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 $L_p + L_q$ and $L_p \cap L_q$ are not isomorphic for all $1 \leq p, q \leq \infty$, $p \neq q$

 $L_p + L_q$ et $L_p \cap L_q$ ne sont pas isomorphes pour tout $1 \leq p, q \leq \infty$, $p \neq q$
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ABSTRACT

We prove that if $1 \leq p, q \leq \infty$, then the spaces $L_p + L_q$ and $L_p \cap L_q$ are isomorphic if and only if $p = q$. In particular, $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic, which is an answer to a question formulated in [2].

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R É S U M É

Nous prouvons que si $1 \leq p, q \leq \infty$, alors les espaces $L_p + L_q$ et $L_p \cap L_q$ sont isomorphes si et seulement si $p = q$. En particulier, $L_2 + L_\infty$ et $L_2 \cap L_\infty$ ne sont pas isomorphes, ce qui est une réponse à une question formulée dans [2].

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1. Preliminaries and main result

Isomorphic classification of symmetric spaces is an important problem related to the study of symmetric structures in arbitrary Banach spaces. A number of very interesting and deep results of such a sort is proved in the seminal work of Johnson, Maurey, Schechtman and Tzafriri [9]. In particular, in [9] (see also [12, Section 2.f]) it was shown that the space $L_2 \cap L_p$ for $2 \leq p < \infty$ (resp. $L_2 + L_p$ for $1 < p \leq 2$) is isomorphic to L_p . A further investigation of various properties of separable sums and intersections of L_p -spaces (i.e. with $p < \infty$) was continued by Dilworth in [6] and [7] and by Dilworth and Carothers in [5]. In contrast to that, in the paper [2] we proved that nonseparable spaces $L_p + L_\infty$ and $L_p \cap L_\infty$ for all $1 \leq p < \infty$ and $p \neq 2$ are not isomorphic. This question was left open for $p = 2$, and this was a motivation to continue this work. Here, we give a solution to this problem and, on the basis of the results of [9] and [2], we prove a more general theorem: $L_p + L_q$ and $L_p \cap L_q$ for all $1 \leq p, q \leq \infty$ are isomorphic if and only if $p = q$.

In this paper, we use the standard notation from the theory of symmetric spaces (cf. [3], [11] and [12]). Let $L_p(0, \infty)$ be the usual Lebesgue space of p -integrable functions $x(t)$ equipped with the norm

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$$\|x\|_{L_p} = \left(\int_0^\infty |x(t)|^p dt \right)^{1/p} \quad (1 \leq p < \infty)$$

and $\|x\|_{L_\infty} = \text{ess sup}_{t>0} |x(t)|$. For $1 \leq p, q \leq \infty$, the space $L_p + L_q$ consists of all sums of p -integrable and q -integrable measurable functions on $(0, \infty)$ with the norm defined by

$$\|x\|_{L_p+L_q} := \inf_{x(t)=u(t)+v(t), u \in L_p, v \in L_q} (\|u\|_{L_p} + \|v\|_{L_q}).$$

The space $L_p \cap L_q$ consists of all both p - and q -integrable functions on $(0, \infty)$ with the norm

$$\|x\|_{L_p \cap L_q} := \max \{ \|x\|_{L_p}, \|x\|_{L_q} \} = \max \left\{ \left(\int_0^\infty |x(t)|^p dt \right)^{1/p}, \left(\int_0^\infty |x(t)|^q dt \right)^{1/q} \right\}.$$

$L_p + L_q$ and $L_p \cap L_q$ for all $1 \leq p, q \leq \infty$ are symmetric Banach spaces (cf. [11, p. 94]). They are separable if and only if both p and q are finite (cf. [11, p. 79] for $p = 1$).

