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Centered Hardy–Littlewood maximal operator on the real line: Lower bounds

Fonction maximale centrée de Hardy–Littlewood : bornes inférieures

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ABSTRACT

For $1 < p < \infty$ and M the centered Hardy–Littlewood maximal operator on $\mathbb{R}$, we consider whether there is some $\varepsilon = \varepsilon(p) > 0$ such that $||Mf||_p \geq (1+\varepsilon)||f||_p$. We prove this for $1 < p < 2$. For $2 \leq p < \infty$, we prove the inequality for indicator functions and for unimodal functions.

RÉSUMÉ

Soient $1 < p < \infty$ et M la fonction maximale de Hardy–Littlewood sur $\mathbb{R}$. Nous étudions l'existence d'un $\varepsilon = \varepsilon(p) > 0$ tel que $||Mf||_p \geq (1+\varepsilon)||f||_p$. Nous l'établissons pour $1 < p < 2$. Pour $2 \leq p < \infty$, nous prouvons l'inégalité pour les fonctions indicatrices et les fonctions unimodales.

1. Introduction

Given a locally integrable real-valued function f on $\mathbb{R}^n$ define its uncentered maximal function $M_u f(x)$ as follows

$$M_u f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy, \tag{1}$$

where the supremum is taken over all balls B in $\mathbb{R}^n$ containing the point x, and $|B|$ denotes the Lebesgue volume of B. In studying *lower operator norms* of the maximal function [4], A. Lerner raised the following question: given $1 < p < \infty$, can one find a constant $\varepsilon = \varepsilon(p) > 0$ such that

$$\|M_u f\|_{L^p(\mathbb{R}^n)} \geq (1+\varepsilon)\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad f \in L^p(\mathbb{R}^n). \tag{2}$$

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The affirmative answer was obtained in [2], i.e. the Lerner's inequality (2) holds for all $1 < p < \infty$ and for any $n \geq 1$. The paper also studied the estimate (2) for other maximal functions. For example, the lower bound (2) persists if one takes the supremum in (1) over the shifts and dilates of a fixed centrally symmetric convex body K. Similar positive results have been obtained for dyadic maximal functions [5]; maximal functions defined over λ-*dense family* of sets, and *almost centered* maximal functions (see [2] for details).

Lerner's inequality for the centered maximal function

$$\|Mf\|_{L^p(\mathbb{R}^n)} \geq (1+\varepsilon(p,n))\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n), \qquad Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} |f|, \tag{3}$$

where the supremum is taken over all balls centered at x, is an open question, and the full characterization of the pairs (p,n), $n \geq 1$, and $1 < p < \infty$, for which (3) holds with some $\varepsilon(p,n) > 0$ and for all $f \in L^p(\mathbb{R}^n)$ seems to be unknown. If $n \geq 3$ and $p > \frac{n}{n-2}$, then one can show that $f(x) = \min\{|x|^{n-2}, 1\} \in L^p(\mathbb{R}^n)$, and $Mf(x) = f(x)$, as f is the pointwise minimum of two superharmonic functions. This gives a counterexample to (3). In fact, Korry [3] proved that the centered maximal operator does not have fixed points unless $n \geq 3$ and $p > \frac{n}{n-2}$, but a lack of fixed points does not imply that (3) holds. On the other hand, for any $n \geq 1$, by comparing $Mf(x) \geq C(n) M_u f(x)$, and using the fact that $\|M_u f\|_{L^p(\mathbb{R}^n)} \geq (1+\frac{B(n)}{p-1})^{1/p}\|f\|_{L^p(\mathbb{R}^n)}$ (see [2]), one can easily conclude that (3) holds true whenever p is sufficiently close to 1. It is natural to ask what is the maximal $p_0(n)$ for which, if $1 < p < p_0(n)$, then (3) holds.

1.1. New results

In this paper, we study the case of dimension $n = 1$ and the centered Hardy–Littlewood maximal operator M. We obtain the following theorem.

Theorem 1. *If $1 < p < 2$ and $n = 1$, then Lerner's inequality (3) holds true, namely*

$$\|Mf\|_p \geq \left(\frac{p}{2\,(p-1)}\right)^{1/p} \|f\|_p.$$

Theorem 2. *For $n = 1$, and any p, $1 < p < \infty$, inequality (3) holds true a) for the class of indicator functions with $\epsilon(p,n) = 1/4^p$, and b) for the class of unimodal functions, with $\epsilon(p,n)$ not explicitly given.*

2. Proof of the main results

2.1. Proof of Theorem 1

First, we prove the following modification of the classical Riesz's sunrise lemma (see Lemma 1 in [1]). Our proof is similar to the proof of the lemma.

