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Combinatorics

A note on small sets of reals

Une remarque sur de petits ensembles de nombres réels

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ABSTRACT

We construct an example of a combinatorially large measure-zero set.
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RÉSUMÉ

Nous construisons un exemple d'un ensemble combinatoirement grand, mais de mesure zéro.
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1. Introduction

We will work in the space 2^ω equipped with standard topology and measure. More specifically, the topology is generated by basic open sets of the form $[s] = \{x \in 2^\omega : s \subset x\}$ for $s \in 2^a$, $a \in \omega^{<\omega}$. The measure is the standard product measure such that $\mu([s]) = 2^{-|\mathrm{dom}(s)|}$; let $\mathcal{N}$ be the collection of all measure-zero sets.

Measure-zero sets in 2^ω admit the following representation (see Lemma 4):

$X \in \mathcal{N}$ iff and only if there exists a sequence $\{F_n : n \in \omega\}$ such that

(1) $F_n \subseteq 2^n$ for $n \in \omega$,
(2) $\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$,
(3) $X \subseteq \{x \in 2^\omega : \exists^\infty n\ x{\restriction}n \in F_n\}$.

The main drawback of this representation is that sets F_n have overlapping domains. The following definitions from [1] and [3] offer a refinement.

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Definition 1.

(1) A set $X \subseteq 2^\omega$ is *small* ($X \in \mathcal{S}$) if there exists a sequence $\{I_n, J_n : n \in \omega\}$ such that

(a) $I_n \in [\omega]^{<\aleph_0}$ for $n \in \omega$,
(b) $I_n \cap I_m = \emptyset$ for $n \neq m$,
(c) $J_n \subseteq 2^{I_n}$ for $n \in \omega$,
(d) $\sum_{n\in\omega} \frac{|J_n|}{2^{|I_n|}} < \infty$
(e) $X \subseteq \{x \in 2^\omega : \exists^\infty n\ x{\restriction}I_n \in J_n\}$

Without loss of generality, we can assume that $\{I_n : n \in \omega\}$ is a partition of ω into finite sets.

(2) We say that X is *small** ($X \in \mathcal{S}^\star$) if, in addition, sets I_n are disjoint intervals, that is, if there exists a strictly increasing sequence of integers $\{k_n : n \in \omega\}$ such that $I_n = [k_n, k_{n+1})$ for each n.

Let $(I_n, J_n)_{n\in\omega}$ denote the set $\{x \in 2^\omega : \exists^\infty n\ x{\restriction}n \in J_n\}$.

It is clear that $\mathcal{S}^\star \subseteq \mathcal{S} \subseteq \mathcal{N}$.

Small sets are useful because of their combinatorial simplicity. To test that $x \in X \in \mathcal{S}$ the real x must pass infinitely many *independent* tests as in Borel–Cantelli's lemma. In section 3 we will show that various structurally simple measure-zero sets are small.

Definition 2. For families of sets $\mathcal{A}$, $\mathcal{B}$, let $\mathcal{A} \oplus \mathcal{B}$ be

$$\{X : \exists a \in \mathcal{A}\ \exists b \in \mathcal{B}\ (X \subset a \cup b)\}$$

Clearly, if $\mathcal{J}$ is an ideal then $\mathcal{J} \oplus \mathcal{J} = \mathcal{J}$. Likewise, $\mathcal{A} \cup (\mathcal{A} \oplus \mathcal{A}) \cup (\mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}) \cup \ldots$ is an ideal for any $\mathcal{A}$.

Theorem 3. *[1]* $\mathcal{S}^\star \oplus \mathcal{S}^\star = \mathcal{S} \oplus \mathcal{S} = \mathcal{N} = \mathcal{N} \oplus \mathcal{N}$.

The main result of this paper is to show that the above result is the best possible, that is, $\mathcal{S}^\star \subsetneq \mathcal{S} \subsetneq \mathcal{N}$. It was known ([1]) that $\mathcal{S}^\star \subsetneq \mathcal{N}$.

2. Preliminaries

To make the paper complete and self contained we present a review of known results.

Lemma 4. *Suppose that $X \subset 2^\omega$. X has measure zero iff and only if there exists a sequence $\{F_n : n \in \omega\}$ such that*

(1) $F_n \subseteq 2^n$ *for* $n \in \omega$,
(2) $\sum_{n\in\omega} \frac{|F_n|}{2^n} < \infty$,
(3) $X \subseteq \{x \in 2^\omega : \exists^\infty n\ x{\restriction}n \in F_n\}$.

Proof. $\longleftarrow$ Note that $\{x \in 2^\omega : \exists^\infty n\ x{\restriction}n \in F_n\} = \bigcap_{m\in\omega} \bigcup_{n\geq m} \{x \in 2^\omega : x{\restriction}n \in F_n\}$. Now,

$$\mu\left(\bigcup_{n\geq m} \{x \in 2^\omega : x{\restriction}n \in F_n\}\right) \leq \sum_{n\geq m} \mu\left(\{x \in 2^\omega : x{\restriction}n \in F_n\}\right) \leq \sum_{n\geq m} \frac{|F_n|}{2^n} \longrightarrow 0.$$

$\longrightarrow$ If X has measure zero, then there exists a sequence of open sets $\{U_n : n \in \omega\}$ such that

(1) $\mu(U_n) \leq 2^{-n}$, for each n,
(2) $X \subseteq \bigcap_{n\in\omega} U_n$.

