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Corrections to “Singular Hochschild cohomology via the singularity category” [C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1106–1111]

Corrigendum à « La cohomologie de Hochschild singulière via la catégorie des singularités » [C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1106–1111]

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ABSTRACT

We correct a mistake that occurred in the proof of the main theorem of “Singular Hochschild cohomology via singularity categories” and some inaccuracies in the proof of the reconstruction theorem.

RÉSUMÉ

Nous corrigeons une erreur qui s'est glissée dans la démonstration du théorème principal de « La cohomologie singulière via la catégorie des singularités », ainsi que des imprécisions dans la démonstration du théorème de reconstruction.

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1. Corrections in the proof of the main theorem

We use the notations from section 2.3 of [8] but for simplicity, we assume that k is a field. The case of a commutative ground ring is an easy extension, cf. [7]. Let A be k-algebra and $\mathsf{Sg}(A)$ the singularity category, i.e. the Verdier quotient of the homotopy category of right bounded complexes of finitely generated projective (right) A-modules with bounded homology by its full subcategory of the bounded complexes of finitely generated projective A-modules. For a dg category $\mathcal{A}$, denote by $X \mapsto Y(X)$ the dg Yoneda functor and by $\mathcal{D}\mathcal{A}$ the derived category. We write $\mathcal{A}^e$ for the enveloping dg category $\mathcal{A} \otimes \mathcal{A}^{op}$ and $I_{\mathcal{A}}$ for the *identity bimodule*

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$$I_{\mathcal{A}} : (X, Y) \mapsto \mathcal{A}(X, Y).$$

By definition, the Hochschild cohomology of $\mathcal{A}$ is the graded endomorphism algebra of $I_{\mathcal{A}}$ in the derived category $\mathcal{D}(\mathcal{A}^e)$. In the case of the algebra A, the identity bimodule is the A-bimodule A. Recall that if $F : \mathcal{A} \to \mathcal{B}$ is a fully faithful dg functor, the restriction $F_* : \mathcal{DB} \to \mathcal{DA}$ is a localization functor admitting fully faithful left and right adjoint functors F^* and $F^!$ given respectively by

$$F^* : M \mapsto M \overset{\boldsymbol{L}}{\otimes}_{\mathcal{A}} {}_F\mathcal{B} \quad \text{and} \quad F^! : N \mapsto \mathsf{RHom}_{\mathcal{A}}(\mathcal{B}_F, N)\,,$$

where ${}_F\mathcal{B} = \mathcal{B}(?, F-)$ and $\mathcal{B}_F = \mathcal{B}(F?, -)$.

Let $\mathcal{M} = \mathcal{C}^{-,b}_{dg}(\mathsf{proj}\,A)$ denote the dg category of right-bounded complexes of finitely generated projective A-modules with bounded homology. Let $\mathcal{S}$ denote the dg quotient $\mathcal{M}/\mathcal{P}$. Then $H^0(\mathcal{S})$ is triangle equivalent to $\mathsf{Sg}(A)$. We have the obvious inclusion and projection dg functors

$$A \xrightarrow{\;i\;} \mathcal{M} \xrightarrow{\;p\;} \mathcal{S}.$$

It was claimed on line 4 of page 1108 in [7] that the functor $(i \otimes i)^* : \mathcal{D}(A^e) \to \mathcal{D}(\mathcal{M}^e)$ takes the identity bimodule A to the identity bimodule $I_{\mathcal{M}}$. This cannot be true, since the image of $I_{\mathcal{M}}$ under $(p \otimes p)^*$ is $I_{\mathcal{S}}$ (because p is a localization), whereas $(p \otimes p)^*(i \otimes i)^*(A)$ vanishes. To correct the mistake, we replace $(i \otimes i)^*$ with another functor: consider the fully faithful dg functors

$$A \otimes A^{op} \xrightarrow{\mathbf{1}\otimes i} A \otimes \mathcal{M}^{op} \xrightarrow{i \otimes \mathbf{1}} \mathcal{M} \otimes \mathcal{M}^{op}\,.$$

The restriction along $G = \mathbf{1} \otimes i$ admits the left adjoint G^* given by

$$G^* : X \mapsto \mathcal{M}_i \overset{\boldsymbol{L}}{\otimes}_A X\,,$$

and the restriction along $F = i \otimes \mathbf{1}$ admits the fully faithful left and right adjoints F^* and $F^!$ given by

$$F^* : Y \mapsto Y \overset{\boldsymbol{L}}{\otimes}_A {}_i\mathcal{M} \quad \text{and} \quad F^! : Y \mapsto \mathsf{RHom}_A(\mathcal{M}_i, Y).$$

Since F^* and $F^!$ are the two adjoints of a localization functor, we have a canonical morphism $F^* \to F^!$.

