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Riemann curvature tensor on RCD spaces and possible applications

Tenseur de courbure de Riemann sur les espaces RCD *et applications possibles*

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ABSTRACT

We show that, on every RCD space, it is possible to introduce, by a distributional-like approach, a Riemann curvature tensor.
Since, after the works of Petrunin and Zhang–Zhu, we know that finite dimensional Alexandrov spaces are RCD spaces, our construction applies in particular to the Alexandrov setting. We conjecture that an RCD space is Alexandrov if and only if the sectional curvature – defined in terms of such abstract Riemann tensor – is bounded from below.

RÉSUMÉ

Nous montrons que, sur chaque espace RCD, il est possible d'introduire, par une approche distributionnelle, un tenseur de courbure de Riemann.
Puisque, d'après les travaux de Petrunin et de Zhang–Zhu, nous savons que les espaces d'Alexandrov de dimension finie sont des espaces RCD, notre construction s'applique en particulier au cadre d'Alexandrov. Nous conjecturons qu'un espace RCD est Alexandrov si et seulement si la courbure sectionnelle – définie en termes de ce tenseur de Riemann abstrait – est bornée par en dessous.

1. Introduction

One of the outcomes of the tensor calculus on RCD spaces built in [5] is the existence of a measure-valued Ricci tensor. An example of application of this object to the study of the geometry of such spaces is given by the paper [7] where, inspired by some more formal computations due to Sturm [16], it is shown that transformations of the metric-measure structure (like, e.g., conformal ones) alter lower Ricci curvature bounds as in the smooth context.

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Stability of the CAT(0) condition under appropriate conformal transformations has been shown in [11] with different techniques, in particular without relying on any sort of tensor calculus, but, notably, the same kind of question in the context of Alexandrov spaces is open, see, e.g., [13]. In this paper, we want to propose an attack plan to such problem: our idea is to leverage on the available calculus tools on the RCD setting to produce a sectional curvature tensor on Alexandrov spaces, so that then, hopefully, appropriate transformation formulas can be studied. In this direction, recall that thanks to Petrunin's [14] and Zhang–Zhu's results [18], we know that an n-dimensional Alexandrov space with curvature bounded from below by k is $\mathsf{CD}(k(n-1),n)$, in line with the smooth setting. Since, moreover, Alexandrov spaces are infinitesimally Hilbertian (see [10]; the notion of Sobolev function used in this work is different from the one – introduced in [2] – typically adopted on metric measure spaces, yet the two are easily seen to be equivalent thanks to the structural properties of Alexandrov spaces pointed out in [10]), we see that a finite dimensional Alexandrov space is always an RCD space and thus all the calculus tools available in the latter setting are at our disposal also in the former.

An interesting fact, and the main point of this current manuscript, is that on an RCD space, it is possible to give a meaning, in a kind of distributional sense, to the full Riemann curvature tensor. To see why, start recalling that the typical term in the definition of $R(X,Y,Z,W)$ on the smooth setting is $\langle\nabla_X\nabla_Y Z, W\rangle$ and observe that it seems hard to give a 'direct' meaning to such expression on RCD spaces, because it is unclear whether there are vector fields regular enough to be covariantly differentiated twice (we believe in general that there are not many of these). Instead, we have 'many' (i.e. L^2-dense) vector fields that are bounded and with covariant derivative in L^2, and this allows us to give a 'weak' meaning to such object: indeed, multiplying $\langle\nabla_X\nabla_Y Z, W\rangle$ by a smooth function f, integrating w.r.t. the volume measure and then integrating by parts, we obtain

$$\int f\langle\nabla_X\nabla_Y Z, W\rangle\,\mathrm{dvol} = -\int \langle\nabla_Y Z, \nabla_X(fW)\rangle + f\,\mathrm{div}X\langle\nabla_Y Z, W\rangle\,\mathrm{dvol},$$

and from what we just said we see that the right-hand side is well defined for 'many' vector fields and functions. Thus, following this line of thought, it is possible to define, for any X,Y,Z,W sufficiently smooth vector fields, the Riemann curvature tensor $\mathcal{R}(X,Y,Z,W)$ as a real-valued operator acting on a space of sufficiently smooth functions. In more precise terms, we shall work with the spaces of *test functions* Test(M) and *test vector fields* TestV(M) – see the beginning of the next section for the definitions.

In particular, we also have a sectional curvature operator on RCD spaces, and a fortiori on finite-dimensional Alexandrov spaces. We conjecture that lower bounds of such sectional curvature are equivalent to the Alexandrov condition in the following sense.

