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Topology

A geometric Morse–Novikov complex with infinite series coefficients

Un complexe de Morse–Novikov géométrique avec des coefficients séries infinis

François Laudenbach [a], Carlos Moraga Ferrándiz [b]

[a] *Laboratoire de mathématiques Jean-Leray, UMR 6629 du CNRS, Faculté des sciences et techniques, Université de Nantes, 2, rue de la Houssinière, 44322 Nantes cedex 3, France*
[b] *36, av. Camille-Guérin, 44000 Nantes, France*

A R T I C L E I N F O

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A B S T R A C T

Let M be a closed n-dimensional manifold, $n > 2$, whose first real cohomology group $H^1(M;\mathbb{R})$ is non-zero. We present a general method for constructing a Morse 1-form α on M, closed but non-exact, and a pseudo-gradient X such that the differential ∂^X of the Novikov complex of the pair (α, X) has at least one incidence coefficient which is an infinite series. This is an application of our previous study of the homoclinic bifurcation of pseudo-gradients of multivalued Morse functions.

R É S U M É

Soit M une variété fermée de dimension $n > 2$ avec $H^1(M;\mathbb{R}) \neq 0$. Nous présentons une méthode générale pour construire une 1-forme fermée α sur M, non exacte, avec un pseudo-gradient X, tels que la différentielle ∂^X du complexe de Novikov de la paire (α, X) ait au moins un coefficient d'incidence qui soit une série infinie. Ceci est une application de notre étude antérieure sur les bifurcations homoclines des pseudo-gradients de 1-formes fermées non exactes.

Version française abrégée

Considérons une variété différentiable M, compacte à bord vide, de dimension n, ayant une cohomologie de de Rham non nulle en degré un. Soit $u \in H^1_{dR}(M)$, $u \neq 0$, et soit α une forme différentielle fermée dans la classe u. D'après le théorème de transversalité de Thom (transversalité sous contrainte) [10,11], génériquement dans la classe u, une 1-forme fermée α

E-mail addresses: francois.laudenbach@univ-nantes.fr (F. Laudenbach), crlsmrgf@gmail.com (C. Moraga Ferrándiz).

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a pour primitives locales des fonctions de Morse. L'ensemble fini $Z(\alpha)$ des zéros de α est gradué par l'indice de Morse : $Z(\alpha) = \cup_{k=0}^{n} Z_k(\alpha)$.

En adoptant le point de vue selon lequel une 1-forme de Morse n'est autre qu'une fonction de Morse multivaluée (bien définie à une constante additive près sur chaque boule de M), S.P. Novikov a developpé une théorie des 1-formes de Morse analogue à la théorie de Morse [5,6], donnant naissance à une homologie notée $H_*(M; u)$. Nous la résumons ci-dessous selon le point de vue de J.-C. Sikorav [9].

La classe de cohomologie u est vue comme le morphisme $\pi_1(M) \to \mathbb{R}$ qui, sur un lacet orienté γ, vaut $\int_\gamma \alpha$. Il lui est alors associé une valuation $\tilde{u} : \mathbb{Z}\pi_1(M) \to \overline{\mathbb{R}}$. Précisément, pour une somme finie $\lambda = \sum_i n_i g_i$, avec $n_i \in \mathbb{Z}^*$ et $g_i \in \pi_1(M)$, on a $\tilde{u}(\lambda) = \max u(g_i)$. L'anneau de Novikov *universel* Λ_u est la complétion de l'anneau $\mathbb{Z}\pi_1(M)$ pour $\tilde{u}(\lambda) \to -\infty$. Le complexe de Novikov $N_*(\alpha, \partial^X)$ est le Λ_u-module libre engendré par $Z_*(\alpha)$.

Pour la différentielle ∂^X, on choisit un pseudo-gradient (descendant) X *adapté* à la 1-forme de Morse α. Par définition, X vérifie ce qui suit :

1) $\langle \alpha(x), X(x) \rangle < 0$ pour tout $x \in M$ en dehors de $Z(\alpha)$;

2) pour tout $p \in Z_k(\alpha)$, le champ $-X$ est le gradient euclidien d'une primitive locale h_p de α dans une carte de Morse autour de p, où la matrice hessienne de h_p en p a ses valeurs propres dans $\{-1, +1\}$.

