to give the dual variational problem and to look for a solution by means of Lax–Milgram's theorem. In order to do so, and in agreement with the modelling, we suppose that $||\phi'||_\infty, ||D_2K||_\infty$

and $||D_1K||_\infty$ are small, in a sense precised in the sequel. We note now

$$b : V^0 \times V^0 \longrightarrow \mathbb{R}$$

$$(u,v) \mapsto \int_\Omega {}^t\overrightarrow{\nabla v} K(\sigma_2, p_2) \overrightarrow{\nabla u}\, \mathrm{d}x + \int_\Omega v\, \overrightarrow{D}_1(\sigma_2) \overrightarrow{\nabla u}\, \mathrm{d}x + \int_\Omega v \frac{\mathrm{d}\beta}{\mathrm{d}z}[u]\, \mathrm{d}x + \sum_i \int_{f_i} (\beta^+[u] - \beta^-[u])\, v\, n_z\, \mathrm{d}\sigma.$$

As $\beta[u]$ and $\sigma[u]$ are linear with respect to u, b is a bilinear form.

Furthermore, as in dimension 2, the injection of $H^1(\Omega)$ in $L^{q'}(\Omega)$ is compact ($\frac{1}{q} + \frac{1}{q'} = 1$), b is continuous on $V^0 \times V^0$, for its natural topology. Thus, for any u in V^0, if $a = n_w \inf(\lambda_a, \lambda_s)$, one has

$$b(u,u) \ge a\,||\overrightarrow{\nabla u}||^2 - c(||\phi'||_\infty + ||D_2K||_\infty + ||D_1K||_\infty)\,||u||^2_{V^0},$$

and as Poincaré's inequality is valid in V^0, one gets,

$$\forall u \in V^0, \quad b(u,u) \ge a||u||^2_{V^0} - c(||\phi'||_\infty + ||D_2K||_\infty + ||D_1K||_\infty)||u||^2_{V^0}.$$

From then on, hypothesis concerning the smallness of $||\phi'||_\infty$, $||D_2K||_\infty$ and $||D_1K||_\infty$, which finds here its implicit formulation, insures the coercitivity of the form b and so, the existence of a solution to the dual system.

In conclusion, there exists (α, β, γ) in $X \times Y_{p_0} \times V^0$, verifying the equations (29) and (30) with the conditions (32) and (33), and for any φ of V^0, the variational equation,

$$\int_\Omega \varphi \left[\frac{\mathrm{d}\beta}{\mathrm{d}z}[\gamma] + \overrightarrow{D}_1(\sigma_2)\overrightarrow{\nabla\gamma} \right] \mathrm{d}x + \int_\Omega {}^t\overrightarrow{\nabla\gamma} K(\sigma_2, p_2) \overrightarrow{\nabla\varphi}\, \mathrm{d}x + \sum_i \int_{f_i} v_2(\phi'^+ \alpha^+[u] - \phi'^- \alpha^-[u])\, v\, n_z\, \mathrm{d}\sigma = \langle \overline{p}, \varphi \rangle.$$

In particular, $\varphi = \overline{p}$ in (28) leads to $||\overline{p}||^2_{L^2(\Omega)} = 0$ and to $p_1 = p_2$ a.e.

As σ and v_s are univocaly obtained for a fixed p_w, one has shown the uniqueness of the solution (σ, v_s, p_w).

3.6.4. *Alternatives*

Another approach consists in applying Fredholm's alternative as presented in [6] and [12]. This theorem, concerning the wellknown compatibility conditions of linear systems in finite dimension, becomes widespread in the case of compact operators. We are going to develop this method of uniqueness in the next section. A third one is the method of artificial compressibility where one suggests to replace the elliptic equation

$$\frac{\partial\beta}{\partial z} + \overrightarrow{D}_1(\sigma_2)\overrightarrow{\nabla\gamma} - \mathrm{Div}\left\{ K(\sigma_2, p_2)\overrightarrow{\nabla\gamma} \right\} = \overline{p},$$

by the parabolic regularization (effect of very weak compressibility)

$$\varepsilon\frac{\partial\gamma}{\partial t} + \frac{\partial\beta}{\partial z} + \overrightarrow{D}_1(\sigma_2)\overrightarrow{\nabla\gamma} - \mathrm{Div}\left\{ K(\sigma_2, p_2)\overrightarrow{\nabla\gamma} \right\} = \overline{p}.$$

On the one hand, one needs less hypotheses on the operator (for the existence of a solution) in the parabolic case than in the elliptic case. On the other hand, one needs *a priori* estimations, on the obtained solution γ_ε, and needs to pass to the limit on ε, as $\varepsilon \to 0^+$.

