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On the norms of quaternionic harmonic projection operators

Sur les normes des opérateurs de projection harmoniques sur la sphère dans l'espace quaternionique

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ABSTRACT

As a consequence of integral bounds for three classes of quaternionic spherical harmonics, we prove some bounds from below for the (L^p, L^2) norm of quaternionic harmonic projectors, for $p \in [1, 2]$.

RÉSUMÉ

En conséquence d'estimations intégrales pour trois classes d'harmoniques sphériques quaternioniques, nous prouvons quelques minorations pour la (L^p, L^2) norme des projecteurs harmoniques quaternioniques, pour $p \in [1, 2]$.

1. Introduction

In this note, we prove some bounds from below for the (L^p, L^2) norm of the quaternionic harmonic projectors $\pi_{\ell\ell'}$, which are the projection operators mapping the space of square integrable functions defined on the quaternionic unit sphere S^{4n-1} in $\mathbb{H}^n$ onto the subspace $\mathcal{H}^{\ell,\ell'}$, consisting of all quaternionic spherical harmonics of bidegree (ℓ, ℓ'). Here $\ell, \ell' \in \mathbb{N}$, $0 \leqslant \ell' \leqslant \ell$, and $p \in [1, 2]$.

Since the transposed operator $\pi^*_{\ell\ell'} : \mathcal{H}^{\ell\ell'} \to L^q(S^{4n-1})$ is the inclusion operator (here $1/p + 1/q = 1$), we have

$$\|\pi_{\ell\ell'}\|_{(p,2)} \geqslant \frac{\|Y_{\ell\ell'}\|_q}{\|Y_{\ell\ell'}\|_2}, \qquad q \geqslant 2, \; Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}. \tag{1.1}$$

Thus, to prove these inequalities, we are led to study the L^q norms of the functions $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, for $q \geqslant 2$. Our estimates are therefore related to the problem of size concentration of the bigraded spherical harmonics. In the real and complex context,

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where the analogous question has been largely investigated (see [11,12] and [4,5]), it is fully understood that two classes of spherical harmonics with competing behaviours, the highest-weight vectors and the zonal functions, play a prominent role in the analysis of the harmonic projectors and also in some related applications (see, e.g., [2,3,7]).

The quaternionic framework turns out to be more interesting: indeed, we identify three classes of spherical harmonics with competing behaviours, giving rise, in the light of (1.1), to different bounds from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ on three subintervals of $p \in [1,2]$. More precisely, for p close to 1, like in the real and complex framework [11,4,5], the estimates for $\|\pi_{\ell\ell'}\|_{(p,2)}$ turn out to be sensitive to a high pointwise concentration. Thus, we obtain bounds from below by considering the quaternionic zonal functions $\mathbb{Z}_{\ell\ell'}$, which are highly concentrated at the North Pole. When p is close to 2, the estimates are more sensitive to a sparse concentration along the Equator; in this case, we prove our bounds by considering the highest-weight spherical harmonics, since these functions spread out in a small neighborhood around the Equator.

Anyway, in a third interval inside $[1,2]$, more precisely when $p \in (4/3, 2(4n-3)/(4n-1))$, the dichotomy between zonal and highest-weight harmonics is partially mitigated; we obtain indeed better bounds from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, by considering a third class of spherical harmonics. We refer to Section 3 for a discussion about these elements of $\mathcal{H}^{\ell\ell'}$, which have no analogous in the real or complex case and are related to representation-theoretic questions on S^{4n-1}.

Finally, in the light of these bounds for the spherical harmonics, in Section 4 we are able to prove $L^p - L^2$ bounds from below for $\pi_{\ell\ell'}$. The proof of the same bounds from above is already under way.

2. Notation and preliminaries

We denote by $\mathbb{H}$ the skew field of all quaternions $q = x_0 + x_1 i + x_2 j + x_3 k$ over $\mathbb{R}$, where x_0, x_1, x_2, x_3 are real numbers and the imaginary units i, j, k satisfy $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $ik = -ki = -j$, $jk = -kj = i$. The conjugate $\overline{q}$ and the modulus $|q|$ are defined by $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and $|q|^2 = q\overline{q} = \sum_{j=0}^{3} x_j^2$, respectively. For $n \geqslant 1$, the symbol $\mathbb{H}^n$ will denote the n-dimensional vector space over $\mathbb{H}$. By abuse of notation, we write q also to denote $(q_1, \ldots, q_n) \in \mathbb{H}^n$. Sometimes we will adopt a complex notation, writing $q = (z_1 + jz_{n+1}, \ldots, z_n + jz_{2n})$, with $z_1, \ldots, z_{2n} \in \mathbb{C}$.