Lemma 3. *For a nonnegative continuous compactly supported f and any $\lambda > 0$, we have*

$$|\{Mf \geq \lambda\}| \geq \frac{1}{2\,\lambda} \int\limits_{\{f \geq \lambda\}} f.$$

Proof. Define an auxiliary function $\varphi(x)$ via

$$\varphi(x) = \sup_{y<x} \int\limits_{y}^{x} f(t)\,\mathrm{d}t - 2\,\lambda\,(x-y).$$

Notice that, if $f(x) > 2\,\lambda$, then $\varphi(x) > 0$. Indeed,

$$\varphi(x) = \sup_{y<x}\,(x-y)\left[\frac{1}{x-y}\int\limits_{y}^{x} f - 2\,\lambda\right] > 0, \tag{4}$$

because we can choose y sufficiently close to x, and use the fact that $\lim_{y\to x}\frac{1}{x-y}\int_y^x f = f(x)$. On the other hand, if $\varphi(x) > 0$, then $Mf(x) > \lambda$. Indeed, it follows from (4) that $\sup_{y<x}\frac{1}{x-y}\int_y^x f > 2\,\lambda$. Therefore,

$$Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} f \geq \frac{1}{2\,(x-y)} \int_{y}^{x} f \geq \lambda.$$

Thus, we obtain

$$\{Mf \geq \lambda\} \supseteq \{f \geq \lambda\} \cup \{\varphi > 0\}; \tag{5}$$

$$\{f > 2\,\lambda\} \subseteq \{\varphi > 0\}. \tag{6}$$

Therefore, it follows that

$$\begin{aligned} |\{Mf \geq \lambda\}| &\geq |\{\varphi > 0\}| + |\{\lambda \leq f \leq 2\,\lambda\} \backslash \{\varphi > 0\}| \\ &\geq \frac{1}{2\,\lambda} \int_{\{\varphi>0\}} f + \frac{1}{2\,\lambda} \int_{\{\lambda \leq f \leq 2\,\lambda\}\backslash\{\varphi>0\}} f \\ &\geq \frac{1}{2\,\lambda} \int_{\{f \geq \lambda\}} f. \quad \square \end{aligned}$$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Take any continuous bounded compactly supported $f \geq 0$. By Lemma 3, for any $\lambda > 0$, we have

$$|\{Mf \geq \lambda\}| \geq \frac{1}{2\,\lambda} \int_{\mathbb{R}} f(x) \mathbb{1}_{[\lambda,\infty)}(f(x))\, \mathrm{d}x. \tag{7}$$

Finally, we multiply both sides of (7) by $p\,\lambda^{p-1}$, and we integrate the obtained inequality in λ on $(0, \infty)$, so we obtain

$$\int_{\mathbb{R}} (Mf)^p \geq \int_0^\infty \int_{\mathbb{R}} \frac{p\,\lambda^{p-2}}{2} f(x) \mathbb{1}_{[\lambda,\infty)}(f(x))\, \mathrm{d}x\, \mathrm{d}\lambda = \frac{p}{2\,(p-1)} \int_{\mathbb{R}} f^p,$$

and $\frac{p}{2p-2} > 1$ precisely when $p < 2$. This finishes the proof of Theorem 1 for continuous compactly supported bounded nonnegative f. To obtain the inequality $\|Mf\|_p \geq (\frac{p}{2\,(p-1)})^{1/p} \|f\|_p$ for an arbitrary nonnegative $f \in L^p(\mathbb{R})$, we can approximate f in L^p by a sequence of compactly supported smooth functions f_n, and use the fact that the operator M is Lipschitz on L^p (since it is bounded and subadditive). $\square$

Remark 4. The argument presented above is a certain modification of the classical Riesz's sunrise lemma, and an adaptation of an argument of Lerner (see Section 4 in [4]). For p less than about 1.53, it is possible to use Lerner's result directly, together with the fact that $Mf \geq (M_u f)/2$. We need the modified sunrise lemma to get the result for all $p < 2$.

2.2. Proof of Theorem 2

2.2.1. Indicator functions

Proof of Theorem 2 for indicator functions $\mathbb{1}_E$. Let $\mathbb{1}_E \in L^p(\mathbb{R})$ and let $\hat{\delta} > 0$. We approximate $\mathbb{1}_E$ arbitrarily well in L^p by a nonnegative continuous compactly supported function f. Then, f approximates $\mathbb{1}_E$ and Mf also approximates $M\mathbb{1}_E$ to within some $\delta \ll \hat{\delta}$ in L^p.

