C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1179–1187

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Partial differential equations/Differential geometry

The Dirichlet problem on a strip for the α-translating soliton equation

Le problème de Dirichlet pour l'équation α-soliton de translation dans une bande

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A R T I C L E I N F O

Article history:
Received 17 April 2018
Accepted 22 October 2018
Available online 29 October 2018

Presented by Haïm Brézis

A B S T R A C T

In this paper, we investigate the Dirichlet problem associated with the α-translating equation. Using the Perron method and a family of grim reapers as barriers, we prove the existence of a solution on a strip of $\mathbb{R}^2$ and the boundary data is formed by two copies of a convex function.

R É S U M É

Dans cette note, nous prouvons l'existence de solutions classiques au problème de Dirichlet pour l'équation de α-soliton de translation définie dans une bande de $\mathbb{R}^2$; les données sur le bord sont deux copies d'une fonction convexe continue. Nous utilisons la méthode de Perron, dans laquelle une famille de *grim reapers* est employée comme barrière pour résoudre le problème de Dirichlet.

Version française abrégée

Nous nous intéressons au problème de Dirichlet (1)–(2), dans lequel Ω est une bande de $\mathbb{R}^2$. L'équation (1) est appelée « équation de α-soliton de translation ». L'intérêt pour le cas où Ω est une bande provient du résultat de [23], dont l'auteur a trouvé des solitons de translation convexes qui sont des graphes sur une bande. Notre approche est différente, et nos preuves sont plus géométriques. Si $\alpha = 1$, le *grim reaper* est le soliton de translation, qui est le graphe de la fonction $u(x, y) = -\log(\cos(y))$, où u est défini dans la bande $\Omega = \{(x, y) \in \mathbb{R}^2 : -\pi/2 < y < \pi/2\}$. Si nous restreignons Ω à $|y| < \pi/2 - m$, $0 < m < \pi/2$, alors les valeurs limites de la fonction u sont constantes. Dans le cas général $\alpha > 0$, nous considérons les solutions w_θ de (1) données par la Définition 2.2, qui ne dépendent que d'une variable. Alors, ces fonctions w_θ, jusqu'aux homothéties et rotations, sont candidates pour être des supersolutions de (1).

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https://doi.org/10.1016/j.crma.2018.10.005
1631-073X/

Dans cet article, nous résolvons (1)–(2) quand le domaine est la bande $\Omega_m = \{(x, y) \in \mathbb{R}^2 : -m < y < m\}$, $m > 0$ et les données de frontière sont une fonction convexe continue. Exactement, soit f une fonction convexe continue définie dans $\mathbb{R}$; nous étendons f à $\partial\Omega_m$ en définissant $\varphi_f = \partial\Omega_m \to \mathbb{R}$ comme $\varphi_f(x, \pm m) = f(x)$. Alors, nous prouvons que, pour chaque f, il existe une solution de (1) pour la valeur $u = \varphi_f$ sur $\partial\Omega_m$ dans les cas suivants : (i) pour tout $\alpha > 1$ et $m > 0$; (ii) si $0 < \alpha \leq 1$, à condition que la largeur de Ω_m vérifie $m < d(\alpha)$, où $d(\alpha) > 0$ ne dépend que de α.

La technique de résolution utilisée est la méthode de Perron, qui consiste à résoudre le problème de Dirichlet dans de petits domaines, à savoir les disques contenus dans la bande Ω_m. En plus de l'utilisation des *grim reapers* w_θ, nous avons besoin, pour notre méthode de Perron, d'une sous-solution de (1). Dans ce but, nous utilisons un résultat de P. Collin prouvant l'existence d'une surface minimale v^0 dans $\mathbb{R}^3$, dont les données limites sont φ_f. Alors, nous allons construire un module de continuité entre les fonctions w_θ et la solution minimale v^0, qui montrera que la solution de Perron atteint la valeur f sur $\partial\Omega_m$.

1. Introduction and statement of results

Let $\Omega \subset \mathbb{R}^2$ be a smooth domain. Given a constant $\alpha > 0$ and $\varphi \in C^0(\partial\Omega)$, we investigate the existence of classical solutions $u \in C^2(\Omega) \cap C^0(\partial\Omega)$ to the Dirichlet problem

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = \left(\frac{1}{\sqrt{1+|Du|^2}}\right)^{\alpha} \quad \text{in } \Omega, \tag{1}$$

$$u = \varphi \quad \text{on } \partial\Omega. \tag{2}$$

We call equation (1) the *α-translating soliton equation* and the graph of u, $\Sigma_u = \{(x, u(x)) : x \in \Omega\}$, an *$\alpha$-translating soliton*. In the limit case $\alpha = 0$, equation (1) is the known constant mean curvature equation. The motivation for studying the α-translating soliton equation comes from the case $\alpha = 1$, where 1-translating solitons, or simply, translating solitons, appear in the theory of mean curvature flow of surfaces in $\mathbb{R}^3$ [10,11,24]. A *translating soliton* is a surface $\Sigma \subset \mathbb{R}^3$ that is a solution to the mean curvature flow such that Σ evolves purely by translations along some direction $\vec{a} \in \mathbb{R}^3 \setminus \{0\}$. In the case of Σ_u, the direction $\vec{a}$ is $e_3 = (0, 0, 1)$. When $\alpha \neq 1$, equation (1) extends this setting to the flow of surfaces by powers of mean curvature [16,17,19].