Conjecture 1.1. *Let* (M,d) *be a complete and separable metric space. Then the following propositions are equivalent:*

i) (M,d) *is a n-dimensional Alexandrov space of curvature bounded from below by* $k\in\mathbb{R}$.
ii) $(\mathrm{M},\mathsf{d},\mathcal{H}^n)$ *is a* $\mathsf{RCD}(k(n-1),n)$ *space,* $\mathrm{supp}(\mathcal{H}^n)=\mathrm{M}$ *and*

$$\mathcal{R}(X,Y,Y,X)(f)\geq k\int f|X\wedge Y|^2\,\mathrm{d}\mathfrak{m}\qquad\forall f\in\mathrm{Test}(\mathrm{M}),\ f\geq 0,\ X,Y\in\mathrm{TestV}(\mathrm{M}).$$

Here and below $\mathcal{H}^n$ is the n-dimensional Hausdorff measure. Let us collect some comments about this conjecture. First of all, the already recalled results by Kuwae–Machigashira–Shioya, Petrunin and Zhang–Zhu grant that if (i) holds, then $(\mathrm{M},\mathsf{d},\mathcal{H}^n)$ is an $\mathsf{RCD}(k(n-1),n)$ space such that $\mathrm{supp}(\mathcal{H}^n)=\mathrm{M}$. This is all is known so far about the relation between (i) and (ii). We also point out that $\mathsf{RCD}(K,N)$ spaces for which the reference measure is $\mathcal{H}^N$ are called *non-collapsed* RCD spaces (see [4]) and are more regular than generic RCD spaces: they are the synthetic analogue of the *non-collapsed Ricci limit spaces* introduced by Cheeger–Colding in [3].

Finally, we remark that the abstract tensor calculus developed in [5] has already been shown to have strong links with the geometry of RCD spaces. As a non-exhaustive list of recent results where it has been used as key (but certainly not exclusive) tool, let us mention: the link between dimension of the first cohomology group and the geometry of the underlying space [6], the constant dimension property of RCD spaces [1], the regularity that comes from imposing both a lower Ricci bound and an upper sectional bound on the space [8].

2. Riemann curvature tensor on $\mathsf{RCD}(K,\infty)$ spaces

To keep this note short, we shall assume the reader familiar with the notion of RCD space and with the calculus developed in [5]. Throughout this note, $(\mathrm{M},\mathsf{d},\mathfrak{m})$ will be a fixed $\mathsf{RCD}(K,\infty)$ space, $K\in\mathbb{R}$.

We recall that, in [15], it has been introduced the space of 'test functions'

$$\mathrm{Test}(\mathrm{M}):=\left\{f\in D(\Delta)\subset W^{1,2}(\mathrm{M})\ :\ \mathrm{Lip}(f)<\infty,\ f\in L^\infty(\mathrm{M}),\ \Delta f\in W^{1,2}(\mathrm{M})\right\}$$

and proved that this is an algebra dense in $W^{1,2}(\mathrm{M})$. Then, in [5], the space of 'test vector fields' has been defined as

$$\mathrm{TestV}(\mathrm{M}) := \Big\{ \sum_{i=1}^{n} f_i \nabla g_i \ : \ n \in \mathbb{N},\ f_i, g_i \in \mathrm{Test}(\mathrm{M}) \Big\}.$$

Let us briefly recall the notion of covariant derivative as introduced in [5]. The first step is to recall the one of Hessian, whose definition is based on the identity

$$2\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) = \langle \nabla\langle \nabla f, \nabla g_1\rangle, \nabla g_2\rangle + \langle \nabla\langle \nabla f, \nabla g_2\rangle, \nabla g_1\rangle - \langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle$$

valid in the smooth world: it can be proved (see [5, Theorem 3.3.8]) that, for any $f \in \mathrm{Test}(\mathrm{M})$, there is a unique element $\mathrm{Hess}\, f \in L^2((T^*)^{\otimes 2}\mathrm{M})$ such that

$$2\int h\mathrm{Hess}\, f(\nabla g_1, \nabla g_2)\,\mathrm{d}\mathfrak{m} = \int -\langle \nabla f, \nabla g_1\rangle \mathrm{div}(h\nabla g_2) - \langle \nabla f, \nabla g_2\rangle \mathrm{div}(h\nabla g_1) - \langle \nabla f, \nabla\langle \nabla g_1, \nabla g_2\rangle\rangle\,\mathrm{d}\mathfrak{m}.$$

