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Probability theory

Kolmogorov distance between the exponential functionals of fractional Brownian motion

Distance de Kolmogorov entre les fonctionnelles exponentielles du mouvement brownien fractionnaire

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A R T I C L E   I N F O

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A B S T R A C T

In this note, we investigate the continuity in law with respect to the Hurst index of the exponential functional of the fractional Brownian motion. Based on the techniques of Malliavin's calculus, we provide an explicit bound on the Kolmogorov distance between two functionals with different Hurst indexes.

R É S U M É

Dans cette Note, nous étudions la continuité en loi relativement à l'indice de Hurst des fonctionnelles exponentielles du mouvement brownien fractionnaire. En nous reposant sur les techniques du calcul de Malliavin, nous donnons des bornes explicites de la distance de Kolmogorov entre deux fonctionnelles d'indices de Hurst différents.

1. Introduction

Let $B^H = (B_t^H)_{t\in[0,T]}$ be a fractional Brownian motion (fBm) with Hurst index $H \in (0,1)$. We recall that fBm admits the Volterra representation

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s, \tag{1.1}$$

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where $(W_t)_{t\in[0,T]}$ is a standard Brownian motion and for some normalizing constants c_H and c'_H, the kernel K_H is given by $K_H(t,s) = c_H s^{1/2-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, \mathrm{d}u$ if $H > \frac{1}{2}$ and

$$K_H(t,s) = c_H\left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}} - (H-\frac{1}{2})\int_s^t \frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}\,\mathrm{d}u\right] \text{ if } H < \frac{1}{2}.$$

Given real numbers a and σ, we consider the exponential functional of the form

$$F_H = \int_0^T \mathrm{e}^{as+\sigma B_s^H}\,\mathrm{d}s.$$

It is known that this functional plays an important role in several domains. For example, it can be used to investigate the finite-time blowup of positive solutions to semi-linear stochastic partial differential equations [1]. In the special case $H = \frac{1}{2}$, fBm reduces to a standard Brownian motion and a lot of fruitful properties of $F_{\frac{1}{2}}$ can be founded in the literature, see, e.g., [4,5,8,11]. In particular, the distribution of $F_{\frac{1}{2}}$ can be computed explicitly. However, to the best our knowledge, it remains a challenge to obtain the deep properties of F_H for $H \neq \frac{1}{2}$.

On the other hand, because of its applications in statistical estimators, the problem of proving the continuity in law with respect to H of certain functionals has been studied by several authors. Among others, we refer the reader to [2,3,9,10] and the references therein for detailed discussions and related results. Motivated by this observation, the aim of the present paper is to investigate the continuity in law of the exponential functional F_H. Intuitively, the continuity of F_H with respect to H is not surprising. However, the interesting point of Theorem 1.1 below is that we are able to give an explicit bound on Kolmogorov distance between two functionals with different Hurst indexes.

Theorem 1.1. *For any $H_1, H_2 \in (0,1)$, we have*

$$\sup_{x\geq 0} |P\left(F_{H_1} \leq x\right) - P\left(F_{H_2} \leq x\right)| \leq C|H_1 - H_2|, \tag{1.2}$$

where C is a positive constant depending on a, σ, T, and H_1, H_2.

2. Proofs

Our main tools are the techniques of Malliavin's calculus. Hence, for the reader's convenience, let us recall some elements of Malliavin's calculus with respect to the Brownian motion W, where W is used to present B^H as in (1.1). We suppose that $(W_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filtration generated by the Brownian motion W. For $h \in L^2[0,T]$, we denote by $W(h)$ the Wiener integral

$$W(h) = \int_0^T h(t)\,\mathrm{d}W_t.$$

Let $\mathcal{S}$ denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(W(h_1), ..., W(h_n)), \tag{2.1}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in L^2[0,T]$. If F has the form (2.1), we define its Malliavin derivative as the process $DF := \{D_t F, t \in [0,T]\}$ given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(W(h_1), ..., W(h_n))h_k(t).$$

