



Algebraic geometry

Equivariant compactifications of vector groups with high index

*Compactifications équivariantes du groupe vectoriel de grand indice*Baohua Fu^{a,b}, Pedro Montero^c^a MCM, AMSS, Chinese Academy of Sciences, 55 ZhongGuanCun East Road, Beijing, 100190, China^b School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing, China^c Departamento de Matemática, Universidad Técnica Federico Santa María, Avenida España 1680, Valparaíso, Chile

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ABSTRACT

In this note, we classify smooth equivariant compactifications of \mathbb{G}_a^n that are Fano manifolds with index $\geq n - 2$.

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R É S U M É

Dans cette note, nous classifions les compactifications équivariantes lisses de \mathbb{G}_a^n qui sont des variétés de Fano d'indice $\geq n - 2$.

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1. Introduction

Let \mathbb{G}_a^n be the complex vector group of dimension n , i.e. \mathbb{C}^n equipped with the additive group structure. A *smooth equivariant compactification* of \mathbb{G}_a^n (SEC in abbreviation) is a \mathbb{G}_a^n -action $A : \mathbb{G}_a^n \times X \rightarrow X$ on a projective manifold X of dimension n with an open orbit $O \subset X$. Since \mathbb{G}_a^n contains no nontrivial finite subgroup, the open orbit O is isomorphic to \mathbb{C}^n . When our interest is on the underlying manifold X , we say that X is a SEC.

The study of SEC is started from [15], where a classification in dimension 3 and of Picard number one is obtained. Recently, a full classification of Fano 3-folds that are SEC is obtained in [16], while it seems difficult to pursue further in higher dimension. In [11], the first author and J.-M. Hwang introduced the notion of Euler-symmetric projective varieties, namely nondegenerate projective varieties admitting many \mathbb{C}^* -actions of Euler type. It is shown in [11] that they are equivariant compactifications of vector groups and they are classified by certain algebraic data (called symbol systems), while it remains the problem to translate the smoothness in terms of these algebraic data.

Recall that, for a Fano manifold X of dimension n , its *index* i_X is the greatest integer such that $-K_X = i_X H$ for some divisor H on X . It is well known that $i_X \leq n + 1$. By a series of works by Fujita ([12], [13], [14]), Mella ([21]), Mukai ([22]), and Wiśniewski ([24]), the classification of Fano n -folds with index $i_X \geq n - 2$ is known. Based on this, we will give a

E-mail addresses: bhfu@math.ac.cn (B. Fu), pedro.montero@usm.cl (P. Montero).

classification of SEC n -folds with index $\geq n - 2$. In the case of a prime Fano variety, that is, of a Fano variety with Picard number one, our result reads as follows (Theorem 1.1).

Theorem 1.1. *Let X be an n -dimensional SEC with Picard number one. Assume that $i_X \geq n - 2$. Then X is isomorphic to one of the following:*

- (1) 6 homogeneous varieties of algebraic groups: \mathbb{P}^n , \mathbb{Q}^n , $\text{Gr}(2, 5)$, $\text{Gr}(2, 6)$, \mathbb{S}_5 , $\text{Lag}(6)$.
- (2) 5 non-homogeneous varieties:
 - (2-a) smooth linear sections of $\text{Gr}(2, 5)$ of codimension 1 or 2.
 - (2-b) \mathbb{P}^4 -general¹ linear sections of \mathbb{S}_5 of codimension 1, 2 or 3.

Here \mathbb{S}_5 and $\text{Lag}(6)$ denote the 10-dimensional spinor variety and the 6-dimensional Lagrangian Grassmannian, respectively. As a corollary of Theorem 1.1, we obtain the following corollary.

Corollary 1.2. *Let X be a SEC n -fold of Picard number one. Assume that the VMRT at a general point of X is smooth (e.g., X is covered by lines) and $n \leq 5$. Then X is isomorphic to one of the following:*

\mathbb{P}^n , \mathbb{Q}^n , smooth linear sections of codimension 1 or 2 of $\text{Gr}(2, 5)$.

It is expected that for a SEC of Picard number one, its VMRT at a general point is always smooth ([11]).

