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Partial differential equations

Blow-up dynamics for the hyperbolic vanishing mean curvature flow of surfaces asymptotic to a Simons cone

Sur la formation de singularités pour le flot hyperbolique de courbure moyenne nulle de surfaces asymptotiques au cône de Simons

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ABSTRACT

In this paper, we establish the existence of a family of surfaces $(\Gamma(t))_{0<t\leq T}$ that evolve by the vanishing mean curvature flow in Minkowski space and, as t tends to 0, blow up towards a surface that behaves like the Simons cone at infinity. This issue amounts to investigate the singularity formation for a second-order quasilinear wave equation. Our constructive approach consists in proving the existence of a finite-time blow-up solution to this hyperbolic equation under the form $u(t,x) \sim t^{\nu+1} Q\left(\frac{x}{t^{\nu+1}}\right)$, where Q is a stationary solution and ν is an irrational number strictly larger than $1/2$. Our strategy roughly follows that of Krieger, Schlag and Tataru in [7–9]. However, contrary to these articles, the equation to be handled in this work is quasilinear. This induces a number of difficulties to face.

RÉSUMÉ

Dans cette note, on établit l'existence d'une famille de surfaces $(\Gamma(t))_{0<t\leq T}$ qui évoluent sous le flot de courbure moyenne nulle dans l'espace de Minkowski et qui explosent lorsque t tend vers 0 vers une surface asymptotique au cône de Simons à l'infini. Ce problème revient à étudier la formation de singularités pour une équation d'ondes quasi-linéaire du second ordre. Notre approche constructive consiste à démontrer l'existence de solutions à cette équation hyperbolique explosant en temps fini sous la forme $u(t,x) \sim t^{\nu+1} Q\left(\frac{x}{t^{\nu+1}}\right)$, où Q est une solution stationnaire et $\nu > 1/2$ est un nombre irrationnel. Notre démarche s'inspire de celle de Krieger, Schlag et Tataru dans [7–9]. Cependant contrairement à ces travaux, l'équation en question dans cette note est quasi-linéaire, ce qui génère des difficultés que l'on doit surmonter.

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On s'intéresse dans cette note à la question de la formation de singularités pour des surfaces qui évoluent sous le flot hyperbolique de courbure moyenne nulle et qui sont asymptotiques au cône de Simons à l'infini. Cette question se ramène à l'étude des phénomènes d'explosion en temps fini d'une équation d'ondes quasi-linéaire du second ordre. Un résultat dans cette direction a été réalisé dans [15] dans le cadre parabolique. Le cône de Simons défini comme suit :

$$C_n = \left\{X = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n}, x_1^2 + \cdots + x_n^2 = x_{n+1}^2 + \cdots + x_{2n}^2\right\},$$

en lien avec le problème de Bernstein, est une surface minimisante uniquement dans le cas où $n \geq 4$ (voir, par exemple, [3,13]). Il est clair qu'il est invariant par l'action du groupe $O(n) \times O(n)$, où $O(n)$ est le groupe orthogonal de $\mathbb{R}^n$ et peut être paramétré de la manière suivante :

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \to (\rho\omega_1, \rho\omega_2).$$

L'équation de surface minimale en géométrie riemannienne a un analogue hyperbolique dans le cadre lorentzien. En particulier, dans l'espace de Minkowski $\mathbb{R}^{1,2n}$ muni de la métrique $\mathrm{d}g = -\mathrm{d}t^2 + \sum_{j=1}^n \mathrm{d}x_j^2$, si on considère des surfaces de type temps qui, pour t fixé, ont une paramétrisation de la forme

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \to (x, u(t,x)\omega) \in \mathbb{R}^{2n}, \tag{1}$$

où u est une fonction strictement positive, on aboutit à l'équation d'ondes quasi-linéaire d'ordre deux suivante :

$$\begin{aligned}(\mathrm{NW})\, u := {} & (1 + |\nabla u|^2)\, u_{tt} - (1 - (u_t)^2 + |\nabla u|^2)\, \Delta u \\ & + \sum_{j,k=1}^n u_{x_j} u_{x_k} u_{x_j x_k} - 2u_t(\nabla u \cdot \nabla u_t) + \frac{(n-1)}{u}(1 - (u_t)^2 + |\nabla u|^2) = 0\cdot\end{aligned} \tag{2}$$