Apart from this interest in the mean curvature flow, Eq. (1) already appeared in the early works on the Dirichlet problem for quasilinear elliptic equations of Bernstein [3, p. 240] in the analytic case, and of Serrin [18] in the smoothness case. Possibly due to the extension of [18] and its focus on the constant mean curvature equation, Eq. (1) seemed be forgotten. Indeed, in [18, p. 477], Serrin considered this equation written as

$$(1+|Du|^2)^{\frac{3}{2}} \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = c(1+|Du|^2)^{\frac{3-\alpha}{2}}, \tag{3}$$

where $c > 0$ is a constant and he investigated the solvability of the Dirichlet problem for mean convex domains. Recall that Ω is said to be mean convex if the mean curvature $H_{\partial\Omega}$ with respect to the inward normal is non-negative. For arbitrary dimension, Serrin proved the following result in [18, p. 478].

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Then there exists a unique solution to (1)–(2) for any continuous function φ if and only if:*

(1) *Ω is mean convex, when $\alpha \geq 1$, or*
(2) *$H_{\partial\Omega} > 0$, when $0 < \alpha < 1$.*

This result has been recently revisited in the literature: see [2,12,13,15]. In the context of translating solitons ($\alpha = 1$), the existence of solutions has been studied in [1,22,23], including also Neumann boundary conditions.

In this article, we study the solvability of (1)–(2) when Ω is a strip of $\mathbb{R}^2$. The interest of studying the case when Ω is a strip comes from the result of Wang in [23], where it was proven the existence of convex translating soliton graphs on a strip. A first example of a translating soliton graph on a strip is the grim reaper $w(x, y) = -\log(\cos(y))$, where w is defined in $\Omega = \{(x, y) \in \mathbb{R}^2 : -\pi/2 < y < \pi/2\}$. Let us observe that $w \to \infty$ as $|y| \to \pm\pi/2$. If we narrow Ω, then the restriction of w on the new strip has constant boundary values. For $\alpha \neq 1$, we extend in Section 2 the notion of the grim reaper by considering solutions to (1) depending only on one variable. In fact, and up to scaling, we can rotate the grim reapers an angle θ, $|\theta| < \pi/2$, about the y-axis, and the boundary values are now linear functions.

Our main result establishes the existence of (1)–(2) when the boundary data is formed by two copies of a continuous convex function of $\mathbb{R}$. This result generalizes the examples of the grim reapers where the boundary values are linear functions. We introduce the following notation. Let Ω_m denote the strip of width $2m > 0$ given by $\Omega_m = \{(x, y) \in \mathbb{R}^2 : -m < y < m\}$. Let f be a continuous convex function defined in $\mathbb{R}$; we extend f to a continuous function φ_f on $\partial\Omega_m$ by defining $\varphi_f(x, \pm m) = f(x)$. The main result of the paper can now be stated.

Theorem 1.2. *Let $\Omega_m \subset \mathbb{R}^2$. For each convex function f defined in $\mathbb{R}$, there exists a solution $u \in C^2(\Omega_m) \cap C^0(\overline{\Omega_m})$ to (1)–(2) for boundary values φ_f on $\partial\Omega_m$ in the following cases:*

(1) *for any $\alpha > 1$ and $m > 0$;*
(2) *if $0 < \alpha \le 1$, provided $m < d(\alpha)$, where $d(\alpha) > 0$ is a constant that depends only on α.*

This result was proven for the constant mean curvature equation ($\alpha = 0$) by Collin in [5], where $d(0) = 1$. Our work is inspired by this paper. We also point out the relevant work of Jian and Ju [12], where they prove the existence of a solution to the Dirichlet problem (1)–(2) when Ω is an unbounded strictly convex domain like U-type or a cone in $\mathbb{R}^2$.

This paper is organized as follows. In Section 2, we find all solutions to (1) depending on one variable. In Section 3, we recall some properties of the solutions to (1) when Ω is a bounded domain. Finally, in Section 4, we prove Theorem 1.2 by means of the Perron process of sub- and supersolutions.

2. The family of α-grim reapers

For the proof of Theorem 1.2 (Section 4), we shall show that the solution u takes the boundary data φ_f by means of appropriate barriers. In the present section, we construct these barriers, which are nothing more than the generalization of the notion of a grim reaper for every $\alpha > 0$. As usual, we let (x, y, z) be canonical coordinates in Euclidean space $\mathbb{R}^3$. It is immediate that any translation of $\mathbb{R}^3$ and a rotation about an axis parallel to e_3 preserves solutions to the α-translating soliton equation. In this section, we find the solutions to (1) depending only on one variable or, in other words, we classify all α-translating solitons that are cylindrical surfaces.