Les zéros de X, qui sont aussi ceux de α, sont *hyperboliques*. Ainsi, à chaque $p \in Z_k(\alpha)$ sont associées une variété *stable* $W^{\mathrm{s}}(p, X)$ de dimension $n-k$ et une variété instable $W^{\mathrm{u}}(p, X)$ de dimension k. D'après le théorème de Kupka–Smale (voir [8]), on peut approximer X de telle sorte que toutes les variétés stables et instables soient mutuellement transverses. Pour $p \in Z_k(\alpha)$, la valeur de ∂^X en p s'écrit :

$$\partial^X p = \sum_{q \in Z_{k-1}(\alpha)} \langle p, q \rangle^X q$$

où le *coefficient d'incidence* $\langle p, q \rangle^X$ est un élément de l'anneau de Novikov Λ_u.

À notre connaissance, il n'y a pas d'exemple publié[1] où cette formule ait des coefficients qui sont des séries infinies comme le permet l'anneau de Novikov. À l'opposé, on sait que si $f : M \to \mathbb{R}$ est une fonction de Morse, pour tout $c \gg 0$, la différentielle du complexe de Novikov $N_*(\alpha + c\,df)$ a ses coefficient d'incidence dans $\mathbb{Z}\pi_1(M)$ [3, Lemma 3.7]. Pour $n \geq 3$, nous proposons une méthode générale pour fabriquer des complexes de Novikov géométriques avec des coefficients séries. Cette méthode est basée sur notre étude [4] des bifurcations *homoclines* du pseudo-gradient X.

Théorème. *Pour tout $n \geq 3$ et tout $1 \leq k \leq n-2$, toute forme de Morse α, dans une classe de cohomologie $u \neq 0$ sans zéro d'indice 0 ou n, peut être modifiée en une 1-forme de Morse α' cohomologue, par naissance d'une paire de zéros (p, q) d'indices respectifs $(k+1, k)$, qui admet un pseudo-gradient X' tel que le coefficient d'incidence $\langle p, q \rangle^{X'}$ soit une série infinie dans l'anneau de Novikov Λ_u.*

Idée de la preuve. Signalons qu'à partir d'une forme de Morse non exacte, il est possible d'en éliminer tous les zéros d'indice extrême [1, Remarque section 1.2] ; on commence donc avec un tel α. Dans ce cas, pour tout pseudo-gradient, presqu'aucune orbite ne tend vers un zéro (dans le passé ou le futur). Une modification C^0-petite du champ dans un voisinage d'un point d'accumulation d'une telle orbite permet, non seulement de créer une orbite périodique, mais aussi tout un tube $T \cong D^{n-1} \times S^1$ d'orbites périodiques $\{pt\} \times S^1$. Notons g la classe d'homotopie de lacets déterminée par T ; cette classe est non nulle, car l'intégrale de α sur chaque orbite est strictement négative.

Dans une petite boîte cylindrique $B \subset T$, disjointe de l'orbite centrale de T et dont le bord *vertical* est tangent au facteur S^1, on fait apparaître une paire de points critiques (p, q) d'indices respectifs $k+1$ et k. Soit α' la 1-forme de Morse ainsi obtenue ; on a $Z(\alpha') = Z(\alpha) \cup \{p, q\}$. On choisit une modification radiale à support dans T de l'application de premier retour sur un méridien de T de telle sorte qu'aucune orbite issue de B n'y revienne jamais dans le futur. Appelons X un pseudo-gradient de ce type. Le bord du tube étant fait d'orbites périodiques, il n'y a aucune interaction des points p et q avec les autres zéros de α'. On a alors (avec des orientations convenables) $\partial^X p = q$ et $\partial^X q = 0$.