To complete the classification of smooth equivariant compactifications of \mathbb{G}_a^n that are Fano manifolds with index $\geq n - 2$, it remains to consider Fano n -folds with index $\geq n - 2$ and Picard number $\rho \geq 2$. The main difficulty lies on Mukai fourfolds with $\rho \geq 2$, which were classified by Wiśniewski in [24]. In the last section, we will go through this list and finally classify those that can be SEC (Proposition 3.4). By our results in the present paper, to classify Fano 4-folds which are SEC, the only remaining case is Fano 4-folds of index 1 and with Picard number at least 2.

2. Picard number one case

Let X be a uniruled projective manifold. An irreducible component \mathcal{K} of the space of rational curves $\text{RatCurves}^n(X)$ on X (see [19, Chap. II, Definition 2.11]) is called a *minimal rational component* if the subvariety \mathcal{K}_x of \mathcal{K} parameterizing curves passing through a general point $x \in X$ is non-empty and proper. Curves parameterized by \mathcal{K} will be called *minimal rational curves*. Let $\rho : \mathcal{U} \rightarrow \mathcal{K}$ be the universal family and $\mu : \mathcal{U} \rightarrow X$ the evaluation map. The tangent map $\tau : \mathcal{U} \dashrightarrow \mathbb{P}T(X)$ is defined by $\tau(u) = [T_{\mu(u)}(\mu(\rho^{-1}\rho(u)))] \in \mathbb{P}T_{\mu(u)}(X)$. The closure $\mathcal{C} \subset \mathbb{P}T(X)$ of its image is the VMRT structure on X . The natural projection $\mathcal{C} \rightarrow X$ is a proper surjective morphism and a general fiber $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is called the *variety of minimal rational tangents* (VMRT for short) at the point $x \in X$. It is well known that $\dim \mathcal{C}_x = -K_X \cdot \ell - 2$, where $\ell \in \mathcal{K}$ is a general minimal rational curve through $x \in X$.

Example 1. An irreducible Hermitian symmetric space of compact type (I.H.S.S. for short) is a homogeneous space $M = G/P$ with a simple Lie group G and a maximal parabolic subgroup P such that the isotropy representation of P on $T_x(M)$ at a base point $x \in M$ is irreducible. The highest-weight orbit of the isotropy action on $\mathbb{P}T_x(M)$ is exactly the VMRT at x . The following table (e.g., [9, Section 3.1]) collects basic information on these varieties. By [1], these are all SEC among rational homogeneous manifolds G/P of Picard number one.

Type	I.H.S.S. M	VMRT S	$S \subset \mathbb{P}T_x(M)$
I	$\text{Gr}(a, a + b)$	$\mathbb{P}^{a-1} \times \mathbb{P}^{b-1}$	Segre
II	\mathbb{S}_n	$\text{Gr}(2, n)$	Plücker
III	$\text{Lag}(2n)$	\mathbb{P}^{n-1}	Veronese
IV	\mathbb{Q}^n	\mathbb{Q}^{n-2}	Hyperquadric
V	$\mathbb{O}\mathbb{P}^2$	\mathbb{S}_5	Spinor
VI	$E_7/(E_6 \times U(1))$	$\mathbb{O}\mathbb{P}^2$	Severi

Example 2. Fix two integers $k \geq 2$, $m \geq 1$. Let Σ be an $(m + 2k)$ -dimensional vector space endowed with a skew-symmetric 2-form ω of maximal rank. The symplectic Grassmannian $M = \text{Gr}_\omega(k, \Sigma)$ is the variety of all k -dimensional isotropic subspaces of Σ , which is not homogeneous if m is odd. Let W and Q be vector spaces of dimensions $k \geq 2$ and m respectively. Let \mathbf{t} be the tautological line bundle over $\mathbb{P}W$. The VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(M)$ of $\text{Gr}_\omega(k, \Sigma)$ at a general point is isomorphic to the projective bundle $\mathbb{P}((Q \otimes \mathbf{t}) \oplus \mathbf{t}^{\otimes 2})$ over $\mathbb{P}W$ with the projective embedding given by the complete linear system

¹ See [10, Definition 2.5].

$$H^0(\mathbb{P}W, (Q \otimes \mathbf{t}^*) \oplus (\mathbf{t}^*)^{\otimes 2}) = (W \otimes Q)^* \oplus \text{Sym}^2 W^*.$$

Alternatively, C_x is isomorphic to the blowup of $\mathbb{P}^{(m+k-1)}$ along some linear space, hence it is a SEC.

Recall that a subvariety $X \subset \mathbb{P}V$ is called *conic-connected* if through two general points there passes an irreducible conic.