We shall denote by $\mathbb{D}^{1,2}$ the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,2}^2 := E|F|^2 + E\left[\int_0^T |D_u F|^2\,\mathrm{d}u\right].$$

An important operator in the Malliavin's calculus theory is the divergence operator δ, it is the adjoint of the derivative operator D. The domain of δ is the set of all functions $u \in L^2(\Omega, L^2[0,T])$ such that

$$E|\langle DF, u\rangle_{L^2[0,T]}| \le C(u)\|F\|_{L^2(\Omega)},$$

where $C(u)$ is some positive constant depending on u. In particular, if $u \in Dom\,\delta$, then $\delta(u)$ is characterized by the following duality relationship

$$E\langle DF, u\rangle_{L^2[0,T]} = E[F\delta(u)] \text{ for any } F \in \mathbb{D}^{1,2}.$$

In order to be able to prove Theorem 1.1, we need two technical lemmas.

Lemma 2.1. *For any $H \in (0,1)$, we have $F_H \in \mathbb{D}^{1,2}$ and*

$$\left(\int_0^T |D_r F_H|^2\,\mathrm{d}r\right)^{-1} \in L^p(\Omega),\ \forall\ p \ge 1.$$

Proof. By the representation (1.1), we have $D_r B_s^H = K_H(s,r)$ for $0 \le r < s \le T$. Hence, $F_H \in \mathbb{D}^{1,2}$ and its derivative is given by

$$D_r F_H = \int_r^T \sigma K_H(s,r)\mathrm{e}^{as+\sigma B_s^H}\,\mathrm{d}s,\ 0 \le r \le T.$$

So we can deduce

$$D_r F_H \ge \mathrm{e}^{-|a|T+\sigma \min\limits_{0\le s\le T} B_s^H}\int_r^T \sigma K_H(s,r)\,\mathrm{d}s,\ 0 \le r \le T.$$

As a consequence,

$$\begin{aligned}
\int_0^T |D_r F_H|^2\,\mathrm{d}r &\ge \mathrm{e}^{-2|a|T+2\sigma \min\limits_{0\le s\le T} B_s^H}\int_0^T \left(\int_r^T \sigma K_H(s,r)\,\mathrm{d}s\right)^2 \mathrm{d}r\\
&= \sigma^2 \mathrm{e}^{-2|a|T+2\sigma \min\limits_{0\le s\le T} B_s^H}\int_0^T \left(\int_r^T K_H(s,r)\,\mathrm{d}s\right)\left(\int_r^T K_H(t,r)\,\mathrm{d}t\right)\mathrm{d}r\\
&= \sigma^2 \mathrm{e}^{-2|a|T+2\sigma \min\limits_{0\le s\le T} B_s^H}\int_0^T\int_0^T \left(\int_0^{s\wedge t} K_H(s,r)K_H(t,r)\,\mathrm{d}r\right)\mathrm{d}s\,\mathrm{d}t\\
&= \sigma^2 \mathrm{e}^{-2|a|T+2\sigma \min\limits_{0\le s\le T} B_s^H}\int_0^T\int_0^T E[B_s^H B_t^H]\,\mathrm{d}s\,\mathrm{d}t\\
&= \frac{T^{2H+2}}{2H+2}\sigma^2\,\mathrm{e}^{-2|a|T+2\sigma \min\limits_{0\le s\le T} B_s^H}.
\end{aligned}$$

In the last equality we used the fact that $E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$. We therefore obtain

$$\left(\int_0^T |D_r F_H|^2\,\mathrm{d}r\right)^{-1} \le \frac{2H+2}{T^{2H+2}\sigma^2}\,\mathrm{e}^{2|a|T+2\sigma \max\limits_{0\le s\le T}(-B_s^H)}.$$

By Fernique's theorem, we have $\mathrm{e}^{2\sigma \max\limits_{0\le s\le T}(-B_s^H)} \in L^p(\Omega)$ for any $p \ge 1$. This completes the proof. □

Lemma 2.2. *For any $H_1, H_2 \in (0,1)$, we have*

$$E|F_{H_1} - F_{H_2}|^2 \le C|H_1 - H_2|^2, \tag{2.2}$$

$$\int_0^T E|D_r F_{H_1} - D_r F_{H_2}|^2\,\mathrm{d}r \le C|H_1 - H_2|^2, \tag{2.3}$$

where C is a positive constant depending on a, σ, T, and H_1, H_2.