Finally, a last approach consists in using for this system, although linear, a fixed point method, as the one proposed in the section: existence of a solution.

It consists, for any fixed γ in V^0, in resolving in V^0

$$\frac{\partial \beta}{\partial z}[\gamma] + \overrightarrow{D}_1(\sigma_2)\overrightarrow{\nabla \theta} - \mathrm{Div}\left\{K(\sigma_2, p_2)\overrightarrow{\nabla \theta}\right\} = \overline{p},$$

and in studying the properties of Application $\mathcal{S} : \gamma \mapsto \theta$.

As the equations are linear, it is the same for $\mathcal{S}$. As the previous study proves that $\mathcal{S}$ is continuous, it is weakly sequentially continuous from V^0 into V^0. So one has to find a non empty bounded closed convex set, invariant by $\mathcal{S}$.

By summing up the study of the second step, it comes at once

$$||\theta||_{V^0} \leq c||\overline{p}|| + c(||\phi'||_\infty + ||D_2 K||_\infty + ||D_1 K||_\infty)\,||\gamma||_{V^0},$$

so that if $\eta = c(||\phi'||_\infty + ||D_2 K||_\infty + ||D_1 K||_\infty) < 1$, the looked for convex is $V^0 \cap \overline{B}(0, \frac{c||\overline{p}||}{1-\eta})$.

Even there, hypothesis concerning the smallness of certain coefficients in needed to guarantee that the invariance of a convex set is gathered.

4. A SECOND CASE: GENERAL AMPLITUDE COMPACTION

In this second approach of the discretisation, one does not need any more the smallness of certain coefficients. On the other hand, one considers that ϕ_i is a regular function of arguments x, z and σ and that the function $\phi : \phi(x, z, \sigma) = \sum_i \phi_i(x, z, \sigma)\,\mathbb{I}_{\Omega_i}(x, z)$, is continuous.

We propose, for this problem, an approach of predictive-corrective type. That is, if one still denotes by h the iterative step and σ_0 the datum of σ at the previous iteration, and if one supposes h large enough[1] to neglect $\frac{\partial \phi}{\partial t}$ (asymptotically stable system), one looks for (σ, v_s, p_w) in $W^0 \times W^{g_s} \times V^{P_{\Sigma_1}}$ solution to

$$\begin{aligned} &\frac{\partial \sigma}{\partial z} = F(.,.,\sigma_0) - \frac{\partial p_w}{\partial z}, \\ &\frac{\partial(1 - \phi(.,.,\sigma))v_s}{\partial z} = 0 \text{ on } \Omega_i, \\ &-\mathrm{div}\left\{K(.,.,\sigma,p_w)\left(\overrightarrow{\nabla p_w} - \overrightarrow{B}\right)\right\} + \frac{\partial}{\partial z}v_s = 0, \end{aligned} \tag{34}$$

with the same boundary and interfaces conditions.

After an integration of the two first equations, one calls solution to the system any (σ, v_s, p_w) in $W^0 \times W^{g_s} \times V^{P_{\Sigma_1}}$ such that:

$$\sigma(x, z) = \int_{\Sigma_1(x)}^{z} F(.,.,\sigma_0(x, s))\,ds - p_w(x, z) + P_{\Sigma_1}(x, \Sigma_1(x)), \tag{35}$$

$$v_s(x, z) = \frac{1 - \phi(x, \Sigma_1(x), 0)}{1 - \phi(x, z, \sigma(x, z))}g_s(x, \Sigma_1(x)), \tag{36}$$

$$\forall \varphi \in V^0, \quad \int_\Omega {}^t\overrightarrow{\nabla \varphi}K(.,.,\sigma,p_w)\left(\overrightarrow{\nabla p_w} - \overrightarrow{B}\right)\,\mathrm{d}x + \int_\Omega \frac{\partial v_s}{\partial z}\varphi\,\mathrm{d}x = \int_{\Sigma_2} \phi(.,.,\sigma)v_s n_z\varphi\,\mathrm{d}\sigma. \tag{37}$$

As for the first case, a fixed point method leads directly to the existence of a solution.