S^{4n-1} is the unit sphere in $\mathbb{H}^n$, that is,

$$S^{4n-1} = \{q = (q_1, \ldots, q_n) \in \mathbb{H}^n : \langle q, q \rangle = 1\};$$

here the inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{H}^n$ is defined as $\langle q, q' \rangle = q_1 \overline{q_1'} + \ldots + q_n \overline{q_n'}$, $q, q' \in \mathbb{H}^n$. S^{4n-1} may be identified with K/M, where $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$ and $M = \mathrm{Sp}(n-1) \times \mathrm{Sp}(1)$, $\mathrm{Sp}(n)$ denoting the group of $n \times n$ matrices A with quaternionic entries, such that $\overline{A^{\mathsf{T}}} A = A \overline{A^{\mathsf{T}}} = I_n$. We introduce on S^{4n-1} the coordinate system

$$\begin{cases} q_1 = \cos\theta\,(\cos t + \tilde{q} \sin t) \\ q_s = \sigma_s \sin\theta\,, \qquad s = 2, \ldots, n, \end{cases} \tag{2.1}$$

where $\theta \in [0, \pi/2]$, $t \in [0, \pi]$, $\sigma_s \in \mathbb{H}$ with $\sum_{s=2}^{n} |\sigma_s|^2 = 1$. Moreover, $\tilde{q} \in \mathbb{H}$ with $|\tilde{q}|^2 = 1$ and $\Re\tilde{q} = 0$; we will write $\tilde{q} = \cos\psi\, i + \sin\psi \cos\varphi\, j + \sin\psi\, \sin\varphi\, k$, with $\psi \in [0, \pi]$ and $\varphi \in [0, 2\pi]$. We remark that $(\sin t \sin\psi\, \sin\varphi, \sin t \sin\psi \cos\varphi, \sin t \cos\psi, \cos t)$ yields a coordinate system for $\mathrm{Sp}(1)$.

The normalized invariant measure $\mathrm{d}\sigma = \mathrm{d}\sigma_{S^{4n-1}}$ on S^{4n-1} with respect to the spherical coordinates (2.1) is, up to a constant $C = C(n)$,

$$\sin^{4n-5}\theta \cos^3\theta\, \mathrm{d}\theta\, \sin^2 t\, \mathrm{d}t\, \mathrm{d}\sigma_{S^{4n-5}}\, \mathrm{d}\sigma(\tilde{q})\,, \tag{2.2}$$

$\mathrm{d}\sigma(\tilde{q})$ denoting the measure on the unit sphere in $\mathbb{R}^3$.

By $L^2(S^{4n-1})$, we denote the Hilbert space of square integrable functions on S^{4n-1}, with respect to the inner product

$$(f, g)_{L^2} = \int_{S^{4n-1}} f(q)\, \overline{g(q)}\, \mathrm{d}\sigma\,.$$

Johnson and Wallach, starting from some earlier work by Kostant [10], proved in [9] that this space may be decomposed as

$$L^2(S^{4n-1}) = \bigoplus_{\ell \geqslant \ell' \geqslant 0} \mathcal{H}^{\ell\ell'}\,, \tag{2.3}$$

where each subspace $\mathcal{H}^{\ell\ell'}$

(1) is irreducible under K;
(2) is generated under K by the "highest-weight vector"

$$\mathrm{P}_{\ell,\ell'}(z, \bar{z}) = \bar{z}_{n+1}^{\ell-\ell'} (z_1 \bar{z}_{n+2} - z_2 \bar{z}_{n+1})^{\ell'}\,; \tag{2.4}$$

(3) is finite dimensional.

