Number theory

Some hypergeometric functions in positive characteristic and transcendence

Quelques fonctions hypergéométriques en caractéristique positive et transcendance

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A R T I C L E   I N F O

A B S T R A C T

In this work, we study some special hypergeometric functions in positive characteristic, introduced by D. S. Thakur. We shall establish functional relationships among them, and deduce, with the help of the function field version of the Schneider–Lang theorem obtained by J. Yu, that at least one of their values at nonzero algebraic arguments is transcendental.

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R É S U M É

Nous étudions dans ce travail certaines fonctions hypergéométriques spéciales en caractéristique positive, introduites par D. S. Thakur. Nous allons établir des relations fonctionnelles auxquelles elles satisfont, et déduire, à l'aide de la version pour les corps de fonctions du théorème de Schneider–Lang obtenue par J. Yu, qu'au moins l'une de leurs valeurs aux arguments algébriques non nuls est transcendante.

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Version française abrégée

Désignons par $\mathbb{F}_q$ le corps fini à q éléments, par $\mathbb{F}_q[T]$ l'anneau intègre des polynômes en T à coefficients dans $\mathbb{F}_q$ et par $k := \mathbb{F}_q(T)$ le corps de fractions de $\mathbb{F}_q[T]$. Pour tous les $P, Q \in \mathbb{F}_q[T]$ avec $Q \neq 0$, nous définissons $|P/Q|_\infty :=$ $q^{\deg P - \deg Q}$, et appelons $|\cdot|_\infty$ la valeur absolue ∞-adique sur $\mathbb{F}_q(T)$. Désignons par $\mathbb{F}_q((T^{-1}))$ le complété topologique de $\mathbb{F}_q(T)$ pour $|\cdot|_\infty$, et par $\mathbf{C}_\infty$ le complété topologique d'une clôture algébrique fixée de $\mathbb{F}_q((T^{-1}))$. Finalement, notons $\bar{k}$ la clôture algébrique de k dans $\mathbf{C}_\infty$.

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Posons $D_0 := 1$, et $D_n := \prod_{k=0}^{n-1}(T^{q^n} - T^{q^k})$ pour tout entier $n \geqslant 1$. Définissons $(a)_n = D_{n+a-1}^{q^{-(a-1)}}$, pour tout $a \in \mathbb{Z}_{>0}$. Soient $r, s \geqslant 0$ des entiers. Pour tous les entiers $a_i, b_j > 0$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$, la fonction hypergéométrique $_r F_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z)$ est définie par

$$_r F_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n}.$$

Elle a été introduite pour la première fois par D. S. Thakur [9] afin d'imiter sa contrepartie dans le cas réel, et elle partage de nombreuses propriétés avec cette dernière. Pour en savoir plus sur cette fonction ainsi que sa motivation, le lecteur intéressé peut consulter [9] (voir aussi [10]).

Si $r = s = 0$, nous obtenons l'exponentielle de Carlitz

$$_0 F_0(z) = e_C(z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n}.$$

Si $r = 0$, $s = 1$, et $b_1 = m + 1$, nous obtenons la fonction de Bessel–Carlitz

$$J_m(z) := \sum_{n=0}^{+\infty} \frac{z^{q^{m+n}}}{D_{m+n} D_n^{q^m}},\ \text{et } _0 F_1(-; m + 1; z) = J_m^{q^{-m}}.$$

Notons que la normalisation ici est légèrement différente de celle de [6].

Nous inspirant du travail de L. Denis [6], nous considérons la fonction hypergéométrique spéciale suivante

$$_0 F_s(-; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n (b_1)_n \cdots (b_s)_n},$$

et nous étudions ses valeurs spéciales aux arguments algébriques non nuls.

Voici les résultats principaux.

Théorème 1. *Soient $s \geqslant 1$ et $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ des entiers. Pour tout $\gamma \in \overline{k}$ non nul, nous avons*

$$\mathrm{tr.deg}_k\big(F_0(\gamma),\ F_0(T\gamma),\ \ldots,\ F_0(T^s\gamma)\big) \geqslant 1,$$

où nous posons $F_0(z) = {}_0F_s(-; b_1, b_2, \ldots, b_s; z)$.