Find a sequence of $\{s^n_m : n, m \in \omega\}$ such that

(1) $s^n_m \in 2^{<\omega}$,
(2) $[s^n_m] \cap [s^n_k] = \emptyset$ when $k \neq m$,
(3) $U_n = \bigcup_{m\in\omega} [s^n_m]$.

For $k \in \omega$ let $F_k = \{s^n_m : n, m \in \omega,\ |s^n_m| = k\}$. Note that $X \subseteq \{x \in 2^\omega : \exists^\infty k\ x{\restriction}k \in F_k\}$ and that $\sum_{k\in\omega} \frac{|F_k|}{2^k} \leq \sum_{n\in\omega} \mu(U_n) \leq 1$. □

Theorem 5. *[1] $\mathcal{S}^\star \oplus \mathcal{S}^\star = \mathcal{S} \oplus \mathcal{S} = \mathcal{N}$.*

Proof. Since $\mathcal{N}$ is an ideal, $\mathcal{N} \oplus \mathcal{N} = \mathcal{N}$. Consequently, it suffices to show that $\mathcal{S}^\star \oplus \mathcal{S}^\star = \mathcal{N}$. The following theorem gives the required decomposition.

Theorem 6 *([1]). Suppose that $X \subseteq 2^\omega$ is a measure-zero set. Then there exist sequences $\langle n_k, m_k : k \in \omega\rangle$ and $\langle J_k, J'_k : k \in \omega\rangle$ such that*

(1) $n_k < m_k < n_{k+1}$ for all $k \in \omega$,
(2) $J_k \subseteq 2^{[n_k,n_{k+1})}$, $J'_k \subseteq 2^{[m_k,m_{k+1})}$ for $k \in \omega$,
(3) the sets $\big([n_k, n_{k+1}), J_k\big)_{k\in\omega}$ and $\big([m_k, m_{k+1}), J'_k\big)_{k\in\omega}$ are small*, and
(4) $X \subseteq \big([n_k, n_{k+1}), J_k\big)_{k\in\omega} \cup \big([m_k, m_{k+1}), J'_k\big)_{k\in\omega}$.

In particular, every null set is a union of two small* sets.

Proof. Let $X \subseteq 2^\omega$ be a null set.

We can assume that $X \subseteq \big\{x \in 2^\omega :\ \exists^\infty n\ x{\restriction}n \in F_n\big\}$ for some sequence $\langle F_n : n \in \omega\rangle$ satisfying conditions of Lemma 4.

Fix a sequence of positive reals $\langle \varepsilon_k : k \in \omega\rangle$ such that $\sum_{k=0}^\infty \varepsilon_k < \infty$.

Define two sequences $\langle n_k, m_k : k \in \omega\rangle$ as follows: $n_0 = 0$,

$$m_k = \min\left\{ j > n_k : 2^{n_k} \cdot \sum_{i=j}^{\infty} \frac{|F_i|}{2^i} < \varepsilon_k \right\},$$

and

$$n_{k+1} = \min\left\{ j > m_k : 2^{m_k} \cdot \sum_{i=j}^{\infty} \frac{|F_i|}{2^i} < \varepsilon_k \right\} \text{ for } k \in \omega.$$

Let $I_k = [n_k, n_{k+1})$ and $I'_k = [m_k, m_{k+1})$ for $k \in \omega$. Define

$$s \in J_k \iff s \in 2^{I_k}\ \&\ \exists i \in [m_k, n_{k+1})\ \exists t \in F_i\ s{\restriction}\mathrm{dom}(t) \cap \mathrm{dom}(s) = t{\restriction}\mathrm{dom}(t) \cap \mathrm{dom}(s).$$

Similarly

$$s \in J'_k \iff s \in 2^{I'_k}\ \&\ \exists i \in [n_{k+1}, m_{k+1})\ \exists t \in F_i\ s{\restriction}\mathrm{dom}(t) \cap \mathrm{dom}(s) = t{\restriction}\mathrm{dom}(t) \cap \mathrm{dom}(s).$$

It remains to show that $(I_k, J_k)_{k\in\omega}$ and $(I'_k, J'_k)_{k\in\omega}$ are small sets and that their union covers X.

Consider the set $(I_k, J_k)_{k\in\omega}$. Notice that for $k \in \omega$

$$\frac{|J_k|}{2^{I_k}} \leq 2^{n_k} \cdot \sum_{i=m_k}^{n_{k+1}} \frac{|F_i|}{2^i} \leq \varepsilon_k.$$

Since $\sum_{n=1}^\infty \varepsilon_n < \infty$ this shows that the set $(I_n, J_n)_{n\in\omega}$ is null. An analogous argument shows that $(I'_k, J'_k)_{k\in\omega}$ is null. Finally, we show that

$$X \subseteq (I_n, J_n)_{n\in\omega} \cup (I'_n, J'_n)_{n\in\omega}.$$

Suppose that $x \in X$ and let $Z = \{n \in \omega : x{\restriction}n \in F_n\}$. By the choice of F_n's, the set Z is infinite. Therefore, one of the sets,

$$Z \cap \bigcup_{k\in\omega} [m_k, n_{k+1}) \quad \text{or } Z \cap \bigcup_{k\in\omega} [n_{k+1}, m_{k+1}),$$

is infinite. Without loss of generality, we can assume that it is the first one. It follows that $x \in (I_n, J_n)_{n\in\omega}$ because, if $x{\restriction}n \in F_n$ and $n \in [m_k, n_{k+1})$, then by the definition there is $t \in J_k$ such that $x{\restriction}[n_k, n_{k+1}) = t$. □

Now lets turn attention to the family of small sets $\mathcal{S}$. Observe that the representation used in the definition of small sets is not unique. In particular, Lemma 7 follows easily.

