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Partial differential equations

Stability for entire radial solutions to the biharmonic equation with negative exponents

Stabilité des solutions radiales entières de l'équation biharmonique avec exposants négatifs

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ABSTRACT

In this note, we are interested in entire solutions to the semilinear biharmonic equation

$$\Delta^2 u = -u^{-p}, \ \ u > 0 \quad \text{in } \mathbb{R}^N,$$

where $p > 0$ and $N \geq 3$. In particular, the stability outside a compact set of the entire radial solutions will be completely studied, which resolves the remaining case in [5].

RÉSUMÉ

Dans cette note, on s'intéresse aux solutions radiales entières de l'équation semilinéaire biharmonique

$$\Delta^2 u = -u^{-p}, \ \ u > 0 \quad \text{dans } \mathbb{R}^N,$$

où $p > 0$ et $N \geq 3$. En particulier, on étudie la stabilité en dehors d'un compact des solutions radiales entières, et on résout un cas ouvert dans [5].

1. Introduction

In this note, we are interested in entire radial solutions to the biharmonic equation

$$\Delta^2 u = -u^{-p}, \ \ u > 0 \quad \text{in } \mathbb{R}^N \tag{1.1}$$

where $p > 0$ and $N \geq 3$.

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Recently, the fourth-order equations have attracted the interest of many researchers. In particular, the existence, multiplicity, stability, and qualitative properties of solutions to equation (1.1) are studied in many works, especially for radial solutions. It has been proved in [6] that, if $0 < p \le 1$, the equation (1.1) admits no entire smooth solution. It is showed in [4,7] that, for any $p > 1$, there exist radial solutions to (1.1).

Definition 1. A solution u to (1.1) is said stable in $\Omega \subseteq \mathbb{R}^N$ if there holds

$$\int_\Omega |\Delta\phi|^2 \mathrm{d}x - p\int_\Omega u^{-p-1}\phi^2 \mathrm{d}x \ge 0 \quad \text{for any } \phi \in C_0^\infty(\Omega).$$

Moreover, a solution u to (1.1) is said stable outside a compact set K if u is stable in $\mathbb{R}^N \setminus K$. For simplicity, we say also that u is stable if $\Omega = \mathbb{R}^N$.

We consider the following initial value problem

$$\begin{cases} \Delta^2 u = -u^{-p} \quad \text{for } r \in [0,\ R_{\alpha,\beta}) \\ u'(0) = u'''(0) = 0, \\ u(0) = \alpha, \quad \Delta u(0) = \beta; \end{cases} \tag{1.2}$$

for any $\alpha, \beta \in \mathbb{R}$, we denote by $u_{\alpha,\beta}$ the (local) solution to (1.2) and by $[0, R_{\alpha,\beta})$ the maximal interval of existence. Notice that the equation (1.2) is invariant under the scaling transformation

$$u_\lambda(x) = \lambda^{-\frac{4}{p+1}} u(\lambda x),\ \lambda > 0.$$

Therefore, we need only to consider the case $\alpha = 1$. We will denote $u_{1,\beta}$ by u_β. Let $p > 1$, it is known from [3,5,7] that

- there is no global solution to (1.2) if $N \le 2$;
- for $N \ge 3$, there exists $\beta_0 > 0$ depending on N such that the solution to (1.2) is globally defined if and only if $\beta \ge \beta_0$. Furthermore, $\lim_{r\to\infty} \Delta u_\beta \ge 0$ and $\lim_{r\to\infty} \Delta u_\beta = 0$ if and only if $\beta = \beta_0$;
- for $N \ge 3$, any entire solution u_β is stable outside a compact set if $\beta > \beta_0$;
- for $N = 4$, u_{β_0} is unstable outside every compact set;
- for $5 \le N \le 12$, there exists a critical value $p_N > 1$ (see below for the precise definition) such that, if $1 < p \le p_N$, u_β is stable for every $\beta \ge \beta_0$, while for $p > p_N$, there exists $\beta_1 > \beta_0$ such that u_β is stable if and only if $\beta \ge \beta_1$, and u_{β_0} is unstable outside every compact set;
- for $N \ge 13$ and any $p > 1$, u_β is stable for every $\beta \ge \beta_0$.

Moreover, Warnault [8] proved that equation (1.1) admits no stable solution (radial no not) for $N \le 4$. So it remains to consider the eventual stability outside a compact set for $N = 3$ and $\beta = \beta_0$.