In the following, we shall use the symbols c and C with $0 < c, C < \infty$ to denote constants that are not necessarily equal at different occurrences. They depend only on the dimension n and on the Lebesgue indices p or q. The symbol $\simeq$ between two positive expressions means that their ratio is bounded above and below by such constants. For two positive quantities a and b, we write $a \lesssim b$ instead of $a \leqslant Cb$ and $a \gtrsim b$ for $b \lesssim a$.

Finally, we will denote by $I_{\mathbb{S}}$ the set of indices $\{(\ell, \ell') \in \mathbb{N} \times \mathbb{N} : 0 \leqslant \ell' \leqslant \ell\}$.

3. The main estimates

In [6], we started studying the $L^p - L^2$ norm of the joint spectral projectors $\pi_{\ell\ell'}$, $(\ell, \ell') \in I_{\mathbb{S}}$, mapping $L^p(S^{4n-1})$ onto $\mathcal{H}^{\ell\ell'}$, $1 \leqslant p \leqslant 2$. We proved sharp bounds for these norms under the additional assumptions $\ell - \ell' \leqslant c_0$ or $\ell' \leqslant c_1$, for some positive constants c_0, c_1. In this note, we prove some crucial estimates from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$ in the general case. As illustrated in the Introduction, we are led to study the L^q norms of the eigenfunctions $Y_{\ell\ell'} \in \mathcal{H}^{\ell\ell'}$, for $q \geqslant 2$.

Estimates for zonal functions. We call *zonal function of bidegree* (ℓ, ℓ') with pole $e_1 = (1, 0 \dots, 0)$ a M-invariant function in $\mathcal{H}^{\ell\ell'}$. An explicit formula for the zonal function $\mathrm{Z}_{\ell\ell'}$ with pole e_1 is given for all $(\ell, \ell') \in I_{\mathbb{S}}$ by

$$\mathrm{Z}_{\ell\ell'}(\theta, t) = \frac{d_{\ell\ell'}}{\omega_{4n-1}} \frac{\sin((\ell - \ell' + 1)t)}{(\ell - \ell' + 1)\sin t} (\cos\theta)^{\ell-\ell'} \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}, \tag{3.1}$$

where $t \in [0, \pi]$, $\theta \in [0, \frac{\pi}{2}]$, ω_{4n-1} denotes the surface area of S^{4n-1}, $P_{\ell'}^{(2n-3,\ell-\ell'+1)}$ is the Jacobi polynomial and $d_{\ell\ell'}$ is the dimension of $\mathcal{H}^{\ell\ell'}$, given by

$$d_{\ell\ell'} = (\ell + \ell' + 2n - 1)(\ell - \ell' + 1)^2 \frac{(\ell + 2n - 2)!}{(\ell + 1)!(2n - 3)!} \frac{(\ell' + 2n - 3)!}{\ell'!(2n - 1)!}, \quad \ell \geqslant \ell' \geqslant 0. \tag{3.2}$$

We recall the Mehler–Heine formula for the so-called disk polynomials, proved in [1, p. 10]. The symbol J_α denotes the Bessel function of the first kind of order α.

Proposition 3.1. *Fix $n \in \mathbb{N}$. Let $j, k \in \mathbb{N}$, $j \leqslant k$. Then*

$$\lim_{\substack{j \to +\infty \\ k \to +\infty}} \left(\cos\left(\frac{\theta}{\sqrt{jk}}\right)\right)^{k-j} \frac{P_j^{(2n-3,k-j)}\left(\cos\left(\frac{2\theta}{\sqrt{jk}}\right)\right)}{P_j^{(2n-3,k-j)}(1)} = \Gamma(2n-2) \frac{J_{2n-3}(2\theta)}{\theta^{2n-3}}.$$

This limit holds uniformly in every compact interval.