The norm in $L_p + L_q$ satisfies the following estimates

$$\left(\int_0^1 x^*(t)^p dt \right)^{1/p} + \left(\int_1^\infty x^*(t)^q dt \right)^{1/q} \leq \|x\|_{L_p+L_q} \leq C_{p,q} \left(\left(\int_0^1 x^*(t)^p dt \right)^{1/p} + \left(\int_1^\infty x^*(t)^q dt \right)^{1/q} \right)$$

if $1 \leq p < q < \infty$, and

$$\left(\int_0^1 x^*(t)^p dt \right)^{1/p} \leq \|x\|_{L_p+L_\infty} \leq C_p \left(\int_0^1 x^*(t)^p dt \right)^{1/p}$$

if $1 \leq p < \infty$ (cf. [4, p. 109], [8, Thm. 4.1] and [13, Example 1]). Here, $x^*(t)$ denotes the decreasing rearrangement of $|x(u)|$, that is,

$$x^*(t) = \inf \{ \tau > 0 : m(\{u > 0 : |x(u)| > \tau\}) < t \}$$

(if $E \subset \mathbb{R}$ is a measurable set, then $m(E)$ is its Lebesgue measure). Note that every measurable function and its decreasing rearrangement are equimeasurable, that is,

$$m(\{u > 0 : |x(u)| > \tau\}) = m(\{t > 0 : |x^*(t)| > \tau\})$$

for all $\tau > 0$.

Now, we state the main result of this paper.

Theorem 1. For every $1 \leq p, q \leq \infty$ the spaces $L_p + L_q$ and $L_p \cap L_q$ are isomorphic if and only if $p = q$.

If $\{x_n\}_{n=1}^\infty$ is a sequence from a Banach space X , by $[x_n]$ we denote its closed linear span in X . As usual, the Rademacher functions on $[0, 1]$ are defined as follows: $r_k(t) = \text{sign}(\sin 2^k \pi t)$, $k \in \mathbb{N}$, $t \in [0, 1]$.

2. $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic

Let x be a measurable function on $(0, \infty)$ such that $m(\text{supp } x) \leq 1$. Then, clearly, x is equimeasurable with the function $x^* \chi_{[0,1]}$. Therefore, assuming that $x \in L_2$ (resp. $x \in L_\infty$), we have $x \in L_2 + L_\infty$ and $\|x\|_{L_2+L_\infty} = \|x\|_{L_2}$ (resp. $x \in L_2 \cap L_\infty$ and $\|x\|_{L_2 \cap L_\infty} = \|x\|_{L_\infty}$).

Theorem 2. The spaces $L_2 + L_\infty$ and $L_2 \cap L_\infty$ are not isomorphic.

Proof. On the contrary, assume that T is an isomorphism of $L_2 + L_\infty$ onto $L_2 \cap L_\infty$.

For every $n, k \in \mathbb{N}$ and $i = 1, 2, \dots, 2^k$, we set

$$\Delta_{k,i}^n = (n - 1 + \frac{i - 1}{2^k}, n - 1 + \frac{i}{2^k}], u_{k,i}^n := \chi_{\Delta_{k,i}^n}, v_{k,i}^n := T(u_{k,i}^n).$$

Clearly, $\|u_{k,i}^n\|_{L_2+L_\infty} = 2^{-k/2}$. Therefore, if $x_{k,i}^n = 2^{k/2}u_{k,i}^n$, $y_{k,i}^n = 2^{k/2}v_{k,i}^n$, then $\|x_{k,i}^n\|_{L_2+L_\infty} = 1$ and

$$\|T^{-1}\|^{-1} \leq \|y_{k,i}^n\|_{L_2 \cap L_\infty} = \max(\|y_{k,i}^n\|_{L_2}, \|y_{k,i}^n\|_{L_\infty}) \leq \|T\| \tag{1}$$

for all $n, k \in \mathbb{N}, i = 1, 2, \dots, 2^k$.

At first, we suppose that, for each $k \in \mathbb{N}$, there are $n_k \in \mathbb{N}$ and $1 \leq i_k \leq 2^k$ such that

$$\|y_{k,i_k}^{n_k}\|_{L_2} \rightarrow 0 \text{ as } k \rightarrow \infty. \tag{2}$$

Denoting $\alpha_k := x_{k,i_k}^{n_k}$ and $\beta_k := y_{k,i_k}^{n_k}$, observe that $m(\bigcup_{k=1}^\infty \text{supp } \alpha_k) = 1$ and so the sequence $\{\alpha_k\}_{k=1}^\infty$ is isometrically equivalent in $L_2 + L_\infty$ to the unit vector basis of l_2 and $[\alpha_k]$ is a complemented subspace of $L_2 + L_\infty$. Then, since $\beta_k = T(\alpha_k), k = 1, 2, \dots$, the sequence $\{\beta_k\}_{k=1}^\infty$ is also equivalent in $L_2 \cap L_\infty$ to the unit vector basis of l_2 . Moreover, if P is a bounded projection from $L_2 + L_\infty$ onto $[\alpha_k]$, then the operator $Q := TPT^{-1}$ is the bounded projection from $L_2 \cap L_\infty$ onto $[\beta_k]$. Thus, the subspace $[\beta_k]$ is complemented in $L_2 \cap L_\infty$.