Proposition 2.1 (J.-M. Hwang). *Let $X \subset \mathbb{P}V$ be a conic-connected smooth subvariety. Then X is a SEC if and only if $X \subset \mathbb{P}V$ is isomorphic to one of the following or their biregular projections:*

- (1) the VMRT of an irreducible Hermitian symmetric space of compact type;
- (2) the VMRT of a symplectic Grassmannian $\text{Gr}_\omega(k, 2k + m)$ for $1 \leq m, 2 \leq k$;
- (3) a nonsingular linear section of $\text{Gr}(2, 5) \subset \mathbb{P}^9$ of codimension ≤ 2 ;
- (4) a \mathbb{P}^4 -general linear section of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ of codimension ≤ 3 .

Proof. Assume first that X is not prime Fano, then by [17, Theorem 2.2], $X \subset \mathbb{P}V$ is projectively equivalent to one of the following or their biregular projections:

- (a1) the second Veronese embedding of \mathbb{P}^n ;
- (a2) the Segre embedding of $\mathbb{P}^a \times \mathbb{P}^{n-a}$ for $1 \leq a \leq n - 1$;
- (a3) the VMRT of the symplectic Grassmannian $\text{Gr}_\omega(k, 2k + m)$ for $1 \leq m, 2 \leq k$;
- (a4) A hyperplane section of the Segre embedding $\mathbb{P}^a \times \mathbb{P}^{n+1-a}$ with $2 \leq \min\{a, n + 1 - a\}$.

The case (a4) is not a SEC by the proof of [9, Proposition 6.3].

Now assume that X is a prime Fano manifold. Then it is an Euler symmetric variety by [11, Corollary 5.6]. Let r be the rank of X . By [11, Theorem 3.7], the r -th fundamental form at a general point $x \in X$ is non-zero. This implies that there exists a hyperplane section H such that $\text{mult}_x(H) = r$. Hence, for any curve $C \not\subset H$ lying on X through x , we have $H \cdot C \geq r$. By our assumption, X is conic-connected, hence the conics through x cover X . Let C be a general such conic, then we get $2 = H \cdot C \geq r$. This implies that $r = 2$, hence $X \subset \mathbb{P}V$ is quadratically symmetric. Our claim follows then from [10, Proposition 7.11, Theorem 7.13].

Recall that for a Fano manifold X of dimension n , its *index* i_X is the greatest integer such that $-K_X = i_X H$ for some divisor H on X . By Kobayashi–Ochiai’s theorem, we have $i_X \geq n$ if and only if X is either \mathbb{P}^n or \mathbb{Q}^n . A Fano manifold X is called *del Pezzo* (resp. *Mukai*) if $i_X = n - 1$ (resp. $i_X = n - 2$).

Proposition 2.2. *Let X be a SEC of $\rho_X = 1$. Assume that $\dim X \geq 2$ and that the VMRT at a general point of X is smooth. Then $i_X \geq 3$.*

Proof. Let $D \subset X$ be the boundary divisor, which is irreducible since $\rho_X = 1$. By [15, Theorem 2.5], we have $\text{Pic}(X) = \mathbb{Z}D$, hence $-K_X = i_X D$. By [15, Theorem 2.7], we have $i_X \geq 2$. Let ℓ be a minimal rational curve through a general point. Then we have $\ell \cdot D = 1$ by [11, Proposition 5.4 (v)]. This implies that $i_X = -K_X \cdot \ell$. Assume that $i_X = 2$. Then one has $-K_X \cdot \ell = 2$. Hence, there exists only finitely many minimal rational curves passing through a general point, that is, the VMRT C_x consists of finitely many points for $x \in X$ general. By [11, Proposition 5.4 (ii)], C_x is irreducible and linearly non-degenerate, which yields a contradiction. This gives $i_X \geq 3$.

Remark 1. It is expected that the assumption on the smoothness of the VMRT at a general point of X in Corollary 1.2 and Proposition 2.2 is always fulfilled. See [11, Conjecture 5.7].

Proposition 2.3. *Let X be an n -dimensional del Pezzo manifold of Picard number one. Then X is a SEC if and only if X is a smooth linear section of $\text{Gr}(2, 5)$ of codimension ≤ 2 .*

Proof. As $\rho_X = 1$, X is isomorphic to one of the following by Fujita’s classification ([12], [13], [14]),

- (1) cubic hypersurface in \mathbb{P}^{n+1} ;
- (2) complete intersection of 2 quadrics in \mathbb{P}^{n+2} ;
- (3) a hypersurface of degree 4 in $\mathbb{P}(2, 1, \dots, 1)$;
- (4) a hypersurface of degree 6 in $\mathbb{P}(3, 2, 1, \dots, 1)$;
- (5) smooth linear sections of $\text{Gr}(2, 5)$.

