C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1131–1155

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Partial differential equations

A spectral inequality for degenerate operators and applications

Une inégalité spectrale pour les opérateurs dégénérés et applications

Rémi Buffe [a], Kim Dang Phung [b,1]

[a] *Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France*
[b] *Institut Denis-Poisson, CNRS, UMR 7013, Université d'Orléans, BP 6759, 45067 Orléans cedex 2, France*

A R T I C L E I N F O

Article history:
Received 15 March 2018
Accepted after revision 7 November 2018
Available online 19 November 2018

Presented by Jean-Michel Coron

A B S T R A C T

In this paper, we establish a Lebeau–Robbiano spectral inequality for a degenerate one-dimensional elliptic operator, and we show how it can be used to study impulse control and finite-time stabilization for a degenerate parabolic equation.

R É S U M É

Dans cet article, on s'intéresse à l'inégalité spectrale de type Lebeau–Robbiano sur la somme de fonctions propres pour une famille d'opérateurs dégénérés. Les applications sont données en théorie du contrôle, comme le contrôle impulsionnel et la stabilisation en temps fini.

1. Introduction and main results

The purpose of this article is to prove spectral properties for a family of degenerate operators acting over the interval $(0,1)$. We shall consider linear operators $\mathcal{P}$ in $L^2(0,1)$, defined by

$$\begin{cases} \mathcal{P} = -\frac{\mathrm{d}}{\mathrm{d}x}\left(x^\alpha \frac{\mathrm{d}}{\mathrm{d}x}\right) \text{, with } \alpha \in (0,2) \text{ ,} \\ D(\mathcal{P}) = \left\{\vartheta \in H^1_\alpha(0,1)\,;\, \mathcal{P}\vartheta \in L^2(0,1) \text{ and } \mathrm{BC}_\alpha(\vartheta) = 0\right\} \text{ ,} \end{cases}$$

where

$$H^1_\alpha(0,1) := \left\{\vartheta \in L^2(0,1);\, \vartheta \text{ is absolutely continuous in } (0,1),\, \int_0^1 x^\alpha |\vartheta'|^2 < \infty,\, \vartheta(1) = 0\right\} \text{ ,}$$

E-mail addresses: remi.buffe@inria.fr (R. Buffe), kim_dang_phung@yahoo.fr (K.D. Phung).
[1] The author was partially supported by the "Région Centre" (France) – THESPEGE Project.

https://doi.org/10.1016/j.crma.2018.11.004

and

$$\mathrm{BC}_\alpha(\vartheta)=\begin{cases}\vartheta_{|x=0}\,, & \text{for } \alpha\in[0,1)\,,\\ (x^\alpha\vartheta')_{|x=0}\,, & \text{for } \alpha\in[1,2).\end{cases}$$

Such class of degenerate parabolic operators has been studied by Cannarsa, Martinez, and Vancostenoble in [8]. We recall that $\mathcal{P}$ is a closed self-adjoint positive densely defined operator, with compact resolvent (see [1]). As a consequence, the following spectral decomposition holds: there exists a countable family of eigenfunctions Φ_j associated with eigenvalues λ_j such that

- $\left\{\Phi_j\right\}_{j\geq 1}$ forms a Hilbert basis of $L^2(0,1)$
- $\mathcal{P}\Phi_j=\lambda_j\Phi_j$
- $0<\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_k\to+\infty$.

An explicit expression of the eigenvalues is given in [20] for the weakly degenerate case $\alpha\in(0,1)$, and in [41] for the strongly degenerate case $\alpha\in[1,2)$, and depends on the Bessel functions of first kind (see [38]). Also, we have the following asymptotic formula: $\lambda_k\sim C(\alpha)\,k^2$ as $k\to\infty$.

We are interested in the spectral inequality for the sum of eigenfunctions. Our main result is as follows.

Theorem 1.1. *Let ω be an open and nonempty subset of $(0,1)$. There exist constants $C>0$ and $\sigma\in(0,1)$ such that*

$$\sum_{\lambda_j\leq\Lambda}|a_j|^2\leq Ce^{C\Lambda^\sigma}\int_\omega\left|\sum_{\lambda_j\leq\Lambda}a_j\Phi_j\right|^2,$$

for all $\left\{a_j\right\}\in\mathbb{R}$ and $\Lambda>0$. Further,

$$\sigma=\begin{cases}3/4\,, & \text{if } \alpha\in(0,2)\setminus\{1\}\,,\\ 3/(2\gamma)\ \text{for any } \gamma\in(0,2)\,, & \text{if } \alpha=1.\end{cases}$$

Two different kinds of approach have been developed to obtain the spectral inequality for the sum of eigenfunctions: a first one is due to Lebeau and Robbiano [30] and is based on a Carleman estimate for an elliptic operator, whereas a second one appears in a remark in [3] and is based on an observation estimate at one point in time for a parabolic equation. Note that in the standard setting of uniformly elliptic operator, $\sigma=1/2$ (see [29], [22], [31], [36], [39], [26], [23]). In the present paper we will establish a new Carleman estimate for an associated degenerate elliptic operator. Because of the degeneracy of the coefficients of the operator $\mathcal{P}$, we make use of a new weight function in the design of the Carleman estimate. The subtle difference between the cases $\alpha\in(0,2)\setminus\{1\}$ and $\alpha=1$ is related to the existence of a Hardy type inequality for the H^1_α norm. Indeed, for $\alpha=1$, the desired Hardy inequality fails to hold.

Many applications to such spectral inequality have been developed, in particular in control theory (see [29], [31], [7], [28], [25], [6]). Let ω be an open and nonempty subset of $(0,1)$ and denote $1_{\widetilde{\omega}}$ the characteristic function of a given subdomain $\widetilde{\omega}$. We present the following two results.

Theorem 1.2. *Let $E\subset(0,T)$ be a measurable set of positive measure. For all $y^0\in L^2(0,1)$, there exists $f\in L^2(\omega\times E)$ such that the solution $y=y(x,t)$ of*

$$\begin{cases}\partial_t y-\partial_x\left(x^\alpha\partial_x y\right)=1_{\omega\times E}f\,, & \text{in } (0,1)\times(0,T)\,,\\ BC_\alpha(y)=0\,, & \text{on } (0,T)\,,\\ y_{|x=1}=0\,, & \text{on } (0,T)\,,\\ y_{|t=0}=y^0\,, & \text{in } (0,1)\,,\end{cases}$$

satisfies $y(\cdot,T)=0$.

Theorem 1.3. *Let $(t_m)_{m\in\mathbb{N}}$ be an increasing sequence of positive real numbers converging to $T>0$ and $(\mathcal{F}_m)_{m\in\mathbb{N}}$ a sequence of linear bounded operators from $L^2(0,1)$ into $L^2(\omega)$ such that, for any $z_0\in L^2(0,1)$, the solution $z=z(x,t)$ to*

$$\begin{cases}\partial_t z-\partial_x\left(x^\alpha\partial_x z\right)=\sum\limits_{m\in\mathbb{N}}\delta_{t=(t_{m+1}+t_m)/2}\otimes 1_\omega\mathcal{F}_m\left(z_{|t=t_m}\right)\,, & \text{in } (0,1)\times(0,T)\,,\\ BC_\alpha(z)=0\,, & \text{on } (0,T)\,,\\ z_{|x=1}=0\,, & \text{on } (0,T)\,,\\ z_{|t=0}=z_0\,, & \text{in } (0,1)\,,\end{cases}$$

satisfies $\lim\limits_{t\to T_-}\|z(\cdot,t)\|_{L^2(\Omega)}=0$.

Here, $\delta_{t=(t_{m+1}+t_m)/2}$ denotes the Dirac measure at $t=(t_{m+1}+t_m)/2$. Note that the above system equivalently reads

$$\begin{cases} \partial_t z - \partial_x\left(x^\alpha \partial_x z\right) = 0\,, & \text{for } t \in \mathbb{R}^+ \setminus \bigcup\limits_{m\geq 0}\left(\frac{t_m+t_{m+1}}{2}\right)\,, \\ z\left(\cdot, \frac{t_m+t_{m+1}}{2}\right) = z\left(\cdot, \left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m\left(z\left(\cdot, t_m\right)\right)\,, & \text{for any integer } m \geq 0\,, \\ \mathrm{BC}_\alpha(z) = 0\,, & \text{on } (0,T)\,, \\ z_{|x=1} = 0\,, & \text{on } (0,T)\,, \\ z_{|t=0} = z_0\,, & \text{in } (0,1). \end{cases}$$

Theorem 1.3 is a new approach to steer the solution to zero at time T and can be seen as a finite-time stabilization for the degenerate heat equation by impulse control. This can be compared with [14] (see [21] for ODE). The standard null-controllability problem is given when $E=(0,T)$ and has been studied in [8]. It is now well known that the null controllability for higher degeneracies ($\alpha \geq 2$) fails to hold (see [9] and the references therein). We also refer to [1], where the null-controllability result has been extended to more general degeneracies at the boundary. When the control is located at the boundary where the degeneracy occurs, we refer to [20,12,37]. We finally refer to the recent book [9] and the references therein for a full description of the field. Note that an analysis of the control cost at a degeneracy point is studied in [10], and an estimation of the cost of controllability for small $T>0$, as well as for $\alpha \to 2^-$ has been recently obtained in [11].

The outline of the paper is as follows. In Section 2, we present the key inequalities needed to prove Theorem 1.1 such as a Hardy inequality and a Carleman inequality. Section 3 is devoted to obtaining the applications of the spectral inequality in control theory as observation estimates, impulse approximate controllability, null controllability on measurable set-in time (see Theorem 3.4), and finite-time stabilization (see Theorem 3.5). Theorem 1.2 and Theorem 1.3 are direct consequence of Theorem 3.4 and Theorem 3.5 respectively.

2. Key inequalities

This section is devoted to the statement of the key inequalities: Hardy inequality and Carleman inequality, that will enable us to prove Theorem 1.1. The proof of the Carleman inequality is given at the end of this section.

2.1. Hardy inequality and boundary conditions

The following Hardy inequality shall play a central role in what follows. The proof can be found in [8], [42].

Lemma 2.1. *Let ϑ be a locally absolutely continuous function on $(0,1)$ such that $\int_0^1 x^\alpha |\vartheta'|^2 < \infty$. Then we have*

$$\int_0^1 x^{\alpha-2}|\vartheta|^2 \leq \frac{4}{(2-\alpha)^2}\int_0^1 x^\alpha |\vartheta'|^2\,,$$

if one of the following assumption holds:

i) $\alpha \in (0,1)$ *and* $\vartheta_{|x=0} = 0\,,$
ii) $\alpha \in (1,2)$ *and* $\vartheta_{|x=1} = 0.$

We also have the following lemma, that shall be useful when estimating the boundary terms arising from integration by parts in the strongly degenerate case $\alpha \in [1,2)$. The proof can be found in [8].

Lemma 2.2. *Let $\alpha \in [1,2)$ and $\vartheta \in H^1_\alpha(0,1)$. Then $(x|\vartheta|^2)_{|x=0} = 0$.*

2.2. Global Carleman estimate near the degeneracy

In this section, we shall state the crucial tool, i.e. a global Carleman estimate near the degeneracy of an elliptic operator. Introduce, for $S_0 > s_0 > 0$,

$$Z = (-S_0, S_0) \times (0,1)\,, \quad Y = (-s_0, s_0) \times (0,1)\,.$$

First, we shall write

$$\mathcal{Q} := -\partial_s^2 + \mathcal{P} = -\partial_s^2 - \partial_x\left(x^\alpha \partial_x\right) , \tag{2.2.1}$$

here $(s, x) \in Z$. The weight function we choose is of the form

$$\varphi(s, x) = \tau \frac{x^{2-\alpha}}{2-\alpha} - \frac{\tau^{\gamma/3}}{\nu} s^2 , \tag{2.2.2}$$

where $\tau, \nu > 0$ are two large parameters, and

$$\begin{cases} \gamma = 2 , & \text{for } \alpha \in (0,2)\setminus\{1\} , \\ \gamma < 2 , & \text{for } \alpha = 1. \end{cases} \tag{2.2.3}$$

Note that this weight function is completely decoupled in the two directions, in particular with respect to the dependence on τ. In the case $\alpha = 1$, the Hardy inequality in Lemma 2.1 does not hold, and this is the reason of our subtle choice of weight (2.2.2). Next, we shall set

$$\mathcal{Q}_\varphi := \mathrm{e}^{\varphi} \mathcal{Q}\, \mathrm{e}^{-\varphi}.$$

Finally, we state a global estimate for functions of $C^\infty((-S_0, S_0), D(\mathcal{P}))$, with the proper weight function φ given by (2.2.2) to handle the degeneracy at $x = 0$.

Theorem 2.1. *There exist $\tau_0 > 0$, and $\nu_0 > 0$ such that for $\gamma > 0$ defined in (2.2.3), there exists $c > 0$ such that*

$$\tau^\gamma ||v||^2_{L^2(Z)} + \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha} |v|^2 + \mathcal{B}(v) \le c ||\mathcal{Q}_\varphi v||^2_{L^2(Z)} ,$$

for all $\tau \ge \tau_0$ and $v \in C^\infty((-S_0, S_0), D(\mathcal{P}))$, where $\mathcal{B}$ is a quadratic form satisfying

$$\begin{aligned} \frac{1}{2}\mathcal{B}(v) \ge & -\tau \int_{-S_0}^{S_0} |\partial_x v_{|x=1}|^2 + 2\frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 \left[s|\partial_s v|^2 \right]_{s=-S_0}^{s=S_0} - 2\frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 \left[v \partial_s v \right]_{s=-S_0}^{s=S_0} \\ & + 8\frac{\tau^\gamma}{\nu_0^3} \int_0^1 \left[s^3 |v|^2 \right]_{s=-S_0}^{s=S_0} - 2\tau \int_0^1 \left[x \partial_s v \partial_x v \right]_{s=-S_0}^{s=S_0} - \tau \int_0^1 \left[v \partial_s v \right]_{s=-S_0}^{s=S_0} \\ & - 2\frac{\tau^{\gamma/3}}{\nu_0} \int_0^1 \left[x^\alpha s |\partial_x v|^2 \right]_{s=-S_0}^{s=S_0} + 2\frac{\tau^{2+\gamma/3}}{\nu_0} \int_0^1 \left[x^{2-\alpha} s |v|^2 \right]_{s=-S_0}^{s=S_0} . \end{aligned}$$

Note that, in the above Theorem 2.1, boundary conditions are prescribed through the membership in the domain of $\mathcal{P}$. The proof will be given at the end of this section.