Soit $\mathcal{S}_g$ la strate (de codimension 1) des pseudo-gradients Y adaptés à α', égaux à X hors de T et ayant une orbite homocline ℓ_Y de p à p avec un défaut minimal de transversalité entre $W^{\mathrm{s}}(p, Y)$ et $W^{\mathrm{u}}(p, Y)$. L'holonomie de ℓ_Y depuis sa sortie du modèle de Morse de p jusqu'à son retour dans ce modèle permet de définir un réel $\chi(Y)$ dépendant continûment de $Y \in \mathcal{S}_g$ [4, Définition 2.11]. Posons $\mathcal{S}_g^+ := \chi^{-1}(]0, +\infty))$ et $\mathcal{S}_g^- := \chi^{-1}((-\infty, 0[)$. Dans [4, Théorème 1.1], nous avons montré que $\mathcal{S}_g$ est canoniquement co-orientée et et que les deux parties $\mathcal{S}_g^+$ et $\mathcal{S}_g^-$ sont non vides. Or, à partir de X, on peut s'approcher de $\mathcal{S}_g$ comme on veut. En particulier, on peut trouver un chemin de traversée positive de $\mathcal{S}_g$ passant par un point de $\mathcal{S}_g^+$. Si X' désigne un pseudo-gradient de ce chemin juste après le temps de traversée, d'après [4, Théorème 3.6], la différentielle de Novikov donne :[2]

$$\partial^{X'} p = (1 + g + g^2 + \cdots + g^l + \cdots)\, q\,.$$

[1] A. Pajitnov a présenté un exemple en dimension 3 pour une classe u rationnelle (voir [7, Section 3]).

[2] Une traversée positive de $\mathcal{S}_g$ par un point de $\mathcal{S}_g^-$ donne $\partial^{X''} p = (1+g)\,q$, où X'' est un champ juste après la traversée.

Comme aucune orbite issue de p ne peut sortir du tube T et que X n'a pas de retour de la boîte B dans elle-même, on prouve que la bifurcation de X à X' est *isolée* ; les précautions de troncature prises dans le théorème cité sont alors inutiles.

1. Introduction

Let us consider an n-dimensional closed smooth manifold M with a non-zero de Rham cohomology in degree one. Let $u \in H^1_{dR}(M)$, $u \neq 0$, and let α be a differential closed form in the class u. By Thom's transversality theorem with constraints [10,11], generically in the class u, the local primitives of a closed 1-form α are Morse functions (such an α is named a Morse 1-form). The finite set $Z(\alpha)$ made of its zeroes is graded by the Morse index: $Z(\alpha) = \cup_{k=0}^{n} Z_k(\alpha)$.

Taking the point of view that a Morse 1-form is nothing but a multivalued function (that is, a function well defined up to an additive constant), S.P. Novikov developed a theory analogous to Morse theory [5,6] (today referred to as *Morse–Novikov theory*), which leads to some homology denoted by $H_*(M;u)$. Here, we summarize it with the point of view adopted by J.-C. Sikorav [9].

The class u is seen as the morphism $\pi_1(M) \to \mathbb{R}$ defined by $u(\gamma) = \int_\gamma \alpha$ for every oriented loop γ. Let $\tilde{u} : \mathbb{Z}\pi_1(M) \to \overline{\mathbb{R}}$ be the associate valuation: for $\lambda \in \mathbb{Z}\pi_1(M)$, that is a finite sum $\sum_i n_i g_i$ with $n_i \in \mathbb{Z}^*$ and $g_i \in \pi_1(M)$, one defines $\tilde{u}(\lambda) = \max u(g_i)$. The *universal* Novikov ring Λ_u is the completion of $\mathbb{Z}\pi_1(M)$ for $\tilde{u}(\lambda) \to -\infty$. It is worth noticing that we get the same completion by replacing $\pi_1(M)$ with the *fundamental groupoid* of M. This choice is made in what follows.