Lemma 7. *Suppose that $(I_n, J_n)_{n\in\omega}$ is a small set and $\{a_k : k \in \omega\}$ is a partition of ω into finite sets. For $n \in \omega$, define $I'_n = \bigcup_{l\in a_n} I_l$ and $J'_n = \{s \in 2^{I'_n} : \exists l \in a_n \ \exists t \in J_l \ s{\restriction}I_l = t{\restriction}I_l\}$. Then $(I_n, J_n)_{n\in\omega} = (I'_n, J'_n)_{n\in\omega}$.*

Lemma 8. *Suppose that $(I_n, J_n)_{n\in\omega}$ and $(I'_n, J'_n)_{n\in\omega}$ are two small sets. If $\{I_n : n \in \omega\}$ is a finer partition than $\{I'_n : n \in \omega\}$, then $(I_n, J_n)_{n\in\omega} \cup (I'_n, J'_n)_{n\in\omega}$ is a small set.*

Proof. Define $I''_n = I'_n$ for $n \in \omega$ and let

$$J''_n = J'_n \cup \left\{ s \in 2^{I'_n} : \exists k \ \exists s \in J_k (I_k \subseteq I'_n \ \& \ s{\restriction}I_k \in J_k) \right\}.$$

It is easy to see that $(I_n, J_n)_{n\in\omega} \cup (I_n, J_n)_{n\in\omega} = (I''_n, J''_n)_{n\in\omega}$. □

Since members of $\mathcal{S}$ do not seem to form an ideal, we are interested in characterizing instances when a union of two sets in $\mathcal{S}$ is in $\mathcal{S}$.

Theorem 9. *Suppose that $(I_n, J_n)_{n\in\omega}$ and $(I'_n, J'_n)_{n\in\omega}$ are two small sets and $(I_n, J_n)_{n\in\omega} \subseteq (I'_n, J'_n)_{n\in\omega}$. Then there exists a set $(I''_n, J''_n)_{n\in\omega}$ such that $(I_n, J_n)_{n\in\omega} \subseteq (I''_n, J''_n)_{n\in\omega} \subseteq (I'_n, J'_n)_{n\in\omega}$ and such that the partition $\{I''_n : n \in \omega\}$ is finer than both $\{I_n : n \in \omega\}$ and $\{I'_n : n \in \omega\}$.*

Proof. Let start with the following:

Lemma 10. *Suppose that $(I_n, J_n)_{n\in\omega}$ and $(I'_n, J'_n)_{n\in\omega}$ are two small sets. The following conditions are equivalent:*

(1) $(I_n, J_n)_{n\in\omega} \subseteq (I'_n, J'_n)_{n\in\omega}$,
(2) for all but finitely many $n \in \omega$ and for every $s \in J_n$, there exist $m \in \omega$ and $t \in J'_m$ such that

  (a) $I_n \cap I'_m \neq \emptyset$,
  (b) $s{\restriction}(I_n \cap I'_m) = t{\restriction}(I_n \cap I'_m)$,
  (c) $\forall u \in 2^{I'_m \setminus I_n} \ t{\restriction}(I_n \cap I'_m)^\frown u \in J'_m$.

Proof. (2) → (1) Suppose that $x \in (I_n, J_n)_{n\in\omega}$. Then, for infinitely many n, $x{\restriction}I_n \in J_n$. For all but finitely many of those $n's$, conditions (b) and (c) of clause (2) guarantee that, for some m such that $I_n \cap I'_m \neq \emptyset$, $x{\restriction}(I_n \cap I'_m)^\frown x{\restriction}(I'_m \setminus I_n) \in J'_m$. Consequently, $x \in (I'_n, J'_n)_{n\in\omega}$.

¬(2) → ¬(1) Suppose that condition (2) fails. Then there exists an infinite set $Z \subseteq \omega$ such that, for each $n \in Z$, there is $s_n \in J_n$ such that, for every m such that $I_n \cap I'_m \neq \emptyset$, exactly one of the following conditions holds:

(1) $s_n{\restriction}(I_n \cap I'_m) \neq t{\restriction}(I_n \cap I'_m)$ for every $t \in J'_m$,
(2) there is $t \in J'_m$ such that $s_n{\restriction}(I_n \cap I'_m) = t{\restriction}(I_n \cap I'_m)$ but for some $u = u_{n,m} \in 2^{I'_m \setminus I_n}$, $t{\restriction}(I_n \cap I'_m)^\frown u_{n,m} \notin J'_m$.

By thinning out the set Z, we can assume that no set I'_m intersects two distinct sets I_n for $n \in Z$. Also, for each $m \in \omega$, fix $t^m \in 2^{I_m}$ such that $t^m \notin J'_m$.

