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The algebraic transfer for the real projective space ☆

Transfert algébrique pour l'espace réel projectif

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ABSTRACT

A chain-level representation of the Singer transfer for any left $\mathcal{A}$-module is constructed. We prove that the image of the Singer transfer $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ for the infinite real projective space is a module over the image of the transfer Tr_* for the sphere. Further, the algebraic Kahn–Priddy homomorphism is an epimorphism from $\mathrm{Im}\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ onto $\mathrm{Im}\mathrm{Tr}_*$ in positive stems. The indecomposable elements $\widehat{h}_i$ for $i \geq 1$ and $\widehat{c}_i$, $\widehat{d}_i$, $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{p}_i$ for $i \geq 0$ are detected, whereas the ones $\widehat{g}_i$ for $i \geq 1$ and $\widehat{D}_3(i)$, $\widehat{p}'_i$ for $i \geq 0$ are not detected by the Singer transfer $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$. This transfer is shown to be not monomorphic in every positive homological degree. The transfer behavior is also investigated near "critical elements". We prove that Kameko's squaring operation on the domain of $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ is eventually isomorphic. This phenomenon leads to the so-called "stability" of the Singer transfer for the infinite real projective space under the iterated squaring operation.

RÉSUMÉ

Une description au niveau des chaînes du transfert de Singer pour tout $\mathcal{A}$-module à gauche est construite. Nous démontrons que l'image du transfert de Singer $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ pour l'espace projectif réel infini est un module sur l'image du transfert Tr_* pour la sphère. De plus, l'homomorphisme algébrique de Kahn–Priddy est un épimorphisme de $\mathrm{Im}\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ sur $\mathrm{Im}\mathrm{Tr}_*$ en degré positif. Les éléments indécomposables $\widehat{h}_i$ pour $i \geq 1$ et $\widehat{c}_i$, $\widehat{d}_i$, $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{p}_i$ pour $i \geq 0$ sont détectés, alors que les $\widehat{g}_i$ pour $i \geq 1$ et $\widehat{D}_3(i)$, $\widehat{p}'_i$ pour $i \geq 0$ ne le sont pas. Ce transfert n'est pas injectif en chaque degré homologique positif. Le transfert est aussi étudié au voisinage des « éléments critiques ». Nous montrons que le morphisme de Kameko sur le domaine de $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ est un isomorphisme sur son image après un nombre suffisant d'itérations. Ce phénomène mène à la « stabilité » du transfert pour l'espace projectif réel infini sous l'action du morphisme de Kameko et sous l'action de l'élévation au carré itérée.

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Let $\mathcal{A}$ be the mod 2 Steenrod algebra. Suppose that X is a pointed CW-complex, whose mod 2 homology H_*X is finitely generated in each degree. Singer defined in [14] the algebraic transfer for X:

$$\mathrm{Tr}_s^X : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*X) \to \mathrm{Ext}^s_{\mathcal{A}}(\Sigma^{-s}\widetilde{H}^*X, \mathbb{F}_2),$$

where $\mathbb{V}_s$ denotes an elementary abelian 2-group of rank s, and $H_*\mathbb{V}_s$ is the mod 2 homology of a classifying space of $\mathbb{V}_s$, while $P(H_*\mathbb{V}_s \otimes \widetilde{H}_*X)$ means the primitive part of $H_*\mathbb{V}_s \otimes \widetilde{H}_*X$ under the action of $\mathcal{A}$.

The Singer transfer is expected to be a useful tool in the study of $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*X, \mathbb{F}_2)$ by means of the Peterson hit problem and the invariant theory.

In the present note, we study the Singer transfer for the infinite real projective space $\mathbb{RP}^\infty$ in connection with the one for the sphere S^0. The latter transfer will simply be denoted by Tr_s. The following is one of the note's main results.

Theorem 1. *The image of the Singer transfer for the real projective space* $\mathrm{Tr}_*^{\mathbb{RP}^\infty} = \oplus_{s\ge 0}\mathrm{Tr}_s^{\mathbb{RP}^\infty}$ *is a module over the image of the Singer transfer for the sphere* $\mathrm{Tr}_* = \oplus_{s\ge 0}\mathrm{Tr}_s$. *Further, the algebraic Kahn–Priddy homomorphism*

$$t_* : \mathrm{Ext}^{s,t}_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s+1,t+1}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$$

is an epimorphism from $\mathrm{Im}\mathrm{Tr}_*^{\mathbb{RP}^\infty}$ *onto* $\mathrm{Im}\mathrm{Tr}_*$ *in stem* $t - s > 0$.