Proof. By the Hölder inequality, we have

$$E|F_{H_1}-F_{H_2}|^2 = E\Big|\int_0^T \big(e^{as+\sigma B_s^{H_1}} - e^{as+\sigma B_s^{H_2}}\big)\,ds\Big|^2$$
$$\le T\int_0^T E\big|e^{as+\sigma B_s^{H_1}} - e^{as+\sigma B_s^{H_2}}\big|^2\,ds.$$

Using the fundamental inequality $|e^x - e^y| \le \frac{1}{2}|x-y|(e^x+e^y)$ for all x, y, we deduce

$$E|F_{H_1}-F_{H_2}|^2 \le \frac{T}{4}\int_0^T E\big|(\sigma B_s^{H_1} - \sigma B_s^{H_2})(e^{as+\sigma B_s^{H_1}} + e^{as+\sigma B_s^{H_2}})\big|^2\,ds$$
$$\le \frac{T\sigma^2}{4}\int_0^T \Big(E|B_s^{H_1}-B_s^{H_2}|^4\Big)^{\frac{1}{2}}\Big(E|e^{as+\sigma B_s^{H_1}} + e^{as+\sigma B_s^{H_2}}|^4\Big)^{\frac{1}{2}}\,ds$$
$$\le \frac{T\sigma^2}{4}\int_0^T \Big(E|B_s^{H_1}-B_s^{H_2}|^4\Big)^{\frac{1}{2}}\Big(8E[e^{4as+4\sigma B_s^{H_1}}] + 8E[e^{4as+4\sigma B_s^{H_2}}]\Big)^{\frac{1}{2}}\,ds$$
$$= \frac{T\sigma^2}{4}\int_0^T \Big(E|B_s^{H_1}-B_s^{H_2}|^4\Big)^{\frac{1}{2}}\Big(8e^{4as+8\sigma^2 s^{2H_1}} + 8e^{4as+8\sigma^2 s^{2H_2}}\Big)^{\frac{1}{2}}\,ds$$
$$\le \frac{T\sigma^2}{4}\int_0^T \Big(E|B_s^{H_1}-B_s^{H_2}|^4\Big)^{\frac{1}{2}}\Big(8e^{4as+8\sigma^2 s^{2H_1}} + 8e^{4as+8\sigma^2 s^{2H_2}}\Big)^{\frac{1}{2}}\,ds.$$

It is known from the proof of Theorem 4 in [7] that there exists a positive constant C such that

$$\sup_{0\le s\le T} E|B_s^{H_1}-B_s^{H_2}|^2 \le C|H_1-H_2|^2. \tag{2.4}$$

On the other hand, we have $E|B_s^{H_1}-B_s^{H_2}|^4 = 3(E|B_s^{H_1}-B_s^{H_2}|^2)^2$ because $B_s^{H_1}-B_s^{H_2}$ is a Gaussian random variable for every $s\in[0,T]$. So we can conclude that there exists a positive constant C such that

$$E|F_{H_1}-F_{H_2}|^2 \le C|H_1-H_2|^2.$$

To finish the proof, let us verify (2.3). By the Hölder and triangle inequalities, we obtain