It’s well-known that for X in (1) and (2), the group $\text{Aut}^\circ(X)$ is trivial, hence X is not a SEC. In case (3), it is a double cover of \mathbb{P}^n ramified along a smooth hypersurface of degree 4, hence its automorphism group is finite by [20, Theorem 4.5]. In case (4), the linear system of the ample generator of $\text{Pic}(X)$ gives a rational map $X \dashrightarrow \mathbb{P}^{n-1}$ by [14], which is not

birational, hence it is not a SEC by [11, Proposition 5.4 (vi)]. In case (5), any smooth linear section of X of codimension ≤ 2 is a SEC by [10, Proposition 2.11] since it is Euler symmetric. By Proposition 2.2, we have $i_X = n - 1 \geq 3$, hence $n \geq 4$. This shows that the smooth linear sections of $\text{Gr}(2, 5)$ of codimension 3 and 4 are not SEC.

Lemma 2.4 ([8, Satz 8.11]). *Let X be a smooth variety of dimension n , which is a complete intersection in a weighted projective space. Then $H^p(X, \Omega_X^q(t)) = 0$ if $p + q < n$ and $t < q - p$.*

Corollary 2.5. *Let X be a Mukai manifold of $\rho_X = 1$. Assume that X is a complete intersection in a weighted projective space. Then $\text{Aut}(X)$ is discrete.*

Proof. By assumption, we have $-K_X = \mathcal{O}_X(n - 2)$. As $T_X \simeq \Omega_X^{n-1} \otimes K_X^* = \Omega_X^{n-1}(n - 2)$, we have $H^0(X, T_X) = 0$ by Lemma 2.4.

The following lemma is well known for general sections, but we need it for any (smooth) section.

Lemma 2.6. *Any linear section of codimension ≤ 5 (resp. ≤ 3) of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ (resp. $\text{Gr}(2, 6) \subset \mathbb{P}^{14}$) is conic-connected.*

Proof. By [26, Chap. III, Proposition 2.19], any two points of $\mathbb{S}_5 \subset \mathbb{P}^{15}$ can be joined by a smooth quadric of dimension 6 contained in \mathbb{S}_5 , which implies the first assertion. Consider the Severi variety $X = \text{Gr}(2, 6) \subset \mathbb{P}^{14}$. By [26, Chap. IV, Theorem 2.4 (b)] any two points $x, y \in X$ such that the line \overline{xy} is not on X are joined by a smooth quadric in X of dimension 4. Take any linear section X' of X of codimension ≤ 3 . If for two general points $x', y' \in X'$, the line $\overline{x'y'}$ is not on X , then x', y' are joined by a \mathbb{Q}^4 on X , and therefore they are joined by a conic on X' , which implies that X' is conic-connected. Now assume that for $x', y' \in X'$ general, the line $\overline{x'y'}$ is on X , then it is also contained in X' as X' is a linear section of X . This implies that X' is a projective space of dimension at least 5, which is not possible since X does not contain any \mathbb{P}^5 . This concludes the proof.

Proposition 2.7. *Let X be an n -dimensional Mukai variety with $\rho_X = 1$. Then X is a SEC if and only if X is one of the following*

- (i) a \mathbb{P}^4 -general linear section of the 10-dimensional spinor variety \mathbb{S}_5 of codimension ≤ 3 ;
- (ii) the 8-dimensional Grassmannian $\text{Gr}(2, 6)$;
- (iii) the 6-dimensional Lagrangian Grassmannian $\text{Lag}(6)$.

Proof. When $n = 3$, then X is isomorphic to \mathbb{P}^3 or \mathbb{Q}^3 by [15]. Now assume $n \geq 4$. By Mukai’s classification [22], X is either a complete intersection in a weighted projective space, or a smooth linear section of one of the following varieties:

- (a) a quadric section of the cone over $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
- (b) the 5-dimensional Fano contact manifold $G_2/P_2 \subset \mathbb{P}^{13}$;
- (c) the 6-dimensional Lagrangian Grassmannian $\text{Lag}(6) \subset \mathbb{P}^{13}$;
- (d) the 10-dimensional spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$;
- (e) the 8-dimensional Grassmannian $\text{Gr}(2, 6) \subset \mathbb{P}^{14}$.