The following is an application of Theorem 1 into the indecomposable elements of $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ defined in Lin [10] and Chen [4] for s small: $\widehat{h}_i$ for $s = 0$, $i \ge 1$; $\widehat{c}_i$ for $s = 2$, $i \ge 0$; $\widehat{d}_i$, $\widehat{e}_i$, $\widehat{f}_i$, $\widehat{g}_i$, $\widehat{p}_i$, $\widehat{D}_3(i)$, $\widehat{p}'_i$, for $s = 3$, $i \ge 0$ in the last seven elements, except $i \ge 1$ in $\widehat{g}_i$. Their images under the algebraic Kahn–Priddy homomorphism are respectively the well-known indecomposable elements $h_i, c_i, d_i, e_i, f_i, g_i, p_i, D_3(i), p'_i \in \mathrm{Ext}^*_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ (see Adams [1], Wang [16], Tangora [15]). (The element h_0 of stem 0 is not in the image of the Kahn–Priddy homomorphism.)

Based on the knowledge of the image of the Singer transfer for the sphere

$$\mathrm{Tr}_s : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s) \to \mathrm{Ext}^s_{\mathcal{A}}(\Sigma^{-s}\mathbb{F}_2, \mathbb{F}_2)$$

in Singer [14] for $s = 1$ and 2, in Boardman [2] for $s = 3$, and in Bruner–Hà–Hưng [3], Hưng [7], Hà [5], Hưng–Quỳnh [8] for $s = 4$, a careful investigation of the Kahn–Priddy homomorphism's kernel in certain degrees leads us to the following.

Corollary 2.

(i) *The elements* $\widehat{h}_i$ *for* $i \ge 1$ *and* $\widehat{c}_i, \widehat{d}_i, \widehat{e}_i, \widehat{f}_i, \widehat{p}_i$ *for* $i \ge 0$ *are in the image of the Singer transfer* $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$.

(ii) *The elements* $\widehat{g}_i$ *for* $i \ge 1$ *and* $\widehat{D}_3(i), \widehat{p}'_i$ *for* $i \ge 0$ *are not in the image of the Singer transfer* $\mathrm{Tr}_*^{\mathbb{RP}^\infty}$.

As it is well known, $H^*\mathbb{V}_s \cong P_s := \mathbb{F}_2[x_1, \dots, x_s]$, where x_i is of degree 1 for $1 \le i \le s$. Let T_s be the Sylow 2-subgroup of GL_s consisting of all upper triangular $s \times s$ matrices with 1 on the main diagonal. In [12], Mùi defines the T_s-invariant

$$V_i = V_i(x_1, \dots, x_i) = \prod_{c_j \in \mathbb{F}_2} (c_1x_1 + \cdots + c_{i-1}x_{i-1} + x_i).$$

Singer sets in [13] $v_1 = V_1$, $v_k = V_k/V_1 \cdots V_{k-1}$ $(k \ge 2)$.

The map $\mathcal{T}_s : \mathbb{F}_2[v_1^{\pm 1}, \dots, v_s^{\pm 1}] \otimes M \to \mathbb{F}_2[x_1^{\pm 1}, \dots, x_s^{\pm 1}] \otimes M$ is defined, for M an unstable $\mathcal{A}$-module, by

$$\mathcal{T}_s(v_1^{a_1} \cdots v_s^{a_s} \otimes z) := Sq^{a_1+1}\Big(x_1^{-1} \cdots Sq^{a_{s-1}+1}(x_{s-1}^{-1} Sq^{a_s+1}(x_s^{-1} \otimes z)) \cdots\Big),$$

where $z \in M$ and $a_1, \dots, a_s$ arbitrary integers. Here, we mean $Sq^i = 0$ for $i < 0$.

In [6, Def. 3.1], the first named author introduced the original version of the above definition of $\mathcal{T}_s$ in order to show the following facts for the special and important case of $M = \widetilde{H}^*S^0 \cong \mathbb{F}_2$.

Let $\partial_s : \Gamma_s^+(\Sigma^{-s}M) \to \Gamma_{s-1}^+(\Sigma^{-s}M)$ be the differential in Singer's complex $\Gamma_*^+(\Sigma^{-s}M)$, whose homology is $\mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}M)$, (see [13]). Let $St_s : M \to P_s \otimes M$ be the Steenrod homomorphism.

We show that, for M an unstable $\mathcal{A}$-module, $\mathcal{T}_s$ satisfies the following two properties.