$$E|D_rF_{H_1}-D_rF_{H_2}|^2 \le \sigma^2 T\int_r^T E\big|K_{H_1}(s,r)e^{as+\sigma B_s^{H_1}} - K_{H_2}(s,r)e^{as+\sigma B_s^{H_2}}\big|^2\,ds$$
$$\le 2\sigma^2 T\int_r^T |K_{H_1}(s,r)-K_{H_2}(s,r)|^2 E[e^{2as+2\sigma B_s^{H_1}}] + K_{H_2}^2(s,r)E\big|e^{2as+2\sigma B_s^{H_1}} - e^{as+\sigma B_s^{H_2}}\big|^2\,ds,$$

and hence,

$$\int_0^T E|D_rF_{H_1}-D_rF_{H_2}|^2\,dr \le 2\sigma^2 T\int_0^T E[e^{2as+2\sigma B_s^{H_1}}]\int_0^s |K_{H_1}(s,r)-K_{H_2}(s,r)|^2\,dr\,ds$$
$$+2\sigma^2 T\int_0^T E\big|e^{2as+2\sigma B_s^{H_1}} - e^{as+\sigma B_s^{H_2}}\big|^2\int_0^s K_{H_2}^2(s,r)\,dr\,ds$$

$$= 2\sigma^2 T \int_0^T e^{2as+2\sigma^2 s^{2H_1}} E|B_s^{H_1} - B_s^{H_2}|^2 \, ds$$

$$+ 2\sigma^2 T \int_0^T E\left|e^{2as+2\sigma B_s^{H_1}} - e^{as+\sigma B_s^{H_2}}\right|^2 s^{2H_2} \, ds.$$

Notice that $\int_0^s K_{H_2}^2(s,r)\,dr = E|B_s^{H_2}|^2 = s^{2H_2}$. Thus the estimate (2.3) follows from (2.2) and (2.4). □

Proof of Theorem 1.1. For simplicity, we write $\langle .,. \rangle$ instead of $\langle .,. \rangle_{L^2[0,T]}$. Borrowing the arguments used in the proof of Proposition 2.1.1 in [6], we let ψ be a nonnegative smooth function with compact support, and set $\varphi(y) = \int_{-\infty}^y \psi(z)\,dz$. Given $Z \in \mathbb{D}^{1,2}$, we know that $\varphi(Z)$ belongs to $\mathbb{D}^{1,2}$ and, making the scalar product of its derivative with DF_{H_2}, we obtain:

$$\langle D\varphi(Z), DF_{H_2} \rangle = \psi(Z) \langle DZ, DF_{H_2} \rangle.$$

Fixed $x \in \mathbb{R}_+$, by an approximation argument, the above equation holds for $\psi(z) = \mathbb{1}_{[0,x]}(z)$. Choosing $Z = F_{H_1}$ and $Z = F_{H_2}$, we obtain

$$\langle D \int_{-\infty}^{F_{H_1}} \mathbb{1}_{[0,x]}(z)\,dz, DF_{H_2} \rangle = \mathbb{1}_{[0,x]}(F_{H_1}) \langle DF_{H_1}, DF_{H_2} \rangle,$$

$$\langle D \int_{-\infty}^{F_{H_2}} \mathbb{1}_{[0,x]}(z)\,dz, DF_{H_2} \rangle = \mathbb{1}_{[0,x]}(F_{H_2}) \langle DF_{H_2}, DF_{H_2} \rangle.$$

Hence, we can get

$$\langle D \int_{F_{H_2}}^{F_{H_1}} \mathbb{1}_{[0,x]}(z)\,dz, DF_{H_2} \rangle = \mathbb{1}_{[0,x]}(F_{H_1}) \langle DF_{H_1}, DF_{H_2} \rangle - \mathbb{1}_{[0,x]}(F_{H_2}) \langle DF_{H_2}, DF_{H_2} \rangle$$
$$= \left(\mathbb{1}_{[0,x]}(F_{H_1}) - \mathbb{1}_{[0,x]}(F_{H_2})\right) \langle DF_{H_2}, DF_{H_2} \rangle + \mathbb{1}_{[0,x]}(F_{H_1}) \langle DF_{H_1} - DF_{H_2}, DF_{H_2} \rangle.$$