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[1]One works with geologic scale and h may represent a century.

4.1. Uniqueness of the solution

The method used in this section is closed to the one already proposed in Section 3.6.

Using the same calculus as in the first step of this duality method, one is led to look for (α, β, γ) *a priori* in $X \times Y_{p_0} \times V^0$ solution to System

$$(1 - \phi(.,.,\sigma_1))\frac{d\alpha}{dz} + \frac{\partial\gamma}{\partial z} = 0,$$

$$\frac{d\beta}{dz} + \phi' v_2 \frac{d\alpha}{dz} + \overrightarrow{D_2}(p_1)\overrightarrow{\nabla\gamma} = 0,$$

$$\frac{\partial\beta}{\partial z} - \text{Div}(K(.,.,\sigma_2,p_2)\overrightarrow{\nabla\gamma}) + \overrightarrow{D_1}(\sigma_2)\overrightarrow{\nabla\gamma} = \overline{p},$$

with the interface conditions on f_i

$$[1 - \phi^+(.,.,\sigma_1)]\alpha^+ = [1 - \phi^-(.,.,\sigma_1)]\alpha^-, \quad \beta^+ - \beta^- = v_2[\phi'^+ \alpha^+ - \phi'^- \alpha^-],$$

and the boundary conditions ([3])

$$\gamma = 0 \text{ on } \Sigma_1,$$

$$\beta = 0, \quad \alpha = -\gamma \text{ on } \Sigma_2,$$

$$-{}^t\overrightarrow{n} K(.,.,\sigma_2,p_2)\nabla\gamma = 0 \text{ on } \partial\Omega\backslash\Sigma_1.$$

But as $\phi^+ = \phi^-$ and $\phi'^+ = \phi'^-$ (ϕ is supposed continuous), the interface conditions on f_i become $\alpha^+ = \alpha^-$ and $\beta^+ = \beta^-$.

From then on, α and β are looked for in W and an algebraic manipulation leads us to look for γ in V^0 solution to

$$-\text{Div}\left(K(.,.,\sigma_2,p_2)\overrightarrow{\nabla\gamma}\right) + \overrightarrow{D_1}(\sigma_2)\overrightarrow{\nabla\gamma} + \frac{\phi' v_2}{1 - \phi(.,.,\sigma_1)}\frac{\partial\gamma}{\partial z} - \overrightarrow{D_2}(p_1)\overrightarrow{\nabla\gamma} = \overline{p},$$

i.e.

$$-\text{Div}(A\overrightarrow{\nabla\gamma}) + \overrightarrow{\mathfrak{B}}.\overrightarrow{\nabla\gamma} = \overline{p},$$

with the boundary conditions of edge

$$\gamma = 0 \text{ on } \Sigma_1, \quad -{}^t\overrightarrow{n} K(.,.,\sigma_2,p_2)\overrightarrow{\nabla\gamma} = 0 \text{ on } \partial\Omega\backslash\Sigma_1.$$

Then, the second step is based on an application of Fredholm's alternative.

To conclude, we consider the following family of bilinear forms $(a_\lambda)_{\lambda \geq 0}$ on $V \times V$

$$a_\lambda : V \times V \to \mathbb{R}, \quad (u,v) \mapsto \int_\Omega {}^t\overrightarrow{\nabla u} A \overrightarrow{\nabla v} \, dx + \int_\Omega \overrightarrow{\mathfrak{B}}.\overrightarrow{\nabla u}\, v \, dx + \lambda \int_\Omega uv \, dx,$$

and a instead of a_0. We notice that as $\overrightarrow{\mathfrak{B}} \in (L^{p_0}(\Omega))^2$, for any u in V, $\overrightarrow{\mathfrak{B}}.\overrightarrow{\nabla u} \in L^{q_0}(\Omega)$. So thanks to the continuous injection of V in $L^{q'_0}(\Omega)$, the trilinear term $\int_\Omega \overrightarrow{\mathfrak{B}}.\overrightarrow{\nabla u}\, v \, dx$ has a sense for any v of V and a_λ is continuous. Furthermore, one shows that there is a value of λ such that a_λ is V^0-elliptic. So, for any f in $L^2(\Omega)$, there exists a unique u in V^0 such that

$$\forall v \in V, \quad a_\lambda(u,v) = \int_\Omega fv \, dx. \tag{38}$$

From then on, a classic application of Fredholm's alternative [6] allows us to show that the problem: find u in V^0 such that

$$\forall v \in V, \quad a(u,v) = \int_\Omega fv \, \mathrm{d}x, \tag{39}$$

admits a solution, moreover unique, if the trivial solution is the only solution to (39) when $f = 0$.