Let $x \in 2^\omega$ be defined as follows:

$$x(l) = \begin{cases} s_n(l) & n \in Z \text{ and } l \in I_n \text{ and } u_{n,m} \text{ is not defined} \\ 0 & \text{if } n \in Z \text{ and } l \in I'_m \setminus I_n \text{ and } I_n \cap I_m \neq \emptyset \text{ and } u_{n,m} \text{ is not defined} \\ s_n(l) & \text{if } n \in Z \text{ and } l \in I_n \cap I'_m \text{ and } u_{n,m} \text{ is defined} \\ u_{n,m}(l) & \text{if } n \in Z \text{ and } l \in I'_m \setminus I_n \text{ and } I_n \cap I_m \neq \emptyset \text{ and } u_{n,m} \text{ is defined} \\ t^m(l) & \text{if } l \in I_m \text{ and } I_m \cap I_n = \emptyset \text{ for all } n \in Z \end{cases}.$$

Observe that the first two clauses define $x{\restriction}I'_m$ when $I'_m \cap I_n \neq \emptyset$ for some $n \in Z$ and $u_{n,m}$ is undefined, the next two clauses define $x{\restriction}I'_m$ when $I'_m \cap I_n \neq \emptyset$ for some $n \in Z$ and $u_{n,m}$ is defined, and finally the last clause defines $x{\restriction}I'_m$ when $I'_m \cap I_n = \emptyset$ for all $n \in Z$. It is easy to see that these cases are mutually exclusive and that $x \in (I_n, J_n)_{n\in\omega}$ since $x{\restriction}I_n = s_n \in J_n$ for $n \in Z$. Finally, note that $x \notin (I'_n, J'_n)_{n\in\omega}$, since by the choice of $u_{n,m}$ (or property of s_n) $x{\restriction}I'_m \notin J'_m$ for all m. □

Suppose that $(I_n, J_n)_{n\in\omega}$ and $(I'_n, J'_n)_{n\in\omega}$ are two small sets and $(I_n, J_n)_{n\in\omega} \subseteq (I'_n, J'_n)_{n\in\omega}$. Consider the partition consisting of sets $\{I_n \cap I'_m : n, m \in \omega\}$. For each non-empty set $I_n \cap I'_m$, we define $J''_{n,m} \subseteq 2^{I_n \cap I'm}$ as follows:

$s \in J''_{n,m}$ if there is $t \in J'_m$ such that $s{\restriction}(I_n \cap I'_m) = t{\restriction}(I_n \cap I'_m)$ and for all $u \in 2^{I'_m \setminus I_n} \ t{\restriction}(I_n \cap I'_m)^\frown u \in J'_m$.

Observe that the definition of $J''_{n,m}$ does not depend on J_n.

Note that

$$\sum_{m,n\in\omega, I_n\cap I'_m\neq\emptyset} \frac{|J''_{m,n}|}{2^{|I_n\cap I'_m|}} = \sum_{m\in\omega}\sum_{n\in\omega, I_n\cap I'_m\neq\emptyset} \frac{|J''_{m,n}|}{2^{|I_n\cap I'_m|}} =$$

$$\sum_{m\in\omega}\sum_{n\in\omega, I_n\cap I'_m\neq\emptyset} \frac{|J''_{n,m}|\cdot 2^{|I'_m\setminus I_n|}}{2^{|I''_k|}\cdot 2^{|I'_m\setminus I_n|}} \leq \sum_{m\in\omega}\frac{|J'_m|}{2^{|I'_m|}} < \infty.$$

To finish the proof, observe that, for $x \in 2^\omega$, whenever $x{\restriction}(I_n \cap I'_m) \in J''_{n,m}$, then $x{\restriction}I'_m \in J'_m$. Similarly, if $x{\restriction}I_n \in J_n$, then by Lemma 10 there is m such that $x{\restriction}(I_n \cap I'_m) \in J''_{m,n}$ It follows that $(I_n, J_n)_{n\in\omega} \subseteq (I_{n,m}, J''_{n,m})_{n,m\in\omega} \subseteq (I'_m, J'_m)_{m\in\omega}$. □

3. When null sets are small?

Small sets are combinatorially simple and this is the main motivation to study them and investigating when measure-zero sets are small.

Theorem 11. *Suppose that $X \subseteq 2^\omega$ is a measure-zero set. Then X is small if*

(1) $|X) < 2^{\aleph_0}$,
(2) X can be covered by a countable family of compact measure zero sets,
(3) X is a Menger set, that is, no continuous image of X into ω^ω is a dominating family.

Proof. Suppose that X has measure zero and use Theorem 6 to find small sets $(I_k, J_k)_{k\in\omega}$ and $(I'_k, J'_k)_{k\in\omega}$ such that

(1) $X \subseteq (I_k, J_k)_{k\in\omega} \cup (I'_k, J'_k)_{k\in\omega}$,
(2) $I_k \subseteq I'_{k-1} \cup I'_k$ and $I'_k \subseteq I_k \cup I_{k+1}$ for each $k > 0$.

For each $x \in X$ let $Z_x = \{k : x{\restriction}I_k \in J_k\}$. Note that $x \rightsquigarrow Z_x$ is a continuous mapping from X into $[\omega]^\omega$ (which is homeomorphic to ω^ω.)

Definition 12. A family $\mathcal{A} \subseteq [\omega]^\omega$ has property **Q** if

$$\forall Z \in [\omega]^\omega\ \exists A \in \mathcal{A}\ A \subseteq Z.$$

Lemma 13. *If $\{Z_x : x \in X\}$ does not have property* **Q** *then X is small.*

Proof. Suppose that Z witnesses that $\{Z_x : x \in X\}$ does not have property **Q**, that is that $Z_x \setminus Z \in [\omega]^\omega$ for every $x \in X$. Let $z_0 < z_1 < z_2 < \dots$ be an increasing enumeration of Z. Note that, for every $x \in X$, if $x \in (I_k, J_k)_{k\in\omega}$, then $x \in (I_k, J_k)_{k\notin Z}$. Consequently, $X \subseteq (I_k, J_k)_{k\notin Z} \cup (I'_k, J'_k)_{k\in\omega}$. We will show that this set is small.