Then the definition of covariant derivative is based on the identity

$$\langle \nabla_{\nabla f} X, \nabla g\rangle = \langle \nabla\langle X, \nabla g\rangle, \nabla f\rangle - \mathrm{Hess}\, g(X, \nabla f).$$

More precisely, one says that $X \in L^2(T\mathrm{M})$ belongs to $W^{1,2}_C(T\mathrm{M})$ provided there is an element $\nabla X \in L^2(T^{\otimes 2}\mathrm{M})$ – which is easily seen to be unique – such that

$$\int h\nabla X : (\nabla f \otimes \nabla g)\,\mathrm{d}\mathfrak{m} = \int -\langle X, \nabla g\rangle\, \mathrm{div}(h\nabla f) - h\,\mathrm{Hess}\, g(X, \nabla f)\,\mathrm{d}\mathfrak{m}$$

and it turns out (see [5, Theorem 3.4.2]) that $\mathrm{TestV}(\mathrm{M}) \subset W^{1,2}_C(T\mathrm{M})$. The covariant derivative so defined is *compatible with the metric* and *torsion free*, in the sense that the identities

$$\langle \nabla\langle X, Y\rangle, Z\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle$$
$$[X, Y] = \nabla_X Y - \nabla_Y X$$

hold for every $X, Y, Z \in \mathrm{TestV}(\mathrm{M})$. Also, it holds $\nabla(\nabla f) : (\nabla g \otimes \nabla h) = \mathrm{Hess}\, f(\nabla g, \nabla h)$ for any $f, g, h \in \mathrm{Test}(\mathrm{M})$.

We will not put any topology on the vector space $\mathrm{TestV}(\mathrm{M})$ (see Remark 2.9 for comments in this direction); here we just notice that the product of a function in $\mathrm{Test}(\mathrm{M})$ and a vector field in $\mathrm{TestV}(\mathrm{M})$ is still an element of $\mathrm{TestV}(\mathrm{M})$, i.e. $\mathrm{TestV}(\mathrm{M})$ is a module over $\mathrm{Test}(\mathrm{M})$.

Definition 2.1 *(The space* $\mathrm{TestV}(\mathrm{M})'$*).* The vector space $\mathrm{TestV}(\mathrm{M})'$ is the space of all linear maps from $\mathrm{TestV}(\mathrm{M})$ to $\mathbb{R}$, i.e. the dual of $\mathrm{TestV}(\mathrm{M})$ in the algebraic sense.

The vector space $\mathrm{TestV}(\mathrm{M})'$ comes with the structure of module over $\mathrm{Test}(\mathrm{M})$, the product of an operator $T \in \mathrm{TestV}(\mathrm{M})'$ and a function $f \in \mathrm{Test}(\mathrm{M})$ being given by the formula

$$(fT)(W) := T(fW), \qquad \forall W \in \mathrm{TestV}(\mathrm{M}).$$

We shall think about the space $\mathrm{TestV}(\mathrm{M})'$ as a kind of 'space of vector-valued distributions' on M: our differentiation operators for objects with low regularity will take value in $\mathrm{TestV}(\mathrm{M})'$.

Notice that the compatibility with the metric of the covariant derivative and the very definition of divergence yield

$$\int \langle \nabla_X Y, W\rangle\,\mathrm{d}\mathfrak{m} = \int -\langle \nabla_X W, Y\rangle - \langle Y, W\rangle\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m} \qquad \forall X, Y, W \in \mathrm{TestV}(\mathrm{M}). \tag{2.1}$$

We therefore propose the following definition.

Definition 2.2 *(Distributional covariant derivative).* Let $X, Y \in L^2(T\mathrm{M})$ be with $X \in D(\mathrm{div})$ and so that at least one of $|X|$ and $|Y|$ is in $L^\infty(\mathrm{M})$. Then $\boldsymbol{\nabla}_X Y \in \mathrm{TestV}(\mathrm{M})'$ is defined as

$$\boldsymbol{\nabla}_X Y(W) := \int -\langle \nabla_X W, Y\rangle - \langle Y, W\rangle \mathrm{div}(X)\,\mathrm{d}\mathfrak{m} \qquad \forall W \in \mathrm{TestV}(\mathrm{M}). \tag{2.2}$$

Notice that it holds

$$| - \langle \nabla_X W, Y\rangle - \langle Y, W\rangle \mathrm{div}(X)| \le |\nabla W|_{\mathsf{HS}}|X||Y| + |Y||W||\mathrm{div} X|,$$

and recall that, for $W \in \mathrm{TestV}(\mathrm{M})$, it holds $|W| \in L^\infty(\mathrm{M})$ and $|\nabla W|_{\mathsf{HS}} \in L^2(\mathrm{M})$, while $X \in D(\mathrm{div})$ means that $\mathrm{div}(X) \in L^2(\mathrm{M})$. Thus the integral in (2.2) is well defined and the definition of $\boldsymbol{\nabla}_X Y$ is well posed.