(i) $\mathcal{T}_s(\mathrm{Ker}\partial_s) \subset P_s \otimes M$. Further, $\mathcal{T}_s|_{\Gamma_s^+(\Sigma^{-s}M)} : \Gamma_s^+(\Sigma^{-s}M) \to \mathbb{F}_2[x_1^{\pm 1}, \dots, x_s^{\pm 1}] \otimes M$ is a chain-level representation of the dual of the algebraic transfer

$$(\mathrm{Tr}_s^M)^* : H_s(\Gamma_*^+(\Sigma^{-s}M)) \cong \mathrm{Tor}_s^{\mathcal{A}}(\mathbb{F}_2, \Sigma^{-s}M) \to (\mathbb{F}_2 \otimes_{\mathcal{A}} (P_s \otimes M))^{GL_s}.$$

(ii) For every $q \in D_s := P_s^{GL_s}$ and any homogeneous element $z \in M$,

$$\mathcal{T}_s(qQ_{s,0}^{|z|} \otimes z) = qSt_s(z),$$

where $Q_{s,0}$ is the highest degree generator in the Dickson algebra D_s.

How far from an isomorphism is the Singer transfer for the infinite real projective space?

Theorem 3. $Tr_s^{\mathbb{RP}^\infty} : \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty) \to \mathrm{Ext}^s_{\mathcal{A}}(\Sigma^{-s}\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ *is not a monomorphism in infinitely many degrees for* $s > 0$.

We prove this theorem by showing that $Tr_s^{\mathbb{RP}^\infty}$ vanishes on certain products of "Adams elements" in $\mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$ for $s > 0$. How is the behavior of $Tr_s^{\mathbb{RP}^\infty}$ with respect to elements, which are not products of Adams elements? In order to give an answer to this question, we define the notion of critical element in $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$.

Suppose that M is an $\mathcal{A}$-coalgebra, and N is an $\mathcal{A}$-algebra. Then, as the Steenrod algebra $\mathcal{A}$ is a cocommutative Hopf algebra, there are squaring operations $Sq^i : \mathrm{Ext}^{s,t}_{\mathcal{A}}(M, N) \to \mathrm{Ext}^{s+i,2t}_{\mathcal{A}}(M, N)$, which share most of the properties with Sq^i on cohomology of spaces. (See May [11].) However, the squaring operation Sq^0 is not the identity in general.

In [9], Kameko defined an endomorphism $\widetilde{Sq}^0$ of $PH_*(\mathbb{V}_s)$, the primitive part of $H_*(\mathbb{V}_s)$ consisting of all elements annihilated by any Steenrod squares. This induces the so-called Kameko Sq^0, an endomorphism of $\mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s)$, which commutes with the classical Sq^0 on $\mathrm{Ext}^s_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ through the Singer transfer.

In this article, we recognize that the Kameko Sq^0 is inherited on $\mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$, which also commutes with the classical Sq^0 on $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ through the Singer transfer for the projective space.

A number is said to be *s-spike* if it can be written as $(2^{n_1} - 1) + \cdots + (2^{n_s} - 1)$, but can not be written as a sum of less than s terms of the form $(2^n - 1)$.

By [7, Lemma 3.5], if $\mathrm{Stem}(z)$ is an $(s+1)$-spike, then so is $2\mathrm{Stem}(z) + (s+1)$.

Definition 4. A nonzero element $z \in \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ is called *critical* if

(a) $Sq^0(z) = 0$,
(b) $2\mathrm{Stem}(z) + (s+1)$ is an $(s+1)$-spike,
(c) $t_*(z) \neq 0$, where $t_* : \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^{s+1}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ is the Kahn–Priddy homomorphism.

The role of *critical elements* is explained as follows. If a critical element $z \in \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ is the image of $y \in \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$ under the Singer transfer $Tr_s^{\mathbb{RP}^\infty}$, then the last two conditions of Definition 4 ensure that the Kameko Sq^0 is a monomorphism on $\mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$ in the degree of y. This, together with the first condition of Definition 4, imply that the Singer transfer $Tr_s^{\mathbb{RP}^\infty}$ is either not epimorphic in the degree of y or not monomorphic in the degree of $Sq^0(y)$.

Let $h_n \in \mathrm{Ext}^1_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$ be the well-known Adams element.

Lemma 5. *If* $z \in \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ *is critical, then* $h_n z \in \mathrm{Ext}^{s+1}_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ *is also critical for every n with* $2^n \geq \max\{4d^2, d + (s+1)\}$, *where* $d = \mathrm{Stem}(z)$.