For our convenience, we consider firstly α-translating solitons Σ that are invariant in one direction orthogonal to e_3, for example, invariant along the x-axis. The surface is then generated by a curve $\gamma(s) = (y(s), z(s))$, $s \in I \subset \mathbb{R}$, contained in the yz-plane. If we suppose γ being parameterized by the arc-length, then $y'(s) = \cos\phi(s)$, $z'(s) = \sin\phi(s)$ for some angle function ϕ. Then Σ is parameterized by $X(x, s) = x(1, 0, 0) + (0, \gamma(s)) = (x, y(s), z(s))$, $x \in \mathbb{R}$, $s \in I$, and its mean curvature H is $H(x, s) = \phi'(s)/2$. Recall that $\phi'(s)$ is the curvature of $\gamma(s)$ as a planar curve. Consequently, Σ is an α-translating soliton if and only if γ satisfies the system of differential equations

$$\begin{cases} y'(s) = \cos\phi(s) \\ z'(s) = \sin\phi(s) \\ \phi'(s) = (\cos\phi(s))^\alpha. \end{cases} \tag{4}$$

We see immediately that a vertical line satisfies (4). Indeed, γ can be expressed as $\gamma(s) = (y_0, z_0 \pm s)$, $(y_0, z_0) \in \mathbb{R}^2$, with $\phi(s) = \pm\pi/2$. In general, and after a translation if necessary, we may suppose that γ goes through the origin, hence $\gamma(0) = (0, 0)$.

Proposition 2.1. *Let $\alpha > 0$. There is a family of solutions $\gamma = \gamma(s)$ of (4) satisfying*

$$y(0) = z(0) = 0,\ \phi(0) = 0, \tag{5}$$

and with the following properties.

(1) *The graphic of γ is a graph of a convex function $w = w(y)$ and symmetric with respect to the z-axis. Furthermore, the function w has a unique global minimum.*
(2) *Let $w : (-d, d) \to \mathbb{R}$, where $(-d, d)$ is the maximal domain of w and $d = d(\alpha)$ depends on α. Then the value $d(\alpha)$ satisfies:*
   (a) *$d(\alpha) = \infty$, if $\alpha > 1$;*
   (b) *$d(\alpha) < \infty$, if $0 < \alpha \le 1$ and γ is asymptotic to the vertical lines $y = \pm d(\alpha)$.*

Proof. The classical theory for ODEs guarantees existence and uniqueness of a solution $\{y, z, \phi\}$ to (4)–(5). The maximal domain of the solution γ is $\mathbb{R}$ because the derivatives y', z' and ϕ' in the differential equations (4) are bounded functions.

(1) The proof that γ is a graph on the y-axis is by contradiction. Suppose that at some point $s = s_0$, the angle function ϕ satisfies $\phi(s_0) = \pi/2$ (similar in case $-\pi/2$). Then it is immediate that the triple of functions $\{\bar{y}, \bar{z}, \bar{\phi}\}$ defined by

$$\bar{y}(s) = y(s_0),\ \bar{z}(s) = s - s_0 + z(s_0),\ \bar{\phi}(s) = \pi/2$$

is a solution to (4). Since they have the same values at $s = s_0$ as the initial solution $\{y, z, \phi\}$, we deduce by uniqueness that $\gamma(s) = (y(s_0), s - s_0 + z(s_0))$ for all $s \in \mathbb{R}$. Thus γ is a vertical straight line and $\phi(s) = \pi/2$ for all $s \in \mathbb{R}$, leading to a contradiction because $\phi(0) = 0$.

It follows that γ is a graph on the y-axis of some function $w = w(y)$. Taking into account that $\phi(s) \in (-\pi/2, \pi/2)$, we find from (4) that $\phi'(s) = \cos(\phi(s))^\alpha > 0$. This implies that the (signed) curvature of γ is positive, which implies the convexity of w. On the other hand, equations (4)–(5) may write now

$$w'' = (1 + w'^2)^{\frac{3-\alpha}{2}}, \quad w(0) = 0, \ w'(0) = 0. \tag{6}$$

Because the function $w(-y)$ satisfies (6), uniqueness of solutions implies $w(-y) = w(y)$ and this proves that w is symmetric about the z-axis. Finally, $w'(0) = 0$ and $w''(0) = 1$ and thus w has a minimum at $y = 0$. This minimum is the unique (global) minimum because w is convex.

(2) The integration of (6) is given in terms of the first hypergeometric function ${}_2F_1(a, b; c; x)$. It is known that, when $0 < \alpha \leq 1$, the domain of w is a bounded symmetric interval $(-d, d)$ with $\lim_{y \to \pm d} w(y) = \pm\infty$ and when $\alpha > 1$, then w is defined on the whole real line.