For a.e. $x \in E$, we have $M\mathbb{1}_E(x) \geq 1$. Additionally, by Lemma 3, we have that

$$|\{Mf \geq 1/4\}| \geq 2 \int_{\{f \geq \frac{1}{4}\}} f \geq 2 \int_{\{f \geq \frac{1}{4}\} \cap E} \mathbb{1}_E - 2 \int_E |\mathbb{1}_E - f|.$$

By making δ is small, we can ensure that $\{|f - \mathbb{1}_E| \geq 3/4\}$ is small, so

$$2 \int_{\{f \geq \frac{1}{4}\} \cap E} \mathbb{1}_E \geq 2\,|E| - \hat{\delta}/2.$$

Also, by Holder's inequality, we can bound $\int_E |\mathbb{1}_E - f|$ in terms of $||\mathbb{1}_E - f||_p < \delta$. Thus, when δ is sufficiently small, we get

$$|\{Mf \geq 1/4\}| \geq 2\,|E| - \hat{\delta},$$

so there is a set of measure at least $|E| - \hat{\delta}$, on which $\mathbb{1}_E = 0$ and $Mf \geq 1/4$. If δ is sufficiently small, we have that $|\{Mf - M\mathbb{1}_E \geq \hat{\delta}\}| < \hat{\delta}$, so there is a set of measure $|E| - 2\hat{\delta}$ on which $\mathbb{1}_E = 0$ and $M\mathbb{1}_E \geq 1/4 - \hat{\delta}$. Taking $\hat{\delta} \to 0$, we get

$$\|M\mathbb{1}_E\|_p^p \geq (1 + 1/4^p)\,\|\mathbb{1}_E\|_p^p. \quad \square$$

2.2.2. Unimodal functions

Next, we obtain lower bounds on L^p norms of the maximal operator over the class of unimodal functions. By unimodal function $f \in L^p(\mathbb{R})$, $f \geq 0$, we mean any function that is increasing until some point x_0 and then decreasing. Without loss of generality, we will assume that $x_0 = 0$.

Proof of Theorem 2 for unimodal functions. We can assume that $\|f\mathbb{1}_{\mathbb{R}^+}\|_p^p \geq \frac{1}{2}\|f\|_p^p$.

Let $\tilde{f} = f\mathbb{1}_{\mathbb{R}^+}$. We define $M^n = \underbrace{M \circ \cdots \circ M}_{n}$ to be the n-th iterate of M. We will find an n, independent of f, such that $\|M^n\tilde{f}\|_p^p > 2^{p+1}\|\tilde{f}\|_p^p$, independent of the function f. First, for $x > 0$, let

$$a(x) = \min_{k\in\mathbb{Z},\, 2^k > x} 2^k.$$

Then let

$$\psi(x) = \tilde{f}(a(x)),$$

that is, $\psi \leq \tilde{f}$, and ψ is a step function approximation from below. Then,

$$2\,\|\psi\|_p^p = 2\sum_{k\in\mathbb{Z}} 2^k \tilde{f}(2^{k+1})^p = \sum_{s\in\mathbb{Z}} 2^s \tilde{f}(2^s)^p \geq \|\tilde{f}\|_p^p.$$

Now let

$$\bar{g}(x) = (1 - \sqrt{x})\,\mathbb{1}_{(0,1]}(x).$$

Then for $0 < x \leq 9/8$, we have that

$$M\,\bar{g}(x) \geq \frac{1}{2x}\int_0^{2x} \bar{g}(y)\,\mathrm{d}y \geq \frac{1}{2x}\int_0^{2x} 1 - \sqrt{y}\,\mathrm{d}y = 1 - \frac{2}{3}\sqrt{2x} = \bar{g}(8x/9),$$

and for all $x \notin (0, 9/8]$, we have $M\,\bar{g}(x) \geq 0 = \bar{g}(8x/9)$. Thus,

$$M^n\bar{g}(x) \geq \bar{g}\left((8/9)^n x\right),$$

so

$$\int_{\frac{1}{2}(9/8)^n}^{(9/8)^n} (M^n\mathbb{1}_{(0,1]})^p \geq \int_{\frac{1}{2}(9/8)^n}^{(9/8)^n} (M^n g)^p \geq (9/8)^n \int_{\frac{1}{2}}^{1} \bar{g}^p = C_p(9/8)^n. \tag{8}$$

Note that for all $k \in \mathbb{Z}$, we have $\psi \geq \tilde{f}(2^{k+1})\mathbb{1}_{(2^k, 2^{k+1}]}$. Thus

$$M^n\psi(x) \geq \tilde{f}(2^{k+1})M^n\mathbb{1}_{(2^k,2^{k+1}]}(x).$$

We will use this lower bound for varying values of k for different x. We use (8) in the third inequality below, since $\mathbb{1}_{(2^k,2^{k+1}]}$ is just a horizontal rescaling and translation of $\mathbb{1}_{(0,1]}$. We have