Comme on s'intéresse ici aux surfaces de type temps qui sont asymptotiques à l'infini au cône de Simons dans le cas où $n = 4$, on introduit, pour tout entier $L > 4$, l'espace X_L des fonctions (u_0, u_1) telles que $\nabla(u_0 - Q)$ et u_1 appartiennent à $H^{L-1}(\mathbb{R}^4)$, $\inf u_0 > 0$ et $\inf(1 + |\nabla u_0|^2 - (u_1)^2) > 0$. Par des arguments classiques qu'on peut trouver, par exemple, dans [1,5,14], on peut montrer que, si $(u_0, u_1) \in X_L$, avec $L > 4$, alors le problème de Cauchy

$$\begin{cases} (\mathrm{NW})\, u = 0 \\ u_{|t=0} = u_0 \\ (\partial_t u)_{|t=0} = u_1, \end{cases}$$

admet une unique solution maximale sur $[0, T^*[$ telle que

$$(u, \partial_t u) \in \mathcal{C}([0, T^*[, X_L).$$

Notre propos dans cette note est de construire dans le cas où $n = 4$ une solution de l'équation d'ondes quasi-linéaire ci-dessus sous la forme $u(t,x) \sim t^{\nu+1} Q\left(\frac{x}{t^{\nu+1}}\right)$, qui explose à l'instant $t = 0$, où Q est une solution stationnaire et $\nu > 1/2$ est un nombre irrationnel. Le résultat d'explosion que nous obtenons dans ce travail est le suivant.

Théorème 1. *Pour tout nombre irrationnel $\nu > \frac{1}{2}$ et tout $\delta > 0$ suffisamment petit, il existe $T > 0$ et $u(t, \cdot)$ une solution radiale de* (NW) *sur l'intervalle de temps* $(0, T]$ *telle que*

$$(u, \partial_t u) \in \mathcal{C}((0, T], X_{L_0}) \ \textit{avec}\ L_0 := 2M + 1, \ \ M = \left[\frac{3}{2}\nu + \frac{5}{4}\right]$$

et qui explose à l'instant $t = 0$ par concentration du profil du soliton : il existe deux fonctions radiales $\mathfrak{g}_0 \in \dot{H}^{s+1}(\mathbb{R}^4)$ et $\mathfrak{g}_1 \in \dot{H}^s(\mathbb{R}^4)$, pour tout $0 \leq s < 3\nu + 2$, telles qu'on ait

$$\begin{aligned} u(t,x) &= t^{\nu+1} Q\left(\frac{x}{t^{\nu+1}}\right) + \mathfrak{g}_0(x) + \eta(t,x), \\ u_t(t,x) &= \mathfrak{g}_1(x) + \eta_1(t,x), \end{aligned}$$

avec

$$\|\nabla \eta(t,\cdot)\|_{H^2(\mathbb{R}^4)} + \|\eta_1(t,\cdot)\|_{H^2(\mathbb{R}^4)} \xrightarrow{t \to 0} 0.$$

De plus, en posant

$$u(t,x) = t^{\nu+1}\Big(Q\Big(\frac{x}{t^{\nu+1}}\Big) + \zeta\Big(t, \frac{x}{t^{\nu+1}}\Big)\Big),$$
$$u_t(t,x) = \zeta_1\Big(t, \frac{x}{t^{\nu+1}}\Big),$$

on a :

$$\|\nabla\zeta(t,\cdot)\|_{\dot H^s(\mathbb{R}^4)} + \|\zeta_1(t,\cdot)\|_{\dot H^s(\mathbb{R}^4)} \xrightarrow{t\to 0} 0\,, \forall\, 2 < s \le L_0 - 1\,.$$

En outre, $\mathfrak{g}_0$, $\mathfrak{g}_1$ *sont à support compact, appartiennent à* $\mathcal{C}^\infty(\mathbb{R}^4 \setminus \{0\})$ *et vérifient pour tout* $0 \le s < 3\nu + 2$

$$\|\nabla\mathfrak{g}_0\|_{H^s(\mathbb{R}^4)} + \|\mathfrak{g}_1\|_{H^s(\mathbb{R}^4)} \le C_s\,\delta^{3\nu+2-s}\,,$$
$$\mathfrak{g}_0(x) \sim c_0|x|^{3\nu+1}\,,\ \mathfrak{g}_1(x) \sim c_1|x|^{3\nu}\,,\ \textit{lorsque}\ x \to 0\,,$$

où c_0 *et* c_1 *sont des constantes strictement positives dépendant de* ν.