Remark 1. Explicit solutions to (6) can be obtained in some cases of α by simple quadratures.

(1) Case $\alpha = 1$. Then $w(y) = -\log(\cos(y))$, $d = \pi/2$ and γ is the grim reaper.
(2) Case $\alpha = 2$. Then $w(y) = \cosh(y) - 1$, $d = \infty$ and γ is the catenary.
(3) Case $\alpha = 3$. Then $w(y) = y^2/2$, $d = \infty$ and γ is the parabola.

In the limit case $\alpha = 0$, we have $w(y) = 1 - \sqrt{1 - y^2}$, $d = 1$ and γ is a half circle.

Recall that each solution γ given in Proposition 2.1 corresponds to an α-translating soliton whose rulings are parallel to the x-axis. In order to find supersolutions in the Perron process, let us further require the rulings to be parallel oblique lines. These new surfaces are obtained by rotating the solutions of Proposition 2.1 through the angle $\theta \in (-\pi/2, \pi/2)$ about the y-axis and then scaling the factor $(\sec\theta)^\alpha$. Then the resulting surfaces are again α-translating solitons, which we define as follows.

Definition 2.2. Let $\alpha > 0$. If $w = w(y)$ is a solution to (6), we define the uniparametric family of α-grim reapers $w_\theta = w_\theta(x, y)$ as

$$w_\theta(x, y) = \frac{1}{(\cos\theta)^{\alpha+1}} w((\cos\theta)^\alpha y) + (\tan\theta)x + a,$$

where $\theta \in (-\pi/2, \pi/2)$, $a \in \mathbb{R}$.

As a consequence of Proposition 2.1, if $\alpha > 1$ the domain of w_θ is $\mathbb{R}^2$, whereas if $0 < \alpha \leq 1$, the maximal domain of w_θ is the strip

$$\Omega_{d,\theta} = \left\{ (x, y) \in \mathbb{R}^2 : -\frac{d}{(\cos\theta)^\alpha} < y < \frac{d}{(\cos\theta)^\alpha} \right\}.$$

In particular, if $0 \leq \theta_1 < \theta_2$, it follows that $\Omega_{\theta_1} \subset \Omega_{\theta_2}$ and thus the domain Ω_d defined by

$$\Omega_d = \Omega_{d,0} = \{(x, y) \in \mathbb{R}^2 : -d < y < d\} \tag{7}$$

is contained in $\Omega_{d,\theta}$ for all $\theta \in (-\pi/2, \pi/2)$.

3. Properties of the solutions to the α-translating soliton equation

This section establishes some properties of the solutions u to (1)–(2), with a special interest in the control of $|u|$ and $|Du|$ when Ω is a bounded domain. Here we make use of explicit examples of α-translating solitons to derive these estimates. Firstly we recall that the difference of two solutions to Eq. (1) satisfies the maximum principle, hence the next result as a consequence of the comparison principle ([8, Th. 10.1]).

Proposition 3.1 *(Touching principle). Let Σ_1 and Σ_2 be two α-translating solitons with possibly non-empty boundaries $\partial\Sigma_1$, $\partial\Sigma_2$. If Σ_1 and Σ_2 have a common tangent interior point and Σ_1 lies above Σ_2 around p, then Σ_1 and Σ_2 coincide at an open set around p. The same statement holds if p is a common boundary point and the tangent lines to $\partial\Sigma_i$ coincide at p.*

As a direct application of the touching principle, *there do not exist compact α-translating solitons* because if Σ were a such surface, we can place a vertical plane Π tangent to Σ leaving Σ in one side of Π: this is impossible by the touching principle, because both Π and Σ are α-translating solitons.

Besides the α-grim reapers defined in Section 2, another useful family of α-translating solitons is formed by those ones that are rotationally symmetric about a vertical axis (this axis may be assumed to be the z-axis). These surfaces correspond to radial solutions $u = u(r)$ to Eq. (1), $r^2 = x^2 + y^2$, and were classified in [4] when $\alpha = 1$ and in [20] for the general case $\alpha > 0$. There are two types of such surfaces: convex entire graphs on $\mathbb{R}^2$ and surfaces of winglike-shape. We are interested in the first ones.

Definition 3.2. For each $\alpha > 0$, there is an entire radially symmetric strictly convex solution to (1), namely, $\mathbf{b} = \mathbf{b}(r)$, with a global minimum in the z-axis. The graph $\mathcal{B}$ of $\mathbf{b}$ is called the α-bowl soliton.

