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Number theory

Odd values of the Rogers–Ramanujan functions

Valeurs impaires des fonctions de Rogers–Ramanujan

Shi-Chao Chen

Institute of Contemporary Mathematics, School of Mathematics and Statistics, Henan University, Kaifeng, 475004, P. R. China

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ABSTRACT

Let $g(n)$ and $h(n)$ be the coefficients of the Rogers–Ramanujan identities. We obtain asymptotic formulas for the number of odd values of $g(n)$ for odd n, and $h(n)$ for even n, which improve Gordon's results. We also obtain lower bounds for the number of odd values of $g(n)$ for even n, and $h(n)$ for odd n.

RÉSUMÉ

Soit $g(n)$ et $h(n)$ les coefficients des identités de Rogers–Ramanujan. Nous obtenons des formules asymptotiques pour le nombre de valeurs impaires de $g(n)$ lorsque n est impair et de $h(n)$ lorsque n est pair. Ces formules améliorent un résultat de Gordon. Nous obtenons également des bornes inférieures pour le nombre de valeurs impaires de $g(n)$ pour n pair et de $h(n)$ pour n impair.

1. Introduction

The Rogers–Ramanujan identities, first discovered by Rogers in 1894, are the pair of q-series identities:

$$G(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

and

$$H(q) := \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1-q)(1-q^2)\cdots(1-q^n)},$$

E-mail address: schen@henu.edu.cn.

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where $|q| < 1$. If $q = e^{2\pi i\tau}$ and $\operatorname{Im}\tau > 0$, then Biagioli [2] showed that $q^{-\frac{1}{60}}G(q)$ and $q^{\frac{11}{60}}H(q)$ are modular functions on $\Gamma_1(5)$ with character. Let $G(q) = \sum_{n=0}^{\infty} g(n)q^n$ and $H(q) = \sum_{n=0}^{\infty} h(n)q^n$. We are interested in finding the arithmetic density of the number of odd values of $g(n)$ and $h(n)$. Let

$$\gamma(N) = \sharp\{1 \le n \le N: \quad n \equiv 1 \pmod 2 \text{ and } g(n) \equiv 1 \pmod 2\},$$

and

$$\delta(N) = \sharp\{1 \le n \le N: \quad n \equiv 0 \pmod 2 \text{ and } h(n) \equiv 1 \pmod 2\}.$$

Gordon [3] proved the order of magnitude of $\gamma(N)$ and $\delta(N)$ is $\frac{N}{\log N}$.

Theorem 1.1 *(Gordon). There exist positive constants A and B such that*

$$A\frac{N}{\log N} < \gamma(N), \delta(N) < B\frac{N}{\log N}$$

for sufficiently large N.

Gordon's proof is based on the fact that $g(n)$ with n odd and $h(n)$ with n even are the coefficients of holomorphic modular forms weight 1 mod 2, and hence each such n can be determined explicitly. We obtain the following asymptotic formulas by refining Gordon's arguments.

Theorem 1.2. *For sufficiently large N,*

$$\gamma(N) = \frac{\pi^2}{5} \cdot \frac{N}{\log N} + O\left(\frac{N \log\log N}{\log^2 N}\right),$$
$$\delta(N) = \frac{\pi^2}{5} \cdot \frac{N}{\log N} + O\left(\frac{N \log\log N}{\log^2 N}\right).$$

Let

$$\gamma'(N) = \sharp\{1 \le n \le N: \quad n \equiv 0 \pmod 2 \text{ and } g(n) \equiv 1 \pmod 2\}$$

and

$$\delta'(N) = \sharp\{1 \le n \le N: \quad n \equiv 1 \pmod 2 \text{ and } h(n) \equiv 1 \pmod 2\}.$$

A similar question is to bound $\gamma'(N)$ and $\delta'(N)$. In this case, $g(n)$ with n even and $h(n)$ with n odd are the coefficients of holomorphic modular forms weight $\frac{3}{2}$ mod 2. Applying the ground-breaking work of Bellaïche, Green, and Soundararajan [1] on the non-divisibility of the coefficients of weakly holomorphic modular forms, we obtain the following lower bounds.