Let $I''_k = \bigcup_{j\in[z_k,z_{k+1})} I'_j$, and use Lemma 7 to find $\{J''_k : k \in \omega\}$ such that $(I'_k, J'_k)_{k\in\omega} = (I''_k, J''_k)_{k\in\omega}$. Now Lemma 8 completes the proof, as the partition $\{I_k : k \notin Z\}$ is finer than partition $\{I''_k : k \in \omega\}$. □

To finish the proof, note that no family of size $< 2^{\aleph_0}$ has property **Q** because there is an almost disjoint family of size continuum. The remaining two cases follow from the fact that every family of subsets of ω with property **Q** is dominating. □

The following result shows that measure-zero sets endowed with sum structure are small as well.

Theorem 14 *([2], [4]). Let $\mathcal{F}$ be a filter on ω. Then if $\mathcal{F}$ is a measurable, then $\mathcal{F}$ can be covered by a small set.*

Proof. Let $\mathcal{F}$ be a measurable filter on ω identified with a subset of 2^ω via characteristic functions of its elements. By virtue of 0-1 law, this means that $\mathcal{F}$ is of measure zero (measure one case is clearly impossible). Fix a sequence $\{\varepsilon_n : n \in \omega\}$ of positive reals such that $\sum_{k=1}^\infty 2^k\varepsilon_k < \infty$.

Since $\mathcal{F}$ has measure zero, we can find sequences $\langle n_k, m_k : k \in \omega\rangle$ and $\langle J_k, J'_k : k \in \omega\rangle$ as in Theorem 6 such that $\mathcal{F} \subseteq \big([n_k, n_{k+1}), J_k\big)_{k\in\omega} \cup \big([m_k, m_{k+1}), J'_k\big)_{k\in\omega}$.

If $\mathcal{F} \subset ([n_k, n_{k+1}), J_k)_{k\in\omega}$ or if $\mathcal{F} \subset ([m_k, m_{k+1}), J'_k)_{k\in\omega}$, then we are done, since both sets are small.

Therefore, assume that neither set covers $\mathcal{F}$.

Define for $k \in \omega$

$$S_k = \{s \in 2^{[n_k,m_k)} : s \text{ has at least } 2^{n_{k+1}-m_k-k} \text{ extensions inside } J_k\}\,.$$

It is easy to check that

$$\frac{|S_n|}{2^{m_k-n_k}} \le 2^k \varepsilon_k$$

holds for $k \in \omega$.

Similarly, if we define

$$S'_k = \{s \in 2^{[n_k,m_m)} : s \text{ has at least } 2^{n_k-m_{k-1}-k} \text{ extensions inside } J'_k\}$$

then, by the same argument, we have that

$$\frac{|S'_k|}{2^{m_k-n_k}} \le 2^k \varepsilon_k$$

for all $k \in \omega$.

Consider the set $([n_k, m_k), S_k \cup S'_k)_{k\in\omega}$. This set is small since $\sum_{k=1}^{\infty} |S_k \cup S'_k| 2^{n_k-m_k} \le \sum_{k=0}^{\infty} 2^k \varepsilon_k < \infty$.

Now we have three small sets

(1) $H_1 = ([n_k, n_{k+1}), J_k)_{k\in\omega}$,
(2) $H_2 = ([m_k, m_{k+1}), J'_k)_{k\in\omega}$,
(3) $H_3 = ([n_k, m_k), S_k \cup S'_k)_{k\in\omega}$.

If $\mathcal{F} \subset H_2 \cup H_3$, we are done since, by Lemma 8, $H_2 \cup H_3$ is a small set. Therefore, assume that there exists $X \in \mathcal{F}$ such that $X \notin H_2 \cup H_3$. Since $\mathcal{F} \subset H_1 \cup H_2$, we get that $X \in H_1$. Let $\{k_u : u \in \omega\}$ be an increasing sequence enumerating set

$$\{k \in \omega : X{\restriction}[n_k, n_{k+1}) \in J_k\}.$$

Define for $u \in \omega$

$$\bar{I}_u = [m_{k_u+1}, n_{k_u+1}) \text{ and}$$
$$\bar{J}_u = \{s \in 2^{I_u} : \ X{\restriction}[n_{k_u}, m_{k_u+1})^\frown s \in J_{k_u} \text{ or } s^\frown X{\restriction}[n_{k_u+1}, m_{k_u+1}) \in J'_{k_u+1}\}.$$

By the choice of X, $X{\restriction}[n_{k_u}, n_{k_u+1}) \in \bar{J}_{k_u}$ but $X{\restriction}[n_{k_u}, n_{k_u+1}) \notin S_{k_u} \cup S'_{k_u}$ for sufficiently large $u \in \omega$. Thus $|\bar{J}_u| 2^{-|\bar{I}_u|} \le 2^{-u}$ for all but finitely many $u \in \omega$. Hence the set $H_4 = (\bar{I}_u, \bar{J}_u)_{u\in\omega}$ is small.

Lemma 15. $\mathcal{F} \subseteq H_4$.

Proof. Suppose that $\mathcal{F}$ is not contained in H_4 and let $Y \in \mathcal{F} \setminus H_4$.

Define $Z \in 2^\omega$ as follows

$$Z(n) = \begin{cases} Y(n) & \text{if } n \in \bigcup_{u\in\omega} \bar{I}_u \\ X(n) & \text{otherwise} \end{cases} \quad \text{for } n \in \omega .$$

Notice that $Z \in \mathcal{F}$ since $X \cap Y \subseteq Z$. We will show that $Z \notin H_1 \cup H_2$, which gives a contradiction.