In [8], the authors established a parabolic Carleman estimate for a class of degenerate operators, in the spirit of [18], with a weight linked to geodesic distance to the singularity $\{x = 0\}$, that is a weight of the form

$$\widetilde{\varphi}(x, t) = \frac{x^{2-\alpha} - 1}{(t(T-t))^4}. \tag{2.2.4}$$

In the present article, the design of the weight function φ is similar to (2.2.4). However, as we have to deal with an additional variable s (see the operator (2.2.1)), we also weaken the weight function in the s direction (see the weight (2.2.2), which is anisotropic with respect to powers of the Carleman large parameter τ).

2.3. Inequality with weight for a specific sum of eigenfunctions

A classical trick on quantitative uniqueness consists in transferring properties for elliptic equation into an estimate for parabolic operator (see [32]). Here, we naturally reproduce this idea for the sum of eigenfunctions (see [29], [22], [30], [31], [13], [36], [24], [28]).

We define the following function space, depending on the frequency parameter $\Lambda \ge 1$, by:

$$\mathcal{X}_\Lambda := \left\{ \mathbf{u}(s, x) = \sum_{\lambda_j \le \Lambda} \frac{\sinh(\sqrt{\lambda_j}(s + S_0))}{\sqrt{\lambda_j}} a_j \Phi_j(x); a_j \in \mathbb{R} \right\}.$$

We then obtain an observability estimate from a term localized in the interior of Z for functions $\mathbf{u} \in \mathcal{X}_\Lambda$. Notice that $\mathcal{Q}\mathbf{u} = 0$.

Corollary 2.1. *Let $\gamma > 0$ defined in (2.2.3). Let $0 < \epsilon < 1/2$, let $s_1 > 0$ be such that $s_0 < s_1 < S_0$. Let*

$$Z_1 := \{(s,x) \in Z;\ s \in (-\tfrac{1}{2}(s_1+S_0), \tfrac{1}{2}(s_1+S_0)),\ x \in (\epsilon, 1-\epsilon)\}.$$

There exists $c > 0$ such that

$$\int_0^\epsilon |\sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 \le c\, e^{c\Lambda^{3/(2\gamma)}} ||\mathbf{u}||^2_{H^1(Z_1)}$$

for all $\mathbf{u} \in \mathcal{X}_\Lambda$ and $\Lambda \ge 1$.

Proof. Let $\chi(s,x) = \chi_1(x)\chi_2(s)$, with $\chi_1 \in C^\infty(0,1)$, $\chi_2 \in C^\infty(-S_0, S_0)$, such that

$$\chi_1(x) := \begin{cases} 1 & \text{if } x \in [0,\epsilon] \\ 0 & \text{if } x \in [1-\epsilon, 1], \end{cases} \qquad \chi_2(s) := \begin{cases} 1 & \text{if } s \in [-s_1, s_1] \\ 0 & \text{if } s \in [-S_0, -\frac{1}{2}(s_1+S_0)] \cup [\frac{1}{2}(s_1+S_0), S_0]. \end{cases}$$

We shall apply the Carleman estimate in Theorem 2.1 to $v = e^\varphi \chi \mathbf{u}$, with $\mathbf{u} \in \mathcal{X}_\Lambda$. Note that for such functions v, all the boundary terms in Theorem 2.1 vanish. Recall that $\mathcal{Q}\mathbf{u} = 0$. Hence, we have $\mathcal{Q}_\varphi v = e^\varphi [\mathcal{Q}, \chi]\mathbf{u}$. By Theorem 2.1, this yields

$$\tau^\gamma \|v\|^2_{L^2(Z)} + \tau \int_Z x^\alpha |\partial_x v|^2 + \tau^3 \int_Z x^{2-\alpha}|v|^2 \le c \int_Z \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 . \tag{2.3.1}$$

We first work with the left-hand side of (2.3.1) (from now on, the notation $A \lesssim B$ means that there exists a constant $c > 0$, independent of the concerned parameters such that $A \le cB$). We have

$$\tau^\gamma \|v\|^2_{L^2(Z)} \ge \tau^\gamma \|v\|^2_{L^2(Y)} \gtrsim \tau^\gamma e^{-\frac{2}{\nu}\tau^{\gamma/3}s_0^2} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 , \tag{2.3.2}$$

and then we obtain that there exists $c > 0$ such that

$$\tau^\gamma e^{-\frac{2}{\nu}\tau^{\gamma/3}s_0^2} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 \le c \int_Z \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 . \tag{2.3.3}$$

Second, we work with the right hand side of (2.3.3). Note that $[\mathcal{Q}, \chi]$ is supported where χ varies, that is, in $\overline{Z_1} \cup \overline{Z_2}$ with

$$Z_2 := \{(s,x) \in Z;\ |s| \in (s_1, \tfrac{1}{2}(s_1+S_0)),\ x \in (0,\epsilon)\}.$$

Thus,

$$\int_Z \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 \le \int_{Z_1} \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 + \int_{Z_2} \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 . \tag{2.3.4}$$

Using the particular form of χ and the particular form of $\mathbf{u}$, we have

$$\int_{Z_2} \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 \lesssim \int_{Z_2} \left|e^\varphi \mathbf{u}\right|^2 + \int_{Z_2} \left|e^\varphi \partial_s \mathbf{u}\right|^2 \lesssim e^{-\frac{2}{\nu}\tau^{\gamma/3}s_1^2} e^{cS_0\sqrt{\Lambda}} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 . \tag{2.3.5}$$

Using (2.3.3), (2.3.4), and (2.3.5), we have

$$\tau^\gamma e^{-\frac{2}{\nu}\tau^{\gamma/3}s_0^2} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 \lesssim e^{-\frac{2}{\nu}\tau^{\gamma/3}s_1^2} e^{cS_0\sqrt{\Lambda}} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 + \int_{Z_1} \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 .$$

Taking $\tau = \tau_0 \Lambda^{3/(2\gamma)}$ yields

$$\begin{aligned} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 &\lesssim \tau_0^{-\gamma} \Lambda^{-3/2} e^{\frac{2}{\nu}\tau_0^{\gamma/3}\sqrt{\Lambda}(s_0^2 - s_1^2)} e^{cS_0\sqrt{\Lambda}} \int_0^\epsilon |e^{\tau \frac{x^{2-\alpha}}{2-\alpha}} \sum_{\lambda_j \le \Lambda} a_j \Phi_j|^2 \\ &\quad + \tau_0^{-\gamma} \Lambda^{-3/2} e^{\frac{2}{\nu}\tau_0^{\gamma/3}\sqrt{\Lambda}s_0^2} \int_{Z_1} \left|e^\varphi [\mathcal{Q}, \chi]\mathbf{u}\right|^2 , \end{aligned} \tag{2.3.6}$$

and as $s_0 < s_1$, we see that the first term in the right-hand side of (2.3.6) can be absorbed by the left-hand side by fixing τ_0 sufficiently large. As a result,

$$\int_0^\epsilon |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 \lesssim \int_0^\epsilon |e^{\tau\frac{x^{2-\alpha}}{2-\alpha}}\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 \lesssim e^{c\sqrt{\Lambda}}\int_{Z_1}\left|e^{\varphi}[\mathcal{Q},\chi]\mathbf{u}\right|^2 . \tag{2.3.7}$$

It remains to work with the right-hand side of (2.3.7). We have

$$\int_{Z_1}\left|e^{\varphi}[\mathcal{Q},\chi]\mathbf{u}\right|^2 \lesssim \int_{Z_1}\left|e^{\varphi}\mathbf{u}\right|^2 + \int_{Z_1}\left|e^{\varphi}\partial_x\mathbf{u}\right|^2 \lesssim e^{c\tau}||\mathbf{u}||^2_{H^1(Z_1)} \lesssim e^{c\Lambda^{3/(2\gamma)}}||\mathbf{u}||^2_{H^1(Z_1)}.$$

Combining this last inequality with (2.3.7) yields the sought result. □

2.4. Proof of Theorems 1.1 and 2.1

2.4.1. Proof of the spectral inequality

This section is devoted to proving Theorem 1.1. Recall that, from Corollary 2.1, we have

$$\int_0^\epsilon |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 \leq ce^{c\Lambda^{3/(2\gamma)}}||\mathbf{u}||^2_{H^1(Z_1)} ,$$

where Z_1 is an open set compactly embedded in Z. Where the operator $\mathcal{Q}$ is uniformly elliptic, it is classical that we can propagate interpolation inequalities (see [30, p. 346] and [24, Theorem 5.3, p. 725]) through the domain

$$Z_3 := \{(s,x)\in Z;\ s\in(-\tfrac{1}{2}(s_1+S_0),\tfrac{1}{2}(s_1+S_0)),\ x\in(\epsilon/2,1)\}$$

to obtain that there exists $\mu\in(0,1)$ such that

$$||\mathbf{u}||_{H^1(Z_3)} \lesssim ||\partial_s\mathbf{u}_{|s=-S_0}||^{\mu}_{L^2(\omega)}||\mathbf{u}||^{1-\mu}_{H^1(Z\cap\{(s,x);x\in[\epsilon/4,1]\})}.$$

At this point, it is enough to observe

$$||\mathbf{u}||_{H^1(Z\cap\{(s,x);x\in[\epsilon/4,1]\})} \leq ce^{c\sqrt{\Lambda}}||\sum_{\lambda_j\leq\Lambda} a_j\Phi_j||_{H^1_\alpha(0,1)} \lesssim e^{c'\sqrt{\Lambda}}||\sum_{\lambda_j\leq\Lambda} a_j\Phi_j||_{L^2(0,1)} ,$$

$$||\mathbf{u}||^2_{H^1(Z_3)} \gtrsim \int_\epsilon^1 |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 ,$$

and, as $\partial_s\mathbf{u}_{|s=-S_0} = \sum_{\lambda_j\leq\Lambda} a_j\Phi_j$,

$$||\partial_s\mathbf{u}_{|s=-S_0}||^2_{L^2(\omega)} = \int_\omega |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2.$$

Summing up, using that $Z_1\subset Z_3$, we finally deduce

$$\begin{aligned}\int_0^1 |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 &\leq \int_0^\epsilon |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 + \int_\epsilon^1 |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2 \\ &\lesssim \left(1+e^{c\Lambda^{3/(2\gamma)}}\right)e^{2c'(1-\mu)\sqrt{\Lambda}}\left(\int_\omega |\sum_{\lambda_j\leq\Lambda} a_j\Phi_j|^2\right)^{\mu}||\sum_{\lambda_j\leq\Lambda} a_j\Phi_j||^{2(1-\mu)}_{L^2(0,1)} ,\end{aligned}$$

and Theorem 1.1 follows. □

2.4.2. Proof of Theorem 2.1

Here, we give the proof of the global Carleman estimate near the degeneracy in Theorem 2.1.

Recall that $\mathcal{Q}_\varphi = e^\varphi \mathcal{Q} e^{-\varphi}$ and, therefore,

$$\begin{aligned}\mathcal{Q}_\varphi &= -(\partial_s - (\partial_s\varphi))^2 + (\partial_x - (\partial_x\varphi))x^\alpha(\partial_x - (\partial_x\varphi)) \\ &= -\partial_s^2 - |\partial_s\varphi|^2 + 2(\partial_s\varphi)\partial_s + \partial_s^2\varphi + \mathcal{P} - x^{2\alpha}|\partial_x\varphi|^2 + 2x^\alpha(\partial_x\varphi)\partial_x + \partial_x\left(x^\alpha\partial_x\varphi\right).\end{aligned}$$

Now, we decompose $\mathcal{Q}_\varphi$ into four parts:

$$\mathcal{Q}_\varphi = \mathcal{S}_x + \mathcal{S}_s + \mathcal{A}_x + \mathcal{A}_s\,,$$

where $\mathcal{S}_x + \mathcal{S}_s$ is the symmetric part and $\mathcal{A}_x + \mathcal{A}_s$ is the skew-symmetric part of the full conjugated operator. Using the definition of the weight function (2.2.2), we have

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha}\,,\quad \mathcal{S}_s = -\partial_s^2 - 4\frac{\tau^{2\gamma/3}}{\nu^2}s^2\,,\quad \mathcal{A}_x = 2\tau x\partial_x + \tau\,,\quad \mathcal{A}_s = -4\frac{\tau^{\gamma/3}}{\nu}s\partial_s - 2\frac{\tau^{\gamma/3}}{\nu}.$$

Let $v \in C^\infty((-S_0, S_0), D(\mathcal{P}))$. Introduce $\mathcal{S} = \mathcal{S}_x + \mathcal{S}_s$ and $\mathcal{A} = \mathcal{A}_x + \mathcal{A}_s$. We begin by noting that

$$\begin{aligned}\left\|\mathcal{Q}_\varphi v\right\|^2_{L^2(Z)} &= ||\mathcal{S}v||^2_{L^2(Z)} + ||\mathcal{A}v||^2_{L^2(Z)} + 2\,(\mathcal{S}v, \mathcal{A}v)_Z \\ &\geq ||\mathcal{S}v||^2_{L^2(Z)} + 2\left[(\mathcal{S}_x v, \mathcal{A}_x v)_Z + (\mathcal{S}_s v, \mathcal{A}_s v)_Z + (\mathcal{S}_x v, \mathcal{A}_s v)_Z + (\mathcal{S}_s v, \mathcal{A}_x v)_Z\right].\end{aligned}$$

The proof is divided into three steps. Each step corresponds to the computation of one of the above scalar products.

First step. We begin with the first scalar product $(\mathcal{S}_x v, \mathcal{A}_x v)_Z$.