Theorem 1.3. *For sufficiently large N, we have*

$$\gamma'(N) \gg \frac{\sqrt{N}}{\log\log N},$$
$$\delta'(N) \gg \frac{\sqrt{N}}{\log\log N}.$$

2. Proof of Theorem 1.2

The parity of $g(n)$ for odd n and that of $h(n)$ for even n were determined by Gordon [3] explicitly, i.e. n is odd and $g(n) \equiv 1 \pmod 2$ if and only if $60n - 1 = p^{4a+1}m^2$, where $a \ge 0$ is an integer and p is a prime not dividing m. n is even and $h(n) \equiv 1 \pmod 2$ if and only if $60n + 11 = p^{4a+1}m^2$. Thus $\gamma(N)$ can be represented as

$$\gamma(N) = \sum_{\substack{1 \le n \le N \\ n \equiv 1 \pmod 2 \\ 60n-1 = p^{4a+1}m^2}} 1.$$

We denote by $M = 60N - 1$ for convenience, and by $\pi(x)$ the number of primes less than x. We split the sum above into two parts according to $a = 0$ and $a \ge 1$. The sum over $a \ge 1$ is bounded by

$$\ll \sum_{1\le a\ll \log N}\sum_{m\le\sqrt{M}} \pi\left(\left(\frac{M}{m^2}\right)^{\frac{1}{4a+1}}\right)$$

$$\ll \log N \sum_{m\le\sqrt{M}}\left(\frac{N}{m^2}\right)^{\frac{1}{5}}$$

$$\ll N^{\frac{1}{2}}\log N,$$

which is negligible. Note that the sum over $a=0$ is equivalent to

$$\sum_{\substack{pm^2\le M\\ pm^2\equiv 59 \pmod{120}}} 1 = \sum_{\substack{m\le\sqrt{M}\\ (m,120)=1}} \sum_{\substack{p\le \frac{M}{m^2}\\ p\equiv 59m^{-2} \pmod{120}}} 1.$$

The contribution for $\log^2 M < m$ is negligible since

$$\sum_{\substack{\log^2 M<m\\ (m,120)=1}} \frac{M}{m^2\log(\frac{M}{m^2})} \ll \sum_{\log^2 M<m}\frac{M}{m^2} \ll \frac{M}{\log^2 M} \ll \frac{N}{\log^2 N}.$$

By the prime number theorem for arithmetic progressions, the sum over $m\le \log^2 M$ is

$$\sum_{\substack{m\le\log^2 M\\ (m,120)=1}} \frac{M}{\phi(120)m^2\log(\frac{M}{m^2})}\left(1+O\left(\frac{1}{\log(\frac{M}{m^2})}\right)\right)$$

$$=\frac{1}{32}\sum_{\substack{m\le\log^2 M\\ (m,120)=1}} \frac{M}{m^2\log M}\left(1+O\left(\frac{\log m}{\log M}\right)\right)$$

$$=\frac{1}{32}\sum_{\substack{m=1\\ (m,120)=1}}^{\infty} \frac{M}{m^2\log M}+O\left(\sum_{m>\log^2 M}\frac{M}{m^2\log M}\right)+O\left(\frac{M\log\log M}{\log^2 M}\right)$$

$$=\frac{1}{32}\sum_{\substack{m=1\\ (m,120)=1}}^{\infty} \frac{M}{m^2\log M}+O\left(\frac{M\log\log M}{\log^2 M}\right),$$

where ϕ is the Euler's ϕ-function. Recalling $M=60N-1$, we conclude that

$$\gamma(N)=\frac{1}{32}\sum_{\substack{m=1\\ (m,120)=1}}^{\infty}\frac{60N-1}{m^2\log(60N-1)}+O\left(\frac{N\log\log N}{\log^2 N}\right)$$

$$=\left(\frac{15}{8}\sum_{\substack{m=1\\ (m,120)=1}}^{\infty}\frac{1}{m^2}\right)\frac{N}{\log N}+O\left(\frac{N\log\log N}{\log^2 N}\right)$$

$$=\frac{\pi^2}{5}\cdot\frac{N}{\log N}+O\left(\frac{N\log\log N}{\log^2 N}\right),$$

where the constant in the main term is computed by

$$\frac{15}{8}\sum_{\substack{m=1\\ (m,120)=1}}^{\infty}\frac{1}{m^2}=\frac{15}{8}\prod_{p\nmid 120}(1-p^{-2})^{-1}=\frac{6}{5}\zeta(2)=\frac{\pi^2}{5}.$$

The proof of the result for $\delta(N)$ is similar, so is omitted.