Consider an interval $I_m = [n_m, n_{m+1})$. It suffices to show that $Z{\restriction}I_m \notin J_m$ when m is large enough.

If $m \ne k_u$ for every $u \in \omega$, then $I_m \cap \bigcup_{u\in\omega} \bar{I}_u = \emptyset$ and $Z{\restriction}I_m = X{\restriction}I_m \notin J_m$.

On the other hand, if $m = k_u$ for some $u \in \omega$ then $X{\restriction}I_m \in \dot{J}_m$, but by the choice of X, $Z{\restriction}[n_{k_u}, m_{k_u}) = X{\restriction}[n_{k_u}, m_{\underline{k}_u})$ has only few extensions inside $J_{n_{k_u}}$ (since $X \notin H_3$). More specifically, if $Z{\restriction}I_m \in J_m$ then $Z{\restriction}I_u$ has to be an element of $\bar{J}_u$. But this is impossible since $Z{\restriction}I_u = Y{\restriction}I_u \ \notin \bar{J}_u$ for sufficiently large $u \in \omega$. The proof that $Z \notin H_2$ is the same and uses the second clause in the definition of set H_4. □

4. Small sets versus measure-zero sets

In this section, we will prove the main result.

Theorem 16. *There exists a null set which is not small, that is $\mathcal{S} \subsetneq \mathcal{N}$.*

Proof. We will use the following.

Lemma 17. *For every $\varepsilon > 0$ and sufficiently large $n \in \omega$, there exists a set $A \subset 2^n$ such that $\frac{|A|}{2^n} < \varepsilon$ and, for every $u \subset n$ such that $\frac{n}{4} \le |u| \le \frac{3n}{4}$, and $B_0 \subset 2^u$ and $B_1 \subset 2^{n\setminus u}$ such that $\frac{|B_0|}{2^{|u|}} \ge \frac{1}{2}$ and $\frac{|B_1|}{2^{|n\setminus u|}} \ge \frac{1}{2}$, we have $(B_0 \times B_1) \cap A \ne \emptyset$.*

Proof. The key case is when ε is very small and sets B_0, B_1 have relative measure approximately $\frac{1}{2}$. In such a case, $B_0 \times B_2$ has relative measure $\frac{1}{4}$, yet it intersects A. Fix large $n \in \omega$ and choose $A \subset 2^n$ randomly. That is, for each $s \in 2^n$, the probability $\mathsf{Prob}(s \in A) = \varepsilon$ and for $s, s' \in 2^n$, events $s \in A$ and $s' \in A$ are independent. It is well known that, for large enough n, the set constructed this way will have measure ε (with negligible error).

Fix $n/4 \le |u| \le 3n/4$ and let

$$\mathcal{B}_u = \left\{(B_0, B_1) : B_0 \subset 2^u,\ B_1 \subset 2^{n\setminus u} \text{ and } \frac{|B_0|}{2^{|u|}}, \frac{|B_1|}{2^{|n\setminus u|}} \ge \frac{1}{2}\right\}.$$

Note that $|\mathcal{B}_u| \le 2^{2^{|u|}+2^{|n\setminus u|}} \le 2^{2^{\frac{3n}{4}+1}}$.

For $(B_0, B_1) \in \mathcal{B}_u$, $\mathsf{Prob}((B_0 \times B_1) \cap A = \emptyset) = (1-\varepsilon)^{|B_0 \times B_1|} \le (1-\varepsilon)^{2^{n-2}}$. Consequently,

$$\mathsf{Prob}(\exists (B_0, B_1) \in \mathcal{B}_u\ (B_0 \times B_1) \cap A = \emptyset) \le |\mathcal{B}_u|(1-\varepsilon)^{2^{n-2}} \le 2^{2^{\frac{3n}{4}+1}}(1-\varepsilon)^{2^{n-2}}.$$

Finally, since we have at most 2^n possible sets u,

$$\begin{aligned}&\mathsf{Prob}(\exists u\ \exists (B_0, B_1) \in \mathcal{B}_u\ (B_0 \times B_1) \cap A = \emptyset) \le \\ &\quad 2^n |\mathcal{B}_u|(1-\varepsilon)^{2^{n-2}} \le 2^{2^{\frac{3n}{4}+n+1}}(1-\varepsilon)^{2^{n-2}} \le 2^{2^{\frac{7n}{8}}}(1-\varepsilon)^{2^{n-2}} \le \\ &\quad 2^{2^{\frac{7n}{8}}}(1-\varepsilon)^{\frac{1}{\varepsilon}\epsilon 2^{n-2}} \le \frac{2^{2^{\frac{7n}{8}}}}{2^{\varepsilon 2^{n-2}}} \longrightarrow 0 \text{ as } n \to \infty.\end{aligned}$$

Therefore, there is a non-zero probability that a randomly chosen set A has the required properties. In particular, such a set must exist. □

Let $\{k_n^0, k_n^1 : n \in \omega\}$ be two sequences defined as $k_n^0 = n(n+1)$ and $k_n^1 = n^2$ for $n > 0$.