Lemma 2.3. *We have*

$$(\mathcal{S}_x v, \mathcal{A}_x v)_Z = \tau(2-\alpha)\int_Z x^\alpha|\partial_x v|^2 + \tau^3(2-\alpha)\int_Z x^{2-\alpha}|v|^2 + \mathcal{B}_0(v)\,, \tag{2.4.1}$$

with

$$\mathcal{B}_0(v) = -\tau\int_{-S_0}^{S_0}\left[x^{\alpha+1}|\partial_x v|^2\right]_{x=0}^{x=1} - \tau\int_{-S_0}^{S_0}\left[x^\alpha v\partial_x v\right]_{x=0}^{x=1} - \tau^3\int_{-S_0}^{S_0}\left[x^{3-\alpha}|v|^2\right]_{x=0}^{x=1}.$$

The proof of this lemma will be provided later. Using the Hardy inequality of Lemma 2.1 in (2.4.1), there exists $c > 0$ such that, for all $\alpha \in (0,2)\backslash\{1\}$,

$$c\,(\mathcal{S}_x v, \mathcal{A}_x v)_Z \geq \tau^2(2-\alpha)\int_Z |v|^2 + \tau(2-\alpha)\int_Z x^\alpha|\partial_x v|^2 + \tau^3(2-\alpha)\int_Z x^{2-\alpha}|v|^2 + \mathcal{B}_0(v).$$

In the particular case $\alpha = 1$, using the Hardy inequality, for all $\alpha' \in (1,2)$, there exists $c' > 0$ such that

$$\tau\int_Z x|\partial_x v|^2 \geq \tau\int_Z x^{\alpha'}|\partial_x v|^2 \geq c'\tau\int_Z x^{\alpha'-2}|v|^2. \tag{2.4.2}$$

As a result, interpolating (2.4.2) with (2.4.1), for all $\gamma \in (0,2)$, there exists $c' > 0$ such that

$$c'\,(\mathcal{S}_x v, \mathcal{A}_x v)_Z \geq \tau^\gamma\int_Z |v|^2 + \tau(2-\alpha)\int_Z x^\alpha|\partial_x v|^2 + \tau^3(2-\alpha)\int_Z x^{2-\alpha}|v|^2 + \mathcal{B}_0(v). \tag{2.4.3}$$

Hence, (2.4.3) holds for all $\alpha \in (0,2)$, with γ defined in (2.2.3), with a constant $c' > 0$ that depends on α. We now focus on boundary terms $\mathcal{B}_0$. We have, using the boundary conditions described in $H^1_\alpha(0,1)$ and Lemma 2.2,

$$\mathcal{B}_0(v) = -\tau\int_{-S_0}^{S_0}|\partial_x v_{|x=1}|^2.$$

Second step. We then compute the second scalar product $(\mathcal{S}_s v, \mathcal{A}_s v)_Z$.

Lemma 2.4. *We have*

$$(\mathcal{S}_s v, \mathcal{A}_s v)_Z = -4\frac{\tau^{\gamma/3}}{\nu}\int_Z |\partial_s v|^2 - 16\frac{\tau^\gamma}{\nu^3}\int_Z s^2|v|^2 + \mathcal{B}_1(v)\,,$$

with

$$\mathcal{B}_1(v) := 2\frac{\tau^{\gamma/3}}{\nu}\int_0^1 \left[s|\partial_s v|^2\right]_{s=-S_0}^{s=S_0} + 2\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0} + 8\frac{\tau^\gamma}{\nu^3}\int_0^1 \left[s^3|v|^2\right]_{s=-S_0}^{s=S_0}.$$

The proof of this lemma will be provided later. The two volume terms in Lemma 2.4 are non-positive, and need a particular attention. We set

$$K_1(v) := -4\frac{\tau^{\gamma/3}}{\nu}\int_Z |\partial_s v|^2\,,\quad K_2(v) := -16\frac{\tau^\gamma}{\nu^3}\int_Z s^2|v|^2.$$

Since $\mathcal{S} = \mathcal{S}_x + \mathcal{S}_s$, we have the following relation

$$-\int_Z |\partial_s v|^2 = -\int_Z (\mathcal{S}v)v - 4\frac{\tau^{2\gamma/3}}{\nu^2}\int_Z s^2|v|^2 + \int_Z (\mathcal{P}v)v - \tau^2\int_Z x^{2-\alpha}|v|^2 - \int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0}\,,$$

and we then deduce

$$\begin{aligned} K_1(v) &= -4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{S}v)v - 16\frac{\tau^\gamma}{\nu^3}\int_Z s^2|v|^2 \\ &\quad + 4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{P}v)v - 4\frac{\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2 - 4\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0} \\ &= -4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{S}v)v + K_2(v) + 4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{P}v)v \\ &\quad - 4\frac{\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2 - 4\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0}. \end{aligned}$$

As a result, using integration by parts and Young's inequality,

$$\begin{aligned} (\mathcal{S}_s v, \mathcal{A}_s v)_Z &= -4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{S}v)v + 2K_2(v) + 4\frac{\tau^{\gamma/3}}{\nu}\int_Z (\mathcal{P}v)v \\ &\qquad - 4\frac{\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2 + \mathcal{B}_1(v) - 4\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0} \\ &\geq -\frac{2}{\nu}||\mathcal{S}v||_{L^2(Z)}^2 - 2\frac{\tau^{2\gamma/3}}{\nu}||v||_{L^2(Z)}^2 + 2K_2(v) \\ &\qquad + 4\frac{\tau^{\gamma/3}}{\nu}\int_Z x^\alpha|\partial_x v|^2 - 4\frac{\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2 + \widetilde{\mathcal{B}}_1(v)\,, \end{aligned}$$

with

$$\widetilde{\mathcal{B}}_1(v) = \mathcal{B}_1(v) - \frac{4\tau^{\gamma/3}}{\nu}\int_{-S_0}^{S_0} \left[x^\alpha v\partial_x v\right]_{x=0}^{x=1} - 4\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0}.$$

Note that, using the boundary conditions, we have

$$\widetilde{\mathcal{B}}_1(v) = \mathcal{B}_1(v) - 4\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0}$$
$$= 2\frac{\tau^{\gamma/3}}{\nu}\int_0^1 \left[s|\partial_s v|^2\right]_{s=-S_0}^{s=S_0} - 2\frac{\tau^{\gamma/3}}{\nu}\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0} + 8\frac{\tau^{\gamma}}{\nu^3}\int_0^1 \left[s^3|v|^2\right]_{s=-S_0}^{s=S_0}.$$

Summing up, fixing $\nu := \nu_0 > 0$ sufficiently large, and taking $\tau \geq \tau_0$, with $\tau_0 > 0$ sufficiently large, there exists $c > 0$ such that

$$c\left(\|\mathcal{S}v\|^2_{L^2(Z)} + 2\,(\mathcal{S}_x v, \mathcal{A}_x v) + 2\,(\mathcal{S}_s v, \mathcal{A}_s v)\right)$$
$$\geq \tau^{\gamma}\,\|v\|^2_{L^2(Z)} + \tau\int_Z x^{\alpha}|\partial_x v|^2 + \tau^3\int_Z x^{2-\alpha}|v|^2 + 2\mathcal{B}_0(v) + 2\widetilde{\mathcal{B}}_1(v).$$

Third step. It remains to estimate the crossed-terms $(\mathcal{S}_x v, \mathcal{A}_s v)_Z + (\mathcal{S}_s v, \mathcal{A}_x v)_Z$.

Lemma 2.5. *We have, on the one hand,*

$$(\mathcal{S}_s v, \mathcal{A}_x v)_Z = \mathcal{B}_2(v) := \tau\int_{-S_0}^{S_0}\left[x|\partial_s v|^2\right]_{x=0}^{x=1} - 2\,\tau\int_0^1 [x\partial_s v\partial_x v]_{s=-S_0}^{s=S_0}$$
$$- \tau\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0} - 4\frac{\tau^{1+2\gamma/3}}{\nu_0^2}\int_{-S_0}^{S_0}\left[s^2 x|v|^2\right]_{x=0}^{x=1},$$

and, on the other hand,

$$(\mathcal{S}_x v, \mathcal{A}_s v)_Z = \mathcal{B}_3(v) := 4\frac{\tau^{\gamma/3}}{\nu_0}\int_{-S_0}^{S_0}[x^{\alpha}s\partial_x v\partial_s v]_{x=0}^{x=1} - 2\frac{\tau^{\gamma/3}}{\nu_0}\int_0^1\left[x^{\alpha}s|\partial_x v|^2\right]_{s=-S_0}^{s=S_0}$$
$$+ 2\frac{\tau^{\gamma/3}}{\nu_0}\int_{-S_0}^{S_0}[x^{\alpha}v\partial_x v]_{x=0}^{x=1} + 2\frac{\tau^{2+\gamma/3}}{\nu_0}\int_0^1\left[x^{2-\alpha}s|v|^2\right]_{s=-S_0}^{s=S_0}.$$

The proof of this lemma will be provided later. Note that using the boundary conditions given in $H^1_{\alpha}(0,1)$ as well as Lemma 2.2, we have

$$\mathcal{B}_2(v) \geq -2\,\tau\int_0^1 [x\partial_s v\partial_x v]_{s=-S_0}^{s=S_0} - \tau\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0},$$

and

$$\mathcal{B}_3(v) = -2\frac{\tau^{\gamma/3}}{\nu_0}\int_0^1\left[x^{\alpha}s|\partial_x v|^2\right]_{s=-S_0}^{s=S_0} + 2\frac{\tau^{2+\gamma/3}}{\nu_0}\int_0^1\left[x^{2-\alpha}s|v|^2\right]_{s=-S_0}^{s=S_0}.$$

Now setting $\mathcal{B} = 2\left(\mathcal{B}_0 + \widetilde{\mathcal{B}}_1 + \mathcal{B}_2 + \mathcal{B}_3\right)$ yields the sought result. □

2.4.3. Proof of Lemma 2.3

We recall that

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha}, \quad \mathcal{A}_x = 2\,\tau x\partial_x + \tau.$$

We shall denote by $\mathcal{I}_{ij}$ the scalar product between the i^{th} term of $\mathcal{S}_x$ with the j^{th} term of $\mathcal{A}_x$. Let us compute first

$$\begin{aligned}\mathcal{I}_{11} &= -2\,\tau \int_Z \partial_x x^\alpha \partial_x v x \partial_x v \\ &= 2\,\tau \int_Z x^\alpha |\partial_x v|^2 + \tau \int_Z x^{1+\alpha} \partial_x \left(|\partial_x v|^2 \right) - 2\,\tau \int_{-S_0}^{S_0} \left[x^{1+\alpha} |\partial_x v|^2 \right]_{x=0}^{x=1} \\ &= (1-\alpha)\tau \int_Z x^\alpha |\partial_x v|^2 - \tau \int_{-S_0}^{S_0} \left[x^{1+\alpha} |\partial_x v|^2 \right]_{x=0}^{x=1} .\end{aligned}$$

Second, we have

$$\mathcal{I}_{12} = -\tau \int_Z v \partial_x x^\alpha \partial_x v = \tau \int_Z x^\alpha |\partial_x v|^2 - \tau \int_{-S_0}^{S_0} \left[x^\alpha v \partial_x v \right]_{x=0}^{x=1} .$$

Third, we see that

$$\mathcal{I}_{21} = -2\,\tau^3 \int_Z x^{3-\alpha} v \partial_x v = -\tau^3 \int_Z x^{3-\alpha} \partial_x \left(|v|^2 \right) = (3-\alpha)\tau^3 \int_Z x^{2-\alpha} |v|^2 - \tau^3 \int_{-S_0}^{S_0} \left[x^{3-\alpha} |v|^2 \right]_{x=0}^{x=1} .$$

Finally, we can check that

$$\mathcal{I}_{22} = -\tau^3 \int_Z x^{2-\alpha} |v|^2 ,$$

and we end the proof of Lemma 2.3 by summing the four quantities above. □

2.4.4. Proof of Lemma 2.4

We recall that

$$\mathcal{S}_s = -\partial_s^2 - 4 \frac{\tau^{2\gamma/3}}{\nu^2} s^2 , \quad \mathcal{A}_s = -4 \frac{\tau^{\gamma/3}}{\nu} s \partial_s - 2 \frac{\tau^{\gamma/3}}{\nu} .$$

We shall denote by $\mathcal{I}_{ij}$ the scalar product between the i^{th} term of $\mathcal{S}_s$ with the j^{th} term of $\mathcal{A}_s$. Let us compute the $\mathcal{I}_{ij}$, $1 \le i$, $j \le 2$, by integrations by parts

$$\mathcal{I}_{11} = \frac{4\tau^{\gamma/3}}{\nu} \int_Z s \partial_s^2 v s \partial_s v = \frac{2\,\tau^{\gamma/3}}{\nu} \int_Z s \partial_s \left(|\partial_s v|^2 \right) = -\frac{2\,\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2 + \frac{2\,\tau^{\gamma/3}}{\nu} \int_0^1 \left[s |\partial_s v|^2 \right]_{s=-S_0}^{s=S_0} ,$$

$$\mathcal{I}_{12} = \frac{2\,\tau^{\gamma/3}}{\nu} \int_Z v \partial_s^2 v = -\frac{2\,\tau^{\gamma/3}}{\nu} \int_Z |\partial_s v|^2 + \frac{2\,\tau^{\gamma/3}}{\nu} \int_0^1 \left[v \partial_s v \right]_{s=-S_0}^{s=S_0} ,$$

$$\mathcal{I}_{21} = \frac{16\tau^\gamma}{\nu^3} \int_Z s^3 v \partial_s v = 8 \frac{\tau^\gamma}{\nu^3} \int_Z s^3 \partial_s \left(|v|^2 \right) = -\frac{24\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 + \frac{8\tau^\gamma}{\nu^3} \int_0^1 \left[s^3 |v|^2 \right]_{s=-S_0}^{s=S_0} ,$$

and

$$\mathcal{I}_{22} = 8 \frac{\tau^\gamma}{\nu^3} \int_Z s^2 |v|^2 .$$

Summing all the $\mathcal{I}_{ij}$ yields the sought result of Lemma 2.4. □

2.4.5. Proof of Lemma 2.5

We recall that

$$\mathcal{S}_x = \mathcal{P} - \tau^2 x^{2-\alpha}\ , \quad \mathcal{S}_s = -\partial_s^2 - 4\frac{\tau^{2\gamma/3}}{\nu^2}s^2\ , \quad \mathcal{A}_x = 2\,\tau x\partial_x + \tau\ , \quad \mathcal{A}_s = -4\frac{\tau^{\gamma/3}}{\nu}s\partial_s - 2\frac{\tau^{\gamma/3}}{\nu}.$$