La preuve de ce résultat repose en premier lieu sur la construction d'une solution approchée $u^{(N)}$ de l'équation d'ondes quasi-linéaire (NW) comme perturbation du profile du soliton $t^{\nu+1}Q\big(\frac{x}{t^{\nu+1}}\big)\cdot$ La seconde étape consiste, par un argument perturbatif, à en déduire une solution exacte u vérifiant la conclusion du théorème.

1. Introduction

In this work, we address the question of singularity formation for the hyperbolic vanishing mean curvature flow of surfaces that are asymptotic to the Simons cone at infinity. The Simons cones, which are linked to Bernstein's problem (we refer for instance to [3,13] and the references therein for an overview on the subject), are defined as follows

$$C_n = \big\{X = (x_1, \cdots, x_{2n}) \in \mathbb{R}^{2n}\,, x_1^2 + \cdots + x_n^2 = x_{n+1}^2 + \cdots + x_{2n}^2\big\}. \tag{3}$$

It is clear that the Simons cone, which has dimension $d = 2n - 1$, is invariant under the action of the group $O(n) \times O(n)$, where $O(n)$ is the orthogonal group of $\mathbb{R}^n$, and that can be parametrized in the following way:

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \to (\rho\omega_1, \rho\omega_2)\,.$$

By the works [3,15], it is known that, for $n \ge 4$, the complementary of C_n (which has two connected components $|x| < |y|$ and $|y| < |x|$) is foliated by two families of smooth minimal surfaces $(M_a)_{a>0}$ and $(\widetilde{M}_a)_{a>0}$, which are asymptotic to the Simons cone at infinity. These families of surfaces are the scaling invariant: $M_a = aM$ and $\widetilde{M}_a = a\widetilde{M}$, with M and $\widetilde{M}$ admitting respectively the following parametrization:

$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (\rho\omega_1, Q(\rho)\omega_2) \in \mathbb{R}^{2n}\,, \tag{4}$$
$$\mathbb{R}_+ \times \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \ni (\rho, \omega_1, \omega_2) \mapsto (Q(\rho)\omega_1, \rho\omega_2) \in \mathbb{R}^{2n}, \tag{5}$$

where Q is a positive radial function which belongs to $\mathcal{C}^\infty(\mathbb{R}^n)$ and satisfies $Q(0) = 1$, $Q(\rho) > \rho$ for any $\rho > 0$, and $Q(\rho) = \rho + \frac{d_\alpha}{\rho^\alpha}\big(1 + \circ(1)\big)$ as ρ tends to infinity, with d_α some positive constant and

$$\alpha = -1 + \frac{1}{2}\Big((2n-1) - \sqrt{(2n-1)^2 - 16(n-1)}\Big).$$

In particular, in the case when $n = 4$, we have, as ρ tends to infinity, with d_2 some positive constant:

$$Q(\rho) = \rho + \frac{d_2}{\rho^2}\big(1 + \circ(1)\big)\cdot \tag{6}$$

The minimal surface equation in Riemannian geometry has a natural hyperbolic analogue in the Lorentzian framework. In particular, working in the Minkowski space $\mathbb{R}^{1,2n}$ equipped with the Lorentzian metric, and considering the time-like surfaces that for fixed t are parametrized under the form (1) with some positive function u, lead to the quasilinear second-order wave equation (2). Note that this equation is invariant by the scaling $u_a(t,x) = a\,u\big(\frac{t}{a}, \frac{x}{a}\big)$ in the sense that, if u solves (2), then u_a is also a solution to (2). In the framework of Sobolev spaces, $\dot H^{\frac{n+2}{2}}(\mathbb{R}^n)$ is invariant under this scaling.[1]

[1] All along this note, we shall denote by $\dot H^s(\mathbb{R}^n)$ the homogeneous Sobolev space. We refer to [1] and the references therein for all necessary definitions and properties of those spaces.