The Novikov complex $N_*(\alpha, \partial^X)$ is the free Λ_u-module based on $Z_*(\alpha)$; the differential ∂^X is detailed right below. For defining ∂^X, one chooses some (descending) pseudo-gradient X *adapted* to α, meaning that X fulfils the following:

1) $\langle \alpha(x), X(x) \rangle < 0$ for every $x \in M$ apart from $Z(\alpha)$;

2) for every $p \in Z_k(\alpha)$, the field $-X$ is the Euclidean gradient of a local primitive h_p of α in Morse coordinates $(x_1, \dots, x_n)$ about p where h_p reads:

$$h_p(x_1, \dots, x_n) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 \dots + x_n^2.$$

The zeroes of X, which coincide with the zeroes of α, are *hyperbolic*. Therefore, for each $p \in Z_k(\alpha)$, we have a stable manifold $W^{\mathrm{s}}(p, X)$ of dimension $n-k$ and an unstable manifold $W^{\mathrm{u}}(p, X)$ of dimension k. According to Kupka–Smale's theorem (see [8]), one can approximate X so that all stable and unstable manifolds are mutually transverse. For $p \in Z_k(\alpha)$, the value of ∂^X in p reads:

$$\partial^X p = \sum_{q \in Z_{k-1}(\alpha)} \langle p, q \rangle^X q$$

where the *incidence coefficient* $\langle p, q \rangle^X$ belongs to the Novikov ring Λ_u.

As far as we know, there are no published example[3] where some of the incidence coefficients are infinite series as the Novikov ring allows it. By contrast, it is known that if $f : M \to \mathbb{R}$ is a Morse function, then for every $c \gg 0$, the differential of the Novikov complex $N_*(\alpha + c\,\mathrm{d}f, X)$ has all its incidence coefficients in $\mathbb{Z}\pi_1(M)$. Here, X denotes a pseudo-gradient adapted to $\alpha + c\,\mathrm{d}f$ which is C^1-close to a pseudo-gradient adapted to f [3, Lemma 3.7]. When $n > 2$, we present a general method for constructing some pairs (α, X) such that the differential ∂^X of $N_*(\alpha)$ have some incidence coefficients that are infinite series. Our method is based on our study [4] of the homoclinic bifurcations of the pseudo-gradient X.

Theorem. *Let M be an n-dimensional closed manifold with $n > 2$. Let $u \in H^1_{dR}(M)$ be a non-zero cohomology class and let α be a Morse 1-form in the class u without centres, that is, with no zeroes of extremal index. For $1 \le k \le n-2$, let α' be any Morse 1-form obtained from α by creating the birth of a pair of zeroes (q, p) of respective indices $(k, k+1)$. Then, it is possible to build a pseudo-gradient X' adapted to α' such that the incidence coefficient $\langle p, q \rangle^{X'}$ is an infinite series in the Novikov ring Λ_u.*

Remark. Observe that, when α is a non-exact Morse 1-form, it is possible to cancel all zeroes of extremal indices [1, Remarque section 1.2]. Therefore, the assumption of having no centres is irrelevant.

2. Proof

2.1. Making a tube of periodic orbits

Let X be a Kupka–Smale pseudo-gradient adapted to α. After the no-centre assumption, almost no orbit of X goes to (or comes from) one of its zeroes. A C^0-approximation of X near an accumulation point of such an orbit allows us to create, not only one periodic orbit, but also a tube $T \cong D^{n-1} \times S^1$ made of periodic orbits $\{pt\} \times S^1$ (still name X the pseudo-gradient after this approximation). The (free) homotopy class of these periodic orbits is denoted by g. Note $u(g) < 0$.

[3] A 3-dimensional example where u is a rational class has been given by A. Pajitnov in [7, Section 3].