We first compute the scalar product $(\mathcal{S}_s v, \mathcal{A}_x v)_Z$. We shall denote by $\mathcal{I}_{ij}$ the scalar product between the i^{th} term of $\mathcal{S}_s$ with the j^{th} term of $\mathcal{A}_x$. We have

$$\begin{aligned}
\mathcal{I}_{11} &= -2\,\tau \int_Z x\partial_s^2 v\partial_x v = \tau \int_Z x\partial_x\left(|\partial_s v|^2\right) - 2\,\tau\int_0^1 [x\partial_s v\partial_x v]_{s=-S_0}^{s=S_0} \\
&= -\tau\int_Z |\partial_s v|^2 + \tau\int_{-S_0}^{S_0}\left[x|\partial_s v|^2\right]_{x=0}^{x=1} - 2\,\tau\int_0^1 [x\partial_s v\partial_x v]_{s=-S_0}^{s=S_0}\ , \\
\mathcal{I}_{12} &= -\tau\int_Z v\partial_s^2 v = \tau\int_Z |\partial_s v|^2 - \tau\int_0^1 [v\partial_s v]_{s=-S_0}^{s=S_0}\ , \\
\mathcal{I}_{21} &= -\frac{8\tau^{2\gamma/3+1}}{\nu^2}\int_Z s^2 x v\partial_x v = -\frac{4\tau^{2\gamma/3+1}}{\nu^2}\int_Z s^2 x\partial_x\left(|v|^2\right) \\
&= \frac{4\tau^{2\gamma/3+1}}{\nu^2}\int_Z s^2|v|^2 - \frac{4\tau^{2\gamma/3+1}}{\nu^2}\int_{-S_0}^{S_0}\left[s^2 x|v|^2\right]_{x=0}^{x=1}\ ,
\end{aligned}$$

and

$$\mathcal{I}_{22} = \frac{4\tau^{2\gamma/3+1}}{\nu^2}\int_Z s^2|v|^2.$$

Summing the above quantities yields the result, by remarking that all the volume terms cancel. We second compute the scalar product $(\mathcal{S}_x v, \mathcal{A}_s v)_Z$. We shall denote by $\mathcal{J}_{ij}$ the scalar product between the i^{th} term of $\mathcal{S}_x$ with the j^{th} term of $\mathcal{A}_s$. Integrations by parts then give

$$\begin{aligned}
\mathcal{J}_{11} &= \frac{4\tau^{\gamma/3}}{\nu}\int_Z s\partial_x\left(x^\alpha\partial_x v\right)\partial_s v = -\frac{2\,\tau^{\gamma/3}}{\nu}\int_Z x^\alpha s\partial_s\left(|\partial_x v|^2\right) + \frac{4\tau^{\gamma/3}}{\nu}\int_{-S_0}^{S_0}\left[x^\alpha s\partial_x v\partial_s v\right]_{x=0}^{x=1} \\
&= \frac{2\,\tau^{\gamma/3}}{\nu}\int_Z x^\alpha|\partial_x v|^2 - \frac{2\,\tau^{\gamma/3}}{\nu}\int_0^1\left[x^\alpha s|\partial_x v|^2\right]_{s=-S_0}^{s=S_0} + \frac{4\tau^{\gamma/3}}{\nu}\int_{-S_0}^{S_0}\left[x^\alpha s\partial_x v\partial_s v\right]_{x=0}^{x=1}\ , \\
\mathcal{J}_{12} &= \frac{2\,\tau^{\gamma/3}}{\nu}\int_Z v\partial_x\left(x^\alpha\partial_x v\right) = -\frac{2\,\tau^{\gamma/3}}{\nu}\int_Z x^\alpha|\partial_x v|^2 + \frac{2\,\tau^{\gamma/3}}{\nu}\int_{-S_0}^{S_0}\left[x^\alpha v\partial_x v\right]_{x=0}^{x=1}\ , \\
\mathcal{J}_{21} &= \frac{4\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha} v s\partial_s v = \frac{2\,\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}s\partial_s\left(|v|^2\right) \\
&= -\frac{2\,\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2 + \frac{2\,\tau^{2+\gamma/3}}{\nu}\int_0^1\left[x^{2-\alpha}s|v|^2\right]_{s=-S_0}^{s=S_0}\ ,
\end{aligned}$$

and

$$\mathcal{J}_{22} = \frac{2\,\tau^{2+\gamma/3}}{\nu}\int_Z x^{2-\alpha}|v|^2.$$

It remains to sum the above $\mathcal{J}_{ij}$ to obtain the sought result of Lemma 2.5. □

3. Applications of the spectral inequality

The second part of this article is devoted to show some applications of the spectral inequality.

Let H be a real Hilbert space, and P a linear self-adjoint operator from $D(P)$ into H, where $D(P)$, being the domain of P, is a subspace of H. Denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product of H, respectively. We assume that P is an isomorphism from $D(P)$ (equipped with the graph norm) onto H, that P^{-1} is a compact linear operator in H and that $\langle P\vartheta,\vartheta\rangle > 0$ $\forall\vartheta\in D(P)$, $\vartheta\neq 0$. Introduce the set $\left\{\lambda_j\right\}_{j\geq 1}$ for the family of all eigenvalues of P such that

$$0<\lambda_1\leq\lambda_2\leq\cdot\cdot\leq\lambda_k\leq\lambda_{k+1}\leq\cdots \text{ and } \lim_{j\to\infty}\lambda_j=\infty\,,$$

and let $\left\{\Phi_j\right\}_{j\geq 1}$ be the family of the corresponding normalized orthogonal eigenfunctions.

It is well known that for $u_0\in H$ given, the initial value problem

$$\begin{cases} u'(t)+Pu(t)=0\,, & t\in(0,+\infty)\,,\\ u(0)=u_0\,, \end{cases}$$

possesses a unique solution $u\in L^2\left(0,T;D\left(P^{1/2}\right)\right)\cap C\left([0,T],H\right)$ for any $T>0$, which satisfies

$$u(t)=\sum_{j\geq 1}\left\langle u_0,\Phi_j\right\rangle \mathrm{e}^{-\lambda_j t}\Phi_j \text{ and } \|u(t)\|\leq \mathrm{e}^{-\lambda_1 t}\|u_0\|\,.$$

In particular, if $u_0=\sum_{j\geq 1}a_j\Phi_j$ with $\sum_{j\geq 1}\left|a_j\right|^2<+\infty$, then $\|u_0\|^2=\sum_{j\geq 1}\left|a_j\right|^2$, $\langle Pu_0,u_0\rangle=\sum_{j\geq 1}\lambda_j\left|a_j\right|^2$ and $\left\langle P^{-1}u_0,u_0\right\rangle=\sum_{j\geq 1}\frac{1}{\lambda_j}\left|a_j\right|^2$. Further, $\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2+2\langle Pu(t),u(t)\rangle=0$, and $\frac{\mathrm{d}}{\mathrm{d}t}\left\langle P^{-1}u(t),u(t)\right\rangle+2\|u(t)\|^2=0$.

Let Ω be a bounded domain of $\mathbb{R}^d$, $d\geq 1$, with boundary $\partial\Omega$ of class C^2. Four examples of operator P are the following:

- the 1d degenerate operator with $d=1$ and $P=-\partial_x\left(x^\alpha\partial_x\right)$ with $\Omega=(0,1)$, $H=L^2(\Omega)$ and $D(P)=\left\{\vartheta\in H^1_\alpha(\Omega)\,;\,\mathcal{P}\vartheta\in L^2(\Omega)\text{ and }\mathrm{BC}_\alpha(\vartheta)=0\right\}$,
- the Laplacian with $P=-\Delta$ with $H=L^2(\Omega)$ and $D(P)=H^2(\Omega)\cap H^1_0(\Omega)$,
- the bi-Laplacian with $P=\Delta^2$ with $H=L^2(\Omega)$ and $D(P)=H^4(\Omega)\cap H^2_0(\Omega)$,
- the Stokes operator with $P=-\mathbb{P}\Delta$ with $H=\left\{\vartheta\in L^2(\Omega)^d\,;\,\mathrm{div}\vartheta=0,\ \vartheta\cdot n_{|\partial\Omega}=0\right\}$ and $D(P)=H^2(\Omega)^d\cap\left\{\vartheta\in H^1_0(\Omega)^d\,;\,\mathrm{div}\vartheta=0\right\}$, where $\mathbb{P}$ is the orthogonal projector in $L^2(\Omega)^d$ onto H.

3.1. Equivalence between observation and spectral inequality

In this section, we present several equivalent inequalities. From now on, suppose that $H=L^2(\Omega)$. Denote by $\|\cdot\|_\omega$ and $\langle\cdot,\cdot\rangle_\omega$ the norm and the inner product of $L^2(\omega)$, respectively, where ω is a subdomain of Ω.

Theorem 3.1. *Let ω be an open and nonempty subset of Ω. Let $\sigma\in(0,1)$. Then the following statements are equivalent:*

(i) *there is a positive constant C_1, depending only on P, Ω, ω and σ, so that, for each $\Lambda>0$ and each sequence of real numbers $\left\{a_j\right\}\subset\mathbb{R}$, it holds*

$$\sum_{\lambda_j\leq\Lambda}\left|a_j\right|^2\leq \mathrm{e}^{C_1\left(1+\Lambda^\sigma\right)}\int_\omega\left|\sum_{\lambda_j\leq\Lambda}a_j\Phi_j\right|^2\ ;$$

(ii) *there is a positive constant C_2, depending only on (P,Ω,ω,σ), so that, for all $\theta\in(0,1)$, $t>0$ and $u(0)\in L^2(\Omega)$,*

$$\|u(t)\|\leq \mathrm{e}^{C_2\left(1+\left(\frac{1}{\theta t}\right)^{\frac{\sigma}{1-\sigma}}\right)}\|u(0)\|^\theta\,\|u(t)\|_\omega^{1-\theta}\ ;$$

(iii) *there is a positive constant C_3, depending only on (P,Ω,ω,σ), so that for all $\varepsilon>0$, $t>0$ and $u(0)\in L^2(\Omega)$,*

$$\|u(t)\|^2\leq p_\sigma(t,\varepsilon)\,\|u(t)\|^2_\omega+\varepsilon\,\|u(0)\|^2\ ,$$

where

$$p_\sigma(t,\varepsilon)=\mathrm{e}^{C_3\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)}\mathrm{e}^{\left(\frac{C_3}{t}\ln\left(e+\frac{1}{\varepsilon}\right)\right)^\sigma}.$$

(iv) *There is a positive constant C_4, depending only on $(P, \Omega, \omega, \sigma)$, so that, for all $t > 0$ and $u(0) \in L^2(\Omega) \setminus \{0\}$,*

$$\|u(t)\| \leq e^{C_4\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_4}{t} \ln\left(\frac{\|u(0)\|}{\|u(t)\|}\right)\right)^{\sigma}} \|u(t)\|_{\omega}.$$

In particular, if $P = -\Delta$, then $\sigma = \frac{1}{2}$ (see [29], [31], [46], [4], [44]); if $P = \Delta^2$, then $\sigma = \frac{1}{4}$ (see [2], [16], [19], [27]); if P is the Stokes operator, then $\sigma = \frac{1}{2}$ (see [13]).

Proof. We organize the proof in several steps.

Step 1: to show that $(i) \Rightarrow (ii)$.

Arbitrarily fix $\lambda > 0$, $t > 0$ and $u(0) = \sum_{j\geq 1} a_j \Phi_j$ with $\{a_j\}_{j\geq 1} \subset \ell^2$. Write

$$u(t) = \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j + \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j.$$

Then by (i), we find that

$$\begin{aligned}
\|u(t)\| &\leq \left\| \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j \right\| + \left\| \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j \right\| \\
&\leq \left(\sum_{\lambda_j \leq \Lambda} \left| a_j e^{-\lambda_j t} \right|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\| \\
&\leq \left(e^{C_1(1+\Lambda^{\sigma})} \int_{\omega} \left| \sum_{\lambda_j \leq \Lambda} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\|.
\end{aligned}$$

This, along with the triangle inequality for the norm $\|\cdot\|_{\omega}$, yields that

$$\begin{aligned}
\|u(t)\| \leq &\left(e^{C_1(1+\Lambda^{\sigma})} \int_{\omega} \left| \sum_{j\geq 1} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} \\
&+ \left(e^{C_1(1+\Lambda^{\sigma})} \int_{\omega} \left| \sum_{\lambda_j > \Lambda} a_j e^{-\lambda_j t} \Phi_j \right|^2 \right)^{1/2} + e^{-\Lambda t} \|u(0)\|.
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
\|u(t)\| &\leq e^{\frac{C_1}{2}(1+\Lambda^{\sigma})} \|u(t)\|_{\omega} + e^{\frac{C_1}{2}(1+\Lambda^{\sigma})} e^{-\Lambda t} \|u(0)\| + e^{-\Lambda t} \|u(0)\| \\
&\leq 2 e^{\frac{C_1}{2}(1+\Lambda^{\sigma})} \left(\|u(t)\|_{\omega} + e^{-\Lambda t} \|u(0)\| \right).
\end{aligned}$$

Since, by the Young inequality,

$$C_1 \Lambda^{\sigma} = \frac{C_1}{(\epsilon t)^{\sigma}} (\epsilon \Lambda t)^{\sigma} \leq \epsilon \Lambda t + \left(\frac{C_1}{(\epsilon t)^{\sigma}} \right)^{\frac{1}{1-\sigma}} \quad \text{for any } \epsilon, t > 0,$$

one can deduce that, for all $\epsilon \in (0, 2)$,

$$\|u(t)\| \leq 2 e^{\frac{C_1}{2}} e^{\frac{1}{2}\left(\frac{C_1}{(\epsilon t)^{\sigma}}\right)^{\frac{1}{1-\sigma}}} \left(e^{\frac{\epsilon}{2}\Lambda t} \|u(t)\|_{\omega} + e^{-\frac{2-\epsilon}{2}\Lambda t} \|u(0)\| \right) \quad \text{for each } \Lambda > 0.$$

Notice that if $\|u(t)\|_{\omega} = 0$ then, $\|u(t)\| = 0$. Next, choose

$$\Lambda = \frac{1}{t} \ln\left(\frac{\|u(0)\|}{\|u(t)\|_{\omega}} \right)$$

(knowing that $\|u(t)\|_{\omega} \leq \|u(0)\|$) to get

$$\|u(t)\| \leq 2e^{\frac{C_1}{2}} e^{\frac{1}{2}\left(\frac{C_1}{(\epsilon t)^{\sigma}}\right)^{\frac{1}{1-\sigma}}} \left(2\|u(t)\|_{\omega}^{1-\frac{\epsilon}{2}} \|u(0)\|^{\frac{\epsilon}{2}}\right)$$

which is the inequality in (ii) with $\theta = \frac{\epsilon}{2}$ and $\ln 4 + \frac{C_1}{2} + \frac{1}{2}\left(\frac{C_1}{(\varepsilon t)^{\sigma}}\right)^{\frac{1}{1-\sigma}} \leq C_2\left(1 + \left(\frac{1}{\theta t}\right)^{\frac{\sigma}{1-\sigma}}\right)$.

Step 2: to show that $(ii) \Rightarrow (iii)$.