This last result ensues from the weak maximum principle [12], generalized in $\Omega \subset \mathbb{R}^N$, as follows:

In a first step, one shows that u is upper bounded. We consider k such that $0 \leq k < \gamma = \sup_\Omega ess \, u$ (possibly infinite) and the test function $v_k = (u-k)^+$ in (39). It follows the estimation,

$$c||\overrightarrow{\nabla v_k}||^2_{L^2(\Omega)^N} \leq ||\overrightarrow{\mathfrak{B}}||_{L^{p_0}(\Omega)}) \, ||\overrightarrow{\nabla v_k}||_{L^2(\Omega)^N} ||v_k||_{L^{q'_0}(\Omega)},$$

and, since by construction v_k is not null, it comes, for a big enough value of p^*,

$$||\overrightarrow{\nabla v_k}||_{L^2(\Omega)^N} \leq C \, ||v_k||_{L^{q'_0}(\Omega)} \leq C \, [\text{meas}(\text{supp}\, v_k)]^{\frac{p^*-q_0}{p^* q'_0}} ||v_k||_{L^{p^*}(\Omega)},$$

where C represents an arbitrary constant. Finally, the countinuous injection of V^0 in $L^{p^*}(\Omega)$ leads to

$$||\overrightarrow{\nabla v_k}||_{L^2(\Omega)^N} \leq C[\text{meas}(\text{supp}\, v_k)]^{\frac{p^*-q_0}{p^* q'_0}} ||\overrightarrow{\nabla v_k}||_{L^2(\Omega)^N}.$$

So $\mathcal{L}^N - \text{meas}(\text{supp}\, v_k) \geq C > 0$, constant independent of k ($||\overrightarrow{\nabla v_k}||_{L^2(\Omega)^N} > 0$). By noticing that $\text{supp}\, v_k \searrow \text{supp}\, v_\gamma$ when $k \nearrow \gamma$, it ensues from Beppo–Lévi's theorem that

$$\text{meas}(\text{supp}\, v_\gamma) > 0.$$

So, there is a non negligeable $\mathcal{L}^N$-measurable subset Ω_0 of Ω, such that $u = \gamma$ a.e. in Ω_0. Moreover, since u in $\mathcal{L}^N$-integrable on Ω, $\gamma < +\infty$.

Second step consists in showing that u is non positive $\mathcal{L}^N$-a.e. in Ω. In order to do so, let us consider, for any $\varepsilon > 0$, the test function $v_\varepsilon = \frac{u^+}{\gamma + \varepsilon - u^+}$. From then on, it comes

$$\int_\Omega \frac{\gamma + \varepsilon}{(\gamma + \varepsilon - u^+)^2} {}^t\overrightarrow{\nabla u^+} A \, \overrightarrow{\nabla u^+} \, \mathrm{d}x + \int_\Omega \overrightarrow{\mathfrak{B}}.\overrightarrow{\nabla u^+} \frac{u^+}{\gamma + \varepsilon - u^+} \, \mathrm{d}x = 0,$$

and

$$c \int_\Omega \frac{|\overrightarrow{\nabla u^+}|^2}{(\gamma + \varepsilon - u^+)^2} \, \mathrm{d}x \leq \int_\Omega |\overrightarrow{\mathfrak{B}}| \frac{|\overrightarrow{\nabla u^+}|}{\gamma + \varepsilon - u^+} \frac{u^+}{\gamma + \varepsilon} \, \mathrm{d}x.$$