We also recall (see [1, p. 12]) that, for all $j, k \in \mathbb{N}$, $j \leqslant k$,

$$\sup_{\theta \in [0,\pi/2]} \left| (\cos\theta)^{k-j} \frac{P_j^{(2n-3,k-j)}(\cos(2\theta))}{P_j^{(2n-3,k-j)}(1)} \right| \leqslant 1. \tag{3.3}$$

For $q \geqslant 2$ set

$$\mathcal{I}_q = \left(\int_0^{\pi/2} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \right|^q (\sin\theta)^{4n-5} (\cos\theta)^3 \, \mathrm{d}\theta \right)^{1/q}. \tag{3.4}$$

Lemma 3.2. *For all $q \geqslant 2$ and for all $(\ell, \ell') \in I_{\mathbb{S}}$ such that ℓ' is sufficiently great, we have*

$$\frac{\mathcal{I}_q}{\mathcal{I}_2} \gtrsim (\ell')^{(2n-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \ell^{(2n-2)(\frac{1}{2}-\frac{1}{q})} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos\left(\frac{2\theta}{\sqrt{\ell\ell'}}\right)\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q([0,1];\, \theta^{4n-5}\mathrm{d}\theta)}.$$

Proof. Observe that

$$(\mathcal{I}_q)^q \gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \right|^q (\sin\theta)^{4n-5} (\cos\theta)^3 \, \mathrm{d}\theta$$

$$= \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'+\frac{3}{q}} \right|^q (\sin\theta)^{4n-5}\, \mathrm{d}\theta$$

$$\gtrsim \int_0^{1/\sqrt{\ell\ell'}} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'+1} \right|^q (\sin\theta)^{4n-5}\, \mathrm{d}\theta,$$

where the last inequality follows from the fact that $\theta \in (0, 1/\sqrt{\ell\ell'})$. Then, after a change of variables, we get

$$(\mathcal{I}_q)^q \gtrsim \int_0^1 \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right|^q (\sin(\theta/\sqrt{\ell\ell'}))^{4n-5} \frac{\mathrm{d}\theta}{\sqrt{\ell\ell'}}$$

$$\simeq \int_0^1 \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos(2\theta/\sqrt{\ell\ell'}))}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right|^q (\theta/\sqrt{\ell\ell'})^{4n-5}\, \mathrm{d}\theta/(\sqrt{\ell\ell'})$$

$$\simeq (\ell\ell')^{-(2n-2)} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|^q_{L^q([0,1];\,\theta^{4n-5}\mathrm{d}\theta)}. \tag{3.5}$$

For $q = 2$, we obtain a more precise estimate. Indeed, from standard properties of zonal harmonics, it follows that $||Z_{\ell\ell'}||_2 \simeq (d_{\ell\ell'})^{1/2}$, that is, by means of (3.1),

$$d_{\ell\ell'} \simeq (d_{\ell\ell'})^2 \int_0^{\pi} \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^2 \sin^2 t\, \mathrm{d}t$$

$$\times \int_0^{\pi/2} \left| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos\theta)^{\ell-\ell'} \right|^2 (\sin\theta)^{4n-5} (\cos\theta)^3\, \mathrm{d}\theta.$$

Since

$$\int_0^{\pi} \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^2 \sin^2 t\, \mathrm{d}t \simeq (\ell-\ell'+1)^{-2}, \tag{3.6}$$

we have

$$(\mathcal{I}_2)^2 \simeq (\ell-\ell'+1)^2 (d_{\ell\ell'})^{-1}. \tag{3.7}$$

Then, combining (3.5) and (3.7), we get, for all $q > 2$

$$\frac{\mathcal{I}_q}{\mathcal{I}_2} \gtrsim (\ell-\ell'+1)^{-1} (d_{\ell\ell'})^{1/2} (\ell\ell')^{-(2n-2)/q} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q([0,1];\,\theta^{4n-5}\mathrm{d}\theta)}$$

$$\gtrsim (\ell')^{(2n-3)/2} \ell^{(2n-2)/2} (\ell\ell')^{-(2n-2)/q} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q([0,1];\,\theta^{4n-5}\mathrm{d}\theta)}$$

$$\gtrsim (\ell')^{(2n-2)(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}} \ell^{(2n-2)(\frac{1}{2}-\frac{1}{q})} \left\| \frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}\left(\cos(\frac{2\theta}{\sqrt{\ell\ell'}})\right)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q([0,1];\,\theta^{4n-5}\mathrm{d}\theta)}. \quad \square$$

Then, for $q \geqslant 2$ set

$$\mathcal{J}_q = \left(\int_0^{\pi} \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^q \sin^2 t\, \mathrm{d}t \right)^{1/q}. \tag{3.8}$$