We write the inequality in (ii) in the following way

$$\|u(t)\|^2 \leq \|u(0)\|^{2\theta} \left(\exp\left(\frac{2C_2}{1-\theta}\left(1 + \left(\frac{1}{\theta t}\right)^{\frac{\sigma}{1-\sigma}}\right)\right)\|u(t)\|_{\omega}^2\right)^{1-\theta}$$

and apply the fact that for any $E, B, D > 0$ and $\theta \in (0,1)$

$$E \leq B^{\theta} D^{1-\theta} \Leftrightarrow E \leq \varepsilon B + (1-\theta)\theta^{\frac{\theta}{1-\theta}} \frac{1}{\varepsilon^{\frac{\theta}{1-\theta}}} D \quad \forall \varepsilon > 0.$$

To prove the above equivalence, one uses the Young inequality and one chooses $\varepsilon = \theta\left(\frac{D}{B}\right)^{1-\theta}$. Therefore,

$$\|u(t)\|^2 \leq \varepsilon\|u(0)\|^2 + \exp\left(\frac{2C_2}{1-\theta}\left(1 + \left(\frac{1}{\theta t}\right)^{\frac{\sigma}{1-\sigma}}\right)\right)\frac{1}{\varepsilon^{\frac{\theta}{1-\theta}}}\|u(t)\|_{\omega}^2.$$

By denoting $\beta = \frac{\theta}{1-\theta}$, it yields

$$\|u(t)\|^2 \leq \varepsilon\|u(0)\|^2 + e^{2C_2(1+\beta)\left(1+\left(\frac{1+\beta}{\beta t}\right)^{\frac{\sigma}{1-\sigma}}\right)}\frac{1}{\varepsilon^{\beta}}\|u(t)\|_{\omega}^2.$$

Now, notice, with $B = K\left(1 + \left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)$ and $D = K\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}$ for some constant $K > 0$, that

$$\frac{1}{\varepsilon^{\beta}} e^{2C_2(1+\beta)\left(1+\left(\frac{1+\beta}{\beta t}\right)^{\frac{\sigma}{1-\sigma}}\right)} \leq e^{B+\beta\left(\ln\left(e+\frac{1}{\varepsilon}\right)+B\right)+\left(\frac{1}{\beta}\right)^{\frac{\sigma}{1-\sigma}}D}.$$

Next, choose $\beta = \left(\frac{D}{\ln\left(e+\frac{1}{\varepsilon}\right)+B}\right)^{1-\sigma}$ to get

$$e^{B+\beta\left(\ln\left(e+\frac{1}{\varepsilon}\right)+B\right)+\left(\frac{1}{\beta}\right)^{\frac{\sigma}{1-\sigma}}D} \leq e^{cB+c\left(\ln\left(e+\frac{1}{\varepsilon}\right)+B\right)^{\sigma}D^{1-\sigma}} \leq e^{c'B+c'\left(\ln\left(e+\frac{1}{\varepsilon}\right)\right)^{\sigma}D^{1-\sigma}}$$

for some constants $c, c' > 0$. Therefore, we obtain the desired inequality

$$\|u(t)\|^2 \leq \varepsilon\|u(0)\|^2 + e^{C_3\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{t}\ln\left(e+\frac{1}{\varepsilon}\right)\right)^{\sigma}}\|u(t)\|_{\omega}^2.$$

Step 3: to show that $(iii) \Rightarrow (iv)$.

Take

$$\varepsilon = \frac{1}{2}\frac{\|u(t)\|^2}{\|u(0)\|^2}$$

in the inequality in (iii) and use the fact that $\|u(t)\| \leq \|u(0)\|$. Therefore, we have

$$\frac{1}{2}\|u(t)\|^2 \leq e^{C_3\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{t}\ln\left((e+2)\frac{\|u(0)\|^2}{\|u(t)\|^2}\right)\right)^{\sigma}}\|u(t)\|_{\omega}^2.$$

Step 4: to show that $(iv) \Rightarrow (i)$.

Apply the Young inequality

$$\left(\frac{C_4}{t}\ln\left(\frac{\|u(0)\|}{\|u(t)\|}\right)\right)^{\sigma} \leq \left(\frac{C_4}{t}\right)^{\frac{\sigma}{1-\sigma}} + \ln\left(\frac{\|u(0)\|}{\|u(t)\|}\right)$$

to deduce the inequality: there are two constants $C > 0$ and $\alpha \in (0,1)$, which depend only on (Ω, ω, σ), so that, for all $t > 0$ and $u(0) \in L^2(\Omega)$,

$$\|u(t)\| \leq e^{C\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)}\|u(0)\|^{\alpha}\|u(t)\|_{\omega}^{1-\alpha}.$$

Arbitrarily fix $\Lambda > 0$ and $\{a_j\} \subset \mathbb{R}$. By applying the above inequality, with $u(0) = \sum_{\lambda_j < \Lambda} a_j e^{\lambda_j t} \Phi_j$, we get that

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq e^{2C\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} \left(\sum_{\lambda_j \leq \Lambda} |a_j e^{\lambda_j t}|^2\right)^{\alpha} \left(\int_\omega \left|\sum_{\lambda_j \leq \Lambda} a_j \Phi_j\right|^2\right)^{1-\alpha},$$

which implies that

$$\sum_{\lambda_j \leq \Lambda} |a_j|^2 \leq e^{\frac{2}{1-\alpha} C\left(1+\left(\frac{1}{t}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\frac{2\alpha}{1-\alpha}\Lambda t} \int_\omega \left|\sum_{\lambda_j \leq \Lambda} a_j \Phi_j\right|^2 \quad \text{for each } t > 0\,.$$

Choose $t = \left(\frac{1}{\Lambda}\right)^{1-\sigma}$ to get the conclusion (i).

This ends the proof. □

3.2. Equivalence between observation and control

Let us recall the classical results of equivalence between observation estimate and controllability with cost. There are at least three ways to establish the cost: one is based on the duality of the control operator in the spirit of the HUM method (see [33]) with a spectral decomposition (see [48], [43]); another one has a geometric point of view using Hahn–Banach's Theorem (see [50], [52]); the last one is based on a minimization of a certain functional (see [17], [39]). The arguments we present are similar to those appearing in [39, lemma 3.2, p. 1475] (see also [15, remark 6.6, p. 3670]).

Denote $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product of $L^2(\Omega)$, respectively.

Theorem 3.2. *Let $0 \leq T_0 < T_1 < T_2$. Let $\ell, \varepsilon > 0$. The following two statements are equivalent.*

$(\mathcal{C})$ *For any $y_e \in L^2(\Omega)$, there is $f \in L^2(\omega)$ such that the solution y to*

$$\begin{cases} y'(t) + Py(t) = 0\,, & t \in (T_0, T_2) \setminus \{T_1\}\,, \\ y(T_0) = y_e\,, \\ y(T_1) = y(T_{1-}) + 1_\omega f\,, \end{cases}$$

satisfies

$$\frac{1}{\ell}\|f\|_\omega^2 + \frac{1}{\varepsilon}\|y(T_2)\|^2 \leq \|y_e\|^2\,.$$

$(\mathcal{O})$ *The solution u to*

$$\begin{cases} u'(t) + Pu(t) = 0\,, & t \in (T_0, T_2)\,, \\ u(T_0) \in L^2(\Omega)\,, \end{cases}$$

satisfies

$$\|u(T_2)\|^2 \leq \ell\,\|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon\,\|u(T_0)\|^2\,.$$

Proof of $(\mathcal{C}) \Rightarrow (\mathcal{O})$. We multiply the equations of $(\mathcal{C})$ by $u(T_0 + T_2 - t)$ to get

$$\langle y(T_2), u(T_0)\rangle - \langle y(T_0), u(T_2)\rangle = \langle f, u(T_0 + T_2 - T_1)\rangle_\omega\,,$$

that is,

$$\langle y_e, u(T_2)\rangle = -\langle f, u(T_0 + T_2 - T_1)\rangle_\omega + \langle y(T_2), u(T_0)\rangle\,.$$

By Cauchy–Schwarz's inequality and using the inequality in $(\mathcal{C})$ one can deduce that

$$\begin{aligned} \langle y_e, u(T_2)\rangle &\leq \|f\|_\omega \|u(T_0 + T_2 - T_1)\|_\omega + \|y(T_2)\|\,\|u(T_0)\| \\ &\leq \frac{1}{2\ell}\|f\|_\omega^2 + \frac{1}{2\varepsilon}\|y(T_2)\|^2 + \frac{\ell}{2}\|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2}\|u(T_0)\|^2 \\ &\leq \frac{1}{2}\|y_e\|^2 + \frac{1}{2}\left(\ell\,\|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon\,\|u(T_0)\|^2\right) \end{aligned}$$

which gives the desired estimate by choosing $y_e = u(T_2)$. □

Proof of $(\mathcal{O}) \Rightarrow (\mathcal{C})$**.** Let $y_e \in L^2(\Omega)$. Consider the functional J defined on $L^2(\Omega)$ given by

$$J(\vartheta) = \frac{\ell}{2} \|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2} \|\vartheta\|^2 - \langle y_e, u(T_2)\rangle \ ,$$

where

$$\begin{cases} u'(t) + Pu(t) = 0 \,, & t \in (T_0, T_2) \ , \\ u(T_0) = \vartheta. \end{cases}$$

Notice that J is strictly convex, C^1 and coercive and therefore J has a unique minimizer $w_0 \in L^2(\Omega)$, i.e. $J(w_0) = \min_{\vartheta \in L^2(\Omega)} J(\vartheta)$. Set

$$\begin{cases} w'(t) + Pw(t) = 0 \,, & t \in (T_0, T_2) \ , \\ w(T_0) = w_0 \,, \end{cases} \text{ and } \begin{cases} h'(t) + Ph(t) = 0 \,, & t \in (T_0, T_2) \ , \\ h(T_0) = h_0. \end{cases}$$

Since $J'(w_0)h_0 = 0$ for any $h_0 \in L^2(\Omega)$, we have

$$\ell \langle w(T_0 + T_2 - T_1), h(T_0 + T_2 - T_1)\rangle_\omega + \varepsilon \langle w_0, h_0\rangle - \langle y_e, h(T_2)\rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

On the other hand, the identity

$$\langle y(T_2), u(T_0)\rangle - \langle y_e, u(T_2)\rangle = \langle f, u(T_0 + T_2 - T_1)\rangle_\omega \quad \forall u(T_0) \in L^2(\Omega)$$

implies

$$-\langle f, h(T_0 + T_2 - T_1)\rangle_\omega + \langle y(T_2), h_0\rangle - \langle y_e, h(T_2)\rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

By choosing $f = -\ell w(T_0 + T_2 - T_1)$, we deduce that the solution y satisfies:

$$\varepsilon w_0 = y(T_2) .$$

Further,

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 = \frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2)\|^2 .$$

Moreover, taking $h_0 = w_0$ into $J'(w_0)h_0 = 0$, we get

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 - \langle y_e, w(T_2)\rangle = 0.$$

By Cauchy–Schwarz's inequality,

$$\begin{aligned} \ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 &\leq \|y_e\|_{L^2(\Omega)} \|w(T_2)\|_{L^2(\Omega)} \\ &\leq \|y_e\|_{L^2(\Omega)} \left(\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2\right)^{1/2} \end{aligned}$$

where, in the last line, we used $(\mathcal{O})$. Therefore, we get

$$\ell \|w(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon \|w_0\|^2 \leq \|y_e\|^2 \ ,$$

that is,

$$\frac{1}{\ell} \|f\|_\omega^2 + \frac{1}{\varepsilon} \|y(T_2)\|^2 \leq \|y_e\|^2$$

where

$$\begin{cases} y'(t) + Py(t) = 0 \,, & t \in (T_0, T_2) \setminus \{T_1\} \ , \\ y(T_0) = y_e \ , & \\ y(T_1) = y(T_{1-}) + 1_\omega(-\ell w(T_0 + T_2 - t)) \ , & \\ w'(t) + Pw(t) = 0 \,, & t \in (T_0, T_2) \ , \\ w(T_0) = \frac{1}{\varepsilon} y(T_2) . & \end{cases}$$

This completes the proof. □

Theorem 3.3. *Let $0 \leq T_0 < T_1 < T_2$. Let $\ell, \varepsilon > 0$. The following two statements are equivalent.*

$(\mathcal{C})$ *For any $y_d \in L^2(\Omega)$ such that $\langle P y_d, y_d\rangle < +\infty$, there is $f \in L^2(\omega)$ such that the solution y to*

$$\begin{cases} y'(t) + Py(t) = 0\,, & t \in (T_0, T_2) \setminus \{T_1\}\,, \\ y(T_0) = 0\,, \\ y(T_1) = y(T_{1-}) + 1_\omega f\,, \end{cases}$$

satisfies

$$\frac{1}{\ell}\|f\|_\omega^2 + \frac{1}{\varepsilon}\|y(T_2) - y_d\|^2 \le \langle P y_d, y_d\rangle .$$

$(\mathcal{O})$ *The solution u to*

$$\begin{cases} u'(t) + Pu(t) = 0\,, & t \in (T_0, T_2)\,, \\ u(T_0) \in L^2(\Omega)\,, \end{cases}$$

satisfies

$$\left\langle P^{-1} u(T_0), u(T_0)\right\rangle \le \ell\,\|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon\,\|u(T_0)\|^2 .$$

Proof of $(\mathcal{C}) \Rightarrow (\mathcal{O})$. We multiply the equations of $(\mathcal{C})$ by $u(T_0 + T_2 - t)$ to get

$$\langle y(T_2), u(T_0)\rangle - \langle y(T_0), u(T_2)\rangle = \langle f, u(T_0 + T_2 - T_1)\rangle_\omega\,,$$

that is,

$$\langle y_d, u(T_0)\rangle = \langle f, u(T_0 + T_2 - T_1)\rangle_\omega - \langle y(T_2) - y_d, u(T_0)\rangle .$$

By Cauchy–Schwarz's inequality and using the inequality in $(\mathcal{C})$, one has

$$\begin{aligned} \langle y_d, u(T_0)\rangle &\le \|f\|_\omega \|u(T_0 + T_2 - T_1)\|_\omega + \|y(T_2) - y_d\|\,\|u(T_0)\| \\ &\le \frac{1}{2\ell}\|f\|_\omega^2 + \frac{1}{2\varepsilon}\|y(T_2) - y_d\|^2 + \frac{\ell}{2}\|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2}\|u(T_0)\|^2 \\ &\le \frac{1}{2}\langle P y_d, y_d\rangle + \frac{1}{2}\left(\ell\,\|u(T_0 + T_2 - T_1)\|_\omega^2 + \varepsilon\,\|u(T_0)\|^2\right) \end{aligned}$$

which gives the desired estimate by choosing $y_d = P^{-1}u(T_0)$. □

Proof of $(\mathcal{O}) \Rightarrow (\mathcal{C})$. Let $y_d \in L^2(\Omega)$ such that $\langle P y_d, y_d\rangle < +\infty$. Consider the functional J defined on $L^2(\Omega)$ given by

$$J(\vartheta) = \frac{\ell}{2}\|u(T_0 + T_2 - T_1)\|_\omega^2 + \frac{\varepsilon}{2}\|\vartheta\|^2 + \langle y_d, \vartheta\rangle\,,$$

where

$$\begin{cases} u'(t) + Pu(t) = 0\,, & t \in (T_0, T_2)\,, \\ u(T_0) = \vartheta. \end{cases}$$