Théorème 2. *Soient $s \geqslant 1$ et $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ des entiers. Pour tout $\gamma \in \overline{k}$ non nul, nous avons*

$$\mathrm{tr.deg}_k\big(F_0(\gamma), F_1(\gamma),\ \ldots,\ F_s(\gamma)\big) \geqslant 1,$$

où nous posons $F_j(z) = {}_0F_s(-; b_1, \ldots, b_{j-1}, b_j + 1, \ldots, b_s + 1; z)$, pour $1 \leqslant j \leqslant s$.

1. Statements of the main results

Let $\mathbb{F}_q$ be the finite field with q element, $k := \mathbb{F}_q(T)$ the rational function field over $\mathbb{F}_q$ with T as the indeterminate, and $k_\infty := \mathbb{F}_q((\frac{1}{T}))$ the field of formal Laurent series in $1/T$ over $\mathbb{F}_q$. The latter is also the topological completion of k for the canonical ∞-adic absolute value. Finally, we let $\mathbf{C}_\infty$ denote the topological completion of a fixed algebraic closure of k_∞, and $\overline{k}$ denote the algebraic closure of k in $\mathbf{C}_\infty$.

Set $D_0 := 1$, and $D_n := \prod_{k=0}^{n-1}(T^{q^n} - T^{q^k})$ for all integers $n \geqslant 1$. Set $(a)_n = D_{n+a-1}^{q^{-(a-1)}}$, for all $a \in \mathbb{Z}_{>0}$. Let $r, s \geqslant 0$ be integers. For all integers $a_i, b_j > 0$ $(1 \leqslant i \leqslant r,\ 1 \leqslant j \leqslant s)$, the hypergeometric function $_r F_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z)$ is defined as

$$_r F_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{(a_1)_n \cdots (a_r)_n}{D_n (b_1)_n \cdots (b_s)_n} z^{q^n}.$$

It was introduced firstly by D. S. Thakur [9] to imitate its classical counterpart in the real case, and shares many properties with the latter. For more details on this function and its motivation, the interested reader can consult [9] (see also [10]).

If $r = s = 0$, then we obtain the Carlitz exponential

$$_0 F_0(z) = e_C(z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n}.$$

If $r = 0$, $s = 1$, and $b_1 = m + 1$, then we get the Bessel–Carlitz function

$$J_m(z) := \sum_{n=0}^{+\infty} \frac{z^{q^{m+n}}}{D_{m+n} D_n^{q^m}}, \text{ and } {}_0F_1(-; m+1; z) = J_m^{q^{-m}}.$$

Note that the normalization here is slightly different from that of [6].

As for the classical real case, a natural question is to determine the algebraic nature of the values of the above functions at nonzero algebraic arguments. L. I. Wade showed in [13] that $e_C(\gamma)$ is transcendental over k for all nonzero $\gamma \in \bar{k}$, and his method is now well known under his name. This result is the exact analog of the classical Hermite–Lindemann theorem (see, for example, [8, p. 37]). L. Denis showed in [6] that $J_m(\gamma)$ is transcendental over k for all nonzero $\gamma \in \bar{k}$. His method is a part of the well-known motivic method (see, for example, [18,4] for more details). For the general case, D. S. Thakur, Z.-Y. Wen, J.-Y. Yao, and L. Zhao obtained in [12] (see also [11]) a new transcendence criterion by Diophantine approximation that unifies Wade's method, and showed that $_rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; \gamma)$ is transcendental over k, if $r < s+1$ and $\gamma \in \mathbf{C}_\infty$ is nonzero and algebraic such that the separable degree $[k(\gamma), k]_s < q$. The last result has been extended recently by J.-Y. Yao [15] to fractional hypergeometric functions, defined by D. S. Thakur [10, p. 226]. Besides the above three methods, the automata method introduced by J.-P. Allouche [1] is also important in the transcendence theory over function fields. For more details on this subject, see also [5,1,2,7,3].

Inspired by the work of L. Denis [6], we shall consider the following special hypergeometric function

$$ {}_0F_s(-; b_1, \ldots, b_s; z) := \sum_{n=0}^{+\infty} \frac{z^{q^n}}{D_n (b_1)_n \cdots (b_s)_n}, $$

and study its special values at algebraic nonzero arguments.

Our main results are the following, which rely heavily on the function field version of the Schneider–Lang theorem obtained by J. Yu [17].