The stability property of entire radial solutions is closely related to their asymptotic behaviors. Let us recall the asymptotic behaviors showed in [2,3,5]. For $N = 3$ and $\beta = \beta_0$, the following hold:

$$\begin{cases} \lim\limits_{r\to\infty} u_{\beta_0}(r) r^{-1} = \ell > 0, & \text{if } p > 3; \\ \lim\limits_{r\to\infty} u_{\beta_0}(r) r^{-1}(\ln r)^{-\frac{1}{4}} = \sqrt[4]{2}, & \text{if } p = 3; \\ \lim\limits_{r\to\infty} u_{\beta_0}(r) r^{-\frac{4}{p+1}} = \left[-Q_4\left(-\dfrac{4}{p+1}\right)\right]^{-\frac{1}{p+1}} =: L_0, & \text{if } 1 < p < 3, \end{cases} \tag{1.3}$$

where Q_4 is defined by

$$Q_4(m) := m(m+2)(N-2-m)(N-4-m). \tag{1.4}$$

Remark that equation (1.1) has a singular solution $u_s(r) \equiv L_0 r^{\frac{4}{p+1}}$, if $Q_4\left(-\frac{4}{p+1}\right) < 0$.

From [2], we know that for $N = 3$, there exist $3 > p_c^+ > p_c > 1$ such that, if $p = p_c$ or $p = p_c^+$, then $-pQ_4(m) = \frac{9}{16}$ with $m = -\frac{4}{p+1}$, and if $p_c < p < p_c^+$ then $-pQ_4(m) > \frac{9}{16}$. For $N \ge 5$, p_N is the unique root of

$$-pQ_4\left(-\frac{4}{p+1}\right) = \frac{N^2(N-4)^2}{16}$$

in $(1, \infty)$.

Theorem A *(Theorem 1.6 in [5]). Let $N = 3$, $p > 1$. We have:*

(i) *if $p_c^+ < p < 3$ or $1 < p < p_c$, then u_{β_0} is stable outside a compact set;*
(ii) *if $p_c < p < p_c^+$, u_{β_0} is unstable outside every compact set;*
(iii) *if $p \geq 3$, then u_{β_0} is stable outside a compact set.*

Open problem: What is the stability behavior outside compact set when $\beta = \beta_0$, $N = 3$, $p = p_c$ or $p = p_c^+$?
The following result gives the definite answer.

Theorem 1.1. *Let $N = 3$, $p = p_c$ or $p = p_c^+$, the solution u_{β_0} to equation (1.2) is stable outside a compact set.*

Indeed, we will prove a refined asymptotic behavior for the radial solution u_{β_0} and use the following Hardy–Rellich inequality with weights, see Corollary 5.4 in [1].

Lemma 1.2. *Let $N \geq 3$, $\Omega = \mathbb{R}^N \setminus B_1$, then the following inequality holds*

$$\int_\Omega |\Delta\phi|^2 \mathrm{d}x - \frac{N^2(N-4)^2}{16}\int_\Omega \frac{\phi^2}{|x|^4}\mathrm{d}x \geq \frac{N^2-4N+8}{8}\int_\Omega \frac{\phi^2}{|x|^4 \ln^2|x|}\mathrm{d}x + \frac{9}{16}\int_\Omega \frac{\phi^2}{|x|^4\ln^4|x|}\mathrm{d}x, \quad \forall\, \phi \in C_c^\infty(\Omega). \tag{1.5}$$

2. Proof of Theorem 1.1

Rewrite the equation (1.1) with the radial coordinate.

$$u^{(4)} + \frac{2(N-1)}{r}u''' + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = -u^{-p}.$$

Denote $\alpha := -m = \frac{4}{p+1}$. Without confusion, from now on we omit the index β_0 and fix $N = 3$, $p \in (1,3)$. Let $v(t) = r^{-\alpha}u - L_0$ with $t = \ln r$, then v satisfies

$$v^{(4)} + 2(2\alpha-1)v''' + (6\alpha^2-6\alpha-1)v'' + 2(2\alpha-1)(\alpha^2-\alpha-1)v' - (p+1)L_0^{-(p+1)}v + g(v) = 0, \tag{2.1}$$

where $g(v) = (v+L_0)^{-p} - L_0^{-p} + pL_0^{-(p+1)}v$. As $1 < p < 3$, by (1.3), we have $\lim_{t\to\infty} v(t) = 0$, so $g(v) = O(v^2)$ as $t \to \infty$.
The corresponding characteristic polynomial of equation (2.1) is

$$\lambda^4 + 2(2\alpha-1)\lambda^3 + (6\alpha^2-6\alpha-1)\lambda^2 + 2(2\alpha-1)(\alpha^2-\alpha-1)\lambda - (p+1)L_0^{-(p+1)} = 0.$$