The identity (2.1) grants consistency, i.e.: if $Y \in W^{1,2}_C(TM)$ and $X \in D(\mathrm{div})$ then

$$\nabla_X Y(W) = \int \langle \nabla_X Y, W\rangle \,\mathrm{d}\mathfrak{m}, \qquad \forall W \in \mathrm{TestV}(\mathrm{M}). \tag{2.3}$$

Having a notion of distributional covariant derivative leads to the one of distributional Lie bracket.

Definition 2.3 *(Distributional Lie bracket).* Let $X, Y \in L^2(TM)$ be with $X, Y \in D(\mathrm{div})$ and such that at least one of $|X|$ and $|Y|$ is in $L^\infty(\mathrm{M})$. Then $\mathbf{[}X, Y\mathbf{]} \in \mathrm{TestV}(\mathrm{M})'$ is defined as

$$\mathbf{[}X, Y\mathbf{]} := \boldsymbol{\nabla}_X Y - \boldsymbol{\nabla}_Y X.$$

The assumptions we made on the X, Y grant that both $\boldsymbol{\nabla}_X Y$ and $\boldsymbol{\nabla}_Y X$ are well defined. Hence, so is the case for $\mathbf{[}X, Y\mathbf{]}$. Also, the identity (2.3) grants that, if $X, Y \in W^{1,2}_C(TM) \cap D(\mathrm{div})$, then

$$\mathbf{[}X, Y\mathbf{]}(W) = \int \langle [X, Y], W\rangle \,\mathrm{d}\mathfrak{m} \qquad \forall W \in \mathrm{TestV}(\mathrm{M}).$$

One of the basic properties of the Lie bracket of smooth vector fields on smooth manifolds is the Jacobi identity, in which two consecutive applications of the brackets occur. One might certainly wonder whether the same holds on RCD spaces, but in such setting we do not have vector fields regular enough to make twice the Lie bracket operation: the notion of distributional Lie bracket helps in this direction. We start with the following lemma.

Lemma 2.4. *For every $X, Y \in \mathrm{TestV}(\mathrm{M})$ and $h : \mathrm{M} \to \mathbb{R}$ Lipschitz and bounded. Then both vector fields*

$$h[X, Y], \quad X\mathrm{div}(hY) - Y\mathrm{div}(hX)$$

are in $D(\mathrm{div})$ and

$$\mathrm{div}(h[X, Y]) = \mathrm{div}(X\mathrm{div}(hY) - Y\mathrm{div}(hX)). \tag{2.4}$$

Proof. Assume for the moment $h \in \mathrm{Test}(\mathrm{M})$ and that $\mathrm{div}(X), \mathrm{div}(Y) \in L^\infty(\mathrm{M})$. In this case, by direct computation, it is easy to see that $X\mathrm{div}(hY) \in D(\mathrm{div})$, with

$$\begin{aligned}\mathrm{div}(X\mathrm{div}(hY)) = {} & \mathrm{div}(X)\langle \nabla h, Y\rangle + \mathrm{Hess}\, h(X, Y) + \langle \nabla_X Y, \nabla h\rangle \\ & + h\,\mathrm{div}(X)\mathrm{div}(Y) + \langle \nabla h, X\rangle \mathrm{div}(Y) + h\langle X, \nabla \mathrm{div}(Y)\rangle\end{aligned} \tag{2.5}$$

and thus that $X\mathrm{div}(hY) - Y\mathrm{div}(hX) \in D(\mathrm{div})$ with

$$\mathrm{div}\big(X\mathrm{div}(hY) - Y\mathrm{div}(hX)\big) = h\big(\langle X, \nabla \mathrm{div}(Y)\rangle - \langle Y, \nabla \mathrm{div}(X)\rangle\big) + \langle \nabla h, [X, Y]\rangle.$$

Since the right-hand side of this last expression is continuous in h w.r.t. the $W^{1,2}(\mathrm{M})$ topology, we see that the same conclusion holds for $h \in W^{1,2}(\mathrm{M})$. Finally, the case of h bounded and Lipschitz can be achieved with a standard cut-off argument. The assumption $\mathrm{div}(X), \mathrm{div}(Y) \in L^\infty(\mathrm{M})$, which ensured that the term $h\,\mathrm{div}(X)\mathrm{div}(Y)$ in (2.5) is in $L^2(\mathrm{M})$, can now be dropped with a similar approximation procedure (see also [5, Equation (3.4.3)]).