Now, let $\varepsilon_k > 0, k = 1, 2, \dots$ and $\sum_{k=1}^\infty \varepsilon_k < \infty$ (the choice of these numbers will be specified a little bit later). Thanks to (2), passing to a subsequence (and keeping the notation), we may assume that

$$\|\beta_k\|_{L_2} < \varepsilon_k \text{ and } m\{s > 0 : |\beta_k(s)| > \varepsilon_k\} < \varepsilon_k, k = 1, 2, \dots$$

(clearly, this subsequence preserves the above properties of the sequence $\{\beta_k\}$). Hence, denoting

$$A_k := \{s > 0 : |\beta_k(s)| > \varepsilon_k\} \text{ and } \gamma_k := \beta_k \chi_{A_k}, k = 1, 2, \dots,$$

we obtain

$$\|\beta_k - \gamma_k\|_{L_2 \cap L_\infty} \leq \max\{\|\beta_k \chi_{(0,\infty) \setminus A_k}\|_{L_\infty}, \|\beta_k\|_{L_2}\} \leq \varepsilon_k, k = 1, 2, \dots$$

Thus, choosing ε_k sufficiently small and taking into account inequalities (1), by the principle of small perturbations (cf. [1, Theorem 1.3.9]), we see that the sequences $\{\beta_k\}$ and $\{\gamma_k\}$ are equivalent in $L_2 \cap L_\infty$ and the subspace $[\gamma_k]$ is complemented (together with $[\beta_k]$) in the latter space.

Denote $A := \bigcup_{k=1}^\infty A_k$. We have $m(A) \leq \sum_{k=1}^\infty m(A_k) \leq \sum_{k=1}^\infty \varepsilon_k < \infty$ and hence the space

$$(L_2 \cap L_\infty)(A) := \{x \in L_2 \cap L_\infty : \text{supp } x \subset A\}$$

coincide with $L_\infty(A)$ (with equivalence of norms). As a result, $L_\infty(A)$ contains the complemented subspace $[\gamma_k]$, which is isomorphic to l_2 . Since this is a contradiction with [1, Theorem 5.6.5], our initial assumption on the existence of a sequence $\{y_{k,i_k}^{n_k}\}_{k=1}^\infty$ satisfying (2) fails.

Thus, there are $c > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|y_{k_0,i}^n\|_{L_2} \geq c \text{ for all } n \in \mathbb{N} \text{ and } i = 1, 2, \dots, 2^{k_0}.$$

Then, by the generalized Parallelogram Law (see [1, Proposition 6.2.9]), we have

$$\int_0^1 \left\| \sum_{i=1}^{2^{k_0}} r_i(s) y_{k_0,i}^n \right\|_{L_2}^2 ds = \sum_{i=1}^{2^{k_0}} \|y_{k_0,i}^n\|_{L_2}^2 \geq c^2 2^{k_0}, n \in \mathbb{N},$$

where $r_i = r_i(s)$ are the Rademacher functions. Hence, there exist $\theta_i^n = \pm 1, n = 1, 2, \dots, i = 1, 2, \dots, 2^{k_0}$ such that

$$\left\| \sum_{i=1}^{2^{k_0}} \theta_i^n y_{k_0,i}^n \right\|_{L_2} \geq c 2^{k_0/2}, n \in \mathbb{N}, \text{ or equivalently } \left\| \sum_{i=1}^{2^{k_0}} \theta_i^n v_{k_0,i}^n \right\|_{L_2} \geq c, n \in \mathbb{N}. \text{ So, setting}$$

$$f_n := \sum_{i=1}^{2^{k_0}} \theta_i^n u_{k_0,i}^n, g_n := \sum_{i=1}^{2^{k_0}} \theta_i^n v_{k_0,i}^n,$$

we have

$$\|f_n\|_{L_2+L_\infty} = 1 \text{ and } \|g_n\|_{L_2} \geq c, n = 1, 2, \dots \tag{3}$$