By Corollary 2.5, we only need to consider cases (a)–(e). In case (a), the smooth linear sections of X are called *Gushel–Mukai varieties*. By [7, Proposition 3.19 (c)], they have finite automorphism groups, hence they are not SEC.

For the remaining cases, X is covered by lines, so it has smooth VMRT at general points. By Proposition 2.2, we have $i_X = n - 2 \geq 3$, hence $n \geq 5$.

In case (b), its VMRT at a general point is linearly degenerate, hence it cannot be a SEC by [11, Proposition 5.4 (ii)].

In case (c), a smooth hyperplane section X of $\text{Lag}(6)$ is a compactification of a symmetric variety with $\text{Aut}^\circ = \text{SL}_3$ by [23, Theorem 3]. As SL_3 does not contain any subgroup isomorphic to \mathbb{G}_a^5 , the variety X is not a SEC. Hence only $\text{Lag}(6)$ itself is a SEC.

In cases (d) and (e), X is conic-connected by Lemma 2.6, hence by Proposition 2.1, X is as in (i) and (ii).

This concludes the proof of Theorem 1.1. Now Corollary 1.2 follows by virtue of Proposition 2.2.

3. Higher-Picard-number case

Proposition 3.1. *Let X be a Fano manifold of dimension n with index $i_X \geq (n + 1)/2$. If $\rho_X \geq 2$, then X is a SEC if and only if it is one of the following:*

$$(*) \quad \mathbb{P}^{\frac{n}{2}} \times \mathbb{P}^{\frac{n}{2}}, \quad \mathbb{P}^{\frac{n-1}{2}} \times \mathbb{Q}^{\frac{n+1}{2}}, \quad \mathbb{P}_{\mathbb{P}^{\frac{n+1}{2}}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n-1}{2}}).$$

Proof. By [25], a Fano manifold with $i_X \geq (n + 1)/2$ and $\rho_X \geq 2$ is one of the varieties in the list (*) or the homogeneous variety $\mathbb{P}T_{\mathbb{P}^{\frac{n+1}{2}}}$, while the latter is not a SEC by [1]. The first two varieties in (*) are SEC. The projective bundle $\mathbb{P}_{\mathbb{P}^{\frac{n+1}{2}}}(\mathcal{O}(1) \oplus \mathcal{O}^{\frac{n-1}{2}})$ is isomorphic to the blowup of \mathbb{P}^n along a linear $\mathbb{P}^{\frac{n-3}{2}}$, which is a SEC.

As immediate corollaries, we have the two following ones.

Corollary 3.2. *Let X be a del Pezzo manifold with $\rho_X \geq 2$, then X is a SEC if and only if X is one of the following:*

- (a) blowup of \mathbb{P}^2 at 1 or 2 points;
- (b) $\mathbb{P}^2 \times \mathbb{P}^2$;
- (c) blowup of \mathbb{P}^3 at 1 point;
- (d) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Corollary 3.3. *Let X be a Mukai manifold with $\rho_X \geq 2$. Assume that $\dim X \geq 5$, then X is a SEC if and only if X is one of the following:*

- (a) $\mathbb{P}^3 \times \mathbb{P}^3$;
- (b) $\mathbb{P}^2 \times \mathbb{Q}^3$;
- (c) blowup of \mathbb{P}^4 at a point.

Notice that the Fano SEC in dimension 3 are fully classified in [16], while the Mukai fourfolds are classified by Wiśniewski [24]. To complete the picture, it remains to determine which Fano fourfolds in the list of [24] are SEC.

Proposition 3.4. *Let X be a Mukai fourfold with $\rho_X \geq 2$. Then X is a SEC if and only if X is one of the following:*

- (a) $\mathbb{P}^1 \times \mathbb{P}^3$;
- (b) $\mathbb{P}^1 \times \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$;
- (c) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- (d) blowup of \mathbb{Q}^4 along a line;
- (e) $\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}(-1) \oplus \mathcal{O})$;
- (f) $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$.