2.2. Birth of a pair of zeroes in cancelling position

Let $B \subset T$, $B \cong D^{n-1} \times D^1$, be a polydisc bi-foliated by leaves of α and orbits of X. Let h be a primitive of $\alpha|_B$. Let us insert a *birth model* into B to create a pair (p, q) of critical points of respective indices $k+1$ and k (see [2, chap. III] and also [12]). We impose this one-parameter deformation starting from h to be supported in B. Denote h' the function ending this deformation; set $\alpha' = \alpha + \mathrm{d}h' - \mathrm{d}h$. Let X_1 be a pseudo-gradient adapted to α' near p and q and coinciding with X outside B. There is exactly one orbit of X_1 descending from p to q inside B. However, the pseudo-gradient X_1 is not convenient for our theorem, as some extra connecting orbit exists in T from p to q. Thus, we change X_1 into X' as follows: make ∂T an attractor of $X'|_T$ (while keeping $X'|_{\partial T} = X_1|_{\partial T}$) and one of the periodic orbits in $T \smallsetminus B$ a repeller in order that any orbit of X' emanating from the bottom of B, namely $\partial^- B := \{h' = h'(q) - \varepsilon\}$, $0 < \varepsilon \ll 1$, never returns to B in positive time. The outcome of this construction is that the Novikov complex $N_*(\alpha', X')$ fulfils the following: $\partial^{X'} q = 0$ and $\partial^{X'} p = q$ for suitable orientations of the unstable manifolds. Moreover, as the boundary of T is invariant by X', no zero of α' other than p has a connecting orbit to q; and no zero is connected to p.

Let $\mathcal{S}_g$ be the stratum of pseudo-gradients adapted to α' that are equal to X' outside T and have one homoclinic connecting orbit from p to p in T in the class g with the minimal defect of transversality. This is a codimension-one stratum, which is canonically co-oriented.

2.3. Notation

Denote by $\partial^+ B := \{h' = h'(p) + \varepsilon\}$ the top of B. It contains the so-called *belt sphere* Σ^+ of p, that is, $\Sigma^+ := W^s(p, X') \cap \partial^+ B$. This sphere is $(n-k-2)$-dimensional and co-oriented by the chosen orientation of $W^{\mathrm{u}}(p, X')$. Moreover, it is the boundary of the closed $(n-k-1)$-disc Δ^+ whose interior is made of points in $\partial^+ B$ whose positive X'-flow goes to q. Similarly, set $\partial^- B := \{h' = h'(q) - \varepsilon\}$. Denote by Δ^- the closure of $W^{\mathrm{u}}(p, X') \cap \partial^- B$ (a k-disc) and set $\Sigma^- := W^{\mathrm{u}}(q, X') \cap \partial^- B$. The sphere Σ^- bounds Δ^-.

Let $\mathcal{L}^\pm$ denote the leaf of α' in T that contains $\partial^\pm B$. The ball $\partial^+ B$ shall be regarded as the union of a normal tube $N(\Sigma^+)$ to Σ^+ and a normal tube $N(\Delta^+)$ to Δ^+ in $\mathcal{L}^+$. Moreover, there exists a larger tube $N'(\Sigma^+)$ that contains all fibres of $N(\Delta^+)$ that meet $N(\Sigma^+)$.

Let $\rho : \mathcal{L}^- \to \mathcal{L}^+$ be the *first* holonomy diffeomorphism of the positive flow of X'. Denote by $Desc : \mathcal{L}^+ \smallsetminus \Delta^+ \to \mathcal{L}^- \smallsetminus \Delta^-$ the *gradient descent* by X'. Set $D := \rho(\Delta^-)$; it is a k-disc in $\mathcal{L}^+$ disjoint from $\partial^+ B$ whose boundary lies in $W^{\mathrm{u}}(q, X')$.

Finally, let μ_a be the fibre in $a \in \Sigma^+$ of the tube $N'(\Sigma^+)$. By descent, the $(k+1)$-disc μ_a descends to Γ_a^-, a cylinder over Δ^-, pinched along $\partial\Delta^-$. The cylinder Γ_a^- is foliated by *rays* that are traced by the normal fibres to Δ^- in $\mathcal{L}^-$. The holonomy diffeomorphism ρ carries Γ_a^- and its foliation to a foliated pinched cylinder Γ_a^+ whose leaves are still named rays.