Let $I_n^0 = [k_n^0, k_{n+1}^0)$ and $I_n^1 = [k_n^1, k_{n+1}^1)$ for $n \in \omega$. Observe that the sequences are selected such that

(1) $|I_n^0| = 2n+2$ and $|I_n^1| = 2n+1$ for $n \in \omega$,
(2) $I_n^0 \subset I_n^1 \cup I_{n+1}^1$ for $n > 0$,
(3) $I_n^1 \subset I_{n-1}^0 \cup I_n^0$ for $n > 1$,
(4) $|I_n^0 \cap I_n^1| = |I_n^1 \cap I_{n-1}^0| = n$ for $n > 1$,
(5) $|I_n^0 \cap I_{n+1}^1| = |I_n^1 \cap I_n^0| = n+1$ for $n > 1$.

Finally, for $n > 0$ let $J_n^0 \subset 2^{I_n^0}$ and $J_n^1 \subset 2^{I_n^1}$ be selected as in Lemma 17 for $\varepsilon_n = \frac{1}{n^2}$. Easy calculation shows that for $n \ge 140$, the sets J_n^0 and J_n^1 are defined and have the required properties.

Suppose that $(I_n^0, J_n^0)_{n\in\omega} \cup (I_n^1, J_n^1)_{n\in\omega} \subset (I_n^2, J_n^2)_{n\in\omega}$.

Case 1 There exists $i \in \{0, 1\}$ and infinitely many $n, m \in \omega$ such that

$$\frac{|I_m^i|}{4} \le |I_m^i \cap I_n^2| \le \frac{3|I_m^i|}{4}.$$

Without loss of generality, $i = 0$. Let $\{a_k : k \in \omega\}$ be a partition of ω into finite sets. For $n \in \omega$ define $I'_n = \bigcup_{l \in a_n} I_l^2$ and $J'_n = \{s \in 2^{I'_n} : \exists l \in a_n\ \exists t \in J_l^2\ s{\restriction}I_l^2 = t{\restriction}I_l^2\}$. By Lemma 7, we know that $(I'_n, J'_n)_{n\in\omega} = (I_n^2, J_n^2)_{n\in\omega}$ no matter what is the choice of the partition $\{a_k : k \in \omega\}$.

Consequently, let us choose $\{a_k : k \in \omega\}$ and an infinite set $Z \subseteq \omega$ such that

(1) for every $m \in Z$ there is $n \in \omega$ such that $\frac{|I_m^0|}{4} \le |I_m^0 \cap I'_n| \le \frac{3|I_m^0|}{4}$,
(2) for every $m \in Z$ there exists $n \in \omega$ such that $I_m^0 \subset I'_n \cup I'_{n+1}$,
(3) for every $n \in \omega$ there is at most one $m \in Z$ such that $I_m^0 \cap I'_n \neq \emptyset$.

To construct the required partition $\{a_k : k \in \omega\}$, we inductively glue together the sets I_l^2 as follows: suppose that m is such that there is n such that $\frac{|I_m^0|}{4} \le |I_m^0 \cap I_n^2| \le \frac{3|I_m^0|}{4}$. Then we define $a_n = \{n\}$ and $a_{n+1} = \{u : I_m^0 \cap I_u^2 \neq \emptyset \text{ and } u \neq n\}$. Let Z be the subset of the collection of m's selected as above, that is, thin enough to satisfy condition (3).

Recall that $(I_n^0, J_n^0)_{n\in\omega} \subseteq (I_n^2, J_n^2)_{n\in\omega} = (I'_n, J'_n)_{n\in\omega}$.

Working towards contradiction, fix $m \in Z$, and let $I^0_m \subseteq I'_n \cup I'_{n+1}$ (in this case $I'_n = I^2_n$). By Lemma 10, it follows that, if m is large enough, then for every $s \in J^0_m$ either

(1) for every $u \in 2^{I'_n \setminus I^0_m}$ we have $s{\restriction}(I^0_m \cap I'_n) {}^\frown u \in J'_n$, or
(2) for every $u \in 2^{I'_{n+1} \setminus I^0_m}$ we have $s{\restriction}(I^0_m \cap I'_{n+1}) {}^\frown u \in J'_{n+1}$.

Let $J''_n = \{s \in 2^{I^0_m \cap I'_n} : \forall u \in 2^{I'_n \setminus I^0_m}\ s^\frown u \in J'_n\}$ and $J''_{n+1} = \{s \in 2^{I^0_m \cap I'_{n+1}} : \forall u \in 2^{I'_{n+1} \setminus I^0_m}\ s^\frown u \in J'_{n+1}\}$.
Clearly, $\dfrac{|J''_n|}{2^{|I'_n \cap I^0_m|}} \le \dfrac{|J'_n|}{2^{|I'_n|}} \le \dfrac{1}{2}$ and $\dfrac{|J''_{n+1}|}{2^{|I'_{n+1} \cap I^0_m|}} \le \dfrac{|J'_{n+1}|}{2^{|I'_{n+1}|}} \le \dfrac{1}{2}$.
Let $B_n = 2^{I^0_m \cap I'_n} \setminus J'_n$ and $B_{n+1} = 2^{I^0_m \cap I'_{n+1}} \setminus J'_{n+1}$.
It follows that $\dfrac{|B_n|}{2^{|I^0_m \cap I'_n|}}, \dfrac{|B_{n+1}|}{2^{|I^0_m \cap I'_{n+1}|}} \ge \dfrac{1}{2}$. By Lemma 17 and the definition of set $(I^0_m, J^0_m)_{m\in\omega}$, there is $s_m \in (B_n \times B_{n+1}) \cap J^0_m$. Consequently, there is $t_m \in 2^{I'_n \cup I'_{n+1}}$ such that $t_m{\restriction}I^0_m = s_m \in J^m_0$, but $t_m{\restriction}I'_n \notin J'_n$ and $t_m{\restriction}I'_{n+1} \notin J'_{n+1}$. For each $n \in \omega$ choose $r_n \in 2^{I'_n} \setminus J'_n$. Define $x \in 2^\omega$ as

$$x{\restriction}I'_n = \begin{cases} t_m{\restriction}I'_n & \text{if } I^0_m \cap I'_n \neq \emptyset \\ r_n & \text{if } I^0_m \cap I'_n = \emptyset \text{ for all } m \in Z \end{cases}.$$

It follows that $x \in (I^0_n, J^0_n)_{n\in\omega}$, but $x \notin (I'_n, J'_n)_{n\in\omega} = (I^2_n, J^2_n)_{n\in\omega}$, which is a contradiction.