Using MATLAB, we have the following four roots of the above polynomial:

$$\begin{cases} \lambda_1 = \frac{1}{2} - \alpha + \frac{1}{2}\sqrt{5 + 4\sqrt{h(p,\alpha)}}, \\ \lambda_2 = \frac{1}{2} - \alpha - \frac{1}{2}\sqrt{5 + 4\sqrt{h(p,\alpha)}}, \\ \lambda_3 = \frac{1}{2} - \alpha + \frac{1}{2}\sqrt{5 - 4\sqrt{h(p,\alpha)}}, \\ \lambda_4 = \frac{1}{2} - \alpha - \frac{1}{2}\sqrt{5 - 4\sqrt{h(p,\alpha)}}, \end{cases}$$

where $h(p,\alpha) = 1 + p\alpha(2-\alpha)(1+\alpha)(\alpha-1)$.
Recall that for $p = p_c$ or p_c^+, there holds $-pQ_4(-\alpha) = \frac{9}{16}$, i.e. $p\alpha(2-\alpha)(\alpha+1)(\alpha-1) = \frac{9}{16}$. Hence $h(p,\alpha) = \frac{25}{16}$ and

$$\lambda_1 = \frac{1}{2} - \alpha + \frac{1}{2}\sqrt{10}, \quad \lambda_2 = \frac{1}{2} - \alpha - \frac{1}{2}\sqrt{10}, \quad \lambda_3 = \lambda_4 = \frac{1-2\alpha}{2}, \quad \text{if } p = p_c \text{ or } p_c^+.$$

As $\alpha \in (1,2)$ for $1 < p < 3$, we see that $\lambda_1 > 0$, $\lambda_2 < \lambda_3 = \lambda_4 < 0$. By the variation of parameters method, the solution v to (2.1) is given by

$$
\begin{aligned}
v(t) &= \sum_{i=1}^{3} A_i e^{\lambda_i t} + \sum_{i=1}^{3} B_i \int_0^t e^{\lambda_i(t-s)} g(v(s))\,ds + A_4 t e^{\lambda_4 t} + B_4 \int_0^t (t-s) e^{\lambda_4(t-s)} g(v(s))\,ds \\
&= A_1' e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + A_3 e^{\lambda_3 t} + A_4 t e^{\lambda_4 t} - B_1 \int_t^\infty e^{\lambda_1(t-s)} g(v(s))\,ds \\
&\quad + \sum_{i=2}^{3} B_i \int_0^t e^{\lambda_i(t-s)} g(v(s))\,ds + B_4 \int_0^t (t-s) e^{\lambda_4(t-s)} g(v(s))\,ds
\end{aligned}
$$

where we used the fact that $e^{-\lambda_1 s} g(v(s)) \in L^1(\mathbb{R}_+)$. As $\lim_{t\to\infty} v(t) = 0$ and $\lambda_1 > 0$, there holds $A_1' = 0$. Therefore, for any $\epsilon \in (0, -\lambda_4)$, there exists $C_\epsilon > 0$ such that, for all $t \geq 0$,

$$
|v(t)| \leq C_\epsilon e^{(\lambda_4+\epsilon)t} + C_\epsilon \int_t^\infty e^{\lambda_1(t-s)} |g(v)(s)|\,ds + C_\epsilon \int_0^t e^{(\lambda_4+\epsilon)(t-s)} |g(v)(s)|\,ds.
$$

Moreover, for any $\delta > 0$, there exists $M > 0$ such that $|g(v)(s)| \leq \delta |v(s)|$ if $s \geq M$. Then for $t \geq M$,

$$
\begin{aligned}
|v(t)| &\leq O(e^{(\lambda_4+\epsilon)t}) + C_\epsilon \delta \int_t^\infty e^{\lambda_1(t-s)} |v(s)|\,ds + C_\epsilon \delta \int_M^t e^{(\lambda_4+\epsilon)t} |v(s)|\,ds, \\
&= O(e^{(\lambda_4+\epsilon)t}) + C_\epsilon \delta K_1(t) + C_\epsilon \delta K_2(t)
\end{aligned}
\tag{2.2}
$$

with

$$
K_1(t) := \int_M^t e^{(\lambda_4+\epsilon)(t-s)} |v(s)|\,ds, \quad K_2(t) := \int_t^\infty e^{\lambda_1(t-s)} |v(s)|\,ds.
$$