Theorem 1. *Let $s \geqslant 1$ and $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ be integers. For all nonzero $\gamma \in \bar{k}$, we have*

$$\text{tr.deg}_k\big(F_0(\gamma), \, F_0(T\gamma), \, \ldots, \, F_0(T^s\gamma)\big) \geqslant 1,$$

where we set $F_0(z) = {}_0F_s(-; b_1, b_2, \ldots, b_s; z)$.

Theorem 2. *Let $s \geqslant 1$ and $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ be integers. For all nonzero $\gamma \in \bar{k}$, we have*

$$\text{tr.deg}_k\big(F_0(\gamma), F_1(\gamma), \, \ldots, \, F_s(\gamma)\big) \geqslant 1,$$

where we set $F_j(z) = {}_0F_s(-; b_1, \ldots, b_{j-1}, b_j + 1, \ldots, b_s + 1; z)$, for $1 \leqslant j \leqslant s$.

2. Some preliminary results

In this section, as above, let $s \geqslant 1$ and $0 < b_1 \leqslant b_2 \leqslant \cdots \leqslant b_s$ be integers. Set $f_0(z) = (F_0(z))^{q^{b_s}-1}$, and $f_j(z) = (F_j(z))^{q^{b_s}}$ for $1 \leqslant j \leqslant s$. Inspired by the work of L. Denis [6] about the Bessel–Carlitz function, we obtain the following result, which can be deduced directly from the work of D. S. Thakur [9,10].

Lemma 1. *The functions $f_j(z)$ ($0 \leqslant j \leqslant s$) satisfy the following functional equation:*

$$\begin{bmatrix} f_0(Tz) \\ f_1(Tz) \\ f_2(Tz) \\ \vdots \\ f_s(Tz) \end{bmatrix} = \begin{bmatrix} T^{q^{b_s}-1} & 1 & \ldots & 0 \\ 0 & T^{q^{b_s}-b_1} & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & T \end{bmatrix} \begin{bmatrix} f_0(z) \\ f_1(z) \\ f_2(z) \\ \vdots \\ f_s(z) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \end{bmatrix} \begin{bmatrix} f_0(z) \\ f_1(z) \\ f_2(z) \\ \vdots \\ f_s(z) \end{bmatrix}^q.$$

Proof. For all functions $f(z)$ and for $a \in \mathbb{Z}$, define

$$\Delta f(z) := f(Tz) - T f(z), \text{ and } \Delta_a f(z) := f(Tz) - T^{q^{-a}} f(z).$$

By the formulas in [9] with $r = 0$ (see also [10, pp. 229–230]), for all integers $c_j > 0$, we have

$$\Delta({}_0F_s(-; c_1, \ldots, c_s; z)) = ({}_0F_s(0; c_1 + 1, \ldots, c_s + 1; z))^q,$$

$$\Delta_{c_j-1}({}_0F_s(-; c_1, \ldots, c_j, \ldots, c_s; z)) = {}_0F_s(-; c_1, \ldots, c_j - 1, \ldots, c_s; z), \text{ if } c_j > 1,$$

from which and by definition, we deduce

$$
\begin{aligned}
f_0(Tz) &= ({}_0F_s(-; b_1, \ldots, b_s; Tz))^{q^{b_s-1}} = \left(T({}_0F_s(-; b_1, \ldots, b_s; z)) + \Delta({}_0F_s(-; b_1, \ldots, b_s; z))\right)^{q^{b_s-1}} \\
&= T^{q^{b_s-1}}({}_0F_s(-; b_1, \ldots, b_s; z))^{q^{b_s-1}} + ({}_0F_s(-; b_1+1, \ldots, b_s+1; z))^{q^{b_s}} \\
&= T^{q^{b_s-1}} f_0(z) + f_1(z), \\
f_j(Tz) &= (F_j(Tz))^{q^{b_s}} = \left(T^{q^{-b_j}} F_j(z) + \Delta_{b_j} F_j(z)\right)^{q^{b_s}} \\
&= T^{q^{b_s-b_j}}(F_j(z))^{q^{b_s}} + (F_{j+1}(z))^{q^{b_s}} \\
&= T^{q^{b_s-b_j}} f_j(z) + f_{j+1}(z), \text{ for } 1 \leqslant j \leqslant s-1, \\
f_s(Tz) &= ({}_0F_s(-; b_1, \ldots, b_s+1; Tz))^{q^{b_s}} \\
&= = \left(T^{q^{-b_s}}({}_0F_s(-; b_1, \ldots, b_s+1; z)) + \Delta_{b_s}({}_0F_s(-; b_1, \ldots, b_s+1; z))\right)^{q^{b_s}} \\
&= T({}_0F_s(-; b_1, \ldots, b_s+1; z))^{q^{b_s}} + ({}_0F_s(-; b_1, \ldots, b_s; z))^{q^{b_s}} \\
&= T f_s(z) + (f_0(z))^q.
\end{aligned}
$$