Lemma 3.3. *For all $q \geqslant 2$ and for all $(\ell, \ell') \in I_{\mathbb{S}}$ such that $\ell - \ell'$ is sufficiently great, we have:*

$$\frac{\mathcal{J}_q}{\mathcal{J}_2} \simeq \begin{cases} (\ell-\ell'+1)^{1-3/q} & \text{for all } q > 3 \\ (\log(\ell-\ell'))^{1/3} & \text{for } q = 3 \\ 1 & \text{for all } q < 3. \end{cases}$$

Proof. We start recalling that

$$\frac{\sin((\ell-\ell'+1)t)}{\sin t} = O((\ell-\ell'+1)^{1/2}) P^{(\frac{1}{2},\frac{1}{2})}_{\ell-\ell'}(\cos t),$$

[13, p. 60]. Thus, using some asymptotic integral estimates in [13, p. 391], we see that

$$(\mathcal{J}_q)^q \simeq \int_0^{\pi/2} \left| \frac{\sin((\ell-\ell'+1)t)}{(\ell-\ell'+1)\sin t} \right|^q \sin^2 t \, dt \simeq (\ell-\ell'+1)^{-3}, \tag{3.9}$$

for $q>3$ and $\ell-\ell'$ sufficiently great. Combining (3.6) and (3.9), we get the expected estimate for $\mathcal{J}_q/\mathcal{J}_2$ for all $q>3$. The other two cases analogously follow from [13, p. 391], and (3.6). □

Combining Lemma 3.2 and Lemma 3.3 gives a bound from below for $\|\pi_{\ell\ell'}\|_{(p,2)}$, with $1 \leqslant p \leqslant 2$.

Proposition 3.4. *Fix $n \geqslant 2$. For all $(\ell,\ell') \in I_{\mathbb{S}}$ such that ℓ' and $\ell-\ell'$ are sufficiently great, and for all $q \geqslant 2$ we have*

$$\frac{\|Z_{\ell\ell'}\|_q}{\|Z_{\ell\ell'}\|_2} \gtrsim \begin{cases} (\ell-\ell'+1)^{1-3/q} (\ell\ell')^{(2n-2)(1/2-1/q)} \ell'^{-1/2} & \text{for all } q>3 \\ (\log(\ell-\ell'))^{1/3} (\ell\ell')^{(2n-2)(1/2-1/q)} \ell'^{-1/2} & \text{for } q=3 \\ (\ell\ell')^{(2n-2)(1/2-1/q)} \ell'^{-1/2} & \text{for all } q<3. \end{cases} \tag{3.10}$$

Proof. As a consequence of Lemma 3.2 for $q>3$, we have:

$$\begin{aligned} \frac{\|Z_{\ell\ell'}\|_q}{\|Z_{\ell\ell'}\|_2} &\gtrsim (\ell-\ell'+1)^{1-3/q} \mathcal{I}_q/\mathcal{I}_2 \\ &\simeq (\ell-\ell'+1)^{1-3/q} (\ell\ell')^{(2n-2)(1/2-1/q)} (\ell')^{-1/2} \\ &\quad \times \left\| \frac{P^{(2n-3,\ell-\ell'+1)}_{\ell'}(\cos(2\theta/\sqrt{\ell\ell'}))}{P^{(2n-3,\ell-\ell'+1)}_{\ell'}(1)} (\cos(\theta/\sqrt{\ell\ell'}))^{\ell-\ell'+1} \right\|_{L^q(\theta^{4n-5}d\theta,[0,1])}. \end{aligned}$$

Then the first inequality in (3.10) follows from a slight variation of Proposition 3.1, (3.3) and some trivial asymptotics for the Bessel function. The proof of the other two inequalities is similar. □

Estimates for the highest-weight spherical harmonics. We will estimate the norm of the highest-weight spherical harmonics $P_{\ell,\ell'}$ in $\mathcal{H}^{\ell\ell'}$, defined in (2.4).