Proposition 6.

(i) *For* $s = 4$, $\widehat{Ph_2} \in \mathrm{Ext}^{4,15}_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ *is critical.*
(ii) *For* $s > 4$, *there are infinitely many critical elements, whose stems are pairwise distinct, in* $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$.

Theorem 7.

(i) $Tr_4^{\mathbb{RP}^\infty}$ *is not an isomorphism either in degree* $\mathrm{Stem}(\widehat{Ph_2}) = 11$ *or in degree* $2 \times 11 + (4+1) = 27$.
(ii) $Tr_s^{\mathbb{RP}^\infty}$ *is not an isomorphism for* $s > 4$ *in infinitely many degrees, each of which is either d or* $2d + (s+1)$ *with d the stem of a critical element in* $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$.

The following theorem shows that the Kameko squaring operation for $\mathbb{RP}^\infty$ is eventually isomorphic, that is it becomes isomorphic after enough times of iteration. It is similar to Theorem 6.1 in the first named author's article [7] on the Kameko squaring operation for S^0.

Note that if $y \in \mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$ is of degree d, then $Sq^0(y)$ is of degree $\delta(d) := 2d + (s+1)$.

Theorem 8. *For an arbitrary non-negative integer d,*

$$(Sq^0)^{i-s} : \mathbb{F}_2 \underset{GL_s}{\otimes} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{\delta^s(d)} \to \mathbb{F}_2 \underset{GL_s}{\otimes} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)_{\delta^i(d)}$$

is an isomorphism for $i \geq s$, *where* $\delta^i(d) = 2^i d + (2^i - 1)(s+1)$.

The article's remaining part investigates the behavior of the Singer transfer $Tr_s^{\mathbb{RP}^\infty}$ on Sq^0-families. All the results of this part are consequences of Theorem 8.

Corollary 9.

(i) *Every finite Sq^0-family in $\mathbb{F}_2 \otimes_{GL_s} P(H_*\mathbb{V}_s \otimes \widetilde{H}_*\mathbb{RP}^\infty)$ has at most s nonzero elements.*

(ii) *If $Tr_s^{\mathbb{RP}^\infty}$ is a monomorphism on the degrees, which equal to the stems of elements in a finite Sq^0-family with at least $s+1$ nonzero elements in $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$, then it does not detect any element of this family.*

Note clearly that, if $a \in \mathrm{Im}\big[(Sq^0)^i : \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2) \to \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)\big]$, then $\mathrm{Stem}(a)$ can be written in the form $\mathrm{Stem}(a) = 2^i d + (2^i - 1)(s+1)$, where d is the stem of a pre-image of a by $(Sq^0)^i$. However, the stem of a may be written in this form even if a is not necessarily in $\mathrm{Im}(Sq^0)^i$.

Definition 10. The *root degree* of an element $a_0 \in \mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ is the maximum nonnegative integer r such that $\mathrm{Stem}(a_0)$ can be written in the form

$$\mathrm{Stem}(a_0) = 2^r d + (2^r - 1)(s+1).$$

The following facts are concerned with the so-called stability of the Singer transfer for the infinite real projective space under the iterated squaring operation.

Proposition 11. *Let $\{a_i \mid i \geq 0\}$ be an Sq^0-family in $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ and let r be the root degree of a_0. If a_n is in the image of $Tr_s^{\mathbb{RP}^\infty}$ for some $n \geq \max\{s-r, 0\}$, then a_i is in the image for $i \geq n$, and a_j modulo $\mathrm{Ker}(Sq^0)^{n-j}$ is also in the image for $\max\{s-r, 0\} \leq j < n$.*

Corollary 12. *Let $\{a_i \mid i \geq 0\}$ be a Sq^0-family in $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ and r the root degree of a_0. Suppose Sq^0 on $\mathrm{Ext}^s_{\mathcal{A}}(\widetilde{H}^*\mathbb{RP}^\infty, \mathbb{F}_2)$ is a monomorphism in the stems of the elements $\{a_i \mid i \geq \max\{s-r, 0\}\}$. If $Tr_s^{\mathbb{RP}^\infty}$ detects a_n for some $n \geq \max\{s-r, 0\}$, then it detects a_i for every $i \geq \max\{s-r, 0\}$.*

One of our basic tools is the commutativity of the Singer transfers, the squaring operations, and the algebraic Kahn–Priddy homomorphism.

The contains of this note will be published in detail elsewhere.

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