Thanks to (2.2), if we fix $\delta > 0$ small enough such that $2C_\epsilon \delta \leq \min(\lambda_1, -\lambda_4 - \epsilon)$, there holds

$$
\begin{aligned}
(K_1 - K_2)'(t) &= 2|v(t)| + (\lambda_4 + \epsilon) K_1(t) - \lambda_1 K_2(t) \\
&\leq 2C_\epsilon \delta (K_1 + K_2) + (\lambda_4 + \epsilon) K_1(t) - \lambda_1 K_2 + O(e^{(\lambda_4+\epsilon)t}) \\
&\leq O(e^{(\lambda_4+\epsilon)t}).
\end{aligned}
$$

Using again $\lim_{t\to\infty} v(t) = 0$, we have readily

$$
\lim_{t\to\infty} K_1(t) = \lim_{t\to\infty} K_2(t) = 0.
$$

Hence, $(K_2 - K_1)(t) \leq O(e^{(\lambda_4+\epsilon)t})$ as $t \to \infty$. Going back to (2.2),

$$
|v(t)| \leq O(e^{(\lambda_4+\epsilon)t}) + 2C_\epsilon \delta K_1(t). \tag{2.3}
$$

Consequently,

$$
K_1'(t) = |v(t)| + (\lambda_4 + \epsilon) K_1(t) \leq O(e^{(\lambda_4+\epsilon)t}) + (2C_\epsilon \delta + \lambda_4 + \epsilon) K_1(t).
$$

So $K_1(t) = O(e^{(\lambda_4+\epsilon+2C_\epsilon\delta)t})$; we get $|v(t)| = O(e^{(\lambda_4+\epsilon+2C_\epsilon\delta)t})$ by (2.3). Let $\sigma = -\lambda_4 - \epsilon - 2C_\epsilon \delta > 0$, we obtain

$$
u(r) = L_0 r^\alpha + r^\alpha O(r^{-\sigma}), \quad \text{as } r \to \infty. \tag{2.4}
$$

Finally, let $R > 0$ be large enough, we apply Lemma 1.2 with $N = 3$. Recall that $p = p_c$ or p_c^+, for any $\phi \in C_c^\infty(\mathbb{R}^3 \setminus B_R)$, we have then

$$\begin{aligned}
&\int_{\mathbb{R}^3\backslash B_R} |\Delta\phi|^2 \mathrm{d}x - p \int_{\mathbb{R}^3\backslash B_R} u^{-p-1}\phi^2 \mathrm{d}x \\
\geq &\int_{\mathbb{R}^3\backslash B_R} |\Delta\phi|^2 \mathrm{d}x - p \int_{\mathbb{R}^3\backslash B_R} r^{-4}\left[L_0^{-(p+1)} - O(r^{-\sigma})\right]\phi^2 \mathrm{d}x \\
= &\int_{\mathbb{R}^3\backslash B_R} |\Delta\phi|^2 \mathrm{d}x - pL_0^{-(p+1)} \int_{\mathbb{R}^3\backslash B_R} r^{-4}\phi^2 - \int_{\mathbb{R}^3\backslash B_R} r^{-4} O(r^{-\sigma})\phi^2 \mathrm{d}x \\
= &\int_{\mathbb{R}^3\backslash B_R} |\Delta\phi|^2 \mathrm{d}x + pQ_4\left(-\frac{4}{p+1}\right) \int_{\mathbb{R}^3\backslash B_R} r^{-4}\phi^2 - \int_{\mathbb{R}^3\backslash B_R} r^{-4} O(r^{-\sigma})\phi^2 \mathrm{d}x \\
= &\int_{\mathbb{R}^3\backslash B_R} |\Delta\phi|^2 \mathrm{d}x - \frac{9}{16} \int_{\mathbb{R}^3\backslash B_R} r^{-4}\phi^2 - \int_{\mathbb{R}^3\backslash B_R} O(r^{-4-\sigma})\phi^2 \mathrm{d}x \geq 0.
\end{aligned}$$

This implies that u is stable outside a compact set. The proof is completed. □

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