Now let $f \in \mathrm{Test}(\mathrm{M})$ be arbitrary, notice that

$$\begin{aligned}\int f\,\mathrm{div}(X\mathrm{div}(hY) - Y\mathrm{div}(hX))\,\mathrm{d}\mathfrak{m} &= \int -\langle \nabla f, X\rangle \mathrm{div}(hY) + \langle \nabla f, Y\rangle \mathrm{div}(hX)\,\mathrm{d}\mathfrak{m} \\ &= \int h\langle \nabla f, \nabla_Y X - \nabla_X Y\rangle \,\mathrm{d}\mathfrak{m} = -\int \langle \nabla f, h[X, Y]\rangle \,\mathrm{d}\mathfrak{m}\end{aligned}$$

and conclude with the density of $\mathrm{Test}(\mathrm{M})$ in $W^{1,2}(\mathrm{M})$ and the very definition of $D(\mathrm{div})$ and divergence. □

A direct consequence of such lemma is that, for $X, Y, Z \in \mathrm{TestV}(\mathrm{M})$, the object $\mathbf{[}[X, Y], Z\mathbf{]} \in \mathrm{TestV}(\mathrm{M})'$ is always well defined. We then have the following proposition.

Proposition 2.5 *(Jacobi identity). For every $X, Y, Z \in \mathrm{TestV}(\mathrm{M})$, we have*

$$\mathbf{[}[X, Y], Z\mathbf{]} + \mathbf{[}[Y, Z], X\mathbf{]} + \mathbf{[}[Z, X], Y\mathbf{]} = 0.$$

Proof. We need to prove that, for a generic $W \in \mathrm{TestV(M)}$, we have

$$\mathbf{[}[X,Y],Z\mathbf{]}(W) + \mathbf{[}[Y,Z],X\mathbf{]}(W) + \mathbf{[}[Z,X],Y\mathbf{]}(W) = 0,$$

and, by linearity, we can assume that $W = g\nabla f$ for generic $f, g \in \mathrm{Test(M)}$.

We now claim that, for every $f, g \in \mathrm{Test(M)}$, it holds:

$$\mathbf{[}X,Y\mathbf{]}(g\nabla f) = \int -X(g)Y(f) - gY(f)\,\mathrm{div}(X) + Y(g)X(f) + gX(f)\,\mathrm{div}(Y)\,\mathrm{d}\mathfrak{m}, \tag{2.6}$$

indeed we have

$$\begin{aligned}\nabla_X Y(g\nabla f) &= \int -\langle \nabla_X(g\nabla f), Y\rangle - gY(f)\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m}\\ &= \int -g\,\mathrm{Hess}(f)(X,Y) - X(g)Y(f) - gY(f)\,\mathrm{div}(X)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

so that the claim follows subtracting the analogous identity for $\nabla_Y X(g\nabla f)$ using the symmetry of the Hessian.

Replacing X with $[X,Y]$ and Y by Z in (2.6) and using (2.4) with $h \equiv 1$, we obtain

$$\begin{aligned}\mathbf{[}[X,Y],Z\mathbf{]}(g\nabla f) = \int &- Z(f)\big(XY(g) - YX(g)\big) - \mathrm{div}(X)\big(Y(g)Z(f) + gYZ(f)\big)\\ &+ \mathrm{div}(Y)\big(X(g)Z(f) + gXZ(f)\big) + Z(g)\big(XY(f) - YX(f)\big)\\ &+ g\,\mathrm{div}(Z)(XYf - YXf)\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Adding up the terms obtained by cyclic permutation of X, Y, Z and using the trivial identity

$$\int X(f)Y(g)\,\mathrm{div}(Z)\,\mathrm{d}\mathfrak{m} = -\int Z(X(f))Y(g) + X(f)Z(Y(g))\,\mathrm{d}\mathfrak{m},$$

and the 'permuted' ones, we get the conclusion. □

We are now ready to give the main definition of this note.