Lemma 1.1. *If P is an arbitrary projective A^e-module, the morphism*

$$F^*G^*(P) \to F^!G^*(P)$$

is invertible.

Proof. Let P be the direct sum of finitely generated projective A^e-modules $P_j \otimes Q_j$, $j \in J$. Since F^* and G^* commute with (arbitrary) coproducts, the left-hand side is the dg module

$$\bigoplus_J \mathcal{M}(i?, -) \overset{\boldsymbol{L}}{\otimes}_A (P_j \otimes Q_j) \overset{\boldsymbol{L}}{\otimes}_A \mathcal{M}(?, i-) = \bigoplus_J \mathcal{M}(P_j^\vee, -) \otimes \mathcal{M}(?, Q_j)\,,$$

where $P_j^\vee = \mathsf{Hom}_{A^{op}}(P_j, A)$. The right-hand side is the dg module

$$\mathsf{Hom}_A(\mathcal{M}_i, \mathcal{M}_i \otimes_A (\bigoplus_J P_j \otimes Q_j)) = \mathsf{Hom}_A(\mathcal{M}_i, \bigoplus_J \mathcal{M}(P_j^\vee, -) \otimes Q_j).$$

Let us evaluate the canonical morphism at $(M, L) \in \mathcal{M} \otimes \mathcal{M}^{op}$. We find the canonical morphism

$$\bigoplus_J \mathcal{M}(P_j^\vee, M) \otimes \mathcal{M}(L, Q_j) \to \mathsf{Hom}_A(L, \bigoplus_J \mathcal{M}(P_j^\vee, M) \otimes Q_j).$$

We may assume that $P_j = A$ for all j. We then find the canonical morphism

$$\bigoplus_J M \otimes \mathsf{Hom}_A(L, Q_j) \to \mathsf{Hom}_A(L, \bigoplus_J M \otimes Q_j).$$

Recall that L and M are right bounded complexes of finitely generated projective modules with bounded homology. We fix M and consider the morphism as a morphism of triangle functors with argument $L \in \mathcal{D}^b(\mathsf{Mod}\,A)$. Then we are reduced to the case where L is in $\mathsf{Mod}\,A$. In this case, the morphism becomes an isomorphism of complexes because the components of L are finitely generated projective. $\square$

Let us put

$$H = F^! G^* : \mathcal{D}(A^e) \to \mathcal{D}(\mathcal{M}^e).$$

It is this functor H that replaces the mistaken functor $(i \otimes i)^*$ of [8]. Let us compute the image of the identity bimodule A under H. We have

$$H(A) = F^!(\mathcal{M}_i \overset{\boldsymbol{L}}{\otimes}_A A) = F^!(\mathcal{M}_i) = \mathsf{RHom}_A(\mathcal{M}_i, \mathcal{M}_i)$$

and when we evaluate at L, M in $\mathcal{M}$, we find

$$H(A)(L, M) = \mathsf{RHom}_A(\mathcal{M}(i?, L), \mathcal{M}(i?, M)) = \mathsf{RHom}_A(\mathcal{M}(A, L), \mathcal{M}(A, M)) = \mathsf{Hom}_A(L, M).$$

Thus, the functor H takes the identity bimodule A to the identity bimodule $I_{\mathcal{M}}$. Since $F^!$ and G^* are fully faithful so is H. Denote by $\mathcal{N}$ the image under H of the closure of $\mathsf{Proj}\, A^e$ under finite extensions. Then H yields a fully faithful functor

$$\widehat{\mathsf{Sg}}(A^e) \to \mathcal{D}(\mathcal{M}^e)/\mathcal{N}$$

taking the bimodule A to the identity bimodule $I_{\mathcal{M}}$. Now one concludes as in [8] by showing that the canonical functor $\mathcal{D}(\mathcal{M}^e)/\mathcal{N} \to \mathcal{D}(\mathcal{S}^e)$ induces a bijection in the morphism spaces from $I_{\mathcal{M}}$ to its suspensions.