We intersect $\mathcal{B}$ by a horizontal plane Π of equation $z = t$ such that $t > \mathbf{b}(0)$. Let $\mathcal{B}_R$ denote the compact portion of $\mathcal{B}$ cut off by Π: here $R > 0$ indicates the radius of the circle $\mathcal{B} \cap \Pi$. Let us observe that $R \to \infty$ as $t \to \infty$. Therefore, and after a vertical translation, we conclude that, for any disk $D_R \subset \mathbb{R}^2$ centered at the origin of radius R, there is a radial solution to (1)–(2) with $\varphi = 0$ on ∂D_R. The bowl soliton caps $\mathcal{B}_R$ allow comparison arguments to derive height estimates.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If u is a solution to (1)–(2), we have:*

(1) *the solution is unique;*
(2) *there is a constant $C > 0$ depending only on φ and Ω such that*

$$C \le u \le \max_{\partial\Omega} \varphi \quad \text{in } \Omega; \tag{8}$$

(3) *the maximum of $|Du|$ in $\overline{\Omega}$ is achieved on $\partial\Omega$, that is,*

$$\sup_{\Omega} |Du| = \max_{\partial\Omega} |Du|.$$

Proof.

(1) The uniqueness of solutions is a consequence of the fact that the right-hand side of (1) is non-decreasing on u ([8, Th. 10.1]).
(2) Since the right-hand side of (1) is non-negative, the maximum principle implies $\sup_\Omega u = \max_{\partial\Omega} u = \max_{\partial\Omega} \varphi$, proving the inequality in the right-hand side of (8).
The lower estimate for u in (8) is obtained by using α-bowl soliton caps as comparison surfaces. Indeed, let us take a round disc $D_R \subset \mathbb{R}^2$ of radius R sufficiently big so that $\overline{\Omega} \subset D_R$. After a vertical translation, consider the α-bowl soliton cap $\mathcal{B}_R$ such that $\partial\mathcal{B}_R = \partial D_R$. We move vertically down $\mathcal{B}_R$ sufficiently far so that Σ_u lies above $\mathcal{B}_R$, that is, if $(x, y, z) \in \Sigma_u$, $(x, y, z') \in \mathcal{B}_R$, then $z > z'$. Then we move up $\mathcal{B}_R$ until the first touching point with Σ_u. If the first contact occurs at some interior point, then the touching principle implies $\Sigma_u \subset \mathcal{B}_R$. The other possibility is that the first contact point occurs when $\mathcal{B}_R$ touches a boundary point of Σ_u. In both cases, we conclude $\mathbf{b}(0) \le u - \min_{\partial\Omega} \varphi$ and, consequently, $C := \mathbf{b}(0) + \min_{\partial\Omega} \varphi \le u$.
(3) Equation (1) can be expressed as

$$(1 + |Du|^2)\Delta u - u_i u_j u_{ij} - (1 + |Du|^2)^{\frac{3-\alpha}{2}} = 0, \tag{9}$$

where $u_i = \partial u/\partial x_i$, $i = 1, 2$, and we assume the summation convention of repeated indices. Define the function $v^k = u_k$, $k = 1, 2$, and we differentiate (9) with respect to the variable x_k, obtaining

$$\left((1 + |Du|^2)\delta_{ij} - u_i u_j\right) v^k_{ij} + 2\left(u_i \Delta u - u_j u_{ij} - \frac{3-\alpha}{2} u_i (1 + |Du|^2)^{\frac{1-\alpha}{2}}\right) v^k_i = 0, \tag{10}$$

for each $k = 1, 2$. Hence, v^k satisfies a linear elliptic equation and by the maximum principle, $|v^k|$ has not a maximum at some interior point. Consequently, the maximum of $|Du|$ on the compact set $\overline{\Omega}$ is attained at some boundary point.

4. Proof of Theorem 1.2

Such as it was announced in the § Introduction, the demonstration of Theorem 1.2 uses firstly the Perron method ([7, pp. 306–312]) to prove the existence of a *Perron solution* v that solves (1) in Ω_m and subsequently, that the function v assumes the assigned boundary values φ_f on $\partial\Omega_m$. Both steps will be done in Proposition 4.6 below. First, we define the operator

$$Q[u] = \operatorname{div}\left(\frac{Du}{\sqrt{1 + |Du|^2}}\right) - \left(\frac{1}{\sqrt{1 + |Du|^2}}\right)^{\alpha}.$$

For the Perron process, we need to have a subsolution to (1)–(2). In the next result, f is not necessarily a convex function.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a strip. If f is a continuous function defined in $\mathbb{R}$, then there exists a solution v^0 to the Dirichlet problem*

$$\begin{aligned} &\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0 && \text{in } \Omega, \\ &u=\varphi_f && \text{on } \partial\Omega, \end{aligned} \tag{11}$$

with the property $f(x) \le v^0(x,y)$ for all $(x,y) \in \Omega$.

This result was proven in [5, Rem. 2], where v^0 is called the minimal solution to (11).