2.4. Some isolated self-slide of p; homoclinic bifurcation

In this setup, a *positive self-slide* of p consists in making an isotopy of D in $\mathcal{L}^+$, relative to ∂D, which crosses Σ^+ exactly once and positively with respect to its co-orientation [4, Remark 3.2]. This isotopy lifts to a deformation $(X'_s)_{s\in[0,1]}$ of $X'_0 = X'$. At the time of crossing (let us say $s = \frac{1}{2}$), the homoclinic connecting orbit from p to p lies in the homotopy class g. The path $(X'_s)_s$ is said to cross the codimension-one stratum $\mathcal{S}_g$ positively in the space of pseudo-gradients adapted to α'.

Up to homotopy, the path $(X'_s)_s$ is well defined by the following data:

- a simple smooth path $\gamma : [0, \frac{1}{2}] \to \mathcal{L}^+$, starting non-tangentially to D from any point $\gamma(0) \in int(D)$, ending in $a^+ := \gamma(\frac{1}{2})$ normally to Σ^+ and which, for every $s \in (0, \frac{1}{2})$, avoids both of $W^{\mathrm{u}}(p, X') \cup \Delta^+$ and the repelling periodic orbit;
- a transverse framing $\tau = (\tau_1, \ldots, \tau_{n-2})$ to γ in $\mathcal{L}^+$ such that $\tau_1^k := \tau_1 \wedge \cdots \wedge \tau_k$ is tangent to D in $\gamma(0)$ and such that $(\dot\gamma(\frac{1}{2}), \tau_1, \ldots, \tau_{n-2})$ is an orthogonal framing positively normal to Σ^+ in a^+.

Note that any point $a^+ \in \Sigma^+$ is reachable from D by such a path. In what follows, we will specify γ more precisely near $s \in \{0, \frac{1}{2}\}$. For further use, we introduce $a^- := \rho^{-1}(\gamma(0))$, a point of Δ^-. Let $\tilde\Sigma^- := W^{\mathrm{u}}(p, X') \cap \{h' = h'(p) - \varepsilon\}$ be the attaching k-sphere associated with p and denote by $\tilde a^- \in \tilde\Sigma^-$ the point that descends to a^- through the flow of $X'|_B$.

Let Y be a vector field on $\mathcal{L}^+$, tangent to Δ^+, whose support S is contained in a neighbourhood of $\gamma([0, \frac{1}{2}]) \cup \partial^+ B$ and whose flow is noted Y^s, $s \in \mathbb{R}$. The triple (γ, τ, Y) may be chosen so that the following holds.

(i) For every $s \in [0, \frac{1}{2}]$, we have $Y^s(\gamma(0)) = \gamma(s)$ and τ_1^k is tangent to $Y^s(D)$ in $\gamma(s)$; moreover, we impose the vector $\dot\gamma(\frac{1}{2})$ to be tangent to Δ^+ pointing inwards.

(ii) Denoting by $\mathcal{T}$ the closure of $\cup_{s=0}^{+\infty} Y^s(D)$, named the *tongue*, the intersection $\mathcal{T} \cap \partial^+ B$ is a corned disc $\hat\nu_{a^+}$ contained in the fibre μ_{a^+} of the tube $N'(\Sigma^+)$. Denote by ν_{a^+} the corresponding fibre of $N(\Sigma^+)$.

(iii) For every $s \in [\frac{1}{2}, +\infty)$, the intersection $Y^s(D) \cap \partial^+ B$ is one fibre of $N(\Delta^+)$ which meets $N(\Sigma^+)$. The distance of this fibre to Σ^+ is noted $r(s)$; set $\eta = \max r(s)$. A suitable choice of Y makes it as small as desired.

(iv) The vector field Y is required to be tangent to the foliation of $\Gamma^+_{a^+}$ by rays. This requirement forces the path γ to be tangent to a ray of $\Gamma^+_{a^+}$ and to follow it entirely.
(v) The support S satisfies the following condition: $(S \setminus \partial^+ B)$ is contained in a small neighbourhood of $\gamma([0, \frac{1}{2}])$.