Definition 2.6 *(Distributional curvature tensor).* For $X, Y, Z \in \mathrm{TestV(M)}$, we define $\mathbf{R}(X,Y)(Z) \in \mathrm{TestV(M)}'$ as

$$\mathbf{R}(X,Y)(Z) := \boldsymbol{\nabla}_X(\nabla_Y Z) - \boldsymbol{\nabla}_Y(\nabla_X Z) - \boldsymbol{\nabla}_{[X,Y]}Z.$$

For $X, Y, Z, W \in \mathrm{TestV(M)}$, we also define $\mathcal{R}(X,Y,Z,W) \in \mathrm{Test(M)}'$ as

$$\mathcal{R}(X,Y,Z,W)(f) := \big(\mathbf{R}(X,Y)(Z)\big)(fW).$$

We conclude pointing out that it is a purely algebraic consequence of the definition, and of the calculus tools developed so far, that the curvature has the same symmetries it has in the smooth setting (we shall follow the arguments of [12]).

Proposition 2.7 *(Symmetries of the curvature). For any $X, Y, Z, W \in \mathrm{TestV(M)}$ and $f \in \mathrm{Test(M)}$, it holds:*

$$\mathcal{R}(X,Y,Z,W) = -\mathcal{R}(Y,X,Z,W) = \mathcal{R}(Z,W,X,Y), \tag{2.7a}$$

$$\mathbf{R}(X,Y)(Z) + \mathbf{R}(Y,Z)(X) + \mathbf{R}(Z,X)(Y) = 0, \tag{2.7b}$$

$$f\mathcal{R}(X,Y,Z,W) = \mathcal{R}(fX,Y,Z,W) = \mathcal{R}(X,fY,Z,W) = \mathcal{R}(X,Y,fZ,W) = \mathcal{R}(X,Y,Z,fW). \tag{2.7c}$$

Proof. The first in (2.7a) is equivalent to the identity $\mathbf{R}(X,Y)(Z) = -\mathbf{R}(Y,X)(Z)$, which in turn is a direct consequence of the definition.

The equality (2.7b) follows from the Jacobi identity for the Lie bracket. Indeed, if for a given trilinear map $T : [\mathrm{TestV(M)}]^3 \to \mathrm{TestV(M)}'$ we put $\mathfrak{S}T(X,Y,Z) := T(X,Y,Z) + T(Y,Z,X) + T(Z,X,Y)$, we have

$$\begin{aligned}\mathfrak{S}\mathbf{R}(X,Y)(Z) &= \mathfrak{S}\boldsymbol{\nabla}_X\nabla_Y Z - \mathfrak{S}\boldsymbol{\nabla}_Y\nabla_X Z - \mathfrak{S}\boldsymbol{\nabla}_{[X,Y]}Z\\ &= \mathfrak{S}\boldsymbol{\nabla}_Z\nabla_X Y - \mathfrak{S}\boldsymbol{\nabla}_Z\nabla_Y X - \mathfrak{S}\boldsymbol{\nabla}_{[X,Y]}Z\\ &= \mathfrak{S}\boldsymbol{\nabla}_Z(\nabla_X Y - \nabla_Y X) - \mathfrak{S}\boldsymbol{\nabla}_{[X,Y]}Z\\ &= \mathfrak{S}\big(\boldsymbol{\nabla}_Z[X,Y] - \boldsymbol{\nabla}_{[X,Y]}Z\big)\\ &= \mathfrak{S}\mathbf{[}Z,[X,Y]\mathbf{]} = 0.\end{aligned}$$

We now claim that

$$\mathcal{R}(X,Y,Z,W) = -\mathcal{R}(X,Y,W,Z). \tag{2.8}$$

To prove this, let $f \in \mathrm{Test(M)}$ and notice that

$$\begin{aligned}\mathcal{R}(X,Y,Z,W)(f) &= \big(\mathbf{R}(X,Y)(Z)\big)(fW)\\ &= \int -\langle\nabla_X(fW),\nabla_Y Z\rangle - f\langle\nabla_Y Z,W\rangle\,\mathrm{div}(X)\\ &\qquad + \langle\nabla_Y(fW),\nabla_X Z\rangle + f\langle\nabla_X Z,W\rangle\,\mathrm{div}(Y) - f\langle\nabla_{[X,Y]}Z,W\rangle\,\mathrm{d}\mathfrak{m}.\end{aligned}$$