Notice that J is strictly convex, C^1 and coercive, and therefore J has a unique minimizer $w_0 \in L^2(\Omega)$, i.e. $J(w_0) = \min_{\vartheta \in L^2(\Omega)} J(\vartheta)$. Set

$$\begin{cases} w'(t) + Pw(t) = 0\,, & t \in (T_0, T_2)\,, \\ w(T_0) = w_0\,, \end{cases} \quad \text{and} \quad \begin{cases} h'(t) + Ph(t) = 0\,, & t \in (T_0, T_2)\,, \\ h(T_0) = h_0. \end{cases}$$

Since $J'(w_0)h_0 = 0$ for any $h_0 \in L^2(\Omega)$, we have

$$\ell\,\langle w(T_0 + T_2 - T_1), h(T_0 + T_2 - T_1)\rangle_\omega + \varepsilon\,\langle w_0, h_0\rangle + \langle y_d, h_0\rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

On the other hand, the identity

$$\langle y(T_2), u(T_0)\rangle - \langle y(T_0), u(T_2)\rangle = \langle f, u(T_0 + T_2 - T_1)\rangle_\omega \quad \forall u(T_0) \in L^2(\Omega)$$

implies

$$-\langle f, h(T_0 + T_2 - T_1)\rangle_\omega + \langle y(T_2) - y_d, h_0\rangle + \langle y_d, h_0\rangle = 0 \quad \forall h_0 \in L^2(\Omega) .$$

By choosing $f = -\ell w(T_0 + T_2 - T_1)$, we deduce that the solution y satisfies:

$$\varepsilon\, w_0 = y(T_2) - y_d .$$

Further,

$$\ell \left\| w\left(T_0+T_2-T_1\right)\right\|^2_\omega + \varepsilon \left\| w_0\right\|^2 = \frac{1}{\ell}\left\| f\right\|^2_\omega + \frac{1}{\varepsilon}\left\| y\left(T_2\right)-y_d\right\|^2 .$$

Moreover, taking $h_0 = w_0$ into $J'(w_0)h_0 = 0$, we get

$$\ell \left\| w\left(T_0+T_2-T_1\right)\right\|^2_\omega + \varepsilon \left\| w_0\right\|^2 + \left\langle y_d, w_0\right\rangle = 0.$$

By Cauchy–Schwarz's inequality,

$$\begin{aligned}\ell \left\| w\left(T_0+T_2-T_1\right)\right\|^2_\omega + \varepsilon \left\| w_0\right\|^2 &\leq \left\langle P y_d, y_d\right\rangle^{1/2} \left\langle P^{-1} w_0, w_0\right\rangle^{1/2} \\ &\leq \left\langle P y_d, y_d\right\rangle^{1/2} \left(\ell \left\| w\left(T_0+T_2-T_1\right)\right\|^2_\omega + \varepsilon \left\| w_0\right\|^2\right)^{1/2}\end{aligned}$$

where in the last line, we used $(\mathcal{O})$. Therefore, we get

$$\ell \left\| w\left(T_0+T_2-T_1\right)\right\|^2_\omega + \varepsilon \left\| w_0\right\|^2 \leq \left\langle P y_d, y_d\right\rangle ,$$

that is,

$$\frac{1}{\ell}\left\| f\right\|^2_\omega + \frac{1}{\varepsilon}\left\| y\left(T_2\right)-y_d\right\|^2 \leq \left\langle P y_d, y_d\right\rangle$$

where

$$\begin{cases} y'(t) + P y(t) = 0 , & t \in (T_0, T_2) \setminus \{T_1\} , \\ y(T_0) = 0 , & \\ y(T_1) = y(T_{1-}) + 1_\omega\left(-\ell w\left(T_0+T_2-t\right)\right) , & \\ w'(t) + P w(t) = 0 , & t \in (T_0, T_2) , \\ w(T_0) = \frac{1}{\varepsilon}\left(y(T_2) - y_d\right) . & \end{cases}$$

This completes the proof. □

3.3. Approximate impulse control

Direct applications of Theorem 3.1 and Theorem 3.2, Theorem 3.3 are given now (see [49] for applications to inverse source problem). Recall that Ω is a bounded domain of $\mathbb{R}^d$, $d \geq 1$, with boundary $\partial\Omega$ of class C^2, and ω is an open and nonempty subset of Ω.

Corollary 3.1. *Let $0 < L < T$ and $\varepsilon > 0$. If one of the statement of Theorem 3.1 holds then for any $y_e \in L^2(\Omega)$, there is $f \in L^2(\omega)$ such that the solution y to*

$$\begin{cases} y'(t) + P y(t) = 0 , & t \in (0, T) \setminus \{L\} , \\ y(0) = y_e , & \\ y(L) = y(L_-) + 1_\omega f , & \end{cases}$$

satisfies

$$\left\| y(T)\right\|^2 \leq \varepsilon \left\| y_e\right\|^2 \text{ and } \left\| f\right\|^2_\omega \leq e^{C_3\left(1+\left(\frac{1}{T-L}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{T-L}\ln\left(e+\frac{1}{\varepsilon}\right)\right)^\sigma} \left\| y_e\right\|^2 .$$

Proof. We apply Theorem 3.2 with $\ell = p_\sigma(T-L,\varepsilon) = e^{C_3\left(1+\left(\frac{1}{T-L}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_3}{T-L}\ln\left(e+\frac{1}{\varepsilon}\right)\right)^\sigma}$ given by Theorem 3.1 and $T_0 = 0$, $T_1 = L$, $T_2 = T$ (knowing that $\left\| u(T)\right\| \leq \left\| u(T-L)\right\|$). □

Corollary 3.2. *Let $0 < L < T$ and $\varepsilon > 0$. If one of the statement of Theorem 3.1 holds, then, for any $y_d \in L^2(\Omega)$ such that $\left\langle P y_d, y_d\right\rangle < +\infty$, there is $f \in L^2(\omega)$ such that the solution y to*

$$\begin{cases} y'(t) + P y(t) = 0 , & t \in (0, T) \setminus \{L\} , \\ y(0) = 0 , & \\ y(L) = y(L_-) + 1_\omega f , & \end{cases}$$

satisfies $\left\| y(T) - y_d\right\|^2 \leq \varepsilon \left\langle P y_d, y_d\right\rangle$ and

$$\|f\|_\omega^2 \le \frac{1}{\lambda_1} e^{2C_4\left(1+\left(\frac{1}{T-L}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{2\left[\frac{T}{\varepsilon}+\left(\frac{C_4}{T-L} ln\left(\sqrt{\frac{1}{\varepsilon\lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma\right]} \langle P y_d, y_d\rangle .$$

Proof. We aim to apply Theorem 3.3 and, to this end, we need to establish the inequality ($\mathcal{O}$) in Theorem 3.3. Recall that $\frac{d}{dt}\left\langle P^{-1}u(t), u(t)\right\rangle + 2\|u(t)\|^2 = 0$ and it can be written as

$$\frac{1}{2}\frac{d}{dt}\left\langle P^{-1}u, u\right\rangle + N(t)\left\langle P^{-1}u, u\right\rangle = 0 \text{ with } N(t) = \frac{\|u(t)\|^2}{\left\langle P^{-1}u(t), u(t)\right\rangle}.$$

In the spirit of [5, p. 12] (see also [43, p. 535]), one can check that $N'(t) \le 0$ by using Cauchy–Schwarz's inequality: $\|u\|^4 \le \left\langle P^{-1}u, u\right\rangle\langle Pu, u\rangle$ and $\frac{d}{dt}\|u\|^2 + 2\langle Pu, u\rangle = 0$. Therefore,

$$\left\langle P^{-1}u(0), u(0)\right\rangle \le e^{2N(0)T}\left\langle P^{-1}u(T), u(T)\right\rangle \le \frac{1}{\lambda_1} e^{2N(0)T}\|u(T)\|^2 .$$

But by Theorem 3.1, it holds

$$\|u(T)\| \le e^{C_4\left(1+\left(\frac{1}{T}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{\left(\frac{C_4}{T}\ln\left(\frac{\|u(0)\|}{\|u(T)\|}\right)\right)^\sigma} \|u(T)\|_\omega .$$

Therefore,

$$\left\langle P^{-1}u(0), u(0)\right\rangle \le \frac{1}{\lambda_1} e^{2N(0)T} e^{2C_4\left(1+\left(\frac{1}{T}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{2\left(\frac{C_4}{T}\ln\left(\frac{\|u(0)\|}{\|u(T)\|}\right)\right)^\sigma} \|u(T)\|_\omega^2$$

which implies, using $\frac{\|u(0)\|}{\|u(T)\|} \le \frac{\|u(0)\|}{\sqrt{\lambda_1}\left\langle P^{-1}u(T), u(T)\right\rangle^{1/2}} \le \sqrt{\frac{N(0)}{\lambda_1}} e^{2N(0)T}$, the following estimate:

$$\left\langle P^{-1}u(0), u(0)\right\rangle \le \frac{1}{\lambda_1} e^{2N(0)T} e^{2C_4\left(1+\left(\frac{1}{T}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{2\left(\frac{C_4}{T}\ln\left(\sqrt{\frac{N(0)}{\lambda_1}} e^{2N(0)T}\right)\right)^\sigma} \|u(T)\|_\omega^2 .$$

One concludes by distinguishing the case $N(0) \le 1/\varepsilon$ and the case $N(0) > 1/\varepsilon$, that is, for any $\varepsilon, T > 0$,

$$\left\langle P^{-1}u(0), u(0)\right\rangle \le \frac{1}{\lambda_1} e^{\frac{2T}{\varepsilon}} e^{2C_4\left(1+\left(\frac{1}{T}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{2\left(\frac{C_4}{T}\ln\left(\sqrt{\frac{1}{\varepsilon\lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma} \|u(T)\|_\omega^2 + \varepsilon\|u(0)\|^2 .$$

It remains to apply Theorem 3.3 with $\ell = \frac{1}{\lambda_1} e^{\frac{2T}{\varepsilon}} e^{2C_4\left(1+\left(\frac{1}{T-L}\right)^{\frac{\sigma}{1-\sigma}}\right)} e^{2\left(\frac{C_4}{T-L}\ln\left(\sqrt{\frac{1}{\varepsilon\lambda_1}} e^{\frac{2T}{\varepsilon}}\right)\right)^\sigma}$ and $T_0 = 0$, $T_1 = L$, $T_2 = T$. $\square$

3.4. Null controllability with measurable set-in time

Recall that Ω is a bounded domain of $\mathbb{R}^d$, $d \ge 1$, with boundary $\partial\Omega$ of class C^2, and ω is an open and nonempty subset of Ω.

Theorem 3.4. *Let $T > 0$ and $E \subset (0, T)$ be a set of positive measure. If one of the statements of Theorem 3.1 holds, then for any $y^0 \in L^2(\Omega)$, there is $f \in L^2(\omega \times E)$ such that the solution y to*

$$\begin{cases} y'(t) + Py(t) = 1_{\omega\times E} f , & t \in (0, T) , \\ y(0) = y^0 , \end{cases}$$

satisfies $y(T) = 0$.

Proof. The proof is divided into three steps.

Step 1: observability estimate with measurable set-in time. Based on a telescoping series method (see [39], [40] and already exploited in [45], [47], [3], [16], [54], [51], [35], [53], [44]), the statement (ii) in Theorem 3.1 implies the following observability: the solution u to

$$\begin{cases} u'(t) + Pu(t) = 0 , & t \in (0, T) , \\ u(0) \in L^2(\Omega) , \end{cases}$$

satisfies

$$\|u(T)\|^2 \le K \int_E \|u(T-t)\|_\omega^2 \, dt.$$

Here, K is a constant depending only on $(P,\Omega,\omega,\sigma,|E|)$. Further, if $E=(0,T)$, then $K=C\exp\left(\frac{C}{T^{\frac{\sigma}{1-\sigma}}}\right)$ for some $C=C\left(P,\Omega,\omega,\sigma\right)$.

Step 2: approximate controllability. Let $\varepsilon>0$. Consider the functional J_ε defined on $L^2(\Omega)$ given by

$$J_\varepsilon\left(u_0\right)=\frac{K}{2}\int_E\left\|u\left(T-t\right)\right\|_\omega^2\mathrm{d}t+\frac{\varepsilon}{2}\left\|u_0\right\|^2-\left\langle y^0,u\left(T\right)\right\rangle,$$

where

$$\begin{cases}u'\left(t\right)+Pu\left(t\right)=0\,, & t\in(0,T)\,,\\ u\left(0\right)=u_0.\end{cases}$$

Notice that J_ε is strictly convex, C^1 and coercive, and therefore J_ε has a unique minimizer $w_{\varepsilon,0}\in L^2(\Omega)$, i.e. $J_\varepsilon(w_{\varepsilon,0})=\min\limits_{u_0\in L^2(\Omega)}J_\varepsilon(u_0)$. Set

$$\begin{cases}w_\varepsilon'\left(t\right)+Pw_\varepsilon\left(t\right)=0\,, & t\in(0,T)\,,\\ w_\varepsilon\left(0\right)=w_{\varepsilon,0}\,,\end{cases}\quad\text{and}\quad\begin{cases}h'\left(t\right)+Ph\left(t\right)=0\,, & t\in(0,T)\,,\\ h\left(0\right)=h_0.\end{cases}$$

Since $J_\varepsilon'(w_{\varepsilon,0})h_0=0$ for any $h_0\in L^2(\Omega)$, we have

$$K\int_E\left\langle w_\varepsilon\left(T-t\right),h\left(T-t\right)\right\rangle_\omega\mathrm{d}t+\varepsilon\left\langle w_{\varepsilon,0},h_0\right\rangle-\left\langle y^0,h\left(T\right)\right\rangle=0\quad\forall h_0\in L^2\left(\Omega\right).$$

But the solution y_ε to

$$\begin{cases}y_\varepsilon'\left(t\right)+Py_\varepsilon\left(t\right)=1_{\omega\times E}f_\varepsilon\,, & t\in(0,T)\,,\\ y_\varepsilon\left(0\right)=y^0\,,\end{cases}$$

satisfies

$$\left\langle y_\varepsilon\left(T\right),u\left(0\right)\right\rangle-\left\langle y^0,u\left(T\right)\right\rangle=\int_E\left\langle f_\varepsilon\left(\cdot,t\right),u\left(T-t\right)\right\rangle_\omega\mathrm{d}t\quad\forall u\left(0\right)\in L^2\left(\Omega\right)$$

which means

$$-\int_E\left\langle f_\varepsilon\left(\cdot,t\right),h\left(T-t\right)\right\rangle_\omega\mathrm{d}t+\left\langle y_\varepsilon\left(T\right),h_0\right\rangle-\left\langle y^0,h\left(T\right)\right\rangle=0\quad\forall h_0\in L^2\left(\Omega\right).$$