Hence the desired functional equation holds. □

Now let $| \cdot |_\infty$ be the canonical ∞-adic absolute value over $\mathbf{C}_\infty$ such that $|T|_\infty = q$. For all $\alpha \in \mathbf{C}_\infty$, set $d(\alpha) := \frac{\log|\alpha|_\infty}{\log q}$. Let $f(z) = \sum_{j=0}^{\infty} \alpha_j z^j$ be an entire function on $\mathbf{C}_\infty$. Define

$$
\mathbf{M}_r(f) = \max_{j \geqslant 0}\left(d(\alpha_j) + rj\right), \text{ and } \rho_f = \limsup_{r \to +\infty} \frac{\log \mathbf{M}_r(f)}{\log q}.
$$

The latter is called the order of f. If f is $\mathbb{F}_q$-linear, i.e., $f(z) = \sum_{j=0}^{\infty} c_j z^{q^j}$, then we obtain (see [16]),

$$
\rho_f = \limsup_{j \to +\infty} \frac{-j q^j}{d(c_j)}. \tag{1}
$$

Let K be a finite extension of k. For all $\alpha \in K$, define

$$
\text{size}(\alpha) = \max\left\{d(\beta) : \beta \in \bar{k} \text{ is a conjugate of } \alpha\right\}.
$$

According to J. Yu [17], an $\mathbb{F}_q$-linear entire function f of finite order is an E_q-function with respect to K, if one can write $f(z) = \sum_{j=0}^{\infty} c_j z^{q^j}$, where $c_j \in K$ and $\exists C > 0$ such that $\text{size}(c_j) \leqslant C$ for all integers $j \geqslant 0$, and there exists a sequence $\alpha_j \in \mathbb{F}_q[T] \setminus \{0\}$ satisfying the following conditions:

(1) there exists a constant $c > 0$ such that $d(\alpha_j) \leqslant cjq^j$, for all integers $j \geqslant 1$;
(2) for all integers j, h $(0 \leqslant j \leqslant h)$, $\alpha_h c_j$ are algebraic integers in K;
(3) if $q^{j_1} + q^{j_2} + \cdots + q^{j_s} < q^n$, then $\alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_s}$ divides α_n in $\mathbb{F}_q[T]$.

From the above definition, we deduce immediately the following result.

Lemma 2. *The function $f_0(z)$ is an E_q-function with respect to k of order $\frac{1}{s+1}$.*

Proof. By definition, $f_0(z) = \sum_{n=0}^{\infty} \frac{z^{q^{n+b_s-1}}}{D_n^{q^{b_s-1}} D_{n+b_1-1}^{q^{b_s-b_1}} \cdots D_{n+b_s-1}}$. Note that $d(D_n) = n q^n$ for all integer $n \geqslant 0$, hence $f_0(z)$ is entire and additive. Moreover by the formula (1), we have

$$
\begin{aligned}
\rho_{f_0} &= \lim_{n \to \infty} \frac{(n + b_s - 1) q^{n+b_s-1}}{q^{b_s-1} n q^n + q^{b_s-b_1}(n + b_1 - 1) q^{n+b_1-1} + \cdots + (n + b_s - 1) q^{n+b_s-1}} \\
&= \lim_{n \to \infty} \frac{n + b_s - 1}{(s+1)n - s + b_1 + \cdots + b_s} = \frac{1}{s+1}.
\end{aligned}
$$

To conclude, it suffices to take $\alpha_n = D_n^{s+1}$, and note that $D_n = \prod_{j=1}^{n}(T^{q^j} - T)^{q^{n-j}}$. □

Finally, we recall without proof the following theorem of J. Yu [17].