Proof. By Wiśniewski’s classification [24] (see [18, Table 12.7]), X is isomorphic to one of the following varieties,

- (1) $\mathbb{P}^1 \times V$, where $V \cong \mathbb{P}T_{\mathbb{P}^2}$ or $V \cong V_d$ is a del Pezzo threefold of degree d with $1 \leq d \leq 5$ and $\rho_{V_d} = 1$;
- (2) $\mathbb{P}^1 \times V$, where V is either \mathbb{P}^3 , or $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(1) \oplus \mathcal{O})$ (the blowup of \mathbb{P}^3 at a point), or $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$;
- (3) a Verra fourfold, that is, a double cover of $\mathbb{P}^2 \times \mathbb{P}^2$ whose branch locus is a divisor of bidegree $(2, 2)$;
- (4) a divisor on $\mathbb{P}^2 \times \mathbb{P}^3$ of bidegree $(1, 2)$;
- (5) an intersection of two divisors of bidegree $(1, 1)$ on $\mathbb{P}^3 \times \mathbb{P}^3$;
- (6) a divisor on $\mathbb{P}^2 \times \mathbb{Q}^3$ of bidegree $(1, 1)$;
- (7) the blowup of \mathbb{Q}^4 along a conic which is not contained in a plane lying on \mathbb{Q}^4 ;
- (8) $\mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$, where \mathcal{N} is the null-correlation bundle on \mathbb{P}^3 ;
- (9) the blowup of \mathbb{Q}^4 along a line;
- (10) $\mathbb{P}_{\mathbb{Q}^3}(\mathcal{O}(-1) \oplus \mathcal{O})$;
- (11) $\mathbb{P}_{\mathbb{P}^3}(\mathcal{O}(-1) \oplus \mathcal{O}(1))$.

In case (1), it follows from Blanchard’s lemma [5, Theorem 7.2.1] that an effective action of \mathbb{G}_a^4 on $X \cong \mathbb{P}^1 \times V$ descends in a unique way to an action of \mathbb{G}_a^4 on V making the second projection an equivariant morphism. The image of the latter action is isomorphic to \mathbb{G}_a^3 making V a SEC, while by [15, Theorem 6.1] the only SEC threefolds with Picard number one are \mathbb{P}^3 and \mathbb{Q}^3 . We conclude therefore that the variety X is not a SEC. In case (2), the listed varieties are clearly SEC.

In case (3), an effective action of \mathbb{G}_a^4 on X induces the inclusions $\mathbb{G}_a^4 \subseteq \text{Aut}_L(B) \subseteq \text{Aut}(\mathbb{P}^2 \times \mathbb{P}^2)$, where $\text{Aut}_L(B)$ stands for the group of automorphisms of the branch locus B induced by automorphisms of $\mathbb{P}^2 \times \mathbb{P}^2$. Indeed, this follows verbatim from the proof of [20, Lemma 4.1, Lemma 4.2, Proposition 4.3] replacing \mathbb{P}^n by $\mathbb{P}^n \times \mathbb{P}^n$. In particular, we obtain in this way an effective action of \mathbb{G}_a^4 on $\mathbb{P}^2 \times \mathbb{P}^2$ that fixes the branch locus B . We know on the other hand, after Hassett and Tschinkel [15, Proposition 3.2], that the boundary divisors for the possible effective actions of \mathbb{G}_a^4 on $\mathbb{P}^2 \times \mathbb{P}^2$ are of bidegree $(1, 0)$ and $(0, 1)$, hence X cannot be a SEC.

In cases (4), (5) and (6), the variety X is isomorphic to the projectivization $\mathbb{P}_Y(\mathcal{B})$ of a reflexive non-locally free sheaf \mathcal{B} on a smooth variety Y . Such sheaves are called Bănică sheaves in [4, Section 2], where it is shown that the canonical map $X \rightarrow Y$ is a Mori contraction with connected but not equidimensional fibers. The general strategy to prove that none of these cases give rise to a SEC will be to analyze the points where the dimension of the fibers of $X \rightarrow Y$ jumps.