By choosing $f_\varepsilon\left(\cdot,t\right)=-Kw_\varepsilon\left(T-t\right)$, we deduce that the solution y_ε satisfies:

$$\varepsilon w_{\varepsilon,0}=y_\varepsilon\left(T\right).$$

Further,

$$K\int_E\left\|w_\varepsilon\left(T-t\right)\right\|_\omega^2\mathrm{d}t+\varepsilon\left\|w_{\varepsilon,0}\right\|^2=\frac{1}{K}\int_E\left\|f_\varepsilon\left(\cdot,t\right)\right\|_\omega^2\mathrm{d}t+\frac{1}{\varepsilon}\left\|y_\varepsilon\left(T\right)\right\|^2.$$

Moreover, taking $h_0=w_{\varepsilon,0}$ into $J_\varepsilon'(w_{\varepsilon,0})h_0=0$, we get

$$K\int_E\left\|w_\varepsilon\left(T-t\right)\right\|_\omega^2\mathrm{d}t+\varepsilon\left\|w_{\varepsilon,0}\right\|^2-\left\langle y^0,w_\varepsilon\left(T\right)\right\rangle=0.$$

By Cauchy–Schwarz's inequality,

$$\begin{aligned}K\int_E\left\|w_\varepsilon\left(T-t\right)\right\|_\omega^2\mathrm{d}t+\varepsilon\left\|w_{\varepsilon,0}\right\|^2&\leq\left\|y^0\right\|\left\|w_\varepsilon\left(T\right)\right\|\\&\leq\left\|y^0\right\|\left(K\int_E\left\|w_\varepsilon\left(T-t\right)\right\|_\omega^2\mathrm{d}t\right)^{1/2}\end{aligned}$$

where in the last line, we used the observability estimate with measurable set-in time. Therefore, we get

$$K\int_E \|w_\varepsilon(T-t)\|_\omega^2\,dt + 2\varepsilon\left\|w_{\varepsilon,0}\right\|^2 \leq \left\|y^0\right\|^2 ,$$

that is,

$$\frac{1}{K}\int_E \|f_\varepsilon(\cdot,t)\|_\omega^2\,dt + \frac{2}{\varepsilon}\|y_\varepsilon(T)\|^2 \leq \left\|y^0\right\|^2$$

where

$$\begin{cases} y_\varepsilon'(t) + Py_\varepsilon(t) = 1_{\omega\times E} f_\varepsilon , & t\in(0,T) , \\ y_\varepsilon(0) = y^0 , & \\ f_\varepsilon(x,t) = -Kw_\varepsilon(x,T-t) , & (x,t)\in\Omega\times(0,T) , \\ w_\varepsilon'(t) + Pw_\varepsilon(t) = 0 , & t\in(0,T) , \\ w_\varepsilon(T) = \frac{1}{\varepsilon}y_\varepsilon(T) . & \end{cases}$$

Step 3: convergence of the control function. We refer to [55, p. 571]. Since $w_\varepsilon(T-\cdot)$ is bounded in $L^2(\omega\times E)$ and $\sqrt{\varepsilon}w_{\varepsilon,0}$ is bounded in $L^2(\Omega)$, one can deduce that, for some function $w(T-\cdot)$ in $L^2(\omega\times E)$, $w_\varepsilon(T-\cdot)$ weakly converge to $w(T-\cdot)$ in $L^2(\omega\times E)$ and $\varepsilon w_{\varepsilon,0}$ tends to zero in $L^2(\Omega)$. Therefore, the identity

$$K\int_E \langle w_\varepsilon(T-t), h(T-t)\rangle_\omega\,dt + \varepsilon\left\langle w_{\varepsilon,0}, h_0\right\rangle - \left\langle y^0, h(T)\right\rangle = 0 \quad \forall h_0\in L^2(\Omega) ,$$

becomes when $\varepsilon\to 0$, as

$$K\int_E \langle w(T-t), h(T-t)\rangle_\omega\,dt - \left\langle y^0, h(T)\right\rangle = 0 \quad \forall h_0\in L^2(\Omega) .$$

But the solution y to

$$\begin{cases} y'(t) + Py(t) = 1_{\omega\times E} f , & t\in(0,T) , \\ y(0) = y^0 , & \end{cases}$$

satisfies

$$-\int_E \langle f(\cdot,t), h(T-t)\rangle_\omega\,dt + \langle y(T), h_0\rangle - \left\langle y^0, h(T)\right\rangle = 0 \quad \forall h_0\in L^2(\Omega) .$$

By choosing $f(\cdot,t) = -Kw(T-t)$, it follows that the solution y satisfies $y(T)=0$.

This completes the proof. □

3.5. Finite-time stabilization

Recall that $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are the norm and the inner product of $L^2(\Omega)$ respectively.

Assume that there are two positive constants $c = c(\Omega)$ and $\rho = \rho(d)$ such that

$$\mathrm{Card}\,\{\lambda_i \leq \Lambda\} = \sum_{\lambda_i\leq\Lambda} 1 \leq c\Lambda^{1/\rho} .$$

Such estimate can be provided by the Weyl asymptotic formula $\lambda_k \sim C(\Omega)k^\rho$ as $k\to\infty$. In particular, if $P=-\Delta$, then $\rho=\frac{2}{d}$; and if $P=\Delta^2$, then $\rho=\frac{4}{d}$ (see [34]). In the case of the one-dimensional degenerate operator $P=\mathcal{P}$, we have $\rho=2$.

Define an increasing sequence $(t_m)_{m\geq 0}$ converging to $T>0$ by

$$t_m = T\left(1-\frac{1}{b^m}\right) \text{ for some } b>1.$$

Introduce a linear bounded operator $\mathcal{F}_m$ from $L^2(\Omega)$ into $L^2(\omega)$ in the following manner:

$$\begin{array}{cccc} \mathcal{F}_m : & L^2(\Omega) & \to & L^2(\omega) \\ & \vartheta & \mapsto & \displaystyle\sum_{\lambda_j\leq\Lambda_m} \left\langle\vartheta,\Phi_j\right\rangle f_j \end{array}$$

where

$$\Lambda_m := \lambda_1 + \left(\frac{\eta}{T}\frac{b}{b-1}\right) b^{(\beta+1)m} \text{ with } \eta > 1\,, \quad \beta := \frac{\sigma}{1-\sigma}\,,$$

and f_j is the impulse control of the heat equation associated with the eigenfunction Φ_j (see Corollary 3.1):

$$\begin{cases} y_j'(t) + P y_j(t) = 0\,, & t \in (t_m, t_{m+1}) \setminus \left\{\frac{t_m+t_{m+1}}{2}\right\}\,, \\ y_j(t_m) = \Phi_j\,, & \\ y_j\left(\frac{t_m+t_{m+1}}{2}\right) = y_j\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega f_j\,, & \end{cases}$$

satisfying

$$\left\|y_j(t_{m+1})\right\|^2 \leq \frac{e^{-\eta b^{\beta m}}}{\sum_{\lambda_i \leq \Lambda_m} 1} \text{ and } \left\|f_j\right\|_\omega^2 \leq e^{C_3\left(1+\left(\frac{2}{t_{m+1}-t_m}\right)^\beta\right)} e^{\left(\frac{2C_3}{t_{m+1}-t_m}\ln\left(e+e^{\eta b^{\beta m}}\sum_{\lambda_i\leq\Lambda_m}1\right)\right)^\sigma}.$$

Here, $C_3 > 0$ and $\sigma \in (0,1)$ are the constants given in Theorem 3.1. Notice that

$$\|\mathcal{F}_m\|^2_{L^2(\Omega)\to L^2(\omega)} \leq \sum_{\lambda_j \leq \Lambda_m} \left\|f_j\right\|^2_\omega.$$

Theorem 3.5. *Let ω be an open and nonempty subset of Ω. Suppose that one of the statements of Theorem 3.1 holds and*

$$Card\,\{\lambda_i \leq \Lambda\} \leq c\Lambda^{1/\rho} \quad \text{for any } \Lambda > 0.$$

Then, for any $T > 0$ there are $b, \eta > 1$ and $C, K > 0$ such that, for any $z_0 \in L^2(\Omega)$, the solution z to

$$\begin{cases} z'(t) + Pz(t) = 0\,, & t \in \mathbb{R}^+ \setminus \bigcup_{m\geq 0}\left(\frac{t_m+t_{m+1}}{2}\right)\,, \\ z\left(\frac{t_m+t_{m+1}}{2}\right) = z\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m(z(t_m))\,, & \text{for any integer } m \geq 0\,, \\ z(0) = z_0\,, & \end{cases}$$

satisfies $\|z(t)\| \leq Ce^{-\frac{1}{K}\left(\frac{T}{T-t}\right)^{\frac{\sigma}{1-\sigma}}} \|z_0\|$ for any $0 \leq t < T_-$. Further, $\lim_{m\to\infty} \|\mathcal{F}_m(z(t_m))\|_\omega = 0$.

Proof. We start to focus on the solution z on interval (t_m, t_{m+1}) with initial data $z(t_m) = \sum_{j\geq 1} a_j \Phi_j$ in $L^2(\Omega)$. Introduce the initial datum $\phi(t_m) = \sum_{\lambda_j > \Lambda_m} a_j \Phi_j$ and $\psi(t_m) = \sum_{\lambda_j \leq \Lambda_m} a_j \Phi_j$ associated with the solution of $\phi'(t) + P\phi(t) = 0$ and

$$\begin{cases} \psi'(t) + P\psi(t) = 0\,, & t \in (t_m, t_{m+1}) \setminus \left\{\frac{t_m+t_{m+1}}{2}\right\}\,, \\ \psi\left(\frac{t_m+t_{m+1}}{2}\right) = \psi\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega \sum_{\lambda_j\leq\Lambda_m} a_j f_j. & \end{cases}$$

Therefore, $\phi(t_{m+1}) = \sum_{\lambda_j > \Lambda_m} a_j e^{-\lambda_j(t_{m+1}-t_m)} \Phi_j$ and

$$\|\phi(t_{m+1})\| \leq e^{-\Lambda_m(t_{m+1}-t_m)} \|z(t_m)\|\,.$$

But we have chosen $\Lambda_m > \lambda_1$ so that $\eta b^{\beta m} \leq \Lambda_m(t_{m+1} - t_m)$. This implies that

$$\|\phi(t_{m+1})\| \leq e^{-\eta b^{\beta m}} \|z(t_m)\|\,.$$

On the other hand, the solution ψ satisfies $\psi = \sum_{\lambda_j \leq \Lambda_m} a_j y_j$ and

$$\|\psi(t_{m+1})\| \leq \sum_{\lambda_j\leq\Lambda_m} \left|a_j\right| \sqrt{\frac{e^{-\eta b^{\beta m}}}{\sum_{\lambda_i\leq\Lambda_m} 1}} \leq e^{-\frac{1}{2}\eta b^{\beta m}} \|z(t_m)\|\,.$$

Consequently, we have

$$\|z(t_{m+1})\| \leq \|\phi(t_{m+1})\| + \|\psi(t_{m+1})\| \leq \mathrm{e}^{1-\frac{1}{2}\eta b^{\beta m}} \|z(t_m)\| \ ,$$

which implies by induction that, for any $m \geq 1$,

$$\|z(t_m)\|^2 \leq \mathrm{e}^{2m-\eta b^{\beta m}} \|z(t_0)\|^2 .$$

Now, we treat the boundedness of the control associated with ψ: notice that $\sum_{\lambda_j \leq \Lambda_m} a_j f_j := \mathcal{F}_m(z(t_m))$, and then, by the Cauchy–Schwarz and Young inequalities,

$$\begin{aligned}
\|\mathcal{F}_m(z(t_m))\|_\omega^2 &\leq \int_\omega \left(\sum_{\lambda_j \leq \Lambda_m} |a_j| \, |f_j| \right)^2 \leq \sum_{\lambda_j \leq \Lambda_m} |a_j|^2 \sum_{\lambda_j \leq \Lambda_m} \|f_j\|_\omega^2 \\
&\leq \|z(t_m)\|^2 \sum_{\lambda_j \leq \Lambda_m} \mathrm{e}^{C_3\left(1+\left(\frac{2}{t_{m+1}-t_m}\right)^\beta\right)} \mathrm{e}^{\left(\frac{2C_3}{t_{m+1}-t_m} \ln\left(e+\sum_{\lambda_i \leq \Lambda_m} 1 \mathrm{e}^{\eta b^{\beta m}}\right)\right)^\sigma} \\
&\leq \|z(t_m)\|^2 \, \mathrm{e}^{C_3\left(1+\left(\frac{2}{t_{m+1}-t_m}\right)^\beta\right)} \sum_{\lambda_j \leq \Lambda_m} \mathrm{e}^{\left(\frac{4C_3}{t_{m+1}-t_m}\right)^\beta + \frac{1}{2}\ln\left(\sum_{\lambda_i \leq \Lambda_m} 1 \mathrm{e}^{\eta b^{\beta m}}\right)} ,
\end{aligned}$$

where in the last line $AB \leq A^p + B^q$ with $A = \frac{4C_3}{t_{m+1}-t_m}$, $B = \frac{1}{2} \ln \left(\sum_{\lambda_i \leq \Lambda_m} 1 \, \mathrm{e}^{\eta b^{\beta m}} \right)$, $p = \frac{1}{1-\sigma}$, $q = \frac{1}{\sigma}$ (recall that $\beta = \frac{\sigma}{1-\sigma}$). Next,

$$\begin{aligned}
\|\mathcal{F}_m(z(t_m))\|_\omega^2 &\leq \|z(t_m)\|^2 \, \mathrm{e}^{C_3\left(1+\left(\frac{2}{t_{m+1}-t_m}\right)^\beta\right)} \mathrm{e}^{\left(\frac{4C_3}{t_{m+1}-t_m}\right)^\beta} \mathrm{e}^{\frac{1}{2}\eta b^{\beta m}} \left(\sum_{\lambda_i \leq \Lambda_m} 1 \right)^{3/2} \\
&\leq \mathrm{e}^{2m-\eta b^{\beta m}} \|z(t_0)\|^2 \, \mathrm{e}^{C_3\left(1+\left(\frac{2}{T}\frac{b^{m+1}}{b-1}\right)^\beta\right)} \mathrm{e}^{\left(\frac{4C_3}{T}\frac{b^{m+1}}{b-1}\right)^\beta} \mathrm{e}^{\frac{1}{2}\eta b^{\beta m}} \left(c \Lambda_m^{1/\rho} \right)^{3/2} \\
&\leq \mathrm{e}^{2m-\frac{1}{2}\eta b^{\beta m}} \|z(t_0)\|^2 \, \mathrm{e}^{C_3} \mathrm{e}^{(C_3+(2C_3)^\beta)\left(\frac{2}{T}\frac{b}{b-1}\right)^\beta b^{\beta m}} \left(c \left(\lambda_1 + \left(\frac{\eta}{T}\frac{b}{b-1} \right) b^{(\beta+1)m} \right) \right)^{\frac{3}{2\rho}} ,
\end{aligned}$$

where in the last line we used the definition of Λ_m. Now, we choose $\eta > 1$, precisely

$$\eta = 1 + 4\left(C_3 + (2C_3)^\beta\right) \left(\frac{2}{T}\frac{b}{b-1} \right)^\beta$$

in order that $-\frac{1}{2}\eta b^{\beta m} + \left(C_3 + (2C_3)^\beta\right) \left(\frac{2}{T}\frac{b}{b-1}\right)^\beta b^{\beta m} \leq -\frac{1}{4}\eta b^{\beta m}$.