In this note, we are interested in time-like surfaces of the form (1) that are asymptotic to the Simons cone as $|x|$ tends to infinity in the case when $n = 4$. To take care of this behavior, we introduce the functional spaces X_L, $L > 4$, which we define as being the set of functions (u_0, u_1) such that $\nabla(u_0 - Q)$ and u_1 belong to $H^{L-1}(\mathbb{R}^4)$, and which satisfy

$$\inf u_0 > 0\,, \inf(1 + |\nabla u_0|^2 - (u_1)^2) > 0\,. \tag{7}$$

By classical arguments, one can prove that the Cauchy problem for the quasilinear wave equation (2) is locally well posed in X_L provided that $L > 4$. More precisely, one has the following theorem.

Theorem 1. *Consider the Cauchy problem*

$$\begin{cases} (\mathrm{NW})\, u = 0 \\ u_{|t=0} = u_0 \\ (\partial_t u)_{|t=0} = u_1\,. \end{cases} \tag{8}$$

Assume that the Cauchy data (u_0, u_1) belongs to the functions class X_L, with $L > 4$, then there exists a unique maximal solution u to (8) on $[0, T^*[$ such that, for any T in $[0, T^*[$,

$$(u, \partial_t u) \in \mathcal{C}([0, T], X_L)\,. \tag{9}$$

In what follows, we assume that u is radial, which implies that, for fixed t, the surfaces defined by (1) are invariant under the action of the group $O(4) \times O(4)$. We readily check that, in that case, the function u satisfies the following equation:

$$(1 + u_\rho^2)\, u_{tt} - (1 - u_t^2)\, u_{\rho\rho} - 2u_t u_\rho u_{\rho t} + 3(1 + u_\rho^2 - u_t^2)\Big(\frac{1}{u} - \frac{u_\rho}{\rho}\Big) = 0\,\cdot \tag{10}$$

Note that the Simons cone and the minimal surfaces M_a are stationary solutions to our model of surfaces (1), with $u(t, \rho) = \rho$ in the case of the Simons cone, and $u(t, \rho) = Q_a(\rho)$, $Q_a(\rho) = aQ\left(\frac{\rho}{a}\right)$ in the case of M_a.

The question we address here is that of blow up, i.e. the description of possible singularities that smooth hypersurfaces may develop as they evolve by the Minkowski zero mean curvature flow. There is by now a considerable literature dealing with the construction of type-II blow up solutions for semilinear heat, wave and Schrödinger-type equations both in critical and supercritical cases (among others, one can consult [4,10,6–9,11,12] and the references therein). The viewpoint we adopt in this work is the one that has been initiated by Krieger, Schlag, and Tataru in [9] where, for the energy critical focusing semilinear wave equation, they constructed type-II blow-up solutions with a continuum of blow-up rates that become singular via a concentration of a stationary state profile. The goal of the present paper is to show that this blow-up mechanism exists as well for the quasilinear wave equation (10). Our main result is given by the following theorem.

Theorem 2. *For all irrational numbers $\nu > \frac{1}{2}$ and any $\delta > 0$ sufficiently small, there exists a positive time T and a radial solution $u(t, \cdot)$ to (10) on the interval $(0, T]$ such that*

$$(u, \partial_t u) \in \mathcal{C}((0, T], X_{L_0}) \text{ with } L_0 := 2M + 1\,,\ M = \Big[\frac{3}{2}\nu + \frac{5}{4}\Big], \tag{11}$$

such that it blows up at time $t = 0$ by concentrating the soliton profile: there exist two radial functions $\mathfrak{g}_0 \in \dot{H}^{s+1}(\mathbb{R}^4)$, $\mathfrak{g}_1 \in \dot{H}^{s}(\mathbb{R}^4)$ for any $0 \le s < 3\nu + 2$, such that one has

$$\begin{aligned} u(t, x) &= t^{\nu+1} Q\Big(\frac{x}{t^{\nu+1}}\Big) + \mathfrak{g}_0(x) + \eta(t, x)\,, \\ u_t(t, x) &= \mathfrak{g}_1(x) + \eta_1(t, x), \end{aligned}$$

with

$$\|\nabla \eta(t, \cdot)\|_{H^2(\mathbb{R}^4)} + \|\eta_1(t, \cdot)\|_{H^2(\mathbb{R}^4)} \xrightarrow{t \to 0} 0\,.$$