Theorem 3. *Let K be a finite extension of k. Let g_1, g_2 be E_q-functions with respect to K that are algebraically independent over $\bar{k}$. Then, there are only finitely many points at which g_1, g_2 simultaneously assume values in K.*

3. Proofs

Proof of Theorem 1. By contradiction, suppose that all the $F_0(\gamma)$, $F_0(T\gamma)$, ..., $F_0(T^s\gamma)$ were algebraic over k. Set $K = k(\gamma, f_0(\gamma), f_0(T\gamma), \ldots, f_0(T^s\gamma))$. Then K is a finite extension of k. Set

$$d_0 = q^{b_s-1}, \text{ and } d_j = q^{b_s-b_j} \ (1 \leqslant j \leqslant s).$$

By Lemma 1, we have $f_j(Tz) = T^{d_j} f_j(z) + f_{j+1}(z)$ $(0 \leqslant j \leqslant s-1)$, and $f_s(Tz) = T^{d_s} f_s(z) + (f_0(z))^q$, from which we deduce directly

$$\sum_{j=0}^{s} f_j(T^{s-j+1}z) = \sum_{j=0}^{s} T^{d_j} f_j(T^{s-j}z) + \sum_{j=0}^{s-1} f_{j+1}(T^{s-j}z) + (f_0(z))^q,$$

and then we obtain $f_0(T^{s+1}z) = \sum_{j=0}^{s} T^{d_j} f_j(T^{s-j}z) + (f_0(z))^q$.

Below, by induction on j $(0 \leqslant j \leqslant s)$, we show that, for $0 \leqslant i \leqslant s-j$, each $f_j(T^i z)$ is a linear combination of $f_0(T^m z)$ $(0 \leqslant m \leqslant s)$ with coefficients in $\mathbb{F}_q[T]$.

The case where $j = 0$ is direct. Now assume that the desired result holds for all integers j $(0 \leqslant j \leqslant n)$, and $n < s$. Note that $f_{n+1}(T^i z) = f_n(T^{i+1}z) - T^{d_n} f_n(T^i z)$ for $0 \leqslant i \leqslant s-n-1$; hence, by induction, the right-hand side is a linear combination of $f_0(T^m z)$ $(0 \leqslant m \leqslant s)$ with coefficients in $\mathbb{F}_q[T]$, so is $f_{n+1}(T^i z)$.

Consequently, we can find $P_0, P_1, \ldots, P_s \in \mathbb{F}_q[T]$ such that

$$f_0(T^{s+1}z) = P_s f_0(T^s z) + P_{s-1} f_0(T^{s-1}z) + \ldots + P_0 f_0(z) + (f_0(z))^q. \tag{2}$$

Set $g_1(z) = f_0(z)$, and $g_2(z) = z$. Then $g_2(z)$ is an E_q-function with respect to k of order 0, and it is algebraically independent with $g_1(z)$ over $\bar{k}$, for $g_1(z)$ is entire and not a polynomial, hence a transcendental function over $\mathbf{C}_\infty(z)$ (see [14]). Then by Theorem 3, we obtain that there are only finitely many points at which g_1, g_2 simultaneously assume values in K. However, by the $\mathbb{F}_q$-additivity of g_1 and the equation (2), we have $g_1(\mathbb{F}_q[T]\gamma) \subseteq K$. Absurd! So Theorem 1 holds. $\square$

Proof of Theorem 2. By contradiction, suppose that all the $F_0(\gamma), F_1(\gamma), \ldots, F_s(\gamma)$ were algebraic over k. Set $K = k(\gamma, f_0(\gamma), f_1(\gamma), \ldots, f_s(\gamma))$. Then K is a finite extension of k. By Lemma 1 and the $\mathbb{F}_q$-linearity of $f_i(z)$ $(0 \leqslant j \leqslant s)$, we obtain directly $f_j(\mathbb{F}_q[T]\gamma) \subseteq K$ $(0 \leqslant j \leqslant s)$. In particular, we have $f_0(\mathbb{F}_q[T]\gamma) \subseteq K$. Absurd, by the same argument as above. So Theorem 2 holds. $\square$

Acknowledgements

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