Add to this the equality obtained exchanging Z and W and observe that

$$\begin{aligned}&-\langle\nabla_X(fW),\nabla_Y Z\rangle - f\langle\nabla_Y Z,W\rangle\,\mathrm{div}(X) + \langle\nabla_Y(fW),\nabla_X Z\rangle + f\langle\nabla_X Z,W\rangle\,\mathrm{div}(Y)\\ &-\langle\nabla_X(fZ),\nabla_Y W\rangle - f\langle\nabla_Y W,Z\rangle\,\mathrm{div}(X) + \langle\nabla_Y(fZ),\nabla_X W\rangle + f\langle\nabla_X W,X\rangle\,\mathrm{div}(Y)\\ &\quad = -X(f)\Big(\langle\nabla_Y Z,W\rangle + \langle\nabla_Y W,Z\rangle\Big) - f\,\mathrm{div}(X)\Big(\langle\nabla_Y Z,W\rangle + \langle\nabla_Y W,Z\rangle\Big)\\ &\qquad + Y(f)\Big(\langle\nabla_X W,Z\rangle + \langle\nabla_X Z,W\rangle\Big) + f\,\mathrm{div}(Y)\Big(\langle\nabla_X Z,W\rangle + \langle\nabla_X W,Z\rangle\Big)\\ &\quad = -Y(\langle Z,W\rangle)\,\mathrm{div}(fX) + X(\langle Z,W\rangle)\,\mathrm{div}(fY),\end{aligned}$$

and that

$$-f\langle\nabla_{[X,Y]}Z,W\rangle - f\langle\nabla_{[X,Y]}W,Z\rangle = -f[X,Y](\langle Z,W\rangle).$$

Therefore, we have, after an integration by parts, that

$$\begin{aligned}&(\mathcal{R}(X,Y,Z,W) + \mathcal{R}(X,Y,W,Z))(f)\\ &\qquad = \int \langle Z,W\rangle\Big(\mathrm{div}\big(Y\,\mathrm{div}(fX) - X\,\mathrm{div}(fY)\big) + \mathrm{div}(f[X,Y])\Big)\,\mathrm{d}\mathfrak{m},\end{aligned}$$

and this latter expression vanishes due to (2.4). This proves our claim (2.8).

The equality between the first and last term in (2.7a) now follows from the first in (2.7a), (2.8) and (2.7b), indeed:

$$\begin{aligned}\mathcal{R}(X,Y,Z,W) &= -\mathcal{R}(Z,X,Y,W) - \mathcal{R}(Y,Z,X,W)\\ &= \mathcal{R}(Z,X,W,Y) + \mathcal{R}(Y,Z,W,X)\\ &= -\mathcal{R}(X,W,Z,Y) - \mathcal{R}(W,Z,X,Y) - \mathcal{R}(Z,W,Y,X) - \mathcal{R}(W,Y,Z,X)\\ &= 2\mathcal{R}(Z,W,X,Y) + \mathcal{R}(X,W,Y,Z) + \mathcal{R}(W,Y,X,Z)\\ &= 2\mathcal{R}(Z,W,X,Y) - \mathcal{R}(Y,X,W,Z)\\ &= 2\mathcal{R}(Z,W,X,Y) - \mathcal{R}(X,Y,Z,W),\end{aligned}$$

i.e. $2\mathcal{R}(X,Y,Z,W) = 2\mathcal{R}(Z,W,X,Y)$.

It remains to prove (2.7c). The chain of equalities

$$(f\mathcal{R}(X,Y,Z,W))(g) = \mathcal{R}(X,Y,Z,W)(fg) = \big(\mathbf{R}(X,Y)(Z)\big)(fgW) = \mathcal{R}(X,Y,Z,fW)(g),$$

valid for any $f,g \in \mathrm{Test(M)}$ shows that the first and last terms in (2.7c) coincide. The equality with the others then follows from (2.7a). □

Remark 2.8. We should not expect the trace of the sectional curvature to be equal to the Ricci curvature in any sense: this is not the case not even on weighted Riemannian manifolds, where the correct notion of Ricci tensor is the Lichnerowicz–Bakry–Émery one. This is so because the former is unchanged if we replace the volume measure with a different one, while the latter, as said, is affected by the weight. This 'invariance of the sectional curvature under modification of the measure' remains true even in our non-smooth context (albeit some care is needed, as the measure certainly plays a role for instance in defining the domain of the various differentiation operators – we omit the details) and is in line with Conjecture 1.1, which relates a bound on such curvature to the genuinely metric concept of Alexandrov space.