This, together with the fact that $\|DF_{H_2}\|^2 := \langle DF_{H_2}, DF_{H_2} \rangle > 0$ $a.s.$ gives us

$$\mathbb{1}_{[0,x]}(F_{H_1}) - \mathbb{1}_{[0,x]}(F_{H_2}) = \frac{\langle D \int_{F_{H_2}}^{F_{H_1}} \mathbb{1}_{[0,x]}(z)\,dz, DF_{H_2} \rangle}{\|DF_{H_2}\|^2} - \frac{\mathbb{1}_{[0,x]}(F_{H_1}) \langle DF_{H_1} - DF_{H_2}, DF_{H_2} \rangle}{\|DF_{H_2}\|^2}.$$

Taking the expectation yields

$$P\left(F_{H_1} \le x\right) - P\left(F_{H_2} \le x\right) = E[\mathbb{1}_{[0,x]}(F_{H_1}) - \mathbb{1}_{[0,x]}(F_{H_2})]$$
$$= E\left[\int_{F_{H_2}}^{F_{H_1}} \mathbb{1}_{[0,x]}(z)\,dz\,\delta\left(\frac{DF_{H_2}}{\|DF_{H_2}\|^2}\right)\right] - E\left[\frac{\mathbb{1}_{[0,x]}(F_{H_1}) \langle DF_{H_1} - DF_{H_2}, DF_{H_2} \rangle}{\|DF_{H_2}\|^2}\right].$$

By the Hölder inequality,

$$\sup_{x \ge 0} |P\left(F_{H_1} \le x\right) - P\left(F_{H_2} \le x\right)| \le E\left|(F_{H_1} - F_{H_2})\delta\left(\frac{DF_{H_2}}{\|DF_{H_2}\|^2}\right)\right| + E\left|\frac{\langle DF_{H_1} - DF_{H_2}, DF_{H_2} \rangle}{\|DF_{H_2}\|^2}\right|$$
$$\le \left(E|F_{H_1} - F_{H_2}|^2\right)^{\frac{1}{2}} \left(E\delta\left(\frac{DF_{H_2}}{\|DF_{H_2}\|^2}\right)^2\right)^{\frac{1}{2}} + E\left|\frac{\|DF_{H_1} - DF_{H_2}\|}{\|DF_{H_2}\|}\right|$$
$$\le \left(E|F_{H_1} - F_{H_2}|^2\right)^{\frac{1}{2}} \left(E\delta\left(\frac{DF_{H_2}}{\|DF_{H_2}\|^2}\right)^2\right)^{\frac{1}{2}} + \left(E\|DF_{H_1} - DF_{H_2}\|^2\right)^{\frac{1}{2}} \left(E\left[\frac{1}{\|DF_{H_2}\|^2}\right]\right)^{\frac{1}{2}}.$$

Recalling Lemma 2.2, we obtain

$$\sup_{x\geq 0}|P\left(F_{H_1}\leq x\right)-P\left(F_{H_2}\leq x\right)|\leq C|H_1-H_2|\left[\left(E\delta\left(\frac{DF_{H_2}}{\|DF_{H_2}\|^2}\right)^2\right)^{\frac{1}{2}}+\left(E\left[\frac{1}{\|DF_{H_2}\|^2}\right]\right)^{\frac{1}{2}}\right].$$

Thanks to Lemma 2.1 we have

$$E\left[\frac{1}{\|DF_{H_2}\|^2}\right]=E\left[\left(\int_0^T|D_rF_{H_2}|^2\,\mathrm{d}r\right)^{-1}\right]<\infty.$$

Thus we can obtain (1.2) by checking the finiteness of $E[\delta(u)^2]$, where

$$u_r:=\frac{D_rF_{H_2}}{\|DF_{H_2}\|^2},\ 0\leq r\leq T.$$

It is known from Proposition 1.3.1 in [6] that

$$E[\delta(u)^2]\leq\int_0^T E|u_r|^2\,\mathrm{d}r+\int_0^T\int_0^T E|D_\theta u_r|^2\,\mathrm{d}\theta\,\mathrm{d}r.$$