$$\begin{aligned}\|M^n\psi\|_p^p &\geq \sum_{-\infty}^{\infty} \int_{2^k+(9/8)^n 2^{k-1}}^{2^k+(9/8)^n 2^k} (M^n\psi)^p \\ &\geq \sum_{-\infty}^{\infty} \tilde{f}(2^{k+1})^p \int_{2^k+(9/8)^n 2^{k-1}}^{2^k+(9/8)^n 2^k} (M^n\mathbb{1}_{(2^k,2^{k+1}]})^p\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{-\infty}^{\infty} \tilde{f}(2^{k+1})^p C_p (9/8)^n 2^k \\
&= C_p (9/8)^n \|\psi\|_p^p \\
&\geq \frac{1}{2} C_p (9/8)^n \|\tilde{f}\|_p^p,
\end{aligned}
$$

so by picking $n = n(p)$ sufficiently large, we get

$$
\|M^n f\|_p^p \geq \|M^n \psi\|_p^p \geq 2^{p+1} \|\tilde{f}\|_p^p \geq 2^p \|f\|_p^p,
$$

so

$$
\|M^n f\|_p \geq 2\|f\|_p. \tag{9}
$$

Now suppose that $\|Mf - f\|_p < \tilde{\epsilon}\|f\|_p$ for some $\tilde{\epsilon}$ to be chosen later. From the subadditivity of the maximal operator, it follows that $\|M\phi_1 - M\phi_2\|_p \leq A_p \|\phi_1 - \phi_2\|_p$, so

$$
\|M^n f - f\|_p \leq \sum_{j=1}^{n} \|M^j f - M^{j-1} f\|_p \leq \sum_{j=1}^{n} A_p^{j-1} \|Mf - f\|_p < \left(\tilde{\epsilon} \sum_{j=1}^{n} A_p^{j-1}\right) \|f\|_p
$$

which contradicts (9) for $\tilde{\epsilon} = \tilde{\epsilon}(p)$ sufficiently small. Thus $\|Mf - f\|_p \geq \tilde{\epsilon}\|f\|_p$, so

$$
\|Mf\|_p^p = \int (Mf)^p \geq \int f^p + (Mf - f)^p = \|f\|_p^p + \|Mf - f\|_p^p \geq \left(1 + \tilde{\epsilon}^p\right) \|f\|_p^p,
$$

which proves the theorem. □

3. Concluding remarks

Take any compactly supported bounded function $f \geq 0$ which is not identically zero. One can show that

$$
(9/8)^{1/p} \leq \liminf_{k\to\infty} \|M^k f\|_{L^p}^{1/k} \leq \limsup_{k\to\infty} \|M^k f\|_{L^p}^{1/k} \leq a_p, \tag{10}
$$

where the number $a_p > 1$ solves $M(|x|^{-1/p}) = a_p |x|^{-1/p}$ (such an a_p can be seen to exist by a calculation, or by scaling considerations). In other words, the growth of $\|M^k f\|_p$ is exponential, which suggests that Theorem 1 is likely to be true for all $1 < p < \infty$. To show (10), let us first illustrate the upper bound. Consider the function $\tilde{f}(x) := f(Cx)/\|f\|_\infty$. For any fixed constant $C \neq 0$, one can easily see that $\limsup_{k\to\infty} \|M^k f\|_{L^p}^{1/k} = \limsup_{k\to\infty} \|M^k \tilde{f}\|_{L^p}^{1/k}$. Therefore, without loss of generality, we can assume that $f \leq 1$ and the support of f is in $[-1, 1]$. Next, take any $\delta \in (0, p-1)$, and consider

$$
h(x) = \begin{cases} 1 & |x| \leq 1, \\ |x|^{-1/(p-\delta)} & |x| > 1. \end{cases}
$$

Clearly, $h \in L^p$, and $f \leq h$. Since $M(|x|^{-1/p}) = a_p |x|^{-1/p}$, it follows that $Mh(x) \leq a_{p-\delta} h(x)$ for all $x \in \mathbb{R}$. Thus

$$
\limsup_{k\to\infty} \|M^k f\|_p^{1/k} \leq \limsup_{k\to\infty} \|M^k h\|_p^{1/k} \leq a_{p-\delta} \limsup_{k\to\infty} \|h\|_p^{1/k} = a_{p-\delta}.
$$

Finally, taking $\delta \to 0$ gives the desired inequality.

To prove the lower bound, we have already seen that the function $\bar{g}(x) = (1 - \sqrt{x})\mathbb{1}_{(0,1]}$ satisfies

$$
M^n \bar{g}(x) \geq \bar{g}\left((8/9)^n x\right),
$$

so we can obtain the growth $(9/8)^{n/p}$ for the function $\bar{g}(x)$. Now it remains to notice that, for any $f \geq 0, f \in L^p$ not identically zero, we can rescale and shift the function $\bar{g}$ so that $Mf(x) \geq A\bar{g}(Bx + C)$ for some constants $A > 0$, $B, C \neq 0$. This finishes the proof of the claim.

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