For the existence of the Perron solution, Eq. (1) fulfills the required properties for the Perron technique; namely, the difference of solutions satisfies the maximum principle (Proposition 3.1) that bounded families of solutions are compact (see [9]) and that the Dirichlet problem is solvable for all sufficiently small disks with arbitrary boundary data (Theorem 1.1). We begin the process by lifting functions. In what follows, we will use the notation Ω for a strip of $\mathbb{R}^2$ when we do not emphasize the value of its width. Let $u \in C^0(\overline{\Omega})$ be a continuous function, and let D be a closed round disk in Ω. Let $\bar{u} \in C^2(D)$ denote the unique solution to the Dirichlet problem

$$\begin{aligned} &Q[\bar{u}]=0 && \text{in } D, \\ &\bar{u}=u && \text{on } \partial D, \end{aligned}$$

whose existence and uniqueness is assured by Theorem 1.1. Then the function $\bar{u}$ may extend to a continuous function $M_D[u]$ in Ω by

$$M_D[u]=\begin{cases} \bar{u} & \text{in } D, \\ u & \text{in } \Omega \setminus D. \end{cases}$$

The function u is said to be a *supersolution* in Ω if u satisfies $M_D[u] \le u$ for every closed round disk D in Ω.

Example 1. For any domain $\Omega \subset \mathbb{R}^2$, the function $u=0$ in $\overline{\Omega}$ is a supersolution. Indeed, if $D \subset \Omega$ is a closed round disk, then $Q[\bar{u}]=0$ in D and $\bar{u}=0$ on ∂D. By the maximum principle, $\bar{u}<0$ in D and consequently, $M_D[u] \le 0 = u$.

On the other hand, for each $p \in \Omega$, there is a supersolution u with $u(p)<0$. Indeed, let $D \subset \Omega$ be a closed round disk centered at p, which is supposed to be the origin of $\mathbb{R}^2$. Let $\mathbf{b}=\mathbf{b}(r)$ be the α-bowl soliton with $\mathbf{b}_{|\partial D}=0$. Then the function u defined by $u=\mathbf{b}$ in D and $u=0$ in $\overline{\Omega} \setminus D$ is a supersolution that satisfies $u(p)=\mathbf{b}(p)<0$.

Definition 4.2. Let $u \in C^0(\overline{\Omega})$. We say that u is a superfunction relative to f if u is a supersolution in Ω and $f \le u$ on $\partial\Omega$. Let $\mathcal{S}_f$ denote the set of superfunctions relative to f,

$$\mathcal{S}_f = \{u \in C^0(\overline{\Omega}) : M_D[u] \le u \text{ for every closed round disk } D \subset \Omega,\ f \le u \text{ on } \partial\Omega\}.$$

Lemma 4.3. *The set $\mathcal{S}_f$ is not empty.*

Proof. We prove that $v^0 \in \mathcal{S}_f$, where v^0 is the minimal solution given in Proposition 4.1. Let $D \subset \Omega$ be a closed round disk. Notice that, since v^0 is a minimal solution, we have $Q[v^0]<0$. As $Q[\overline{v^0}]=0$ and $\overline{v^0}=v^0$ on ∂D, the maximum principle gives $\overline{v^0} \le v^0$ in D, hence that $M_D[v^0]=\overline{v^0} \le v^0$. Finally, since $v^0=f$ on $\partial\Omega$, it follows that $v^0 \in \mathcal{S}_f$.

We now state some properties of superfunctions. Their proofs are straightforward and therefore they are omitted: for the constant mean curvature equation, we refer to [14]; in the context of α-translating solitons, see [12, Lems. 4.2–4.4].

Lemma 4.4.

(1) *If $\{u_1,\dots,u_n\} \subset \mathcal{S}_f$, then $\min\{u_1,\dots,u_n\} \in \mathcal{S}_f$.*
(2) *The operator M_D is increasing in $\mathcal{S}_f$.*
(3) *If $u \in \mathcal{S}_f$ and D is a closed round disk in Ω, then $M_D[u] \in \mathcal{S}_f$.*

We begin with the proof of Theorem 1.2. We take $d(\alpha)$ the number given in Proposition 2.1 and let f be a continuous convex function on $\mathbb{R}$.

Consider the family of α-grim reaper w_θ defined in Section 2. Recall that the domain of w_θ is $\mathbb{R}^2$ if $\alpha>1$ or the strip $\Omega_{d,\theta}$ if $\alpha \le 1$. In particular, by the definition of Ω_d in (7), we find $\Omega_m \subset \Omega_d \subset \Omega_{d,\theta}$ for any θ because $m<d$. Thus, it makes sense to restrict w_θ to the strip Ω_m (we keep the same notation for its restriction on Ω_m). Consequently, w_θ is a linear function on $\partial\Omega_m$ and the boundary of Σ_{w_θ} consists of two parallel oblique lines.

Consider the subfamily of α-grim reapers

$$\mathcal{G} = \{w_\theta : w_\theta \le f \text{ on } \partial\Omega_m, \theta \in (-\pi/2, \pi/2)\}.$$

Two observations are to be stated:

(1) the set $\mathcal{G}$ is not empty because f is convex;
(2) if v^0 is the minimal solution, then $Q[v^0] < 0 = Q[w_\theta]$ for all $w_\theta \in \mathcal{G}$. The comparison principle asserts that $w_\theta < v^0$ in Ω_m for all $\theta \in (-\pi/2, \pi/2)$. Thus the function v^0 will play the role of a subsolution to (1)–(2).