It follows from [4, Theorem 6.8] that, for X as in case (4), we have $Y \cong \mathbb{P}^3$ and the canonical fibration $X \rightarrow \mathbb{P}^3$ has eight fibers isomorphic to \mathbb{P}^2 . By Blanchard's lemma, an effective action of \mathbb{G}_a^4 on X induces a unique action on \mathbb{P}^3 for which $X \rightarrow \mathbb{P}^3$ is equivariant. As before, there is an induced effective action of \mathbb{G}_a^3 making \mathbb{P}^3 a SEC. On the one hand, we notice that the eight points $p_1, \dots, p_8 \in \mathbb{P}^3$ having 2-dimensional fibers are fixed and hence contained in the boundary hyperplane divisor $H \cong \mathbb{P}^2 \subset \mathbb{P}^3$. On the other hand, if we write

$$X = \left\{ \sum_{i=0}^2 x_i q_i(y_0, y_1, y_2, y_3) = 0 \right\} \subset \mathbb{P}_{\mathbf{x}}^2 \times \mathbb{P}_{\mathbf{y}}^3,$$

where the q_i are quadratic forms, we have that $X \rightarrow \mathbb{P}^3$ is induced by the second projection and hence it has 2-dimensional fibers over the set $S = \{q_0(\mathbf{y}) = q_1(\mathbf{y}) = q_2(\mathbf{y}) = 0\} = \{p_1, \dots, p_8\}$. We claim that S is not contained in a hyperplane H and hence X is not a SEC. Indeed, if we assume that $S \subset H$ and we denote by Q_i the hyperquadric $\{q_i(\mathbf{y}) = 0\}$, then $L_i = Q_i \cap H$ is a (possibly reducible) curve of degree 2 in H and $L_{ij} = Q_i \cap Q_j$ is a curve of degree 4 in \mathbb{P}^3 for $i \neq j$. Since $S \subset L_{ij} \cap H$, and S is a 0-dimensional scheme of length 8, it follows that L_{ij} has a common component, say N_{ij} , contained in H and thus L_i and L_j have a common component for $i \neq j$. Since each of the L_i is a curve of degree 2, we have that if $L_0 \cap L_1 \cap L_2$ is 0-dimensional, then each L_i is reducible and given by the union of two lines in H . We can easily verify that $L_0 \cap L_1 \cap L_2$ is of length 3 and contains S , which is absurd. We conclude therefore that $Q_0 \cap Q_1 \cap Q_3$ have a common component. However, this is not possible as X is irreducible. The cases (5) and (6) are similar but easier: in the former case, we have $Y \cong \mathbb{P}^3$ and the canonical fibration $X \rightarrow \mathbb{P}^3$ has four fibers isomorphic to \mathbb{P}^2 by [4, Theorem 6.8]. More precisely, for each $p_1, \dots, p_4 \in \mathbb{P}^3$ having 2-dimensional fiber, the fiber is given by the dual hyperplane H_{p_i} in $(\mathbb{P}^3)^\vee$ determined by p_i . Since each of the points p_i is fixed, we deduce that each H_{p_i} is invariant under the induced action of \mathbb{G}_a^3 in $(\mathbb{P}^3)^\vee$. Thus, we get four different invariant divisors, a contradiction. In the latter case, we have $Y \cong \mathbb{Q}^3$, and the canonical fibration $X \rightarrow \mathbb{Q}^3$ has two fibers isomorphic to \mathbb{P}^2 . Hence, the result follows from the fact [15, Theorem 6.1] that there is a unique effective action of \mathbb{G}_a^3 making \mathbb{Q}^3 a SEC with a unique fixed point.

In cases (7) and (9), the variety X is isomorphic to the blowup of \mathbb{Q}^4 along a smooth curve $C \subset \mathbb{Q}^4$. By Blanchard's lemma, X is a SEC if and only if C is invariant under the unique effective action of \mathbb{G}_a^4 on \mathbb{Q}^4 . A simple computation in coordinates shows that the only invariant smooth curves on \mathbb{Q}^4 are lines. Thus in case (7) X is not a SEC, while in case (9) it is.

In case (8), it follows from [6, Theorem 3.1] that $X \cong \mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$ is isomorphic a homogeneous space G/P . Hence it follows from [1] that X is not a SEC.

In cases (10) and (11), there is a blowdown $X \rightarrow Z$ sending the divisor corresponding to a section of the \mathbb{P}^1 -bundle structure of X to a point $z \in Z$, where $Z \cong \mathbb{Q}_0^4$ is the cone over \mathbb{Q}^3 in \mathbb{P}^5 and $Z \cong \mathbb{P}(1, 1, 1, 1, 2)$, respectively, and $z \in Z$ is the only singular point of each of these varieties. In both cases, Z is a SEC (cf. [2, Section 6], [3, Proposition 2]) and $z \in Z$ is a fixed point under the respective actions, since it is the only singular point. We conclude therefore that in both cases (10) and (11) X is a SEC.

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