Moreover, by the definition of the norm in $L_2 + L_\infty$ and the fact that

$$\left| \sum_{n=1}^m f_n \right| = \left| \sum_{n=1}^m \sum_{i=1}^{2^{k_0}} \theta_i^n u_{k_0,i}^n \right| = \sum_{n=1}^m \sum_{i=1}^{2^{k_0}} \chi_{\Delta_{k_0,i}^n} = \chi_{(0,m)},$$

we obtain

$$\left\| \sum_{n=1}^m f_n \right\|_{L_2+L_\infty} = \|f_1\|_{L_2} = 1, \quad m = 1, 2, \dots \tag{4}$$

On the other hand, since $\{f_n\}$ is an 1-unconditional sequence in $L_2 + L_\infty$, for each $t \in [0, 1]$ we have

$$\left\| \sum_{n=1}^m f_n \right\|_{L_2+L_\infty}^2 = \left\| \sum_{n=1}^m r_n(t) f_n \right\|_{L_2+L_\infty}^2 \geq \frac{1}{\|T\|^2} \left\| \sum_{n=1}^m r_n(t) g_n \right\|_{L_2 \cap L_\infty}^2.$$

Integrating this inequality, by the generalized Parallelogram Law and (3), we obtain

$$\begin{aligned} \left\| \sum_{n=1}^m f_n \right\|_{L_2+L_\infty}^2 &\geq \frac{1}{\|T\|^2} \int_0^1 \left\| \sum_{n=1}^m r_n(t) g_n \right\|_{L_2 \cap L_\infty}^2 dt \geq \frac{1}{\|T\|^2} \int_0^1 \left\| \sum_{n=1}^m r_n(t) g_n \right\|_{L_2}^2 dt \\ &= \frac{1}{\|T\|^2} \sum_{n=1}^m \|g_n\|_{L_2}^2 \geq \left(\frac{c}{\|T\|} \right)^2 \cdot m, \quad m = 1, 2, \dots \end{aligned}$$

Since the latter inequality contradicts (4), the proof is completed. \square

Remark 1. Using the same arguments as in the proof of the above theorem, we can show that the spaces $L_p + L_\infty$ and $L_p \cap L_\infty$ are not isomorphic for every $1 \leq p < \infty$. This gives a new proof of Theorem 1 from [2]. However, note that in the latter paper (see Theorems 3 and 5), it is proved the stronger result, saying that the space $L_p \cap L_\infty$, $p \neq 2$, does not contain any complemented subspace isomorphic to $L_p(0, 1)$.

3. $L_p + L_q$ and $L_p \cap L_q$ are not isomorphic for $1 < p, q < \infty$, $p \neq q$

Both spaces $L_p + L_q$ and $L_p \cap L_q$ for all $1 \leq p, q \leq \infty$ are special cases of Orlicz spaces on $(0, \infty)$.

A function $M: [0, \infty) \rightarrow [0, \infty]$ is called a *Young function* (or *Orlicz function* if it is finite-valued) if M is convex, non-decreasing with $M(0) = 0$; we assume also that $\lim_{u \rightarrow 0^+} M(u) = M(0) = 0$ and $\lim_{u \rightarrow \infty} M(u) = \infty$.

The *Orlicz space* $L_M = L_M(I)$ with $I = (0, 1)$ or $I = (0, \infty)$ generated by the Young function M is defined as

$$L_M(I) = \{x \text{ measurable on } I: \rho_M(x/\lambda) < \infty \text{ for some } \lambda = \lambda(x) > 0\},$$

where $\rho_M(x) := \int_I M(|x(t)|) dt$. It is a Banach space with the *Luxemburg–Nakano norm*

$$\|x\|_{L_M} = \inf\{\lambda > 0: \rho_M(x/\lambda) \leq 1\}$$

and is a symmetric space on I (cf. [3], [10–15]). Special cases of Orlicz spaces on $I = (0, \infty)$ are the following (cf. [14, pp. 98–100]):

- (a) for $1 \leq p, q < \infty$, let $M(u) = \max(u^p, u^q)$, then $L_M = L_p \cap L_q$;
- (b) for $1 \leq p < \infty$, let

$$M(u) = \begin{cases} u^p & \text{if } 0 \leq u \leq 1, \\ \infty & \text{if } 1 < u < \infty, \end{cases} \text{ then } L_M = L_p \cap L_\infty;$$