In [6, Lemma 5.3] we proved that for all $\zeta_1 \in \mathbb{R}$, $\zeta_1 > 0$, and for all $\zeta_2 \in \mathbb{N}$ one has

$$\int_{S^{4n-1}} |\bar{z}_{n+1}|^{2\zeta_1} |z_1\bar{z}_{n+2} - z_2\bar{z}_{n+1}|^{2\zeta_2} d\sigma = \frac{c_n \, \Gamma(\zeta_1+\zeta_2+2)\Gamma(\zeta_2+1)}{\Gamma(\zeta_1+2\zeta_2+2n)\,(\zeta_1+1)}. \tag{3.11}$$

We also proved that, as a consequence of (3.11), the following bound holds

$$\|P_{\ell,\ell'}\|_2 \simeq \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)} \right)^{\frac{1}{2}}. \tag{3.12}$$

Proposition 3.5. *Let $P_{\ell\ell'}$ be the highest-weight vector defined by (2.4). For all $q \geqslant 2$, we have:*

$$\limsup_{\ell'\to+\infty} \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)} \right)^{\frac{1}{2}-\frac{1}{q}} \frac{\|P_{\ell,\ell'}\|_q}{\|P_{\ell,\ell'}\|_2} > 0. \tag{3.13}$$

Proof. Fix any $q \geqslant 2$ and let $(\ell,\ell') \in I_{\mathbb{S}}$. First of all, we choose $2\zeta_1 = (\ell-\ell')q$. Then, if $\ell' q \in 2\mathbb{N}$, (3.11) applied to $P_{\ell\ell'}$ with $2\zeta_2 = \ell' q$ yields:

$$\|P_{\ell,\ell'}\|_q^q = \frac{c_n \, \Gamma(\frac{q}{2}\ell+2)\,\Gamma(\frac{q}{2}\ell'+1)}{\Gamma(\frac{q}{2}(\ell+\ell')+2n)\,\left(\frac{q}{2}(\ell-\ell')+1\right)}.$$

Then a standard application of Stirling's estimate leads to

$$\|P_{\ell,\ell'}\|_q \simeq \frac{(\frac{q}{2}\ell+1)^{\frac{1}{2}\ell+(1+\frac{1}{2})/q} (\frac{q}{2}\ell'+1)^{\frac{1}{2}\ell'+1/(2q)}}{(\frac{q}{2}(\ell+\ell')+2n-1)^{\frac{1}{2}(\ell+\ell')+(2n-1+\frac{1}{2})/q} \left(\frac{q}{2}(\ell-\ell')+1\right)^{1/q}},$$

which, combined with (3.12), yields:

$$\frac{\|\mathrm{P}_{\ell,\ell'}\|_q}{\|\mathrm{P}_{\ell,\ell'}\|_2} \simeq \left(\frac{(\ell'+1)^{\frac{1}{2}}}{(\ell+\ell')^{2n-2}\,(\ell-\ell'+1)}\right)^{\frac{1}{q}-\frac{1}{2}}. \tag{3.14}$$

This proves the assertion under the assumption $\ell' q \in 2\mathbb{N}$.

If $q = \frac{m_0}{n_0}$, for some $m_0, n_0 \in \mathbb{N}^*$, it suffices to replace ℓ' with $2n_0\ell'$ and then choose $\zeta_2 = m_0\ell'$. By considering $(\ell, \ell') \in I_{\mathbb{S}}$ such that $\ell \geqslant 2n_0\ell'$, we get an estimate analogous to (3.14) for $\|\mathrm{P}_{\ell,2n_0\ell'}\|_q$, yielding (3.13).

Finally, if q is not rational, the desired estimate follows from the continuity of the L^q norms and the previous arguments for rational values of q. □

Estimates for mixed spherical harmonics. We consider the function $\mathrm{Q}_{\ell\ell'}$, given by

$$\mathrm{Q}_{\ell\ell'}(\theta,\varphi,t) = \left(\sin t\,\sin\psi\,\mathrm{e}^{\mathrm{i}\varphi}\right)^{\ell-\ell'}(\cos\theta)^{\ell-\ell'}\frac{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(\cos 2\theta)}{P_{\ell'}^{(2n-3,\ell-\ell'+1)}(1)}, \tag{3.15}$$

for all $(\ell, \ell') \in I_{\mathbb{S}}$, with $t, \psi \in [0, \pi]$, $\varphi \in [0, 2\pi]$, $\theta \in [0, \frac{\pi}{2}]$. Observe that $\mathrm{Q}_{\ell\ell'}$ is obtained replacing the factor $\sin((\ell-\ell'+1)t)/\big((\ell-\ell'+1)\sin t\big)$ in (3.1) with the highest-weight spherical harmonic of degree $\ell-\ell'$ in Σ^3, the unit sphere in $\mathbb{R}^4$. For a discussion about the role of Σ^3 (or, equivalently, of $\mathrm{Sp}(1)$) in our analysis, we refer the reader to [6, Remark 2.3].