The last requirement needs some preparation. By (i), the vector $\dot\gamma(\frac{1}{2})$ is tangent to Δ^+ and points in the direction marked by the point $\Delta^+ \cap \partial\nu_{a^+}$. This vector determines a co-oriented *equator* E in the sphere $\partial\nu_{a^+}$ (see [4, subsection 2.4]). Recall that the Morse model of p produces a canonical diffeomorphism from the meridian sphere $\partial\nu_{a^+}$ to $\tilde\Sigma^-$. It maps E to an equator $\tilde E \subset \tilde\Sigma^-$ that avoids the connecting orbit from p to q by the choice of $\dot\gamma(\frac{1}{2})$. Thus, $\tilde E$ descends through the flow of $X'|_B$ to a $(k-1)$-sphere E^- contained in the interior of Δ^-. This E^- makes a decomposition $\Delta^- = \delta^- \cup_{E^-} \delta^+$ where δ^- is a k-disc and δ^+ is diffeomorphic to $S^{k-1} \times [0, 1]$. The signs of $\delta^\pm$ reflect the co-orientation of E in $\partial\nu_{a^+}$, and hence, of E^- in Δ^-. By moving $\gamma(0)$ in D, we can fulfil the last requirement below.

(vi) The point a^- lies in δ^+. Moreover, we make the support S of Y disjoint from $\rho(\delta^-)$.

Finally, the isotopy $(Y^s)_{s\in[0,1]}$ lifts to the desired one-parameter deformation (X'_s) of X' realizing a self-slide of p in the homotopy class g.

Lemma 1. *With the above requirements* (i–vi)*, the self-slide at $s = \frac{1}{2}$ is* isolated. *More precisely, for $s \neq \frac{1}{2}$, the pseudo-gradient X'_s has no homoclinic orbit.*

Proof. The dynamics of ρ, which moves ∂D away from S, makes it clear that $W^u(q, X'_s)$ never meets Δ^+. This implies that no $s \in [0, 1]$ is a time of self-slide of q.

Concerning the self-slides of p, we argue as follows. By (ii), the corned disc $\hat\nu_{a^+}$ descends to a corned cylinder $C^- \subset \Gamma^-_{a^+}$, also based on Δ^- and pinched along $\partial\Delta^-$. Then, the holonomy diffeomorphism ρ carries C^- to a pinched corned cylinder $C^+ \subset \Gamma^+_{a^+}$. By (iv–v), the intersection $C^+ \cap S$ is foliated by (truncated) rays of C^+ starting from $D \cap S$. In particular, $S \cap \mathcal{T}$ may be viewed as a *sub-tongue* of $\mathcal{T}$.

We are now ready to prove Lemma 1. We limit ourselves to prove that X'_s is never a self-slide of p in the class g^2 when $s \neq \frac{1}{2}$, that is:

$$(*) \quad \forall s \neq \frac{1}{2}, \quad [(Y^s\rho) \circ Desc \circ_1 (Y^s\rho)(\Delta^-)] \cap \Sigma^+ = \emptyset\,.$$

Here, the operator $\circ_1$ stands for *Desc* where it is defined, that is, in $\mathcal{L}^+ \smallsetminus \Delta^+$. The proof concerning the classes g^k, $k > 2$, is similar with the same choices (i)–(vi).

When $s < \frac{1}{2}$ and as long as the disc $Y^s(D)$ does not touch the ball $\partial^+ B$, the dynamics of $\rho \circ Desc$ implies $(*)$. When $Y^s(D) \cap N(\Sigma^+)$ is non-empty, we define $D^-_s := Desc \circ_1 Y^s(D)$. When $s < \frac{1}{2}$, one checks that D^-_s is contained in the union of a thin tube $Desc(S \smallsetminus \partial^+ B)$ and the part of the cylinder C^- *over* δ^- (in the sense of the rays). Thus, by (vi), $\rho(D^-_s)$ does not approach the support S of Y and $(*)$ holds for every $s \in [0, \frac{1}{2})$.