3. Proof of Theorem 1.3

Let $f=\sum_{n=n_0}^{\infty}a(n)q^n$ be a weakly holomorphic modular form of half-integral weight k and level $\Gamma_1(N)$ with integer coefficients. Bellaïche, Green, and Soundararajan [1] showed that if $f\not\equiv 0 \pmod 2$, then

$$\sharp\{n\le N: a(n)\equiv 1 \pmod 2\}\gg \frac{\sqrt{N}}{\log\log N}.$$

Thus, to prove Theorem 1.3, it suffices to show the generating functions for $g(2n)$ and $h(2n+1)$ mod 2 are weakly holomorphic modular forms of half-integral weight for some levels. The generating functions for $g(2n)$ and $h(2n+1)$ were computed by Hirschhorn in [4]. In view of Theorem 1 of [4], we have

$$\sum_{n=0}^{\infty}g(2n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^n)(1-q^{40n-8})(1-q^{40n-32})},$$
$$\sum_{n=0}^{\infty}h(2n+1)q^n=q\prod_{n=1}^{\infty}\frac{(1-q^{4n})}{(1-q^n)(1-q^{40n-16})(1-q^{40n-24})}.$$

Let $\eta(\tau)=q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n)$ be the Dedekind's eta function and denote by $f_1(\tau)=q^{-\frac{1}{60}}G(q)$ and $f_2(\tau)=q^{\frac{11}{60}}H(q)$, where $q=e^{2\pi i\tau}$. We see that

$$\sum_{n=0}^{\infty}g(2n)q^n\equiv\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{5n-1})^8(1-q^{5n-4})^8}=q^{\frac{1}{120}}f_1^8(\tau)\,\eta^3(\tau)\pmod 2,$$
$$\sum_{n=0}^{\infty}h(2n+1)q^n\equiv q\prod_{n=1}^{\infty}\frac{(1-q^n)^3}{(1-q^{5n-2})^8(1-q^{5n-3})^8}=q^{-\frac{71}{120}}f_2^8(\tau)\,\eta^3(\tau)\pmod 2.$$

If $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_1(5)$, by Proposition 2.5 of [2], the transformations of $f_1(\tau)$ and $f_2(\tau)$ are given by

$$f_1(A\tau)=e^{\frac{4\pi iab}{5}}\nu_1(A)f_1(\tau),$$
$$f_2(A\tau)=e^{\frac{-4\pi iab}{5}}\nu_1(A)f_2(\tau),$$

where $\nu_1(A)$ denotes the multiplier system of $\eta^{14}(\tau)$, i.e.

$$\nu_1\begin{pmatrix}a&b\\c&d\end{pmatrix}=\begin{cases}\exp\left(\frac{7\pi i}{6}\left(-3c-bd(c^2-1)+c(a+d)\right)\right), & c\ \text{odd},\\ \exp\left(\frac{7\pi i}{6}\left(3d-3-ac(d^2-1)+d(b-c)\right)\right), & d\ \text{odd}.\end{cases}$$

Using these formulas, we can easily verify that $f_1(120\tau)$ and $f_2(120\tau)$ are invariant on $\Gamma_1(2880)$. Since $\eta(24\tau)$ is a modular form weight $\frac{1}{2}$ on $\Gamma_0(576)$ with character $(\frac{12}{\cdot})$ (see, for example, [5, Corollary 1.62]), it follows that $\sum_{n=0}^{\infty}g(2n)q^{120n-1}\pmod 2$ and $\sum_{n=0}^{\infty}h(2n+1)q^{120n+71}\pmod 2$ are weakly holomorphic form weight $\frac{3}{2}$ and level $\Gamma_1(2880)$, as desired.

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