Since $b > 1$ and $b^{(\beta+1)m(3/2)/\rho} \leq \left(\frac{8(\beta+1)(3/2)}{\beta\rho\eta}\right)^{\frac{(\beta+1)(3/2)}{\beta\rho}} \mathrm{e}^{\frac{1}{8}\eta b^{\beta m}}$, we obtain, for some constant $C_5 := \mathrm{e}^{C_3} \left(\frac{12(\beta+1)}{\beta\rho\eta}\right)^{\frac{3(\beta+1)}{2\beta\rho}} \left(c \left(\lambda_1 + \left(\frac{\eta}{T}\frac{b}{b-1} \right) \right) \right)^{\frac{3}{2\rho}} > 0$, that, for any $m \geq 1$,

$$\|\mathcal{F}_m(z(t_m))\|_\omega^2 \leq C_5 \mathrm{e}^{2m-\frac{1}{8}\eta b^{\beta m}} \|z(0)\|^2 .$$

Finally, let $t \geq 0$, then there is $m \geq 0$ such that $t \in [t_m, t_{m+1}]$. We distinguish four cases: if $t \in [0, t_1/2)$, then

$$\|z(t)\|^2 \leq \|z(0)\|^2 \ ;$$

if $t \in [t_1/2, t_1)$, then

$$\|z(t)\|^2 \leq \left\| z\left((t_1/2)_-\right) + 1_\omega \mathcal{F}_0(z(t_0)) \right\|^2 \leq 2\left(1 + \|\mathcal{F}_0\|^2\right) \|z(0)\|^2 \ ;$$

if $t \in \left[t_m, \frac{t_m+t_{m+1}}{2}\right)$ and $m \geq 1$, then

$$\|z(t)\|^2 \leq \|z(t_m)\|^2 \leq \mathrm{e}^{2m-\eta b^{\beta m}} \|z(0)\|^2 \ ;$$

if $t \in \left[\frac{t_m+t_{m+1}}{2}, t_{m+1}\right)$ and $m \geq 1$, then

$$\begin{aligned}\|z(t)\|^2 &\leq \left\| z\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right) + 1_\omega \mathcal{F}_m(z(t_m))\right\|^2 \\ &\leq 2\left\| z\left(\left(\frac{t_m+t_{m+1}}{2}\right)_-\right)\right\|^2 + 2\|1_\omega \mathcal{F}_m(z(t_m))\|^2 \\ &\leq 2\,\mathrm{e}^{2m-\eta b^{\beta m}}\|z(0)\|^2 + 2\,C_5 \mathrm{e}^{2m-\frac{1}{8}\eta b^{\beta m}}\|z(0)\|^2 \\ &\leq 2(1+C_5)\,\mathrm{e}^{2m-\frac{1}{8}\eta b^{\beta m}}\|z(0)\|^2 .\end{aligned}$$

Consequently, for any $t \in [t_m, t_{m+1}]$, it holds

$$b^{\beta m} \leq \left(\frac{T}{T-t}\right)^\beta \leq b^{\beta m+\beta}$$

and

$$\|z(t)\|^2 \leq 2\left(1 + C_5 + \|\mathcal{F}_0\|^2\right)\mathrm{e}^{-\frac{1}{16}\eta b^{\beta m}}\|z(0)\|^2$$

by choosing $b = \mathrm{e}^{32/(\beta\eta)}$. One can conclude that $\mathrm{e}^{-\frac{1}{16}\eta b^{\beta m}} \leq \mathrm{e}^{-\frac{1}{16}\eta\left(\frac{T}{b}\frac{1}{T-t}\right)^\beta}$ and

$$\|z(t)\|^2 \leq 2\left(1 + C_5 + \|\mathcal{F}_0\|^2\right)\mathrm{e}^{-\frac{1}{16}\eta \mathrm{e}^{32/\eta}\left(\frac{T}{T-t}\right)^\beta}\|z(0)\|^2 .$$

This completes the proof. □

References

[1] F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, J. Evol. Equ. 6 (2) (2006) 161–204.
[2] J. Apraiz, L. Escauriaza, Null-control and measurable sets, ESAIM Control Optim. Calc. Var. 19 (1) (2013) 239–254.
[3] J. Apraiz, L. Escauriaza, G. Wang, C. Zhang, Observability inequalities and measurable sets, J. Eur. Math. Soc. 16 (11) (2014) 2433–2475.
[4] C. Bardos, K.D. Phung, Observation estimate for kinetic transport equations by diffusion approximation, C. R. Acad. Sci. Paris, Ser. I 355 (6) (2017) 640–664.
[5] C. Bardos, L. Tartar, Sur l'unicité retrograde des équations paraboliques et quelques questions voisines, Arch. Ration. Mech. Anal. 50 (1973) 10–25.
[6] K. Beauchard, K. Pravda-Starov, Null controllability of hypoelliptic quadratic equations, J. Éc. Polytech. Math. 5 (2018) 1–43.
[7] A. Benabdallah, M.G. Naso, Null controllability of a thermoelastic plate, Abstr. Appl. Anal. 7 (11) (2002) 585–599.
[8] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, SIAM J. Control Optim. 47 (1) (2008) 1–19.
[9] P. Cannarsa, P. Martinez, J. Vancostenoble, Global Carleman estimates for degenerate parabolic operators with applications, Mem. Amer. Math. Soc. 239 (2016) 1133, ix+209 p.
[10] P. Cannarsa, P. Martinez, J. Vancostenoble, The cost of controlling weakly degenerate parabolic equations by boundary controls, Math. Control Relat. Fields 7 (2017) 171–211.
[11] P. Cannarsa, P. Martinez, J. Vancostenoble, The cost of controlling strongly degenerate parabolic equations, arXiv:1801.01380.
[12] P. Cannarsa, J. Tort, M. Yamamoto, Unique continuation and approximate controllability for a degenerate parabolic equation, Appl. Anal. 91 (8) (2012) 1409–1425.
[13] F.W. Chaves-Silva, G. Lebeau, Spectral inequality and optimal cost of controllability for the Stokes system, ESAIM Control Optim. Calc. Var. 22 (4) (2016) 1137–1162.
[14] J.-M. Coron, H.-M. Nguyen, Null controllability and finite time stabilization for the heat equations with variable coefficients in space in one dimension via backstepping approach, Arch. Ration. Mech. Anal. 225 (3) (2017) 993–1023.
[15] T. Duyckaerts, L. Miller, Resolvent conditions for the control of parabolic equations, J. Funct. Anal. 263 (11) (2012) 3641–3673.
[16] L. Escauriaza, S. Montaner, C. Zhang, Observation from measurable sets for parabolic analytic evolutions and applications, J. Math. Pures Appl. (9) 104 (5) (2015) 837–867.
[17] E. Fernandez-Cara, E. Zuazua, The cost of approximate controllability for heat equations: the linear case, Adv. Differ. Equ. 5 (4–6) (2000) 465–514.
[18] A.V. Fursikov, O.Yu. Imanuvilov, Controllability of Evolution Equations, Lecture Notes Series, vol. 34, Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, Seoul, 1996.
[19] P. Gao, The Lebeau–Robbiano inequality for the one-dimensional fourth order elliptic operator and its application, ESAIM Control Optim. Calc. Var. 22 (3) (2016) 811–831.
[20] M. Gueye, Exact boundary controllability of 1-D parabolic and hyperbolic degenerate equations, SIAM J. Control Optim. 52 (4) (2014) 2037–2054.
[21] X. Huang, W. Lin, B. Yang, Global finite-time stabilization of a class of uncertain nonlinear systems, Automatica 41 (5) (2005) 881–888.
[22] D. Jerison, G. Lebeau, Nodal sets of sums of eigenfunctions, in: Harmonic Analysis and Partial Differential Equations, Chicago, IL, 1996, in: Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, USA, 1999, pp. 223–239.
[23] J. Le Rousseau, M. Léautaud, L. Robbiano, Controllability of a parabolic system with a diffuse interface, J. Eur. Math. Soc. 15 (4) (2013) 1485–1574.
[24] J. Le Rousseau, G. Lebeau, On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations, ESAIM Control Optim. Calc. Var. 18 (2012) 712–747.
[25] J. Le Rousseau, I. Moyano, Null-controllability of the Kolmogorov equation in the whole phase space, J. Differ. Equ. 260 (4) (2016) 3193–3233.
[26] J. Le Rousseau, L. Robbiano, Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations, Arch. Ration. Mech. Anal. 195 (3) (2010) 953–990.
[27] J. Le Rousseau, L. Robbiano, Spectral inequality and resolvent estimate for the bi-Laplace operator, J. Eur. Math. Soc. (2018), in press, arXiv:1509.02098.
[28] M. Léautaud, Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems, J. Funct. Anal. 258 (2010) 2739–2778.

[29] G. Lebeau, Introduction aux inégalités de Carleman (Introduction to Carleman inequalities), in: Control and Stabilization of Partial Differential Equations, in: Sémin. Congr., vol. 29, Soc. Math. France, Paris, 2015, pp. 51–92 (in French).
[30] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Commun. Partial Differ. Equ. 20 (1–2) (1995) 335–356.
[31] G. Lebeau, E. Zuazua, Null-controllability of a system of linear thermoelasticity, Arch. Ration. Mech. Anal. 141 (4) (1998) 297–329.
[32] F-H. Lin, A uniqueness theorem for parabolic equations, Commun. Pure Appl. Math. 43 (1) (1990) 127–136.
[33] J.-L. Lions, Remarks on approximate controllability, J. Anal. Math. 59 (1992) 103–116.
[34] G.Q. Liu, Some inequalities and asymptotic formulas for eigenvalues on Riemannian manifolds, J. Math. Anal. Appl. 376 (2011) 349–364.
[35] H. Liu, C. Zhang, Observability from measurable sets for a parabolic equation involving the Grushin operator and applications, Math. Methods Appl. Sci. 40 (10) (2017) 3821–3832.
[36] Q. Lü, A lower bound on local energy of partial sum of eigenfunctions for Laplace–Beltrami operators, ESAIM Control Optim. Calc. Var. 19 (1) (2013) 255–273.
[37] P. Martin, L. Rosier, P. Rouchon, Null controllability of one-dimensional parabolic equations by the flatness approach, SIAM J. Control Optim. 54 (1) (2016) 198–220.
[38] J. Mcmahon, On the roots of the Bessel and certain related functions, Ann. of Math. (2) 9 (1–6) (1894/1895) 23–30.
[39] L. Miller, A direct Lebeau–Robbiano strategy for the observability of heat-like semigroups, Discrete Contin. Dyn. Syst., Ser. B 14 (2010) 1465–1485.
[40] L. Miller, Spectral inequalities for the control of linear PDEs, in: K. Ammari, G. Lebeau (Eds.), PDE's, Dispersion, Scattering Theory and Control Theory, in: Séminaires et Congrès, vol. 30, Société mathématique de France, Paris, 2017, pp. 81–98.
[41] I. Moyano, Flatness for a strongly degenerate 1-D parabolic equation, Math. Control Signals Syst. 28 (4) (2016) 28.
[42] B. Opic, A. Kufner, Hardy-Type Inequalities, Pitman Research Notes in Mathematics Series, vol. 219, Longman Scientific & Technical, Harlow, UK, 1990.
[43] K.D. Phung, Note on the cost of the approximate controllability for the heat equation with potential, J. Math. Anal. Appl. 295 (2) (2004) 527–538.
[44] K.D. Phung, Carleman commutator approach in logarithmic convexity for parabolic equations, Math. Control Relat. Fields 8 (3–4) (2018) 899–933.
[45] K.D. Phung, G. Wang, An observability estimate for parabolic equations from a measurable set-in time and its applications, J. Eur. Math. Soc. 15 (2) (2013) 681–703.
[46] K.D. Phung, G. Wang, Y. Xu, Impulse output rapid stabilization for heat equations, J. Differ. Equ. 263 (8) (2017) 5012–5041.
[47] K.D. Phung, L. Wang, C. Zhang, Bang-bang property for time optimal control of semilinear heat equation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 31 (3) (2014) 477–499.
[48] L. Robbiano, Fonction de coût et contrôle des solutions des équations hyperboliques, Asymptot. Anal. 10 (1995) 95–115.
[49] T.M.N. Vo, The local backward heat problem, arXiv:1704.05314.
[50] G. Wang, M. Wang, Y. Zhang, Observability and unique continuation inequalities for the Schrödinger equation, J. Eur. Math. Soc. (2018), in press, arXiv:1606.05861.
[51] G. Wang, C. Zhang, Observability inequalities from measurable sets for some abstract evolution equations, SIAM J. Control Optim. 55 (3) (2017) 1862–1886.
[52] G. Wang, D. Yang, Y. Zhang, Time optimal sampled-data controls for the heat equation, C. R. Acad. Sci. Paris, Ser. I 355 (12) (2017) 1252–1290.
[53] X. Yu, L. Zhang, The bang-bang property of time and norm optimal control problems for parabolic equations with time-varying fractional Laplacian, ESAIM Control Optim. Calc. Var. (2018), in press, https://doi.org/10.1051/cocv/2017075.
[54] Y. Zhang, Unique continuation estimates for the Kolmogorov equation in the whole space, C. R. Acad. Sci. Paris, Ser. I 354 (4) (2016) 389–393.
[55] E. Zuazua, Controllability and Observability of Partial Differential Equations: Some Results and Open Problems, Handbook of Differential Equations: Evolutionary Equations, vol. 3, Elsevier/North-Holland, Amsterdam, 2007, pp. 527–621.