Moreover, writing

$$\begin{aligned} u(t, x) &= t^{\nu+1}\Big(Q\Big(\frac{x}{t^{\nu+1}}\Big) + \zeta\Big(t, \frac{x}{t^{\nu+1}}\Big)\Big), \\ u_t(t, x) &= \zeta_1\Big(t, \frac{x}{t^{\nu+1}}\Big), \end{aligned}$$

we have

$$\|\nabla \zeta(t, \cdot)\|_{\dot{H}^s(\mathbb{R}^4)} + \|\zeta_1(t, \cdot)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t \to 0} 0\,, \forall 2 < s \le L_0 - 1\,.$$

Besides, $\mathfrak{g}_0$, $\mathfrak{g}_1$ *are compactly supported, belong to* $C^\infty(\mathbb{R}^4 \setminus \{0\})$ *and verify for all* $0 \le s < 3\nu + 2$

$$\|\nabla \mathfrak{g}_0\|_{H^s(\mathbb{R}^4)} + \|\mathfrak{g}_1\|_{H^s(\mathbb{R}^4)} \le C_s\, \delta^{3\nu+2-s}\,,$$

$$\mathfrak{g}_0(x) \sim \frac{d_2}{3\nu+4} |\sqrt{2}\,x|^{3\nu+1}\,,\ \mathfrak{g}_1(x) \sim d_2 |\sqrt{2}\,x|^{3\nu}\,,\ as\, x \to 0\,,$$

where d_2 *denotes the constant involved in* (6)*.*

Remark 1. Combining Theorem 2 with the asymptotic (6), we readily gather that the blow-up solution u to (10) given by Theorem 2 satisfies

1. $\|\nabla(u(t,\cdot) - Q)\|_{L^\infty((0,T],\dot{H}^s(\mathbb{R}^4))} \lesssim 1\,, \forall\, 0 \le s < 2,$
2. $\|\nabla(u(t,\cdot) - |x| - \mathfrak{g}_0)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t\to 0} 0\,, \forall\, 0 \le s < 2,$
3. $\|\nabla(u(t,\cdot) - Q)\|_{\dot{H}^s(\mathbb{R}^4)} \xrightarrow{t\to 0} \infty\,, \forall\, 2 \le s \le L_0 - 1\cdot$

2. Ideas of the proof

We just give some basic ideas of our strategy, and we refer the reader to [2] for further details. The proof of Theorem 1 is inspired by the classical theory of strictly hyperbolic equations that can be found, for instance, in [1,5,14]. The proof of Theorem 2 is done in two main steps. The first step is dedicated to the construction of an approximate solution as a perturbation of the concentrating soliton profile $t^{\nu+1} Q\left(\frac{\rho}{t^{\nu+1}}\right)$, where $\nu > 1/2$ is a fixed irrational number. The second step is to complement this approximate solution to an actual solution u by a perturbative argument.

To build the approximate solution, we analyze separately the three regions that correspond to three different space scales: the inner region corresponding to $\frac{\rho}{t} \le t^{\epsilon_1}$, the self-similar region corresponding to the region $\frac{1}{10} t^{\epsilon_1} \le \frac{\rho}{t} \le 10 t^{-\epsilon_2}$, and finally the remote region defined by $\frac{\rho}{t} \ge t^{-\epsilon_2}$, where $\epsilon_1 < \nu$ and $\epsilon_2 < 1$ are two fixed positive real numbers. The inner region is the region where the blow up concentrates. In this region, the solution will be constructed as a perturbation of the profile $t^{\nu+1} Q\left(\frac{\rho}{t^{\nu+1}}\right)\cdot$ Here the asymptotic behavior of the soliton Q at infinity that can be easily checked and the properties of the linearized operator of (10) around Q are essential. In that region, we look for an approximate solution as a power expansion in $t^{2\nu}$ of the form:

$$u_{\rm in}^{(N)}(t,\rho) = t^{\nu+1} \sum_{k=0}^{N} t^{2\nu k} V_k\Big(\frac{\rho}{t^{\nu+1}}\Big)\,, \tag{12}$$

where $V_0 = Q$, and where the functions V_k, for $1 \le k \le N$, are obtained recursively, by solving a recurrent system. It turns out that these functions V_k grow at infinity as follows:

$$V_k(y) = \sum_{\ell=0}^{k} (\log y)^\ell \sum_{n \ge 2-2(k-\ell)} d_{n,k,\ell}\, y^{-n}\,.$$

Thus, to obtain a good approximate solution, we are then constrained to restrict the construction under this form to the region $\frac{\rho}{t} \le t^{\epsilon_1}$.

In the self-similar region, we extend the approximate solution given by (12). In that region, the profile of the solution, which is determined uniquely by the matching conditions coming out of the inner region, is a perturbation of ρ built under the form

$$u_{\rm ss}^{(N)}(t,\rho) = \rho + \lambda(t) \sum_{k=3}^{N} t^{\nu k} \sum_{\ell=0}^{\ell(k)} \big(\log t\big)^\ell\, w_{k,\ell}\Big(\frac{\rho}{\lambda(t)}\Big)\,, \tag{13}$$

where $\lambda(t)$ is a suitable function that behaves like t near $t = 0$, and where the functions $w_{k,\ell}$ are determined by induction using again the quasilinear wave equation (10). Actually, the natural idea is rather to look for $u_{\rm ss}^{(N)}$ under the form (13), where $\lambda(t)$ is replaced by t. However, it turns out that the operator coming to play in this region, which is essentially the linearized of (10) around ρ written with respect to the variable $(t, \frac{\rho}{t})$, is degenerate on the corresponding light cone $\frac{\rho}{t} = \frac{1}{\sqrt{2}}$, which induces a loss of regularity at each step. To overcome this difficulty linked to the quasilinear character of the equation to be dealt with, we construct concurrently the solution with the real light cone involved, which is rather given by $\frac{\rho}{\lambda(t)} = \frac{1}{\sqrt{2}}\cdot$ For that purpose, we will require that the function $\lambda(t)$ defining the light cone associated with the quasilinear wave equation (10) (which is a perturbation of t, for t small) satisfies a condition that implies the vanishing on $\frac{\rho}{\lambda(t)} = \frac{1}{\sqrt{2}}$ of some coefficient in the equation at hand.

In the remote region, we construct an approximate solution $u_{\text{out}}^{(N)}$ which extends $u_{\text{ss}}^{(N)}$ by solving the quasilinear wave equation (10) associated with an adapted Cauchy data in that region. More precisely, we look for the approximate solution in the remote region under the following form:

$$u_{\text{out}}^{(N)}(t,\rho) = \rho + \mathfrak{g}_0(\rho) + t\mathfrak{g}_1(\rho) + \sum_{k=2}^{N} t^k \mathfrak{g}_k(\rho)\,, \tag{14}$$

where the Cauchy data $(\cdot + \mathfrak{g}_0, \mathfrak{g}_1)$ is determined by the matching conditions coming out of the self-similar region, and where for $k \geq 2$ the functions $\mathfrak{g}_k$ are determined successively by solving a recurrent relation. Actually, these functions $\mathfrak{g}_k$, $k \geq 0$, are compactly supported and behave as $\rho^{1-k+3\nu}$ close to 0, which ensures that (14) provides us with a good approximate solution in the remote region, for N sufficiently large.

Finally to determine the blow up solution u to (10), we write

$$u = u^{(N)} + \varepsilon^{(N)},$$

where $u^{(N)}$ is the approximate solution built in the first step, and then we derive the equation satisfied by the remainder term $\varepsilon^{(N)}$ with respect to the variable $\big(t, \frac{\rho}{t^{1+\nu}}\big)\cdot$ The study of the equation for $\varepsilon^{(N)}$ is based on continuity arguments coupled with suitable energy estimates.

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