We only recall here that $\mathcal{H}^{\ell\ell'}$ is a joint eigenspace for the spherical Laplacian $\Delta_{S^{4n-1}}$ and for an operator Γ, which essentially coincides with the Casimir operator on $\mathrm{Sp}(1)$ and, in our coordinates, reads as

$$\Gamma = \frac{1}{\sin^2 t}\frac{\partial}{\partial t}\sin^2 t\frac{\partial}{\partial t} + \frac{1}{\sin^2 t\,\sin\psi}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi} + \frac{1}{\sin^2 t}\frac{1}{\sin^2\psi}\frac{\partial^2}{\partial^2\varphi}.$$

We refer to [9] and [8, p. 696] for a discussion about the role of this operator. Then it is easily seen that $\mathrm{Q}_{\ell\ell'}$ belongs to $\mathcal{H}^{\ell\ell'}$, since it is an eigenvector both for $\Delta_{S^{4n-1}}$ and for Γ.

Proposition 3.6. *Fix $n \geqslant 2$. For all $(\ell, \ell') \in I_{\mathbb{S}}$, such that ℓ' and $\ell-\ell'$ are sufficiently great, and for all $q > 2$ we have:*

$$\frac{||\mathrm{Q}_{\ell\ell'}||_q}{||\mathrm{Q}_{\ell\ell'}||_2} \gtrsim (\ell-\ell'+1)^{1/2-1/q}(\ell\ell')^{(2n-2)(1/2-1/q)}\ell'^{-1/2}.$$

Proof. It follows from Lemma 3.2, Proposition 3.1 and some basic estimates for the spherical harmonics in Σ^3 (see [11, Theorem 4.1]). □

4. Bounding the harmonic projections

A comparison between Proposition 3.4, Proposition 3.5, and Proposition 3.6 leads to the following estimate.

Proposition 4.1. *Let $n \geqslant 2$, $1 \leqslant p \leqslant 2$. Set $p_n = 2(4n-3)/(4n-1)$. Then there exists some constant C, only depending on n and p, such that the following estimate holds*

$$||\pi_{\ell\ell'} f||_2 \geqslant C(n,p)\,(1+\ell)^{\alpha(\frac{1}{p},n)}(1+\ell')^{\beta(\frac{1}{p},n)}(\ell-\ell'+1)^{\gamma(\frac{1}{p},n)}\,||f||_p\,, \tag{4.1}$$

where

$$\alpha\Big(\frac{1}{p},n\Big) := 2(n-1)\Big(\frac{1}{p}-\frac{1}{2}\Big) \quad \text{for all } 1 \leqslant p \leqslant 2,$$

$$\beta\Big(\frac{1}{p},n\Big) := \begin{cases} 2(n-1)\Big(\frac{1}{p}-\frac{1}{2}\Big)-\frac{1}{2} & \text{if } 1 \leqslant p \leqslant p_n \\ \frac{1}{2}\big(\frac{1}{2}-\frac{1}{p}\big) & \text{if } p_n \leqslant p \leqslant 2, \end{cases}$$

and

$$\gamma\Big(\frac{1}{p},n\Big) := \begin{cases} 3\big(\frac{1}{p}-\frac{1}{2}\big)-\frac{1}{2} & \text{if } 1 \leqslant p \leqslant \frac{4}{3} \\ \frac{1}{p}-\frac{1}{2} & \text{if } \frac{4}{3} \leqslant p \leqslant 2, \end{cases}$$

for all $(\ell, \ell') \in I_{\mathbb{S}}$, such that $\ell-\ell'$ and ℓ' are sufficiently great.

The proof of (4.1) from above, which involves both real and analytic interpolation arguments, multiplier theorems for $\Delta_{S^{4n-1}}$, Γ and for $\mathcal{L}$, and a very detailed analysis of the Jacobi polynomials, is quite long and tangled. This work is already under way.

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