For $s > \frac{1}{2}$, D^-_s is now an annulus whose boundary is made of two $(k-1)$-spheres, one of which is $Desc(\partial D)$ and the other is $\partial\Delta^-$. By [4, Lemma 3.8], when s goes to $\frac{1}{2}$ from above, it approaches any compact domain given in the interior of the annulus δ^+ in the C^1-topology. As a consequence of (iv), when $s - \frac{1}{2}$ is small enough, the intersection of $\rho(D^-_s)$ with S is the graph G_{u_s} in the cylinder C^+ of some positive function $u_s : \rho(\delta^+) \cap S \to \mathbb{R}_{>0}$. Since the flow of Y preserves the order on each orbit, $[(Y^s\rho)(D^-_s)] \cap S$ (which lies in the tongue) is in front position with respect to $[Y^{1/2}(D)] \cap S$ on each flow line of Y. This claim holds true when $s > \frac{1}{2}$ is very close to $\frac{1}{2}$ and it remains true up to 1 by the flow rule. Therefore, $[(Y^s\rho)(D^-_s)] \cap S$ is still beyond $[Y^{1/2}(D)] \cap S$ in the tongue, and hence, it cannot meet Σ^+. □

2.5. The character function

This function is a natural continuous function $\chi : \mathcal{S}_g \to \mathbb{R}$ that takes positive *and* negative values on each connected component of $\mathcal{S}_g$ [4, Theorem 1.1]. For a pseudo-gradient $Y \in \mathcal{S}_g$, the value $\chi(Y)$ only depends on the 1-jet of the holonomy $\mathfrak{h}$ of Y along the unique homoclinic orbit ℓ_Y from a zero of Y to itself in the homotopy class g. The construction and properties of $\chi(Y)$ are given in [4, Section 2]. Denote by $\mathcal{S}^+_g$ (resp. $\mathcal{S}^-_g$) the non-empty open set in $\mathcal{S}_g$ where $\chi > 0$ (resp. $\chi < 0$). When crossing $\mathcal{S}_g$ transversely, the change of the differential in the Morse–Novikov complex is different according to the crossing: it can be positive or negative, through $\mathcal{S}^+_g$ or $\mathcal{S}^-_g$. A positive crossing through $\mathcal{S}^+_g$ implies a change by multiplication with the infinite series $1 + g + g^2 + \cdots$.

In our setting, the considered pseudo-gradient is $X'_{1/2}$ and the involved homoclinic orbit ℓ is the orbit of $X'_{1/2}$, which leaves the box B at a^- and enters B at a^+. We wish $\chi(X'_{1/2}) > 0$. By the construction, the character reads $\chi(X'_{1/2}) =$

$\lambda\omega_+ + \omega_-$ with $\lambda > 0$ and $\omega_\pm \in [-1, +1]$. Each term[4] in this formula depends on $\mathfrak{h}$ only. The term ω_- is positive by the choice of $\dot\gamma(\frac{1}{2})$ (which fixes the decomposition of $\Delta^- = \delta^- \cup_{E^-} \delta^+$) and by taking $a^- \in \delta^+$. The term ω_+ is not estimated, but the factor λ may be made as small as desired by the choice of the holonomy of Y along $\gamma([0, \frac{1}{2}])$ once $\gamma(0) = \rho(a^-)$ and $\gamma(\frac{1}{2}) = a^+$ are fixed. And this choice is free. As a consequence, by [4, Theorem 3.6], we have:

$$(*) \quad \begin{cases} \partial^{X'_1} q = 0, \\ \partial^{X'_1} p = (1 + g + g^2 + g^3 + \cdots)(\partial^{X'_0} p) = (1 + g + g^2 + g^3 + \cdots)\, q\,. \end{cases}$$

Some comment about $(*)$ is necessary here. In the theorem that is referred to, $(*)$ is stated only for the germ of $(X'_s)_s$ at $s = \frac{1}{2}$ with truncation of the incidence coefficients. The support of a representative of the germ depends on the order of the truncation, the higher the order of truncation the smaller the domain of the representative. Here, we are in a very particular situation: by Lemma 1, the self-slide is isolated. As a consequence, the truncation in question is not needed. This finishes the proof of our theorem.

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[4] The terms ω^+ and ω^- are named the *latitudes* in [4] and the factor λ is called the *holonomy factor*.