Case 2 For every $i \in \{0,1\}$, almost every $n \in \omega$ and every $m \in \omega$,

$$|I^2_n \cap I^i_m| \le \frac{|I^i_m|}{4}.$$

This is quite similar to the previous case.

We inductively choose $\{a_k : k \in \omega\}$ and define I'_n's and J'_n's as before. Next construct an infinite set $Z \subseteq \omega$ such that

(1) for every $m \in Z$ there exists $n \in \omega$ such that $I^0_m \subset I'_n \cup I'_{n+1}$ and $\dfrac{|I^0_m|}{4} \le |I^0_m \cap I'_n|, |I^0_m \cap I'_{n+1}| \le \dfrac{3|I^0_m|}{4}$;
(2) for every $n \in \omega$ there is at most one $m \in Z$ such that $I^0_m \cap I'_n \neq \emptyset$.

Since $|I^2_k \cap I^i_m| \le \dfrac{|I^i_m|}{4}$ for each k, m we can get (1) by careful splitting $\{k : I^0_m \cap I^2_k \neq \emptyset\}$ into two sets.

The rest of the proof is exactly as before.

To conclude the proof, it suffices to show that these two cases exhaust all possibilities. To this end, we check that if, for some $i \in \{0,1\}$, $m, n \in \omega$, $|I^2_n \cap I^i_m| > \dfrac{3|I^1_m|}{4}$, then for some $j \in \{0,1\}$ and $k \in \omega$,

$$\frac{3|I^j_k|}{4} \le |I^2_n \cap I^j_k| \le \frac{3|I^j_k|}{4}.$$

This will show that the potential remaining cases are already included in Case 1.

Fix $i = 0$ and $n \in \omega$ (the case $i = 1$ is analogous.)

By the choice of intervals I^0_m and I^1_m, it follows that, if $|I^2_n \cap I^0_m| > \dfrac{3|I^0_m|}{4}$, then $|I^2_n \cap I^1_m| > \dfrac{|I^1_m|}{4}$. If $|I^2_n \cap I^1_m| \le \dfrac{3|I^1_m|}{4}$, then we are in Case 1. Otherwise, $|I^2_n \cap I^1_m| > \dfrac{3|I^1_m|}{4}$ and so $|I^2_n \cap I^0_{m+1}| > \dfrac{|I^1_{m+1}|}{4}$. Continue inductively until the construction terminates after finitely many steps settling on j and k. $\square$

Theorem 18. *Not every small set is small***, that is* $\mathcal{S}^\star \subsetneq \mathcal{S}$.

Proof. The proof is a modification of the previous argument.

Let I^0_n, I^1_n, J^0_n and J^1_n for $n \in \omega$ be like in the proof of 16. Let $\bar{I}^0_n = \{2k : k \in I^0_n\}$ and $\bar{I}^1_n = \{2k+1 : k \in I^1_n\}$ for $n \in \omega$ and let $\bar{J}^0_n \subset 2^{\bar{I}^0_n}$, $\bar{J}^1_n \subset 2^{\bar{I}^1_n}$ for $n \in \omega$ be the induced sets. Note that $(\{\bar{I}^0_n, \bar{I}^1_n\}, \{\bar{J}^0_n, \bar{J}^1_n\})_{n\in\omega}$ is a small set. We will show that this set is not small*. Suppose that $(\{\bar{I}^0_n, \bar{I}^1_n\}, \{\bar{J}^0_n, \bar{J}^1_n\})_{n\in\omega} \subseteq (I_n, J_n)_{n\in\omega}$, where $I_n = [k_n, k_{n+1})$ for an increasing sequence $\{k_n : n \in \omega\}$.

Without loss of generality we can assume that for every $n \in \omega$ there exists $i \in \{0,1\}$ and $m \in \omega$ such that

(1) $I^i_m \subseteq I_n \cup I_{n+1}$,
(2) $\dfrac{|I^i_m|}{4} \le |I_n \cap I^i_m| \le \dfrac{3|I^i_m|}{4}$,

(3) $\frac{|I^i_m|}{4} \leq |I_{n+1} \cap I^i_m| \leq \frac{3|I^i_m|}{4}$.

To get (1), we combine consecutive intervals I_n to make sure that each I^i_m belongs to at most two of them. Points (2) and (3) are a consequence of the properties of the original sequences $\{I^0_n, I^1_n : n \in \omega\}$, namely that each integer belongs to exactly two of these intervals and that intersecting intervals cut each other approximately in half. The following example illustrates the procedure for finding i and m: if k_n is even, then $k_n/2$ belongs to $I^0_j \cap I^1_k$ with $k - j$ equal to 0 or 1. The values of i and m depend on whether $k_n/2$ belongs to the lower or upper half of the said interval. The case when k_n is odd is similar.

The rest of the proof is exactly like in Case 1 of Theorem 16. □

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