(c) for $1 \leq p, q < \infty$, let $M(u) = \min(u^p, u^q)$, then M is not a convex function on $[0, \infty)$, but $M_0(u) = \int_0^u \frac{M(t)}{t} dt$ is convex and $M(u/2) \leq M_0(u) \leq M(u)$ for all $u > 0$, which gives $L_M = L_{M_0} = L_p + L_q$;

- (d) for $1 \leq p < \infty$, let

$$M(u) = \begin{cases} 0 & \text{if } 0 \leq u \leq 1, \\ u^p - 1 & \text{if } 1 < u < \infty, \end{cases} \text{ then } L_M = L_p + L_\infty.$$

A Young (Orlicz) function M satisfies the Δ_2 -condition if $0 < M(u) < \infty$ for $u > 0$ and there exists a constant $C \geq 1$ such that $M(2u) \leq CM(u)$ for all $u > 0$. An Orlicz space $L_M(0, \infty)$ is separable if and only if M satisfies the Δ_2 -condition (cf. [10, pp. 107–110], [14, Thm. 4.2 (b)], [15, p. 88]). With each Young function M one can associate another convex function M^* , i.e. the *complementary function* to M , which is defined by $M^*(v) = \sup_{u>0} [uv - M(u)]$ for $v \geq 0$. Then M^* is also a Young function and $M^{**} = M$. An Orlicz space $L_M(0, \infty)$ is reflexive if and only if M and M^* satisfy the Δ_2 -condition (cf. [14, Thm. 9.3], [15, p. 112]).

Theorem 3. Let M and N be two Orlicz functions on $[0, \infty)$ such that both spaces $L_M(0, \infty)$ and $L_N(0, \infty)$ are reflexive. Suppose that $L_M(0, \infty)$ and $L_N(0, \infty)$ are isomorphic. Then, the functions M and N are equivalent for $u \geq 1$, that is, there are constants $a, b > 0$ such that $aM(u) \leq N(u) \leq bM(u)$ for all $u \geq 1$.

Proof. If both functions M and N are equivalent to the function u^2 for $u \geq 1$, then nothing has to be proved. So, suppose that the function M is not equivalent to u^2 . Then, clearly, $L_M(0, 1)$ is a complemented subspace of $L_M(0, \infty)$ and $L_M(0, 1)$ is different from $L_2(0, 1)$, even up to an equivalent renorming. By hypothesis, $L_N(0, \infty)$ contains a complemented subspace isomorphic to $L_M(0, 1)$. Then, by [12, Corollary 2.e.14(ii)] (see also [9, Thm. 7.1]) $L_M(0, 1) = L_N(0, 1)$ up to equivalent norm. This implies that M and N are equivalent for $u \geq 1$ (cf. [10, Thm. 8.1], [14, Thm. 3.4]). \square

Corollary 1. Let $1 < p, q < \infty$, $p \neq q$, then $(L_p + L_q)(0, \infty)$ and $(L_p \cap L_q)(0, \infty)$ are not isomorphic.

Proof. For such p, q , the Orlicz spaces $(L_p + L_q)(0, \infty)$ and $(L_p \cap L_q)(0, \infty)$ are reflexive, and are generated by the Orlicz functions $M(u) = \min(u^p, u^q)$ and $N(u) = \max(u^p, u^q)$ respectively, which are not equivalent for $u \geq 1$ whenever $p \neq q$. Thus, by Theorem 3, these spaces cannot be isomorphic. \square

4. Proof of Theorem 1

Proof of Theorem 1. We consider four cases.

- (a) For $p \in [1, 2) \cup (2, \infty)$ and $q = \infty$, it was proved in [2, Theorem 1].
- (b) For $p = 2$ and $q = \infty$, it is proved in Theorem 2.
- (c) Let $p = 1$ and $1 < q < \infty$. If we assume that $L_1 + L_q$ and $L_1 \cap L_q$ are isomorphic, then the dual spaces will be also isomorphic. The dual spaces are $(L_1 + L_q)^* = L_{q'} \cap L_\infty$ and $(L_1 \cap L_q)^* = L_{q'} + L_\infty$, where $1/q + 1/q' = 1$. By (a) and (b), the spaces $L_{q'} + L_\infty$ and $L_{q'} \cap L_\infty$ are not isomorphic, thus their preduals cannot be isomorphic.
- (d) For $1 < p, q < \infty$, $p \neq q$, the desired result follows from Corollary 1, and the proof is completed. \square

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