One observes that, $0 \leq \frac{u^+}{\gamma + \varepsilon} \leq 1$. So, if one notes $w_\varepsilon = \ln\left[\frac{\gamma + \varepsilon}{\gamma + \varepsilon - u^+}\right]$, then $w_\varepsilon \in V^0$, the chain rule gives $\overrightarrow{\nabla w_\varepsilon} = \frac{\overrightarrow{\nabla u^+}}{\gamma + \varepsilon - u^+}$ and one gets

$$c \int_\Omega |\overrightarrow{\nabla w_\varepsilon}|^2 \, \mathrm{d}x \leq \int_\Omega |\overrightarrow{\mathfrak{B}}| \, |\overrightarrow{\nabla w_\varepsilon}| \, \mathrm{d}x \leq ||\overrightarrow{\mathfrak{B}}||_{L^2(\Omega)} \, ||\overrightarrow{\nabla w_\varepsilon}||_{L^2(\Omega)^N}.$$

In conclusion, by means of Poincaré's inequality, for any $\varepsilon > 0$, $||w_\varepsilon||_{L^2(\Omega)} \leq c ||\overrightarrow{\nabla w_\varepsilon}||_{L^2(\Omega)^N} \leq C$ (constant independent of ε).

If one supposes, by contradiction, the negation of the assertion: $u \leq 0$ a.e., $\gamma > 0$ and it ensues directly that

$$\left(\ln[\frac{\gamma + \varepsilon}{\varepsilon}]\right)^2 \mathrm{meas}(\Omega_0) = \int_{\Omega_0} \left(\ln[\frac{\gamma + \varepsilon}{\varepsilon}]\right)^2 \mathrm{d}x$$

$$= \int_{\Omega_0} \left(\ln[\frac{\gamma + \varepsilon}{\gamma + \varepsilon - u^+}]\right)^2 \mathrm{d}x$$

$$\leq ||w_\varepsilon||_{L^2(\Omega)}^2 \leq C \text{ (independent of } \varepsilon).$$

This situation is impossible since $\left(\ln\left[\frac{\gamma + \varepsilon}{\varepsilon}\right]\right)^2$ converges towards $+\infty$ if ε converges to 0^+. Thus, $\gamma \leq 0$ and by definition of γ, $u \leq 0$ a.e. in Ω.

The last step represents the conclusion. It is enough to notice, because of the linear character of the homogeneous equation, that if u is a solution, $-u$ is also a solution. Thus $u \leq 0$ and $u \geq 0$, *i.e.* $u = 0$.

Thus, $f = \overline{p}$ in (39) leads to the uniqueness of the solution to the primal problem, as it was presented in the study of the first case.

5. THE CASE OF DIMENSION 3: CONTROL OF p_0

It is possible to give the same study for a domain of $\mathbb{R}^3$ of the shape

$$\Omega = \left\{(x, y, z) \in \mathbb{R}^3, \ \alpha < x < \beta, \ \varsigma < y < \tau, \ \gamma_1(x, y) < z < \gamma_2(x, y)\right\},$$

where γ_i are Lipschitzian functions. Problems are concentrated around the possibility of achieving the technique of uniqueness. In particular, the treatment of $\int_\Omega w \overrightarrow{D}_1(\sigma_2) \overrightarrow{\nabla \gamma} \, \mathrm{d}x$ when $w \in V^0$. That is to find again the result of Proposition 3.3 and more exactly, to know (for technical reasons) if $p_0 > 3$ is possible.

If one resumes the demonstration of this result, such as proposed in [19], one looks, page 46, for a value of $p_0 > 2$, close to 2, such as, according to the notations of the context: $A_q^{1-\alpha(p_0)} \frac{\lambda_2 - \lambda_1}{\lambda_2 + \lambda_1} n^{\frac{1}{2} - \frac{1}{p_0}} \leq 1$.

Another method consists, for a fixed value of p_0 ($p_0 > 3$ for example), to suppose that λ_1 is close to λ_2. That is, the conditioning of the matrice $(a_{ij})_{ij}$, for the matrix norm subordinate to the euclidian norm, is close to 1.

However, in practice, the relative value of coefficients λ_s and λ_a which specify Kozeny–Carman's tensor of permeability in relation (8) limits the reach of this theoretical remark. The control of p_0, according to the spirit of the method, requires that the elliptic problem with symmetric and bounded coefficients is a "small" disturbance of Poisson's equation. What is not always turned out (case of strong anisotropy). As soon as the studied environment gives an importance to a space direction (in this particular case, the models of sedimentary basins enters in this frame *via* the hypothesis of vertical line compaction), the control of p_0 is very checked. In practice, p_0 remain very close to 2^+ (*cf.* [4]).

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