We have

$$\int_0^T E|u_r|^2\,\mathrm{d}r=E\left[\frac{1}{\|DF_{H_2}\|^2}\right]<\infty.$$

Furthermore, by the chain rule for Malliavin derivative, we have

$$D_\theta u_r=\frac{D_\theta D_rF_{H_2}}{\|DF_{H_2}\|^2}-2\frac{D_rF_{H_2}\langle D_\theta DF_{H_2},DF_{H_2}\rangle}{\|DF_{H_2}\|^4},\ 0\leq\theta\leq T.$$

Hence, by the Hölder inequality,

$$\begin{aligned}\int_0^T\int_0^T E|D_\theta u_r|^2\,\mathrm{d}\theta\,\mathrm{d}r&\leq 2E\left[\frac{\int_0^T\int_0^T|D_\theta D_rF_{H_2}|^2\,\mathrm{d}\theta\,\mathrm{d}r}{\|DF_{H_2}\|^4}\right]+8E\left[\frac{\int_0^T\int_0^T|D_\theta D_rF_{H_2}|^2\,\mathrm{d}\theta\,\mathrm{d}r}{\|DF_{H_2}\|^4}\right]\\&\leq 10\left(E\left[\left(\int_0^T\int_0^T|D_\theta D_rF_{H_2}|^2\,\mathrm{d}\theta\,\mathrm{d}r\right)^2\right]\right)^{\frac{1}{2}}\left(E\left[\frac{1}{\|DF_{H_2}\|^8}\right]\right)^{\frac{1}{2}}.\end{aligned}$$

We now observe that

$$D_\theta D_rF_{H_2}=\int_{r\vee\theta}^T\sigma^2K_{H_2}(s,r)K_{H_2}(s,\theta)\mathrm{e}^{as+\sigma B_s^{H_2}}\,\mathrm{d}s,\ 0\leq r,\theta\leq T.$$

Hence,

$$|D_\theta D_rF_{H_2}|^2\leq T\sigma^4\int_{r\vee\theta}^T K_{H_2}^2(s,r)K_{H_2}^2(s,\theta)\mathrm{e}^{2as+2\sigma B_s^{H_2}}\,\mathrm{d}s,\ 0\leq r,\theta\leq T$$

and we obtain

$$\int_0^T\int_0^T|D_\theta D_rF_{H_2}|^2\,\mathrm{d}\theta\,\mathrm{d}r\leq T\sigma^4\int_0^T s^{4H_2}\mathrm{e}^{2as+2\sigma B_s^{H_2}}\,\mathrm{d}s,$$

which implies that

$$E\left[\left(\int_0^T\int_0^T|D_\theta D_rF_{H_2}|^2\,\mathrm{d}\theta\,\mathrm{d}r\right)^2\right]\leq T^4\sigma^8\int_0^T s^{8H_2}\mathrm{e}^{4as+8\sigma^2s^{2H_2}}\,\mathrm{d}s<\infty.$$

Finally, we have $E\left[\frac{1}{\|DF_{H_2}\|^8}\right] < \infty$ due to Lemma 2.1. So we can conclude that $E[\delta(u)^2]$ is finite. This finishes the proof of Theorem 1.1. □

Remark 2.1. Given a bounded and continuous function ψ, with the exact proof of Theorem 1.1, we also have

$$|E[\psi(F_{H_1})] - E[\psi(F_{H_2})]| \le C|H_1 - H_2|.$$

This kind of estimates has been investigated by Richard and Talay for the solution to fractional stochastic differential equations. However, Theorem 1.1 in [9] requires $H_2 = \frac{1}{2}$ and ψ to be Hölder continuous of order $2 + \beta$ with $\beta > 0$.

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