The fact that the Ricci curvature is not the trace of the sectional curvature should be compared to the fact that the trace of the Hessian of a function is not its Laplacian, in general. We believe that the Ricci curvature tensor coincides with the trace of the sectional curvature if and only if the Laplacian is the trace of the Hessian, but the verification of this fact is outside the scope of this note. We just remark that finite-dimensional RCD spaces for which the latter condition holds are called *weakly non-collapsed* RCD *spaces* and that they are conjectured to be *non-collapsed* RCD *spaces* (see [4, Remark 1.13] for the precise formulation and [7] for very recent important results in this direction).

Finally, let us mention that in the smooth category, in [17] it has been introduced a sectional curvature tensor on Riemannian manifolds, which is affected by a weight on the volume measure and for which comparison results are at our disposal, see also [9] for more recent results and relevant bibliography. It is outside the scope of this note to investigate whether there is an analogous of such construction in the RCD setting. ■

Remark 2.9. The use of the terminology 'distributional' that we made here is quite an abuse. Indeed, not only we certainly do not have C^∞ functions in this setting but, most importantly, we did not put any topology on the spaces $\mathrm{Test(M)}, \mathrm{TestV(M)}$, so that their dual have been considered only in the algebraic sense.

Still, we chose to stick to the use of 'distributional' because, in our opinion, it gives the idea of what is happening: by throwing derivatives on the appropriate test object, we can give a meaning to the Riemann tensor. In any case, it is not hard to put appropriate norms on both Test(M) and TestV(M), so that all the operators we considered are continuous. ■

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References

[1] E. Bruè, D. Semola, Constancy of the dimension for RCD(K, N) spaces via regularity of Lagrangian flows, Commun. Pure Appl. Math. (2019), in press, arXiv:1804.07128.
[2] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, Geom. Funct. Anal. 9 (1999) 428–517.
[3] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below. I, J. Differ. Geom. 46 (1997) 406–480.
[4] G. De Philippis, N. Gigli, Non-collapsed spaces with Ricci curvature bounded from below, J. Éc. Polytech. Math. 5 (2018) 613–650, arXiv:1708.02060.
[5] N. Gigli, Nonsmooth differential geometry – an approach tailored for spaces with Ricci curvature bounded from below, Mem. Amer. Math. Soc. 251 (2014), v+161.
[6] N. Gigli, C. Rigoni, Recognizing the flat torus among $RCD^*(0, N)$ spaces via the study of the first cohomology group, Calc. Var. Partial Differ. Equ. 57 (4) (2018) 104.
[7] B.-X. Han, Conformal transformation on metric measure spaces, Potential Anal. (2018).
[8] V. Kapovitch, C. Ketterer, CD meets CAT, Preprint, arXiv:1712.02839.
[9] L. Kennard, W. Wylie, D. Yeroshkin, The weighted connection and sectional curvature for manifolds with density, J. Geom. Anal. 29 (2019) 957–1001.
[10] K. Kuwae, Y. Machigashira, T. Shioya, Sobolev spaces, Laplacian, and heat kernel on Alexandrov spaces, Math. Z. 238 (2001) 269–316.
[11] A. Lytchak, S. Stadler, Conformal deformations of CAT(o) spaces, Math. Ann. 373 (1–2) (2019) 155–163.
[12] P. Petersen, Riemannian Geometry, third edition, Graduate Texts in Mathematics, vol. 171, Springer, Cham, Switzerland, 2016.
[13] A. Petrunin, Semiconcave functions in Alexandrov's geometry, in: Surveys in Differential Geometry. Vol. XI, in: Surv. Differ., vol. 11, Geom., Int. Press, Somerville, MA, USA, 2007, pp. 137–201.
[14] A. Petrunin, Alexandrov meets Lott–Villani–Sturm, Münster J. Math. 4 (2011) 53–64.
[15] G. Savaré, Self-improvement of the Bakry–Émery condition and Wasserstein contraction of the heat flow in RCD(K, ∞) metric measure spaces, Discrete Contin. Dyn. Syst. 34 (2014) 1641–1661.
[16] K.-T. Sturm, Ricci tensor for diffusion operators and curvature-dimension inequalities under conformal transformations and time changes, J. Funct. Anal. 275 (2018) 793–829.
[17] W. Wylie, Sectional curvature for Riemannian manifolds with density, Geom. Dedic. 178 (2015) 151–169.
[18] H.-C. Zhang, X.-P. Zhu, Ricci curvature on Alexandrov spaces and rigidity theorems, Commun. Anal. Geom. 18 (2010) 503–553.