We are going to construct a solution to Eq. (1) between the α-grim reapers of $\mathcal{G}$ and the minimal solution v^0. Let

$$\mathcal{S}_f^* = \{u \in \mathcal{S}_f : w_\theta \le u \le v^0, \ \forall w_\theta \in \mathcal{G}\}.$$

We notice that $\mathcal{S}_f^*$ is not empty because $v^0 \in \mathcal{S}_f^*$.

Lemma 4.5. *The set $\mathcal{S}_f^*$ is stable with respect to the operator M_D, that is, if $u \in \mathcal{S}_f^*$, then $M_D[u] \in \mathcal{S}_f^*$.*

Proof. Let $u \in \mathcal{S}_f^*$. We know by Lemma 4.4 that $M_D[u] \in \mathcal{S}_f$. On the other hand, since $w_\theta \le u \le v^0$ and because M_D is increasing (Lemma 4.4 again), it follows that

$$M_D[w_\theta] \le M_D[u] \le M_D[v^0] \tag{12}$$

for every closed round disk $D \subset \Omega_m$. Because w_θ satisfies (1), then $M_D[w_\theta] = w_\theta$. On the other hand, since v^0 is a super-solution, we have $M_D[v^0] \le v^0$. Therefore, inequalities (12) give now $w_\theta \le M_D[u] \le v^0$, and this proves the result.

We are going to complete the proof of Theorem 1.2.

Proposition 4.6 *(Perron process). The function $v : \Omega_m \to \mathbb{R}$ defined by*

$$v(x, y) = \inf\{u(x, y) : u \in \mathcal{S}_f^*\}$$

is a solution to (1)–(2).

Proof. The function v is the so-called Perron solution. The proof is divided into two parts.

Step 1. *The function v is a solution to Eq. (1).*

The proof is standard and here we follow [8]. Let $p \in \Omega_m$ be an arbitrary fixed point of Ω_m. Consider a sequence $\{u_n\} \subset \mathcal{S}_f^*$ such that $u_n(p) \to v(p)$ when $n \to \infty$. Let $D \subset \Omega_m$ be a closed round disk centered at p. For each n, define the function

$$v_n(q) = \min\{u_1(q), \ldots, u_n(q)\}, \quad q \in \overline{\Omega_m}.$$

Then $v_n \in \mathcal{S}_f^*$ by Lemma 4.4. By definition of M_D, we deduce $M_D[v_n](p) \to v(p)$ as $n \to \infty$ (Lemma 4.5). Let $V_n = M_D[v_n]$. Then $\{V_n\}$ is a decreasing sequence of functions satisfying (1) in D and $\{V_n\}$ is bounded from below by w_θ for all $w_\theta \in \mathcal{G}$. Consequently the functions V_n are uniformly bounded on compact sets K in D. In each compact set K, the norms of the gradients $|DV_n|$ are bounded by a constant depending only on K and, using Hölder estimates of Ladyzhenskaya and Ural'tseva, there exist uniform $C^{1,\beta}$ estimates for the sequence $\{V_n\}$ on K. See [21] for interior estimates for the mean curvature type equations in two variables; in the context of Eq. (1), these estimates were proven in [9, Th. 1.4].

By the Schauder theory, the second derivatives of V_n are uniformly Hölder continuous on K ([8, Th. 13.1]). By a standard argument (as in [8, Th. 16.8]) using the Arzela–Ascoli theorem, we can find a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ such that $\{V_{n_k}\} \to V$ in $C^\gamma(\overline{\Omega})$, $\gamma > 0$, for some function V. By continuity, the function V satisfies (1). Moreover, by the construction of V_n, at the fixed point p we have $V(p) = v(p)$.

Once proven that V is a solution to (1) and $V(p) = v(p)$, it remains to show that $V = v$ in $\mathrm{int}(D)$, not only at the fixed point p. Let q be an arbitrary fixed point of $\mathrm{int}(D)$. If we proceed with the same argument as in the previous paragraphs, there is a decreasing sequence $\{\tilde{u}_n\} \subset \mathcal{S}_f^*$, such that $\tilde{u}_n(q) \to v(q)$. Let $\tilde{v}_n = \min\{V_n, \tilde{u}_n\}$ and $\tilde{V}_n = M_D[\tilde{v}_n]$. Again, $\{\tilde{V}_n\}$ converges on D in the C^2 topology to a C^2 function $\tilde{V}$ satisfying (1) and $\tilde{V}(q) = v(q)$. By construction, $\tilde{V}_n \le \tilde{v}_n \le V_n$, hence by letting $n \to \infty$, we find $\tilde{V} \le V$. Since $v \le \tilde{V}$, we infer that $v(p) \le \tilde{V}(p) \le V(p)$. Because $v(p) = V(p)$, we deduce $\tilde{V}(p) = V(p)$. Thus the functions V and $\tilde{V}$ satisfy Eq. (1) and coincide at an interior point of D, namely, the point p. Because $\tilde{V} \le V$, the touching principle implies $V = \tilde{V}$ in $\mathrm{int}(D)$. In particular, $V(q) = \tilde{V}(q) = v(q)$. This shows that $V = v$ in $\mathrm{int}(D)$ and the claim is proven.