2. Proof of the reconstruction theorem

The proof given in [8] neglected the subtleties arising from the fact that $A = P/(Q)$ is a complete algebra and did not give enough details in the reference to [6]. We correct this in the following argument: by the Weierstrass preparation theorem, we may assume that Q is a polynomial. Let $P_0 = k[x_1, \dots, x_n]$ and $S = P_0/(Q)$. Then S has isolated singularities, but may have singularities other than the origin. Let $\mathfrak{m}$ be the maximal ideal of P_0 generated by the x_i and let R be the localization of S at $\mathfrak{m}$. Now R is local with an isolated singularity at $\mathfrak{m}$ and A is isomorphic to the completion $\widehat{R}$.

By Theorem 3.2.7 of [6], in sufficiently high degrees r, the Hochschild cohomology of S is isomorphic to the homology in degree r of the complex

$$k[u] \otimes K(S, \partial_1 Q, \dots, \partial_n Q),$$

where u is of degree 2 and K denotes the Koszul complex. Now S is isomorphic to $K(P_0, Q)$ and so $K(S, \partial_1 Q, \dots, \partial_n Q)$ is isomorphic to

$$K(P_0, Q, \partial_1 Q, \dots, \partial_n Q).$$

Since Q has isolated singularities, the $\partial_i Q$ form a regular sequence in P_0. So

$$K(P_0, Q, \partial_1 Q, \dots, \partial_n Q)$$

is quasi-isomorphic to $K(M, Q)$, where $M = P_0/(\partial_1 Q, \dots, \partial_n Q)$. Therefore, in high even degrees $2r$, the Hochschild cohomology of S is isomorphic to

$$T = k[x_1, \dots, x_n]/(Q, \partial_1 Q, \dots, \partial_n Q)$$

as an S-module. Since S and S^e are Noetherian, this implies that the Hochschild cohomology of R in high even degrees is isomorphic to the localization $T_{\mathfrak{m}}$. Since $R \otimes R$ is Noetherian and Gorenstein (cf. Theorem 1.6 of [11]), by Theorem 6.3.4 of [2], the singular Hochschild cohomology of R coincides with Hochschild cohomology in sufficiently high degrees. By the main theorem, the Hochschild cohomology of $\mathsf{Sg}_{dg}(R)$ is isomorphic to the singular Hochschild cohomology of R and thus isomorphic to $T_{\mathfrak{m}}$ in high even degrees. Since R is a hypersurface, the dg category $\mathsf{Sg}_{dg}(R)$ is isomorphic, in the homotopy category of dg categories, to the underlying differential $\mathbb{Z}$-graded category of the differential $\mathbb{Z}/2$-graded category of matrix factorizations of Q, cf. [4], [10] and Theorem 2.49 of [1]. Thus, it is 2-periodic and so is its Hochschild cohomology. It follows that the zeroth Hochschild cohomology of $\mathsf{Sg}_{dg}(R)$ is isomorphic to $T_{\mathfrak{m}}$ as an algebra. The completion functor $? \otimes_R \widehat{R}$ yields an embedding $\mathsf{Sg}(R) \to \mathsf{Sg}(A)$, through which $\mathsf{Sg}(A)$ identifies with the idempotent completion of the triangulated category $\mathsf{Sg}(R)$, cf. Theorem 5.7 of [3]. Therefore, the corresponding dg functor $\mathsf{Sg}_{dg}(R) \to \mathsf{Sg}_{dg}(A)$ induces an equivalence in the derived categories and an isomorphism in Hochschild cohomology. So we find an isomorphism

$$HH^0(\mathsf{Sg}_{dg}(A), \mathsf{Sg}_{dg}(A)) \xrightarrow{\sim} T_{\mathfrak{m}}.$$

Since $Q \in k[x_1, \dots, x_n]_{\mathfrak{m}}$ has an isolated singularity at the origin, we have an isomorphism

$$T_{\mathfrak{m}} \xrightarrow{\sim} k[[x_1, \dots, x_n]]/(Q, \partial_1 Q, \dots, \partial_n Q)$$

with the Tyurina algebra of $A = P/(Q)$. Now by the Mather–Yau theorem [9], more precisely by its formal version [5, Prop. 2.1], in a fixed dimension, the Tyurina algebra determines A up to isomorphism.

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