Step 2. *The function v is continuous up to $\partial\Omega_m$ with $v = \varphi_f$ on $\partial\Omega_m$.*

In order to finish the proof of Theorem 1.2, it remains to show that the function v actually takes the value φ_f on $\partial\Omega_m$. This can be accomplished by the technique of barriers at each boundary point p of Ω_m. Let us observe that the graph of φ_f consists of two copies of f:

$$\Gamma_{\varphi_f} = \Gamma_1 \cup \Gamma_2 = \{(x, m, f(x)) : x \in \mathbb{R}\} \cup \{(x, -m, f(x)) : x \in \mathbb{R}\}.$$

Let $p = (x_0, m) \in \partial\Omega_m$ be a boundary point of Ω_m (similar if $p = (x_0, -m)$). Because of the convexity of f, the tangent line L_p to the planar curve Γ_1 contained in the plane of equation $y = m$ leaves Γ_1 above L_p. Taking into account the symmetry of φ_f and the convexity of f, there is an α-grim reaper w_θ^p such that $w_\theta^p(p) = f(x_0)$ and $w_\theta^p < f$ in $\Gamma_{\varphi_f} \setminus \{(x_0, m, f(x_0)), (x_0, -m, f(x_0))\}$. In particular, $\partial\Sigma_{w_\theta^p}$ lies strictly below $\partial\Sigma_v$ except at the points $(x_0, m, f(x_0))$ and $(x_0, -m, f(x_0))$, where both surfaces $\Sigma_{w_\theta^p}$ and Σ_v coincide.

Therefore, the function w_θ^p and the minimal solution v^0 form a modulus of continuity in a neighborhood of p, namely, $w_\theta^p \leq v \leq v^0$. Because $w_\theta^p(p) = v^0(p) = f(p)$, we deduce that $v(p) = f(p)$. Consequently, v is continuous up to $\partial\Omega_m$, proving that $v \in C^2(\Omega_m) \cap C^0(\overline{\Omega_m})$, and this completes the proof of Theorem 1.2.

Finally, we conclude this paper with two final remarks and some open questions for future research. Let us clarify why in Theorem 1.2, when $0 < \alpha \leq 1$, the width of the strip is restricted (for $\alpha > 1$, the solution exists on any strip of arbitrary width, as it can be seen in the case of α-grim reapers, Proposition 2.1). To this end, we first notice that the width of Ω_m can not be arbitrary large, because the maximum domain of the α-grim reaper w is a strip of bounded width $d(\alpha)$. The case $\alpha = 1$ clearly illustrates this situation. More precisely, in this case, $d = \pi/2$ and $w(x, y) = -\log(\cos(y))$ (see Remark 1). Suppose that Σ is a translating soliton graph over Ω_m, with $m > d$, so that $\Omega_d \subsetneq \Omega_m$ and let us recall that the grim reaper Σ_w is asymptotic to the vertical planes of equations $y = \pm d$. The proof would be then similar to that of Proposition 3.1 by using the touching principle starting with Σ_w sufficiently far from Σ and moving it until when it intersects Σ. The key fact in the reasoning is that one may 'insert' Σ_w in the slab $|y| < m$ and arrive to Σ at a first contact point: since $\Omega_d \subsetneq \Omega_m$, the contact point can not be a boundary point, but an interior point, so that we achieve a contradiction. However, this argument might fail, as we will explain below.

Our second remark is concerned with the uniqueness of solutions to problem (1)–(2) in Theorem 1.2. We recall that uniqueness holds if Ω is a bounded domain (Proposition 3.1). Suppose now that Ω is a strip of $\mathbb{R}^2$ and let Σ_1 and Σ_2 be two α-translating soliton graphs over Ω, with $\partial\Sigma_1 = \partial\Sigma_2$. Again, we follow the reasoning described above by lifting Σ_1 and then moving it down until when it touches Σ_2 for the first time. One would desire to apply the touching principle to obtain $\Sigma_1 = \Sigma_2$. However, the comparison technique may not work. First, one might not be able to get started because it is possible that Σ_i does not have a bounded height (even when it has bounded boundary values). Even in such a case, it is possible that the first time Σ_1 touches Σ_2, the contact point occurs at some point 'at infinity'. In any case, we can not proceed with the conclusion. We refer the reader to [6] for a similar setting in the case of the minimal surface equation, where the uniqueness of the Dirichlet problem for unbounded domains is investigated.

Acknowledgements

The authors thank the referee for many valuable suggestions and corrections. This work has been partially supported by the grant No. MTM2017-89677-P, MINECO/AEI/FEDER, UE.

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